Theoretical and Mathematical Physics

Gerd Rudolph
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Differential Geometry and Mathematical Physics

Part II. Fibre Bundles, Topology and Gauge Fields

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Contents

Introduction

This is the second part of our book on Differential Geometry and Mathematical Physics. It is based on our teaching of these subjects at the University of Leipzig to students of physics and of mathematics and on our research in gauge field theory over many years.

As in Part I, let us start with some historical remarks. The concept of gauge invariance first appeared in the famous papers [660] and [661] of Hermann Weyl from the year 1918.[1] In this work, Weyl extended Einstein's principle of general relativity by postulating that, additionally, the scale of length can vary smoothly from point to point in spacetime. In more detail, Weyl's basic idea was to develop a purely infinitesimal geometry. Behind that concept was his belief that 'a true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point …', see page 25 in [660]. In this context, the notion of connection appeared for the first time in the mathematical literature.[2] In a modern geometric language, he was led to a generalization of Riemannian geometry characterized by a pair consisting of a conformal Riemannian structure and a connection in a line bundle over spacetime. Weyl proposed to identify the connection form with the electromagnetic gauge potential and, consequently, its curvature with the electromagnetic field tensor. Thus, he obtained a unification of general relativity with electromagnetism. However, it quickly became clear that this model was not compatible with basic physical principles. It was Einstein who observed that if this theory was correct, then the behaviour of clocks would depend on their history. This is in contradiction with empirical evidence.[3] Although this model did not survive, the gauge principle did though. In 1929 Weyl proposed to apply it to quantum mechanics. He recognized

[1] In these papers, the term 'gauge invariance' appears in German as 'Maßstab-Invarianz'.

[2] Of course, there were predecessors, notably Christoffel, Ricci and Levi-Civita. The latter had a clear mathematical understanding of parallel transport and of the covariant derivative operator, but up to our knowledge, he did not invent the term 'connection'.

[3] See the postscript by Einstein in [660] and the author's reply. This started a long discussion between Weyl and Einstein. For further reference, see also [604] and [496].

that it is the phase of the Schrödinger wave function which should be gauged, see [663]. In more detail, the idea of Weyl was as follows: since only the absolute value of the wave function has a physical interpretation, the wave function itself may be multiplied by an arbitrary point-dependent phase factor.[4] However, the transformed wave function obviously does not satisfy the Schrödinger equation any more. In order to restore invariance, Weyl proposed to replace the partial derivatives with respect to space and time coordinates occurring in the Schrödinger equation by the covariant derivatives obtained by adding to the partial derivatives the components of the electromagnetic potential. This modified Schrödinger equation is invariant under simultaneous gauge transformations of the wave function and of the electromagnetic potential. This way, the first quantum mechanical model of a U(1)-gauge theory was born.

The combination of this U(1)-gauge principle with the quantum theory of fields led to Quantum Electrodynamics (QED). For an exhaustive historical introduction to that theory we refer to Volume I of [654], see Sect. 1.2. The early contributions to the development of QED date back to the late 1920s and are due to Dirac [152], Weisskopf and Wigner [658], Jordan and Pauli [350], Jordan and Wigner [351] and Heisenberg and Pauli [292]. In the 1930s, QED was studied intensively leading to a further development of the formalism as well as to successful applications. This period culminated in the famous Solvay report by Pauli in 1939, see [505]. Clearly, the biggest puzzle was the emergence of infinities in various kinds of calculations. Amongst a number of approaches to tackle this problem, in the end, the concept of renormalization of the parameters of the theory became the widely accepted strategy. In this spirit, in the late 1940s, Schwinger [579], Tomonaga [629] and Feynman [194] brought QED to its final manifestly relativistic form.[5]

The first non-Abelian gauge theory was proposed by Yang and Mills in 1954, see [685].[6] Their work was based on the idea that the forces between the nucleons were mediated by the exchange of pions and that the interaction was invariant under the isospin group SU(2). In this model, the proton and the neutron form an isospin doublet and the three charged states of the pion form a triplet in the adjoint representation. Yang and Mills postulated the principle of local isotopic gauge invariance. As a consequence, they were led to introduce an SU(2)-gauge potential. They found the field equations of this system, proposed a generalization of the Lorenz gauge fixing condition and made preliminary remarks on the quantum theory of their model. The paper by Yang and Mills dealt with the special gauge group SU(2) only, but from their presentation it was clear how to generalize the

[4]In the group theoretical language, such a transformation is given by a function on spacetime with values in the Abelian group U(1).

[5]For this work, Feynman, Schwinger and Tomonaga received the Nobel Prize in Physics in 1965. Initially, Feynman's diagrammatic technique seemed quite different from the operator-based approach of Schwinger and Tomonaga, but Dyson [171] showed that the two approaches were equivalent.

[6]There was an earlier paper by Klein [378] written in the spirit of Kaluza-Klein theory which already contained a non-Abelian gauge potential.

model to an arbitrary non-Abelian gauge group, see [639] and [236]. It took over ten more years before this seminal paper came into prominence. In 1964 Gell-Mann and Ne'eman [235], [237] proposed SU(3) as the gauge group of strong interactions and in the years 1964–1967 Brout and Englert [106], Higgs [298–300] and Kibble [364] discovered a symmetry-breaking mechanism which gave a mass to some components of the Yang–Mills field. Based on this work and on earlier work by Glashow [247] and others, in the years 1967–1968 Weinberg [654] and Salam [552] unified the electromagnetic and the weak interactions.[7] At the beginning of the 1970s, Gross and Wilczek [264], Politzer [513] and Weinberg [656] created the theory of strong interactions called Quantum Chromodynamics. These theories became the two basic building blocks of the standard model of elementary particle physics.[8]

In the period just described, Weyl's original ingenious understanding that the gauge principle is closely related to the notion of connection did not play any role.[9] The development of the theory of connections evolved in a completely separate way as part of modern geometry and was generally unknown to the physics community. In the beginning of the 1920s, on the basis of his deep expertise in Lie theory and under the influence of Einstein's theory of general relativity and of Klein's Erlangen programme, Élie Cartan started building a general theory of connections with respect to various groups. In contrast to Weyl, who used the absolute differential calculus of Levi-Civita and Ricci, Cartan relied on the calculus of differential forms. In the context of what he called 'generalized spaces',[10] Cartan developed the theory of connections (including torsion) for various types of geometries (Riemannian, Lorentzian, Weylian, affine, conformal, projective and others), see [115–120] and further references in [130] and [568].[11] The next step forward was taken at the beginning of the 1940s by Ehresmann, a student of Cartan, who proposed to use fibre bundles as the natural geometric structure allowing for a global description of a connection, see [174–176] and [410] for further references.[12] As a matter of fact, the very notion of a fibre bundle existed already at that time. It was invented by Seifert [584] as early as in 1932. In the 1930s and 1940s, the study of fibre bundles

[7]For this work, Glashow, Weinberg and Salam received the Nobel prize in 1979.

[8]For an exhaustive presentation of the history of the standard model see [657].

[9]However, inspired by the work of Einstein, Weyl, Yang, Mills and Utiyama, as early as in 1963, Lubkin [411] made a first step towards the analysis of the geometric content of the gauge concept in terms of connection theory in fibre bundles.

[10]A generalized space in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure clearly defines a connection. We note that the tangent space is an abstract notion here, it may not coincide with the space of tangent vectors.

[11]The paper [130] by Chern and Chevalley contains a description of the work of Cartan as a whole. The paper [568] by Scholz gives some interesting insight into the scientific interrelation between Weyl and Cartan.

[12]The paper [410] by Libermann describes the influence of Ehresmann on the development of modern differential geometry in detail.

became a quickly developing field of topology.[13] The main steps were taken by Whitney [665, 666], Hopf and Stiefel [602], Hurewicz and Steenrod [330, 331], Ehresmann and Feldbau (already cited above), Chern[14] [126–129] and Pontryagin [516]. This period culminated in the textbooks on the topology of fibre bundles by Steenrod [599] and on the geometry of connections in fibre bundles by Nomizu [491]. By that time, the theory of fibre bundles was settled as a classical field of geometry and topology. It is beyond the scope of this introduction to describe the further development of this field up until the present time.

The first full description of gauge theory in the language of fibre bundles and connections was presented by Trautman in 1970 [630]. Thereafter, the study of the geometric structure of gauge theories quickly became part of mathematical physics and, within the next decade, quite a number of papers propagating this geometric point of view have been written, see e.g. [161], [173] and [147]. This was related to the fact that, at that time, mathematicians became excited about questions posed by physicists, notably by the question of how to find all self-dual solutions of the Yang–Mills equations. This problem was solved by Atiyah, Drinfeld, Hitchin and Manin [36] using methods of algebraic geometry. In our eyes, this is one of the most fascinating interactions of geometry and physics in the second half of the twentieth century. Via the study of the moduli space of the solutions, it led to deep new insight into the topology of differentiable four-manifolds, see [159]. In the middle of the 1990s, guided by the study of the vacuum structure of $N = 2$ supersymmetric Yang–Mills theory, Seiberg and Witten [582, 583] arrived at a U(1)-gauge model coupled to a spinor field. The investigation of this model gave a new impetus to the study of the topology of differentiable four-manifolds. Within a few months, many of the results obtained via instanton theory were reproved within this new theory and new results, notably in the theory of symplectic manifolds, were obtained. Yet another fruitful interaction of physics and geometry happened in the theory of magnetic monopoles. The three fields of research just mentioned will be discussed in some detail in Chaps. 6 and 7. By the end of the 1970s and the beginning of the 1980s, geometrical and topological methods also started playing a role in quantum gauge theory. This applies, in particular to the study of the Gribov problem and to anomalies. Both of these aspects will be discussed in Chap. 9. Moreover, starting from the beginning of the 1990s, a number of observations, conjectures and results concerning the relevance of the stratified structure of the gauge orbit space for quantum gauge theory appeared. This is one of our fields of research, so we will discuss the structure of the gauge orbit stratification, together with a concept how to implement it on quantum level, in detail in Chaps. 8 and 9.

We continue with a few remarks on the structure and the content of this volume. This volume consists of three building blocks: in the first four chapters we present the geometry and topology of fibre bundles, in Chap. 5 we study the theory of Dirac operators and the remaining four chapters are devoted to gauge theory. In more

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[13] See [434] for a history of the theory of fibre bundles.

[14] See [309] for a detailed description of his mathematical work.

detail, in Chap. 1, we study principal and associated bundles and develop the theory of connections. This includes elementary bundle reduction theory, the theory of holonomy and the theory of invariant connections. In Chap. 2, we study linear connections in the frame bundle of a manifold and their reductions. This leads us to *H*-structures[15] allowing for a unified view on possible geometric structures manifolds may be endowed with. From this perspective, Riemannian geometry occurs as an important special example. In this context, we study compatible connections, the relation of curvature and holonomy and we give an introduction to the theory of symmetric spaces. Moreover, we present elementary Hodge theory and discuss some aspects of 4-dimensional Riemannian manifolds. In Chap. 3, we study the homotopy theory of fibre bundles. We prove the Covering Homotopy Theorem and develop the concept of universal bundles. Using this tool, we prove the fundamental classification theorem for principal bundles in terms of homotopy classes of mappings. We also include a discussion of universal connections. In Chap. 4, we present the basics of the cohomology theory of fibre bundles. We study the cohomology rings of characteristic classes for the classical groups, derive the Whitney Sum Formula and the Splitting Principle and discuss the effect of field restrictions and field extensions. Next, we present the characteristic classes in terms of de Rham cohomology via the Weil homomorphism and discuss the related genera. Finally, we discuss the concept of Postnikov tower and show how it may be used to classify bundles over low-dimensional manifolds. Chap. 5 is devoted to the study of Dirac operators. Given their great importance in gauge theory, we provide the reader with a systematic and quite exhaustive presentation. We start with Clifford algebras, spinor groups and their representations. Next, we discuss spin structures, Dirac bundles and Dirac operators. Since we are going to use the Atiyah–Singer Index Theorem in gauge theory a number of times, we give a full proof of this theorem via the heat kernel method. In the remaining four chapters, we present topics in gauge theory. Clearly, we had to make a choice here, that is, we had to omit a number of interesting topics like, say, topological field theory. In Chap. 6, we study pure gauge theories. We start by deriving the Yang–Mills equations from the variational principle for the Yang–Mills action and show that (anti-)self-dual solutions correspond to absolute minima of the action. We then present a systematic study of instantons: we discuss the BPST-instanton family in detail, present the ADHM-construction and give a partial proof that via that construction one obtains all solutions. In our presentation, we limit our attention to the base space S^4 and to the gauge group $G = SU(2)$. Next, we study the moduli space and outline how it is used for the study of the topology of differentiable 4-manifolds. Finally, we present the classical stability analysis of the Yang–Mills Equation and include a short discussion of non-minimal solutions. In Chap. 7, we include matter fields. We start with the theory of Yang–Mills–Higgs models: we discuss the Higgs mechanism, present a topological classification of static finite-energy configurations and address the problem of constructing asymptotic as

[15]In the literature, the term *G*-structure is common as well.

well as exact solutions to the Yang–Mills–Higgs equations. In particular, we focus on magnetic monopole solutions including the Bogomolnyi–Prasad–Sommerfield model. Next, we pass to the Seiberg–Witten model. We discuss the basic properties of this model in detail and outline some of the topological consequences. Next, we present the (classical) standard model of elementary particle physics in the geometric language. In the remaining two sections, we give an introduction to the method of dimensional reduction in the context of gauge theories including some of our own results. Chap. 8 is devoted to the study of the gauge orbit stratification. In the first part, we provide the reader with the classical geometrical and topological results on that structure. In the second part, we present our own results on the classification of gauge orbit types in some detail. For clearness of presentation, we limit our attention to the case $G = \mathrm{SU}(n)$. The classification is in terms of characteristic classes (fulfilling a number of algebraic relations) of certain reductions of the principal bundle under consideration. We also show how to derive the natural partial ordering of strata. Finally, in Chap. 9, we come to some elements of quantum gauge theory with the main emphasis on those aspects which are related to the structure of the classical gauge orbit space in one or the other way. In the first part, we present the classical Faddeev–Popov path integral quantization procedure, address the Gribov problem in the language of differential geometry and discuss the classical results of Singer concerning the obstruction against the existence of a global gauge fixing. Next, we discuss anomalies within the geometric setting. In the second part, we present some of our results on non-perturbative quantum gauge theory for (finite) lattice models in the Hamiltonian framework. We construct the quantum model via canonical quantization, derive the field algebra and the observable algebra of the system and discuss the Gauß law. Next, we explain how to include the non-generic gauge orbit strata on the quantum level and discuss their possible physical relevance for a toy model.

We assume that the reader is familiar with the calculus on manifolds as presented in Chaps. 1–4 of Part I and with the theory of Lie groups and Lie group actions as presented in Chaps. 5 and 6 of Part I. For the understanding of Chaps. 3 and 4, basic knowledge in homotopy theory and some elements of algebraic topology are needed. In Chap. 9, we use elements of the theory of C^*-algebras. For the convenience of the reader we have added a number of appendices.

Chapter 1
Fibre Bundles and Connections

In this chapter, we present the basics of the theory of fibre bundles and connections. In the first part, we discuss principal and asssociated bundles and the theory of connections including the Koszul calculus. The text is illustrated by many examples which will be taken up later on. In the second part, we focus on topics which are particularly important in this book. We study bundle reductions, discuss the theory of holonomy in some detail and analyze the transformation laws of connection and curvature under bundle automorphisms. Finally, we present the theory of invariant connections for the case of group actions which are not necessarily transitive on the base manifold, that is, we go beyond the classical Wang Theorem.

1.1 Principal Bundles

In a gauge theory describing the fundamental interaction of elementary particles, the interaction is assumed to be mediated by a gauge potential. In geometric terms, a gauge potential is the local (spacetime) representative of a connection form, which naturally lives on a principal fibre bundle over spacetime.

Let us recall the following definition from Sect. 6.5 of Part I.

Definition 1.1.1 (*Principal bundle*) Let (P, G, Ψ) be a free Lie group action, let M be a manifold and let $\pi : P \to M$ be a smooth mapping. The tuple (P, G, M, Ψ, π) is called a principal bundle if for every $m \in M$ there exists an open neighbourhood U of m and a diffeomorphism $\chi : \pi^{-1}(U) \to U \times G$ such that

1. χ intertwines Ψ with the G-action on $U \times G$ by translations[1] on the factor G,
2. $\mathrm{pr}_U \circ \chi(p) = \pi(p)$ for all $p \in \pi^{-1}(U)$.

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[1]Left (right) translations if Ψ is a left (right) action.

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For simplicity, we will sometimes use the short-hand notation $P(M, G)$ or just P. If not otherwise stated, we will consider right principal bundles. If there is no danger of confusion, sometimes we will simply write $\Psi_g(p) = p \cdot g$. For a right action, denoting

$$\kappa := \mathrm{pr}_G \circ \chi : \pi^{-1}(U) \to G, \tag{1.1.1}$$

condition 1 can be rewritten as

$$\kappa(\Psi_a(p)) = \kappa(p)a, \quad p \in \pi^{-1}(U),\ a \in G. \tag{1.1.2}$$

The group G is called the structure group of P. If G is fixed, P is referred to as a principal G-bundle. The pair (U, χ) is called a local trivialization. A local trivialization (U, χ) with $U = M$ is called a global trivialization. If there exists a global trivialization, then P is called trivial. The existence of local trivializations implies that π is a surjective submersion. Hence, by Proposition I/1.7.6, the subsets $\pi^{-1}(m)$, $m \in M$, are embedded submanifolds, called the fibres of P. They are diffeomorphic to the group manifold G.

Remark 1.1.2 Let (P, G, Ψ) be a free proper Lie group action. Let M be the orbit space, equipped with the smooth structure provided by Corollary I/6.5.1, and let $\pi : P \to M$ be the natural projection to orbits. Every tubular neighbourhood of an orbit defines a local trivialization over a neighbourhood of the corresponding point of M. Hence, the Tubular Neighbourhood Theorem I/6.4.3 implies that (P, G, M, Ψ, π) is a principal bundle. Conversely, if (P, G, M, Ψ, π) is a principal bundle, then (P, G, Ψ) is a free proper Lie group action, M is diffeomorphic to the orbit space P/G and π corresponds, via this diffeomorphism, to the natural projection to orbits. ♦

We will also need the general notion of fibre bundle.

Definition 1.1.3 (*General fibre bundle*) Let E and M be manifolds and let $\pi : E \to M$ be a smooth surjection. The triple (E, M, π) is called a fibre bundle if there exists a manifold F such that the following holds. Every $m \in M$ admits an open neighbourhood U and a diffeomorphism $\chi : \pi^{-1}(U) \to U \times F$ fulfilling $\mathrm{pr}_U \circ \chi = \pi$. The manifold F is called the typical fibre of π.

The details of the following example are left to the reader (Exercise 1.1.1).

Example 1.1.4

1. Let M be a manifold, let G be a Lie group and let $\mathrm{pr}_M : M \times G \to M$ be the natural projection. Then, $(M \times G, G, M, \Psi, \mathrm{pr}_M)$, with Ψ given by right translation of G on the second factor of $M \times G$, is a principal bundle, called the product principal bundle. It is obviously trivial.
2. Let (P, G, M, Ψ, π) be a principal bundle and let $U \subset M$ be open. Define $P_U := \pi^{-1}(U)$ and take the restrictions $\pi_U : P_U \to U$ of π and $\Psi_U : P_U \times G \to P_U$ of Ψ. By intersection, any local trivialization of P induces a local trivialization of P_U. Thus, $(P_U, G, U, \Psi_U, \pi_U)$ is a principal G-bundle over U.

3. Let $P_1(M_1, G_1)$ and $P_2(M_2, G_2)$ be principal bundles. Then, the direct product $P_1 \times P_2$ carries the structure of a principal $(G_1 \times G_2)$-bundle over $M_1 \times M_2$.
4. Let G be a Lie group and let $H \subset G$ be a closed subgroup. Consider the free action of H on G by right translation. By Example I/6.3.8/3, this action is proper. Thus, Remark 1.1.2 implies that G carries the structure of a principal H-bundle over the homogeneous space G/H.[2] ♦

Definition 1.1.5 Let (P, G, M, Ψ, π) be a principal bundle. A section of P is a smooth mapping $s\colon M \to P$ such that $\pi \circ s = \mathrm{id}_M$. A local section of P over an open subset $U \subset M$ is a section of the principal bundle P_U.

Proposition 1.1.6 *Local trivializations of P are in one-to-one correspondence with local sections. In particular, a principal bundle is trivial iff it admits a global section.*

Proof If $\chi : P \to M \times G$ is a global trivialization, then we set $s(m) := \chi^{-1}(m, \mathbb{1})$, where $\mathbb{1}$ is the unit element in G. This is a smooth global section of P. Conversely, given a global section $s\colon M \to P$, for every point $p \in P$ there exists a unique group element $\kappa(p)$ such that $p = \Psi_{\kappa(p)}(s(\pi(p)))$. This defines a smooth mapping $\kappa : P \to G$, which fulfils $\kappa(\Psi_a(p)) = \kappa(p)\, a$. Thus, $(M, \pi \times \kappa)$ is a global trivialization. ■

Next, we introduce the notion of morphism of principal bundles.

Definition 1.1.7 (*Morphism*) Let $(P_1, G_1, M_1, \Psi^1, \pi_1)$ and $(P_2, G_2, M_2, \Psi^2, \pi_2)$ be principal fibre bundles.

1. A morphism from P_1 to P_2 is a pair of mappings (ϑ, λ), where $\vartheta\colon P_1 \to P_2$ is smooth and $\lambda\colon G_1 \to G_2$ is a homomorphism of Lie groups such that for all $g \in G_1$
$$\vartheta \circ \Psi^1_g = \Psi^2_{\lambda(g)} \circ \vartheta. \tag{1.1.3}$$
2. A morphism (ϑ, λ) is called an isomorphism if ϑ is a diffeomorphism and λ is an isomorphism of Lie groups. In particular, an isomorphism of a principal bundle onto itself is called an automorphism.

We note that, by Definition I/6.6.1, a morphism of principal bundles P_1 and P_2 is a morphism of the Lie group actions (P_1, G_1, Ψ^1) and (P_2, G_2, Ψ^2).

Remark 1.1.8 (*Special morphisms*)

1. By condition (1.1.3), ϑ maps fibres to fibres. Thus, it induces a mapping $\tilde\vartheta : M_1 \to M_2$ such that the following diagram commutes.

$$\begin{tikzcd} P_1 \arrow[r, "\vartheta"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\tilde\vartheta"] & M_2 \end{tikzcd}$$

[2] This statement also follows from Theorem I/5.7.2 and Remark I/5.7.3.

By local triviality, $\tilde{\vartheta}$ is smooth. We say that ϑ projects to $\tilde{\vartheta}$, or that ϑ covers $\tilde{\vartheta}$. If (ϑ, λ) is an isomorphism, then $\tilde{\vartheta}$ is a diffeomorphism. Given isomorphisms $\vartheta_1 : P_1 \to P_2$ and $\vartheta_2 : P_2 \to P_3$, we have (Exercise 1.1.3)

$$(\vartheta_2 \circ \vartheta_1)^{\sim} = \tilde{\vartheta}_2 \circ \tilde{\vartheta}_1, \quad \left(\vartheta_1^{-1}\right)^{\sim} = \tilde{\vartheta}_1^{-1}. \tag{1.1.4}$$

2. If the principal bundles P and Q have the same base manifold M and if $\tilde{\vartheta} = \mathrm{id}_M$, then (ϑ, λ) is said to be vertical. If P and Q have the same structure group G and if $\lambda = \mathrm{id}_G$, then ϑ is called a G-morphism. By local triviality, every vertical G-morphism is a diffeomorphism and hence an isomorphism.
3. If $\tilde{\vartheta}$ and λ are injective immersions, then ϑ is an injective immersion, too. In this case, P_1 is called a subbundle of P_2.
   (a) If, additionally, $\tilde{\vartheta}$ and λ are embeddings, then P is called an embedded subbundle. In this case, P_1 may be identified with the image of the morphism ϑ in P_2.
   (b) If, additionally, $M_1 = M_2 = M$ and $\tilde{\vartheta} = \mathrm{id}_M$, then P_1 is referred to as a λ-reduction or, simply, a reduction of P_2. In this case, one says that G_1 is a reduction of the structure group G_2. Two reductions are said to be equivalent if they differ by a vertical automorphism of P. ♦

Remark 1.1.9 (*Pullback of principal bundles*)

1. In complete analogy to vector bundles, see Sect. 2.6 of Part I, given a principal G-bundle P over M with canonical projection π, we define its pullback by a smooth mapping $\varphi : N \to M$:

$$\varphi^* P := \{(y, p) \in N \times P : \varphi(y) = \pi(p)\}.$$

   This is a principal G-bundle over N and the canonical projection $N \times P \to P$ restricts to a morphism $\varphi^* P \to P$ covering φ. One can show the following (Exercise 1.1.4).
   (a) If P is vertically isomorphic to some principal G-bundle Q over M, then $f^* P$ is vertically isomorphic to $f^* Q$.
   (b) If $\psi : K \to N$ is a further smooth mapping, then $\psi^*(\varphi^* P)$ is vertically isomorphic to $(\varphi \circ \psi)^* P$.
   (c) Let $\vartheta : P \to Q$ be a principal G-bundle morphism covering $\tilde{\vartheta} : M \to N$. The induced mapping

$$P \to \tilde{\vartheta}^* Q, \quad p \mapsto \big(\pi(p), \vartheta(p)\big),$$

   is a vertical isomorphism and ϑ decomposes into the composition of this isomorphism with the natural principal G-bundle morphism $\tilde{\vartheta}^* Q \to Q$.
2. The following is an important special class of pullback bundles. Let $P_1(M, G_1)$ and $P_2(M, G_2)$ be principal bundles. By Example 1.1.4/3, $P_1 \times P_2$ carries the

structure of a principal $(G_1 \times G_2)$-bundle over $M \times M$. Let $\Delta : M \to M \times M$ be the diagonal embedding. Then, $\Delta^*(P_1 \times P_2)$ is a principal bundle with structure group $G_1 \times G_2$ over M. It will be denoted by $P_1 \times_M P_2$ and will be called the fibre product of P_1 and P_2.[3] ♦

In complete analogy with vector bundles, principal fibre bundles can be characterized and studied in terms of transition mappings associated with a chosen covering of the base manifold. Let there be given a principal bundle $P(M, G)$. By definition, one can choose a countable open covering $\{U_i\}_{i\in I}$ of M such that there exists a system of local trivializations

$$\chi_i : \pi^{-1}(U_i) \to U_i \times G.$$

The collection $\{(U_i, \chi_i)\}_{i\in I}$ will sometimes also be called a bundle atlas of P. Let $\kappa_i : \pi^{-1}(U_i) \to G$ be the corresponding system of mappings defined by (1.1.1). Then,

$$\kappa_i(\Psi_a(p)) \cdot \kappa_j(\Psi_a(p))^{-1} = \kappa_i(p) \cdot a \cdot a^{-1} \cdot \kappa_j(p)^{-1} = \kappa_i(p) \cdot \kappa_j(p)^{-1},$$

that is, the mappings $\pi^{-1}(U_i \cap U_j) \ni p \to \kappa_i(p) \cdot \kappa_j(p)^{-1} \in G$ are constant on fibres. Thus, they induce smooth mappings

$$U_i \cap U_j \ni m \mapsto \rho_{ij}(m) := \kappa_i(p) \cdot \kappa_j(p)^{-1} \in G, \quad p \in \pi^{-1}(m), \tag{1.1.5}$$

which are called the transition mappings of P. They fulfil

$$\rho_{ij}(m) = \rho_{ik}(m) \cdot \rho_{kj}(m), \quad m \in U_i \cap U_j \cap U_k. \tag{1.1.6}$$

This condition implies, in particular,

$$\rho_{ii}(m) = \mathbb{1}, \quad m \in U_i,$$
$$\rho_{ij}(m) = (\rho_{ji}(m))^{-1}, \quad m \in U_i \cap U_j.$$

Proposition 1.1.10 *Let M be a manifold and let G be a Lie group. Then, for every countable open covering $\{U_i\}_{i\in I}$ of M and every system of smooth mappings $\rho_{ij} : U_i \cap U_j \to G$ fulfilling condition (1.1.6) there exists a principal G-bundle over M admitting a system of local trivializations with transition mappings $\{\rho_{ij}\}$.*

Proof Take the topological direct sum

$$X := \bigsqcup_{i\in I} U_i \times G.$$

[3] Some authors call it the spliced product [83].

We define

$$(i, m, g) \sim (i', m', g') \quad \text{iff} \quad m' = m, \; g' = \rho_{ii'}(m) \cdot g,$$

which, by condition (1.1.6), yields an equivalence relation on X. We denote by $P = X/\sim$ the topological quotient and by $\mathrm{pr} : X \to P$ the canonical projection. Since X admits a countable basis, P admits a countable basis, too. Moreover, it is easy to show that P is Hausdorff, see Exercise 1.1.2. We endow P with the structure of a principal G-bundle. For that purpose, we define

$$\Psi : P \times G \to P, \quad \Psi([(i, m, a)], b) := [(i, m, a \cdot b)].$$

Clearly, this definition does not depend on the choice of the representative (i, m, a). Thus, Ψ defines a topological right group action, which is obviously free. By construction, $[(i, m, a)] = [(i', m', a')]$ implies $m' = m$. Thus, we can define a continuous mapping

$$\pi : P \to M, \quad \pi([(i, m, a)]) := m.$$

Since the U_i cover M, π is surjective. Let pr_i be the restriction of pr to $U_i \times G$. By construction, for every $i \in I$, it defines a bijection

$$\mathrm{pr}_i \colon U_i \times G \to \pi^{-1}(U_i).$$

We endow P with the structure of a differentiable manifold by observing that $\pi^{-1}(U_i)$ is an open subset and by postulating that pr_i be a diffeomorphism for every $i \in I$. With respect to this differentiable structure, the action Ψ is smooth. Finally, putting

$$\chi_i := \mathrm{pr}_i^{-1} : \pi^{-1}(U_i) \to U_i \times G$$

we get a system of local trivializations whose transition mappings coincide with the mappings ρ_{ij}. Moreover, by the definition of Ψ, the induced mappings $\kappa_i : \pi^{-1}(U_i) \to G$ fulfil condition (1.1.2). ■

Our next aim is to show that vertical isomorphism classes of principal G-bundles over M can be labelled in terms of the first Čech cohomology of M. Thus, let P_1 and P_2 be isomorphic principal G-bundles over M via a morphism (ϑ, λ). Let $\{(U_i, \chi_i^1)\}$ and $\{(U_i, \chi_i^2)\}$ be local trivializations and let $\{\rho_{ij}^1\}$ and $\{\rho_{ij}^2\}$ be the corresponding transition mappings of P_1 and P_2, respectively. Here, again without loss of generality, we have assumed that both trivializations are associated with one and the same covering of M. Let $m \in U_i \cap U_j$. Since $p \in \pi_1^{-1}(m)$ implies $\vartheta(p) \in \pi_2^{-1}(m)$, using (1.1.5), we obtain

$$\begin{aligned}
\rho_{ij}^2(m) &= \kappa_i^2(\vartheta(p))\, \kappa_j^2(\vartheta(p))^{-1} \\
&= \left(\kappa_i^2(\vartheta(p))\, \left(\kappa_i^1(p)\right)^{-1}\right) \left(\kappa_i^1(p)\right)\left(\kappa_j^1(p)\right)^{-1} \left(\kappa_j^2(\vartheta(p))\, \left(\kappa_j^1(p)\right)^{-1}\right)^{-1}.
\end{aligned}$$

Since, for every $a \in G$, we have

$$\kappa_i^2(\vartheta(\Psi_a(p))) \left(\kappa_i^1(\Psi_a(p))\right)^{-1} = \kappa_i^2(\vartheta(p)) \left(\kappa_i^1(p)\right)^{-1},$$

we can define a smooth mapping

$$\rho_i : U_i \to G, \quad \rho_i(m) := \kappa_i^2(\vartheta(p)) \left(\kappa_i^1(p)\right)^{-1}, \quad p \in \pi^{-1}(m).$$

Thus, for every $m \in U_i \cap U_j$, we obtain

$$\rho_{ij}^2(m) = \rho_i(m)\, \rho_{ij}^1(m)\, \rho_j(m)^{-1}. \tag{1.1.7}$$

To summarize, if the principal G-bundles P_1 and P_2 are vertically isomorphic, then there exists a family of smooth mappings $\rho_i : U_i \to G$ such that their transition mappings are related by (1.1.7).

It turns out that the converse is also true.

Theorem 1.1.11 *Two principal G-bundles over M are vertically isomorphic iff there exists a family of smooth mappings $\rho_i : U_i \to G$ such that the corresponding transition mappings fulfil* (1.1.7).

Proof It remains to show that condition (1.1.7) implies that P_1 and P_2 are isomorphic. Thus, let there be given a family of mappings $\{\rho_i\}_{i\in I}$ fulfilling (1.1.7). In the above notation, we define

$$\vartheta_i : \pi_1^{-1}(U_i) \to \pi_2^{-1}(U_i), \quad \vartheta_i := \left(\chi_i^2\right)^{-1} \circ (\mathrm{id}_M \times \rho_i) \circ \chi_i^1$$

for every $i \in I$. Obviously, this is a family of diffeomorphisms fulfilling $\pi_2(\vartheta_i(p)) = \pi_1(p)$, $p \in \pi_1^{-1}(U_i)$. By (1.1.5), we have

$$\chi_j^\alpha \circ \left(\chi_i^\alpha\right)^{-1} = \mathrm{id}_M \times \rho_{ji}^\alpha, \quad \alpha = 1, 2.$$

Using this and condition (1.1.7), we obtain $\vartheta_i = \vartheta_j$ on $\pi_2^{-1}(U_i \cap U_j)$ for every pair (i, j) such that $U_i \cap U_j \neq \varnothing$. Thus, the family $\{\vartheta_i\}_{i\in I}$ defines a diffeomorphism $\vartheta : P_1 \to P_2$ fulfilling $\pi_2 \circ \vartheta = \pi_1$. By (1.1.2), it also fulfils the equivariance property (1.1.3). We conclude that ϑ is a vertical isomorphism of principal G-bundles. ■

Remark 1.1.12 (*Čech cohomology*) The systems of mappings $\{\rho_i\}$ and $\{\rho_{ij}\}$ are, respectively, referred to as a 0-cocyle and a 1-cocyle on M with values in G, relative to a chosen covering $\mathfrak{U} = \{U_i\}_{i\in I}$. Formula (1.1.7) defines an equivalence relation in the set of 1-cocycles. The corresponding set of equivalence classes $H^1_{\check{c}}(\mathfrak{U}, G)$ is referred to as the first cohomology set $H^1_{\check{c}}(\mathfrak{U}, G)$ in the sense of Čech, relative to a chosen covering. The set of open coverings of M forms a directed system with respect to refinement, that is, $\mathfrak{U} \leq \mathfrak{V}$ if each $V_\alpha \in \mathfrak{V}$ is contained in some $U_i \in \mathfrak{U}$. By restriction, we get a mapping $H^1_{\check{c}}(\mathfrak{U}, G) \to H^1_{\check{c}}(\mathfrak{V}, G)$. The cohomology set $H^1_{\check{c}}(M, G)$ is the direct limit of the sets $H^1_{\check{c}}(\mathfrak{U}, G)$ with respect to the restriction mappings, as $\mathfrak{U}$ runs through all open coverings of M, cf. [304] for further details.

Note that there is a distinguished element $1 \in H^1_{\check{C}}(M, G)$, given by the constant 1-cocycle. More precisely, for any open covering, we put $\rho_{ij}(m) = \mathbb{1}$ for every pair (i, j). Note, however, that $H^1_{\check{C}}(M, G)$ is in general not a group. ♦

Using this terminology, Theorem 1.1.11 can be reformulated as follows.

Corollary 1.1.13 *The vertical isomorphism classes of principal G-bundles over M are in one-one correspondence with the elements of the cohomology set $H^1_{\check{C}}(M, G)$. Thereby, the product bundle $M \times G$ corresponds to the distinguished element.* ■

We close this section with a number of examples. All of them will be taken up again later on.

Example 1.1.14 (*Frame bundle of a manifold*) Let M be an n-dimensional manifold. A linear n-frame at $m \in M$ is an ordered basis $u = (u_1, \dots, u_n)$ of the tangent space $\mathrm{T}_m M$. Let $L(M)$ be the set of all linear n-frames on M. For an n-frame $u = (u_1, \dots, u_n)$ at $m \in M$ and an element $a = (a^i{}_j) \in \mathrm{GL}(n, \mathbb{R})$, the ordered set $ua := u_i\, a^i{}_j$ is again an n-frame. Thus, we get a right action of $\mathrm{GL}(n, \mathbb{R})$ on $L(M)$,

$$\Psi : L(M) \times Gl(n, \mathbb{R}), \quad \Psi(u, a) := ua,$$

which is obviously free. Clearly, the orbit space of this action is M and the corresponding canonical projection $\pi : L(M) \to M$ coincides with the mapping which assigns to an n-frame u at $m \in M$ the point m.

Let (U, φ) be a local chart of M. Then, on U, every basis vector u_i belonging to $u = (u_1, \dots, u_n) \in \pi^{-1}(U)$ can be represented by $u_i = (u_i)^{\varphi, j} \partial^{\varphi}_j$, that is, locally u is given by the matrix $u^{\varphi} = \big((u_i)^{\varphi, j}\big) \in Gl(n, \mathbb{R})$. Thus,

$$\chi : \pi^{-1}(U) \to U \times \mathrm{GL}(n, \mathbb{R}), \quad \chi(u) := (\pi(p), u^{\varphi}) \tag{1.1.8}$$

is a bijection fulfilling $\mathrm{pr}_U \circ \chi = \pi$. By the definition of Ψ, it also fulfils the equivariance property (1.1.2). We equip $L(M)$ with a differentiable structure by postulating that all the mappings (1.1.8) be diffeomorphisms. Then, $(L(M), \mathrm{GL}(n, \mathbb{R}), M, \Psi, \pi)$ is a principal fibre bundle and the family of mappings (1.1.8) forms a system of local trivializations. ♦

Example 1.1.15 (*Frame bundle of a vector bundle*) Let E be a $\mathbb{K}$-vector bundle of rank k over M, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let L_m be the set of bases in the fibre E_m. Then,

$$L(E) := \bigcup_{m \in M} L_m$$

carries the structure of a principal fibre bundle over M with structure group $\mathrm{GL}(k, \mathbb{K})$. The details are analogous to the previous example and are, therefore, left to the reader (Exercise 1.1.5). ♦

Definition 1.1.16 Let E be a $\mathbb{K}$-vector bundle of rank k over M, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We say that E is endowed with a fibre metric h, if there exists a non-degenerate inner product[4] on each fibre $\pi^{-1}(m)$ of E depending smoothly on m. The pair (E, h) will be called (pseudo-)Riemannian for $\mathbb{K} = \mathbb{R}$ and Hermitean for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$.

Remark 1.1.17 Every vector bundle over a manifold admits a fibre metric. This can be easily shown using a partition of unity of the base manifold (Exercise 1.1.6). Moreover, note that a fibre metric in the tangent bundle of a manifold M is the same as a (pseudo-)Riemannian metric on M. ♦

Example 1.1.18 For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, let E be a $\mathbb{K}$-vector bundle of rank k over M endowed with a fibre metric h. Let O_m be the set of h-orthonormal bases in the fibre E_m. Then,

$$O(E) := \bigcup_{m \in M} O_m$$

carries the structure of a principal fibre bundle over M with structure group being the isometry group of the metric. Details are left to the reader (Exercise 1.1.5).

In the special case where $\mathbb{K} = \mathbb{R}$ and E is the tangent bundle of an n-dimensional Riemannian manifold M, this construction yields the orthonormal frame bundle $O(M)$ of M with the structure group $\mathrm{O}(n)$. If M is in addition oriented, the subset $O_+(M) \subset O(M)$ of ordered orthonormal frames is a reduction to the subgroup $\mathrm{SO}(n) \subset \mathrm{O}(n)$.

As an example, consider $M = \mathrm{S}^n$, realized as the unit sphere in $\mathbb{R}^{n+1}$. Since for $\mathbf{x} \in \mathrm{S}^n$, the tangent space $\mathrm{T}_{\mathbf{x}}\mathrm{S}^n$ may be identified with the subspace of vectors in $\mathbb{R}^{n+1}$ orthogonal to $\mathbf{x}$, every orthonormal frame in $\mathrm{T}_{\mathbf{x}}\mathrm{S}^n$ complements $\mathbf{x}$ to an orthonormal basis in $\mathbb{R}^{n+1}$. Since every such basis corresponds to an orthogonal transformation, we obtain a mapping $O(\mathrm{S}^n) \to \mathrm{O}(n+1)$. We leave it to the reader to check that this mapping is an isomorphism of principal $\mathrm{O}(n)$-bundles, where $\mathrm{O}(n)$ acts on $\mathrm{O}(n+1)$ by right translation via the blockwise embedding $\mathrm{O}(n) \to \mathrm{O}(n+1)$ defined by the decomposition $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$ (Exercise 1.1.8). Clearly, this isomorphism restricts to an isomorphism of principal $\mathrm{SO}(n)$-bundles between $O_+(\mathrm{S}^n)$ with respect to the standard orientation (pointing outwards) and $\mathrm{SO}(n+1)$. ♦

Definition 1.1.19 The principal bundle $L(E)$ constructed in Example 1.1.15 is called the frame bundle of E. The principal bundle $O(E)$ constructed in Example 1.1.18 is called the orthonormal frame bundle of E. More precisely, it is called the bundle of orthogonal, unitary and symplectic frames for, respectively, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

For the following two examples, the reader should recall the notion of projective space, cf. Example I/1.1.15.

[4]In our convention, for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, the inner product is assumed to be anti-linear in the first and linear in the second component.

Example 1.1.20 (*Complex Hopf bundle*) Consider the natural free action of $\mathrm{U}(1)$ on $\mathbb{C}^2$ given by

$$\Psi : \mathbb{C}^2 \times \mathrm{U}(1) \to \mathbb{C}^2, \quad \Psi((z_1, z_2), e^{i\alpha}) := (e^{i\alpha} z_1, e^{i\alpha} z_2).$$

Since the embedded submanifold

$$\mathrm{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

is invariant under Ψ, we have an induced free action of $\mathrm{U}(1)$ on S^3. Since the Lie group $\mathrm{U}(1)$ is compact, this action is proper. Thus, by Corollary I/6.5.1, the orbit space $\mathrm{S}^3/\mathrm{U}(1)$ admits a unique differentiable structure such that the canonical projection $\pi : \mathrm{S}^3 \to \mathrm{S}^3/\mathrm{U}(1)$ is a submersion. According to Example I/6.5.4, the orbit space endowed with this smooth structure coincides with the 1-dimensional complex projective space $\mathbb{C}\mathrm{P}^1$. Finally, the Tubular Neighbourhood Theorem I/6.4.3 implies the existence of local trivializations. Thus, the above action Ψ defines on S^3 the structure of a principal $\mathrm{U}(1)$-bundle over $\mathbb{C}\mathrm{P}^1$. This bundle is called the complex Hopf bundle. For later purposes, we construct a system of local trivializations.

(a) Let $U_1 := \mathbb{C}\mathrm{P}^1 \setminus \{\pi(0, 1)\}$. Then, $\pi^{-1}(U_1) = \{(z_1, z_2) \in \mathrm{S}^3 : z_1 \neq 0\}$. Thus, we can define

$$\chi_1 : \pi^{-1}(U_1) \to U_1 \times \mathrm{U}(1), \quad \chi_1(z_1, z_2) := \left(\pi(z_1, z_2), \frac{z_1}{|z_1|}\right).$$

Then, $\kappa_1(z_1, z_2) = \frac{z_1}{|z_1|}$. Obviously, κ_1 is smooth and $\mathrm{U}(1)$-equivariant, that is,

$$\kappa_1((z_1 e^{i\alpha}, z_2 e^{i\alpha})) = \frac{z_1}{|z_1|} e^{i\alpha}.$$

Thus, χ_1 is a local trivialization.

(b) Analogously, we put $U_2 := \mathbb{C}\mathrm{P}^1 \setminus \{\pi(1, 0)\}$. Then, $\pi^{-1}(U_1) = \{(z_1, z_2) \in \mathrm{S}^3 : z_2 \neq 0\}$ and we define

$$\chi_2 : \pi^{-1}(U_2) \to U_2 \times \mathrm{U}(1), \quad \chi_2(z_1, z_2) := \left(\pi(z_1, z_2), \frac{z_2}{|z_2|}\right).$$

Thus, $\kappa_2(z_1, z_2) = \frac{z_2}{|z_2|}$ and χ_2 is also a local trivialization.

Since $U_1 \cup U_2 = \mathbb{C}\mathrm{P}^1$, the collection $\{(U_i, \chi_i)\}_{i=1,2}$ defines a system of local trivializations. Its transition mapping $\rho_{12} : U_1 \cap U_2 \to \mathrm{U}(1)$ is given by

$$\rho_{12}(\pi(z_1, z_2)) = \kappa_1(z_1, z_2)\kappa_2(z_1, z_2)^{-1} = \frac{z_1}{|z_1|} \left(\frac{z_2}{|z_2|}\right)^{-1}.$$

♦

Remark 1.1.21

1. We show that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to the 2-sphere. For that purpose, consider the smooth mapping

$$\mathrm{S}^3 \ni (z_1, z_2) \mapsto (2\bar{z}_1 z_2, |z_1|^2 - |z_2|^2) \in \mathbb{C} \times \mathbb{R}.$$

Since

$$(|z_1|^2 - |z_2|^2)^2 + |2\overline{z_1} z_2|^2 = (|z_1|^2 + |z_2|^2)^2 = 1,$$

its image is contained in $\mathrm{S}^2 \subset \mathbb{C} \times \mathbb{R}$. Thus, it induces a smooth mapping $f : \mathrm{S}^3 \to \mathrm{S}^2$. Since f is $\mathrm{U}(1)$-invariant, it induces a mapping $\tilde{f} : \mathbb{C}\mathrm{P}^1 \to \mathrm{S}^2$. The local triviality of the Hopf bundle implies that $\tilde{f}$ is smooth. It remains to show that $\tilde{f}$ is invertible and that the inverse mapping is smooth. For that purpose, we put $V_+ := \mathrm{S}^2 \setminus \{(0, 1)\}$ and define

$$g_+ : V_+ \to \mathbb{C}^2, \quad g_+(z, t) := \left(\frac{\bar{z}}{\sqrt{2(1-t)}}, \sqrt{\frac{1-t}{2}} \right).$$

Since the image of g_+ is contained in S^3, it induces a smooth mapping $g_+ : V_+ \to \mathrm{S}^3$. Composition with π then yields a smooth mapping $\tilde{g}_+ := \pi \circ g : V_+ \to \mathbb{C}\mathrm{P}^1$. We continue $\tilde{g}_+$ to a mapping $\tilde{g} : \mathrm{S}^2 \to \mathbb{C}\mathrm{P}^1$ by setting $\tilde{g}(0, 1) := [(1, 0)]$. Then, $\tilde{g} \circ \tilde{f} = \mathrm{id}_{\mathbb{C}P^1}$ and $\tilde{f} \circ \tilde{g} = \mathrm{id}_{\mathrm{S}^2}$, see Exercise 1.1.7. Thus, $\tilde{g}$ is inverse to $\tilde{f}$. It remains to show smoothness of $\tilde{g}$ at the point $(0, 1)$. This is left to the reader, see Exercise 1.1.7.

2. The Hopf bundle is clearly nontrivial, because otherwise S^3 would have to be diffeomorphic to $\mathrm{S}^2 \times \mathrm{U}(1)$. This fact can be also read off from the transition mappings as follows: it is enough to prove that ρ_{12} is not homotopic to the constant mapping $U_1 \cap U_2 \ni \pi(z_1, z_2) \mapsto 1 \in \mathrm{U}(1)$. To show this, it is enough to find a continuous path $t \mapsto \gamma(t)$ in $U_1 \cap U_2$ such that the path $\rho_{12} \circ \gamma$ in $\mathrm{U}(1)$ is not contractible to a point. We put

$$\tilde{\gamma}(t) := \left(\frac{1}{\sqrt{2}} e^{\frac{1}{2} it}, \frac{1}{\sqrt{2}} e^{-\frac{1}{2} it} \right), \quad t \in [0, 2\pi],$$

and $\gamma(t) := \pi(\tilde{\gamma}(t))$. Clearly, γ is continuous and its image is contained in $U_1 \cap U_2$. We have $\rho_{12}(\gamma(t)) = e^{it}$, with t running from 0 to 2π. Obviously, this path is not contractible in $\mathrm{U}(1)$ showing that the Hopf bundle is nontrivial, indeed. By construction, $U_1 \cap U_2$ is homeomorphic to $\mathrm{S}^1 \times (0, 1)$ and the path γ runs through the S^1-factor exactly once. Since $\rho_{12} \circ \gamma$ also runs through $\mathrm{U}(1) \cong \mathrm{S}^1$ exactly once, the mapping degree of ρ_{12} is 1. Later on, we will see that the mapping degree of the transition mapping yields a useful tool for the study of isomorphism classes of principal bundles over spheres.

3. In complete analogy to the Hopf bundle, the natural action of $\mathrm{U}(1)$ on $\mathbb{C}^n$ yields principal $\mathrm{U}(1)$-bundles over the complex projective spaces $\mathbb{C}\mathrm{P}^{n-1}$.[5] ♦

Example 1.1.22 (*Quaternionic Hopf bundle*) Recall the skew field $\mathbb{H}$ of quaternions, cf. Remark I/1.1.13. Consider the natural right action of the classical Lie group[6] Sp(1) of quaternions of norm 1 on $\mathbb{H}^2$,

$$\Psi : \mathbb{H}^2 \times \mathrm{Sp}(1) \to \mathbb{H}^2, \quad \Psi((\mathbf{q}_1, \mathbf{q}_2), \mathbf{u}) = (\mathbf{q}_1\mathbf{u}, \mathbf{q}_2\mathbf{u}).$$

Clearly, Ψ is free and leaves the embedded submanifold

$$\mathrm{S}^7 = \{(\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{H}^2 : \ \parallel \mathbf{q}_1 \parallel^2 + \parallel \mathbf{q}_2 \parallel^2 = 1\}$$

invariant. Thus, it induces a right free action on S^7. Since Sp(1) is compact, this action is proper. By the same arguments as in Example 1.1.20, the sphere S^7 endowed with the above action carries the structure of a principal Sp(1)-bundle over the quaternionic projective space $\mathbb{H}\mathrm{P}^1$. This bundle is called the quaternionic Hopf bundle. By Example I/5.1.10, the Lie group Sp(1) is isomorphic to the special unitary group $\mathrm{SU}(2)$ and, by completely analogous arguments as in Remark 1.1.21/1, the base manifold $\mathbb{H}\mathrm{P}^1$ is diffeomorphic to S^4 via the mapping (B.1). Thus, the quaternionic Hopf bundle may be viewed as a principal $\mathrm{SU}(2)$-bundle over S^4. Let $\pi : \mathrm{S}^7 \to \mathrm{S}^4$ be the canonical projection. Again, in complete analogy to the complex Hopf bundle, one constructs a system of local trivializations $\{(U_i, \chi_i)\}_{i=1,2}$ as follows: take $U_1 = \mathbb{H}\mathrm{P}^1 \setminus \{\pi(0, 1)\}$ and $U_2 = \mathbb{H}\mathrm{P}^1 \setminus \{\pi(1, 0)\}$ and define

$$\chi_1(\mathbf{q}_1, \mathbf{q}_2) := \left(\pi(\mathbf{q}_1, \mathbf{q}_2), \frac{\mathbf{q}_1}{\parallel \mathbf{q}_1 \parallel}\right), \quad \chi_2(\mathbf{q}_1, \mathbf{q}_2) := \left(\pi(\mathbf{q}_1, \mathbf{q}_2), \frac{\mathbf{q}_2}{\parallel \mathbf{q}_2 \parallel}\right). \tag{1.1.9}$$

♦

Remark 1.1.23

1. Using the criterion given in Remark 1.1.21/2, one can prove that the quaternionic Hopf bundle is nontrivial (Exercise 1.1.9/c).
2. The construction of the quaternionic Hopf bundle obviously generalizes to the case of the natural action of Sp(1) on $\mathbb{H}^n$. This way one obtains a family of principal Sp(1)-bundles with bundle space S^{4n-1} and base space $\mathbb{H}\mathrm{P}^{n-1}$. ♦

Example 1.1.24 (*Stiefel bundles*) Recall from Example I/5.7.5 that the Stiefel manifold $S_{\mathbb{K}}(k, n)$, with $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, is the set of k-frames in $\mathbb{K}^n$ which are orthonormal with respect to the standard scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} \overline{x}_i y_i .$$

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[5] Sometimes, they are also called Hopf bundles.

[6] For the terminology, see Example I/1.2.6.

As shown there, $S_{\mathbb{K}}(k,n)$ aquires its manifold structure by identifying it with the homogeneous space obtained by taking the quotient of the isometry group $\mathrm{U}_{\mathbb{K}}(n)$ of the scalar product with respect to the stabilizer $\mathrm{U}_{\mathbb{K}}(n-k)$ of a chosen frame, that is,

$$S_{\mathbb{K}}(k,n) \cong \mathrm{U}_{\mathbb{K}}(n)/\mathrm{U}_{\mathbb{K}}(n-k). \tag{1.1.10}$$

Here,

$$\mathrm{U}_{\mathbb{K}}(n) = \begin{cases} \mathrm{O}(n) & \textit{if} \ \ \mathbb{K} = \mathbb{R}, \\ \mathrm{U}(n) & \textit{if} \ \ \mathbb{K} = \mathbb{C}, \\ \mathrm{Sp}(n) & \textit{if} \ \ \mathbb{K} = \mathbb{H}. \end{cases}$$

Correspondingly, consider the Graßmann manifold $G_{\mathbb{K}}(k,n)$, which is the set of k-dimensional subspaces of $\mathbb{K}^n$, cf. Example I/5.7.6. One has

$$G_{\mathbb{K}}(k,n) \cong \mathrm{U}_{\mathbb{K}}(n)/(\mathrm{U}_{\mathbb{K}}(n-k) \times \mathrm{U}_{\mathbb{K}}(k)). \tag{1.1.11}$$

Clearly, $\mathrm{U}_{\mathbb{K}}(k)$ acts smoothly on $S_{\mathbb{K}}(k,n)$. By Corollary I/6.5.3, this action is free and proper. Thus, by the arguments given in Remark 1.1.2, $S_{\mathbb{K}}(k,n)$ carries the structure of a principal fibre bundle over $G_{\mathbb{K}}(k,n)$ with structure group $\mathrm{U}_{\mathbb{K}}(k)$. The principal bundles so obtained are called, respectively, the real, complex and quaternionic Stiefel bundles. ♦

Remark 1.1.25 Consider the special case $S_{\mathbb{K}}(1,n)$. Then, one has the following diffeomorphisms (Exercise 1.1.10):

$$S_{\mathbb{K}}(1,n) \cong \mathrm{S}^{dn-1}, \quad G_{\mathbb{K}}(1,n) \cong \mathbb{K}\mathrm{P}^{n-1}, \tag{1.1.12}$$

with $d = \dim_{\mathbb{R}} \mathbb{K}$. Thus, S^{n-1}, S^{2n-1} and S^{4n-1} carry the structure of principal fibre bundles with structure groups $\mathrm{O}(1)$, $\mathrm{U}(1)$ and $\mathrm{Sp}(1)$ and base spaces $\mathbb{R}\mathrm{P}^{n-1}$, $\mathbb{C}\mathrm{P}^{n-1}$ and $\mathbb{H}\mathrm{P}^{n-1}$, respectively. They are isomorphic to the real, complex and quaternionic Stiefel bundles with $k=1$, respectively. In particular, the Hopf bundles of Examples 1.1.20 and 1.1.22 coincide with the Stiefel bundles $S_{\mathbb{K}}(1,2)$ with $\mathbb{K}=\mathbb{C}$ and $\mathbb{K}=\mathbb{H}$, respectively. ♦

Example 1.1.26 (*Universal Covering*) Consider the universal covering space $\tilde{M}$ of a manifold M. Then, $\tilde{M}$ is a principal fibre bundle over M whose (discrete) structure group is the first homotopy group $\pi_1(M)$ (Exercise 1.1.12). ♦

Exercises

1.1.1 Prove the statements of Example 1.1.4.

1.1.2 Complete the proof of Proposition 1.1.10 by showing that P is Hausdorff.
Hint. By elementary set topology, it is enough to prove that pr is open and that the graph of the equivalence relation is closed in $X \times X$.

1.1.3 Prove Eq. (1.1.4).

1.1.4 Prove the assertion of point 1 of Remark 1.1.8.

1.1.5 Construct the principal bundle structures for Examples 1.1.15 and 1.1.18.

1.1.6 Prove the statement of Remark 1.1.17.

1.1.7 Complete the arguments in Remark 1.1.21/1.
Hint. Consider $V_- := \mathrm{S}^2 \setminus \{(0, -1)\}$ and define a second mapping

$$g_- : V_- \to \mathbb{C}^2 \quad g_-(z,t) := \left(\sqrt{\frac{1+t}{2}}\,,\ \frac{z}{\sqrt{2(1+t)}}\right).$$

Show that g_- induces a smooth mapping $\tilde{g}_- : V_- \to \mathbb{C}\mathrm{P}^1$ and prove that $\tilde{g}_{\restriction V_-} = \tilde{g}_-$.

1.1.8 Prove that the mapping $O(\mathrm{S}^n) \to \mathrm{O}(n+1)$ constructed in Example 1.1.18 is an isomorphism of principal $\mathrm{O}(n)$-bundles.

1.1.9 Consider the quaternionic Hopf bundle defined in Example 1.1.22.

(a) By analogous arguments as in Remark 1.1.21/1, show that the base manifold $\mathbb{H}\mathrm{P}^1$ is diffeomorphic to S^4.
(b) Show that the mappings defined in (1.1.9) yield a system of local trivializations.
(c) Using the criterion given in Remark 1.1.21/2, prove that the quaternionic Hopf bundle is nontrivial.
Hint. The group manifold of $\mathrm{SU}(2)$ is diffeomorphic to S^3.

1.1.10 Prove the statements made in Remark 1.1.25.

1.1.11 Construct systems of local trivializations for the Stiefel bundles.

1.1.12 Prove the statement of Example 1.1.26.

1.2 Associated Bundles

First, we recall the notion of associated bundle from Sect. 6.5 in Part I. Let (P, G, M, Ψ, π) be a principal bundle and let (F, G, σ) be a left Lie group action. Let $\check{\sigma}$ be the right action associated with σ,

$$\check{\sigma} : F \times G \to F, \quad (f, a) \mapsto \check{\sigma}_a(f) := \sigma_{a^{-1}}(f).$$

Since the G-action Ψ on P is free, the direct product action $\Psi \times \check{\sigma}$ is free, too. According to Remark I/6.3.9/2, it is proper. Thus, by Corollary I/6.5.1, the orbit space

$$P \times_G F := (P \times F)/G$$

inherits a unique smooth structure. Since the natural projection $P \times F \to P$ is equivariant, it induces a smooth surjective mapping

$$\pi_F : P \times_G F \to P/G = M, \quad \pi_F([(p, f)]) = \pi(p)\,.$$

This endows $P \times_G F$ with a natural bundle structure. Finally, the local triviality of P induces the local triviality of this bundle. To see this, recall from Proposition 1.1.6 that (local) trivializations of P are in one-to-one correspondence with (local) sections. Thus, let $s : U \to P$ be a local section corresponding to a local trivialization (U, χ) of P. Then, the mapping

$$U \times F \to \pi_F^{-1}(U), \quad (m, f) \mapsto [(s(m), f)]$$

is a diffeomorphism projecting to the identical mapping on U. The inverse mapping $\xi : \pi_F^{-1}(U) \to U \times F$ yields a local trivialization of $P \times_G F$. Thus, we have constructed a fibre bundle over M with typical fibre F.

Definition 1.2.1 The fibre bundle $(P \times_G F, M, \pi_F)$ is said to be associated with the principal bundle (P, G, M, Ψ, π) and the G-manifold (F, σ).

The proof of the following observation is left to the reader (Exercise 1.2.1).

Proposition 1.2.2 *For given principal bundles $P_1(M_1, G_1)$ and $P_2(M_2, G_2)$ and representations (F_1, G_1, σ_1) and (F_2, G_2, σ_2), let (ϑ, λ) be a morphism from P_1 to P_2 and let $T : F_1 \to F_2$ be a homomorphism of the representations σ_1 and σ_2. Then, $\vartheta \times T : P_1 \times F_1 \to P_2 \times F_2$ induces a vector bundle morphism $P_1 \times_{G_1} F_1 \to P_2 \times_{G_2} F_2$ projecting to $\tilde{\vartheta}$.* ■

This proposition applies, in particular, to the case where $F_1 = F_2$ and $T = \mathrm{id}$.

Remark 1.2.3

1. Let us express the local trivialization (U, ξ) constructed above explicitly in terms of the local trivialization (U, χ). As usual, denote $\kappa = \mathrm{pr}_G \circ \chi$ and let s be the associated local section of P. Recall that, for any $p \in \pi^{-1}(U)$, we have $p = \Psi_{\kappa(p)} s(\pi(p))$. Using this, we calculate

   $$\xi([(p, f)]) = \xi([(\Psi_{\kappa(p)} s(m), f)]) = \xi([(s(m), \sigma_{\kappa(p)} f)]) = (m, \sigma_{\kappa(p)} f),$$

   with $m = \pi(p)$. Since $\pi(p) = \pi_F([(p, f)])$, we obtain

   $$\xi([(p, f)]) = (\pi_F([(p, f)]), \sigma_{\kappa(p)} f). \tag{1.2.1}$$

2. The natural projection $\iota : P \times F \to P \times_G F$ induces for every $p \in P$ a mapping

$$\iota_p : F \to P \times_G F, \quad \iota_p(f) := [(p, f)] \tag{1.2.2}$$

whose image is contained in the fibre over $\pi(p)$. Moreover, since

$$\iota_{\Psi_a(p)}(f) = [(\Psi_a(p), f)] = [(\Psi_a(p), \sigma_{a^{-1}} \circ \sigma_a(f))] = [(p, \sigma_a(f))],$$

ι_p is equivariant,

$$\iota_{\Psi_a(p)} = \iota_p \circ \sigma_a. \tag{1.2.3}$$

From these properties it is clear that, viewed as a mapping from F to the fibre over $\pi(p)$, ι_p is bijective. Finally, from (1.2.1) we read off

$$\mathrm{pr}_2 \circ \xi \circ \iota_p = \sigma_{\kappa(p)}, \tag{1.2.4}$$

for any local trivialization (U, ξ) such that $\pi(p) \in U$. Since this is a diffeomorphism of F, we conclude that ι_p is a diffeomorphism identifying F with the fibre $\pi_F^{-1}(\pi(p))$.

3. Let $\{(U_i, \chi_i)\}$ be a system of local trivializations of P and let $\{\rho_{ij}\}$ be the corresponding system of transition mappings. Let $\{(U_i, \xi_i)\}$ be the induced system of local trivializations of $P \times_G F$. Let us find the corresponding system of transition mappings. For $m \in U_i \cap U_j, f \in F$ and $p \in \pi^{-1}(m)$, we calculate

$$\xi_i \circ \xi_j^{-1}(m, f) = \xi_i([(p, \sigma_{\kappa_j(p)^{-1}}(f))]) = (m, \sigma_{\kappa_i(p)} \circ \sigma_{\kappa_j(p)^{-1}}(f)).$$

Since $\rho_{ij}(m) = \kappa_i(p)\kappa_j(p)^{-1}$, we obtain

$$\xi_i \circ \xi_j^{-1}(m, f) = (m, \sigma_{\rho_{ij}(m)}(f)), \tag{1.2.5}$$

that is, the transition mappings of $P \times_G F$ are given by $\sigma_{\rho_{ij}} : U_i \cap U_j \to \mathrm{Diff}(F)$. Then, in complete analogy to Proposition 1.1.10, one can reconstruct $P \times_G F$ from the transition mappings $\sigma_{\rho_{ij}}$. ♦

Example 1.2.4

1. Let $P(M, G)$ be a principal fibre bundle and let $H \subset G$ be a closed subgroup. Then, by Theorem I/5.6.8, H is an embedded Lie subgroup of G. Consider the action of G on the homogeneous space G/H by left translation. Then, $P \times_G G/H$ is an associated bundle over M with typical fibre being a transitive G-manifold. One can show the following, see Exercise 1.2.2:
   (a) As a fibre bundle over M, the associated bundle $P \times_G G/H$ is isomorphic to the quotient P/H, endowed with the natural fibre bundle structure induced from P.
   (b) P may be viewed as a principal H-bundle over $P \times_G G/H$.
2. Let $P(M, G)$ be a principal bundle, let $E = P \times_G F$ be an associated bundle and let $\varphi : N \to M$ be a smooth mapping of manifolds. Consider the pullback bundle

$\varphi^* E$,

$$\begin{tikzcd} \varphi^* E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi_F"] \\ N \arrow[r, "\varphi"] & M \end{tikzcd}$$

In this notation, $\varphi^* E = \{(y, e) \in N \times E : \varphi(y) = \pi_F(e)\}$. It is easy to show that the mapping

$$\varphi^* E \to \varphi^* P \times_G F, \quad (y, [(p, f)]) \mapsto [((y, p), f)] \tag{1.2.6}$$

is well defined and an isomorphism of fibre bundles (Exercise 1.2.3). Thus, $\varphi^* E$ is naturally associated with $\varphi^* P$.

3. Consider the fibre product $P_1 \times_M P_2$ of two principal bundles $P_1(M, G_1)$ and $P_2(M, G_2)$, cf. Remark 1.1.9/2. Let (F_i, G_i, σ_i), $i = 1, 2$, be Lie group representations and let $E_i = P_i \times_{G_i} F_i$ be associated vector bundles. Taking the tensor product representation $\sigma_1 \otimes \sigma_2 : G_1 \times G_2 \to \operatorname{Aut}(F_1 \otimes F_2)$ of $G_1 \times G_2$, defined by
$$(\sigma_1 \otimes \sigma_2)_{(g_1, g_2)} (f_1 \otimes f_2) := (\sigma_1)_{g_1}(f_1) \otimes (\sigma_2)_{g_2}(f_2), \tag{1.2.7}$$
we can build the associated bundle $(P_1 \times P_2) \times_{(G_1 \times G_2)} (F_1 \otimes F_2)$ over $M \times M$. We take the pullback of this bundle under the diagonal mapping
$$\Delta : M \to M \times M.$$
Using point 2, we obtain
$$\Delta^* \big((P_1 \times P_2) \times_{(G_1 \times G_2)} (F_1 \otimes F_2)\big) = (P_1 \times_M P_2) \times_{(G_1 \times G_2)} (F_1 \otimes F_2).$$
It is easy to show that this bundle is isomorphic to the tensor product $E_1 \otimes E_2$ (Exercise 1.2.4), that is, $E_1 \otimes E_2$ is naturally associated with the fibre product $P_1 \times_M P_2$. Moreover, one can prove [472] that every finite-dimensional irreducible representation of $G_1 \times G_2$ is equivalent to the tensor product of irreducible representations of G_1 and G_2, that is, by the above construction we exhaust all finite-dimensional irreducible representations of $G_1 \times G_2$. ♦

Let P be a principal G-bundle over M and let $\lambda : G \to H$ be a Lie group homomorphism. Consider the associated bundle

$$P^{[\lambda]} := P \times_G H, \tag{1.2.8}$$

where G acts on H by left translations via λ. Since left and right translations on H commute, the action of H on $P \times H$ by right translation on the second factor descends to a free right action of H on $P^{[\lambda]}$. Clearly, this action turns $P^{[\lambda]}$ into a principal H-bundle over M, called the principal H-bundle associated with P via λ. The proof of the following proposition is left to the reader (Exercise 1.2.5).

Proposition 1.2.5 (Associated principal bundles) *Let $\lambda : G \to H$ be a Lie group homomorphism and let P, P_1, P_2 be principal G-bundles over, respectively, M, M_1, M_2.*

1. *If $\vartheta : P_1 \to P_2$ is a morphism of principal G-bundles, then the mapping*

$$P_1{}^{[\lambda]} \to P_2{}^{[\lambda]}, \quad [(p, a)] \mapsto \big[\big(\vartheta(p), a\big)\big]$$

is a morphism of principal H-bundles having the same projection as ϑ.

2. *If $f : N \to M$ is a smooth mapping, then the induced mapping*

$$f^*(P^{[\lambda]}) \to (f^*P)^{[\lambda]}, \quad \big(m, [(p, a)]\big) \mapsto \big[\big((m, p), a\big)\big]$$

is a vertical isomorphism.

3. *For $i = 1, 2$, let G_i and H_i be Lie groups and let $\lambda_i : G_i \to H_i$ be Lie group homomorphisms. By restriction, the rearrangement*

$$(P_1 \times P_2) \times (H_1 \times H_2) \to (P_1 \times H_1) \times (P_2 \times H_2)$$

induces a vertical isomorphism $(P_1 \times P_2)^{[\lambda_1 \times \lambda_2]} \cong P_1{}^{[\lambda_1]} \times P_2{}^{[\lambda_2]}$. ■

Next, we study the structure of the set of smooth sections $\Gamma^\infty(P \times_G F)$. For that purpose, let $\mathrm{Hom}_G(P, F)$ be the set of smooth equivariant mappings $\tilde{\Phi} : P \to F$,

$$\tilde{\Phi} \circ \Psi_a = \sigma_{a^{-1}} \circ \tilde{\Phi}\ . \tag{1.2.9}$$

Proposition 1.2.6 *For every $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$, there exists a unique element $\Phi \in \Gamma^\infty(P \times_G F)$ such that the following diagram commutes.*

$$\begin{tikzcd} P \arrow[r, "(\mathrm{id}_P, \tilde{\Phi})"] \arrow[d, "\pi"'] & P \times F \arrow[d, "\iota"] \\ M \arrow[r, "\Phi"] & P \times_G F \end{tikzcd} \tag{1.2.10}$$

The assignment $\tilde{\Phi} \mapsto \Phi$ defines a bijection from $\mathrm{Hom}_G(P, F)$ *onto $\Gamma^\infty(P \times_G F)$.*

Proof For $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$, we define

$$\Phi(m) := [(p, \tilde{\Phi}(p))], \tag{1.2.11}$$

where $p \in \pi^{-1}(m)$. This is a well-defined section of $P \times_G F$, because the equivariance property (1.2.9) implies

$$[(\Psi_a(p), \tilde{\Phi}(\Psi_a(p)))] = [(\Psi_a(p), \sigma_{a^{-1}}\tilde{\Phi}(p))] = [(p, \tilde{\Phi}(p))]$$

for all $a \in G$. By definition of Φ, the above diagram commutes. Conversely, since

$$\Phi(m) = [(p, \tilde{\Phi}(p))] = \iota_p \tilde{\Phi}(p),$$

$\tilde{\Phi}$ can be uniquely reconstructed from Φ. ■

Proposition 1.2.6 applies, in particular, to the case where $F = Q$ is a principal G-bundle[7] and thus yields a bijective correspondence between morphisms $P \to Q$ of principal G-bundles and sections of $P \times_G Q$. In the special case where P and Q have the same base manifold M, this correspondence can be refined to describe vertical morphisms as follows. The direct product mapping $\pi_P \times \pi_Q : P \times Q \to M \times M$ defined by the projections $\pi_P : P \to M$ and $\pi_Q : Q \to M$ descends to a surjective submersion

$$\pi_P \times_G \pi_Q : P \times_G Q \to M \times M\,. \tag{1.2.12}$$

This is a fibre bundle with typical fibre G: given local trivializations (U_P, χ_P) of P and (U_Q, χ_Q) of Q, one can check that the mapping $\chi_P \times_G \chi_Q$ defined by

$$\begin{tikzcd}
\pi_P^{-1}(U_P) \times \pi_Q^{-1}(U_Q) \arrow[r, "\chi_P \times \chi_Q"] \arrow[d] & U_P \times U_Q \times G \times G \arrow[d, "\mathrm{id}_{U_P} \times \mathrm{id}_{U_Q} \times \mu"] \\
(\pi_P \times_G \pi_Q)^{-1}(U_P \times U_Q) \arrow[r, "\chi_P \times_G \chi_Q"] & U_P \times U_Q \times G
\end{tikzcd}$$

with $\mu(a, b) = ab^{-1}$ is a diffeomorphism. Let $P \times_{G,M} Q$ denote the restriction[8] of the fibre bundle (1.2.12) to the diagonal $M \subset M \times M$. Then, $P \times_{G,M} Q$ is an embedded submanifold of $P \times_G Q$ and the induced projection $P \times_{G,M} Q \to M$ coincides with the restriction of the associated bundle projection $P \times_G Q \to M$ to this submanifold. Thus, $P \times_{G,M} Q$ is an embedded vertical subbundle of the associated bundle $P \times_G Q$.

Corollary 1.2.7 *By restriction, the bijection between G-morphisms $P \to Q$ and sections of the associated bundle $P \times_G Q$ induces a bijection between vertical G-morphisms $P \to Q$ and sections of the vertical subbundle $P \times_{G,M} Q$.*

Proof Let $\vartheta : P \to Q$ be a G-morphism. Proposition 1.2.6 assigns to ϑ a section s of $P \times_G Q$ via $s(m) = [\big(p, \vartheta(p)\big)]$, where $p \in \pi_P^{-1}(m)$. We compute

$$(\pi_P \times_G \pi_Q) \circ s(m) = \big(m, \tilde{\vartheta}(m)\big)\,.$$

Thus, $\tilde{\vartheta} = \mathrm{id}_M$ iff s takes values in the submanifold $P \times_{G,M} Q \subset P \times_G Q$, and hence is a section in the vertical subbundle $P \times_{G,M} Q$. ■

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[7] Clearly, the definition of associated bundle carries over to right G-manifolds F.

[8] Defined by the pullback via the diagonal mapping $M \to M \times M$.

For the remainder of this section, we assume that F is a finite-dimensional vector space carrying a representation of the structure group G.

Proposition 1.2.8 *Let $P(M, G)$ be a principal G-bundle and let (F, G, σ) be a Lie group representation. Then,*

1. *the associated bundle $P \times_G F$ is a vector bundle,*
2. *the bijection between $\mathrm{Hom}_G(P, F)$ and $\Gamma^\infty(P \times_G F)$ given by Proposition 1.2.6 is an isomorphism of vector spaces.*
3. *If $\vartheta : P_1 \to P_2$ is a morphism of principal G-bundles with projection $\tilde{\vartheta}$, then the mapping*
$$P_1 \times_G F \to P_2 \times_G F, \quad [(p, f)] \mapsto \big[\big(\vartheta(p), f\big)\big]$$
is a morphism of vector bundles with projection $\tilde{\vartheta}$.
4. *If $f : N \to M$ is a smooth mapping, then the induced mapping*
$$f^*(P \times_G F) \to (f^*P) \times_G F, \quad \big(m, [(p, f)]\big) \mapsto \big[\big((m, p), f\big)\big]$$
is a vertical vector bundle isomorphism.

Proof 1. We endow the fibres of $P \times_G F$ with a vector space structure by requiring that the diffeomorphisms ι_p be linear (and thus vector space isomorphisms) for all $p \in P$. Since the mappings σ_a are vector space automorphisms, formula (1.2.3) implies that for every pair of points p, p' belonging to the same fibre, $\iota_{p'}$ is linear iff ι_p is linear. Thus, this vector space structure is well defined. Now, let (U, χ) be a local trivialization of P, let s be the corresponding local section of P and let (U, ξ) be the corresponding local trivialization of $P \times_G F$. We have to show that, with respect to the above defined linear structure, the induced mappings

$$\mathrm{pr}_2 \circ \xi\restriction_{\pi_F^{-1}(m)} : \pi_F^{-1}(m) \to F, \quad m \in U,$$

are linear. Using (1.2.4) and $\kappa(s(m)) = \mathbb{1}$, we obtain

$$\mathrm{pr}_2 \circ \xi \circ \iota_{s(m)}(f) = \sigma_{\kappa(s(m))}(f) = f.$$

Thus, $\mathrm{pr}_2 \circ \xi\restriction_{\pi_F^{-1}(m)} = \iota_{s(m)}^{-1}$ and the assertion follows.

2. This is an immediate consequence of the linearity of ι_p.

3 and 4. This is analogous to points 1 and 2 of Proposition 1.2.5. ■

Remark 1.2.9

1. By definition of ι_p, the linear structure on the fibre through $[(p, f)] \in P \times_G F$ is given as follows:
$$\lambda_1[(p, f_1)] + \lambda_2[(p, f_2)] = [(p, \lambda_1 f_1 + \lambda_2 f_2)], \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Thus, to calculate the sum of two elements one has to choose representatives with the same p.

2. Let E be a $\mathbb{K}$-vector bundle of rank k over M, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $L(E)$ be its principal $\mathrm{GL}(k, \mathbb{K})$-bundle of linear frames, cf. Example 1.1.15. Let L_m be the set of bases $s_m = (s_1, \ldots, s_k)$ in the fibre E_m. Clearly, the vector bundle E is associated with $L(E)$, that is, there exists a vector bundle isomorphism $L(E) \times_{\mathrm{GL}(k,\mathbb{K})} \mathbb{K}^k \cong E$, given by

$$[((s_1, \ldots, s_k), \mathbf{x})] \mapsto \sum_{i=1}^{k} x_i s_i .$$

This shows that any vector bundle may be viewed as a bundle associated with a principal bundle. If E carries a fibre metric, we have an analogous isomorphism between E and the bundle $O(E) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ associated with the orthonormal frame bundle $O(E)$, cf. Definition 1.1.19, via the standard representation of $\mathrm{U}_{\mathbb{K}}(k)$ on $\mathbb{K}^k$. ♦

In the sequel, we denote $E = P \times_G F$. Since E is a vector bundle, we can form the tensor product $\bigwedge^k(\mathrm{T}^*M) \otimes E$ and we may consider sections of this bundle.

Definition 1.2.10 A section in $\bigwedge^k(\mathrm{T}^*M) \otimes E$ is called a differential k-form on M with values in E. The vector space of these sections will be denoted by $\Omega^k(M, E)$.

Since $\bigwedge^0(\mathrm{T}^*M) = M \times \mathbb{R}$ and $(M \times \mathbb{R}) \otimes E = E$, we may identify $\Omega^0(M, E)$ with $\Gamma^\infty(E)$. In analogy to the case of sections, elements of $\Omega^k(M, E)$ may be viewed as differential forms on P.

Definition 1.2.11 Let $P(M, G)$ be a principal bundle and let (F, G, σ) be a finite-dimensional representation of G. A differential k-form $\tilde{\alpha}$ on P with values in F is called horizontal of type σ if it is annihilated by any vector tangent to the fibres and if it fulfils

$$\Psi_a^* \tilde{\alpha} = \sigma_{a^{-1}} \circ \tilde{\alpha}$$

for every $a \in G$. The vector space of horizontal k-forms of type σ will be denoted by $\Omega^k_{\sigma,\mathrm{hor}}(P, F)$.

Correspondingly, the space of ordinary horizontal differential k-forms on P will be denoted by $\Omega^k_{\mathrm{hor}}(P)$.

Proposition 1.2.12 *To every element $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ there corresponds a unique element $\alpha \in \Omega^k(M, E)$ such that the following diagram commutes.*

$$\begin{array}{ccc} \bigwedge^k(\mathrm{T}P) & \xrightarrow{\mathrm{pr} \times \tilde{\alpha}} & P \times F \\ {\scriptstyle \Lambda^k \pi'} \downarrow & & \downarrow {\scriptstyle \iota} \\ \bigwedge^k(\mathrm{T}M) & \xrightarrow{\alpha} & E \end{array}$$

Here, $\mathrm{pr} : \bigwedge^k(\mathrm{T}P) \to P$ *denotes the natural projection. The assignment* $\tilde{\alpha} \mapsto \alpha$ *defines a vector space isomorphism from* $\Omega^k_{\sigma,\mathrm{hor}}(P, F)$ *onto* $\Omega^k(M, E)$*.*

Proof Let $m \in M$ and $X_i \in \mathrm{T}_m M$, $i = 1, \dots, k$. Choose $p \in P$ fulfilling $\pi(p) = m$ and $Y_i \in \mathrm{T}_p P$ such that $\pi'(Y_i) = X_i$. We define

$$\alpha_m(X_1, \dots, X_k) := \iota_p \circ \tilde{\alpha}_p(Y_1, \dots, Y_k). \tag{1.2.13}$$

We must show that this definition does neither depend on the choice of p nor on the choice of the Y_i. Thus, take $p' = \Psi_a(p)$ and tangent vectors Y'_i at p' which also project onto the X_i. Then, there exist vertical vectors $Z_i \in \mathrm{T}_{p'}P$ such that $Y'_i = \Psi'_a(Y_i) + Z_i$ and we obtain

$$\begin{aligned}
\iota_{p'} \circ \tilde{\alpha}_{p'}(Y'_1, \dots, Y'_k) &= \iota_{\Psi_a(p)} \circ \tilde{\alpha}_{\Psi_a(p)}(\Psi'_a(Y_1) + Z_1, \dots, \Psi'_a(Y_k) + Z_k) \\
&= \iota_p \circ \sigma_a \circ \left(\Psi_a^* \tilde{\alpha}\right)_p (Y_1, \dots, Y_k) \\
&= \iota_p \circ \tilde{\alpha}_p(Y_1, \dots, Y_k).
\end{aligned}$$

Here, we have used (1.2.3) together with the horizontality and equivariance of $\tilde{\alpha}$. Bijectivity and linearity of the assignment $\tilde{\alpha} \mapsto \alpha$ follow from the bijectivity and linearity of ι_p. ■

Note that, conversely, we have

$$\tilde{\alpha}_p = \iota_p^{-1} \circ (\pi^* \alpha)_p \,. \tag{1.2.14}$$

The following is left to the reader (Exercise 1.2.6).

Remark 1.2.13 Let $\alpha \in \Omega^k(M, E)$, let $\beta \in \Omega^l(M)$ and let $\tilde{\alpha}$ and $\tilde{\beta}$ be the corresponding horizontal forms on P with values in F and in $\mathbb{R}$, respectively. Clearly, $\tilde{\beta} = \pi^* \beta$. Then,

$$\tilde{\beta} \wedge \tilde{\alpha} = \widetilde{\beta \wedge \alpha} \,.$$

Thus, the direct sums

$$\Omega^*(M, E) = \bigoplus_{k=0}^{\infty} \Omega^k(M, E) \quad \text{and} \quad \Omega^*_{\sigma,\mathrm{hor}}(P, F) = \bigoplus_{k=0}^{\infty} \Omega^k_{\sigma,\mathrm{hor}}(P, F)$$

carry the structure of modules over the Cartan algebra $\Omega^*(M)$. ♦

We close this section by giving the local description of the above notions. Let (U, χ) be a local trivialization of P, let $\kappa : P \to G$ be the corresponding equivariant mapping and let $s \colon U \to P$ be the associated local section. We define the local representative of $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ by

$$\tilde{\alpha}^{\chi} := s^* \tilde{\alpha}. \tag{1.2.15}$$

This is a k-form on U with values in F. The following proposition shows that a horizontal form of type σ may be reconstructed from its local representatives.

Proposition 1.2.14 *Let $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ and let $\tilde{\alpha}^\chi$ be its representative in a local trivialization (U, χ) given by (1.2.15). Then, for every $p \in \pi^{-1}(U)$, we have*

$$\tilde{\alpha}_p = \sigma_{\kappa(p)^{-1}} \circ (\pi^* \tilde{\alpha}^\chi)_p. \tag{1.2.16}$$

Proof By the equivariance of $\tilde{\alpha}$, for every $p \in \pi^{-1}(U)$ und $Y_i \in \mathrm{T}_p P$, we obtain

$$\begin{aligned}\sigma_{\kappa(p)} \circ \tilde{\alpha}_p(Y_1, \ldots, Y_k) &= \left(\Psi^*_{\kappa(p)^{-1}} \tilde{\alpha}\right)_p (Y_1, \ldots, Y_k) \\ &= \tilde{\alpha}_{\Psi_{\kappa(p)^{-1}}(p)}(\Psi'_{\kappa(p)^{-1}}(Y_1), \ldots, \Psi'_{\kappa(p)^{-1}}(Y_k)).\end{aligned}$$

Since $\Psi_{\kappa(p)^{-1}}(p) = s(\pi(p))$, we have $\Psi'_{\kappa(p)^{-1}}(Y_i) \in \mathrm{T}_{s(\pi(p))} P$ and

$$\pi' \left(\Psi'_{\kappa(p)^{-1}}(Y_i) - s' \circ \pi'(Y_i)\right) = 0.$$

Thus, using the horizontality of $\tilde{\alpha}$, in the above formula we may replace the tangent vectors $\Psi'_{\kappa(p)^{-1}}(Y_i)$ by $s' \circ \pi'(Y_i)$. This yields

$$\begin{aligned}\sigma_{\kappa(p)} \circ \tilde{\alpha}_p(Y_1, \ldots, Y_k) &= \tilde{\alpha}_{s(\pi(p))}\left(s' \circ \pi'(Y_1), \ldots, s' \circ \pi'(Y_k)\right) \\ &= (\pi^*(s^* \tilde{\alpha}))_p(Y_1, \ldots, Y_k),\end{aligned}$$

and, thus, the assertion. ■

Remark 1.2.15

1. Let $\{(U_i, \chi_i)\}_{i \in I}$ be a system of local trivializations of P and let (U_j, χ_j) and (U_k, χ_k) be elements of this system fulfilling $U_j \cap U_k \neq \varnothing$. Then, (1.2.16) and (1.1.5) imply
$$\tilde{\alpha}^{\chi_j}_m = \sigma_{\rho_{jk}(m)} \tilde{\alpha}^{\chi_k}_m, \quad m \in U_j \cap U_k.$$
It is easy to show that a system of k-forms $\{\tilde{\alpha}^{\chi_i}\}_{i \in I}$ fulfilling these relations defines a unique element of $\Omega^k_{\sigma,\mathrm{hor}}(P, F)$ with local representatives $\tilde{\alpha}^{\chi_i}$ (Exercise 1.2.7).
2. Let α be the k-form on M with values in E corresponding to $\tilde{\alpha}$ and let (U, ξ) be the local trivialization of E induced by (U, χ) via (1.2.1). We define the local representative of α by
$$\alpha^\xi = \mathrm{pr}_2 \circ \xi \circ \alpha_{\restriction U}. \tag{1.2.17}$$
Using (1.2.13), (1.2.16) and (1.2.1), we calculate
$$\alpha_m(X_1, \ldots, X_k) = \left[\left(s(m), \tilde{\alpha}^\chi_m(X_1, \ldots, X_k)\right)\right] = \xi^{-1}\left(m, \tilde{\alpha}^\chi_m(X_1, \ldots, X_k)\right) \tag{1.2.18}$$
for $m \in U$ and $X_i \in \mathrm{T}_m M$. Thus, $\alpha^\xi = \tilde{\alpha}^\chi$ as expected.

3. In particular, for a section $\Phi \in \Gamma^{\infty}(E)$ we denote $\varphi := \Phi^{\xi} = \tilde{\Phi}^{\chi}$. This is a function on U with values in F. Here, the reconstruction formulae read

$$\Phi(m) = [(s(m), \varphi(m))], \quad \tilde{\Phi}(p) = \sigma_{\kappa(p)^{-1}}\varphi(\pi(p)),$$

for any $m \in U$ and $p \in \pi^{-1}(U)$. ♦

Exercises

1.2.1 Prove Proposition 1.2.2.

1.2.2 Prove the assertions of Example 1.2.4/1.
Hint. To prove point (a), show that the mapping

$$i \colon P \times_G G/H \to P/H, \quad i([(p, gH)]) := [\Psi_g(p)]$$

is bijective. To prove point (b), construct local trivializations of $P \times_G G/H$ from local trivializations of the principal G-bundle $P \to M$ and of the principal H-bundle $G \to G/H$.

1.2.3 Prove that formula (1.2.6) defines a vector bundle isomorphism.

1.2.4 Prove the statements of Example 1.2.4/3.

1.2.5 Prove Proposition 1.2.5.

1.2.6 Prove the statements of Remark 1.2.13.

1.2.7 Prove the statement of Remark 1.2.15/1.

1.3 Connections

The notion of connection will play a fundamental role throughout this book, because it yields the mathematical model for a gauge potential.

To start with, we recall the notion of Killing vector field, cf. Sect. 6.2 of Part I. Given a Lie group action (P, G, Ψ), every element A of the Lie algebra $\mathfrak{g}$ of G defines a vector field A_* via the flow $\Psi_{\exp(tA)}$, that is,

$$(A_*)_p = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 \Psi_{\exp(tA)}(p) = \Psi'_p(A)\,.$$

A_* is called the Killing vector field generated by A.

Now, consider a principal fibre bundle (P, G, M, Ψ, π). We denote the vertical distribution spanned by the Killing vector fields of the G-action by V and call $V_p \subset \mathrm{T}_p P$ the vertical subspace of $\mathrm{T}_p P$ at $p \in P$.

Lemma 1.3.1 *The vertical distribution V has the following properties.*

1. *It is equivariant, that is,* $V_{\Psi_a(p)} = \Psi_a'(V_p)$.
2. *The mapping*
$$\psi : P \times \mathfrak{g} \to V, \quad (p, A) \mapsto \Psi_p'(A)$$
is an isomorphism of vector bundles. In particular, the mappings $\Psi_p' : \mathfrak{g} \to V_p$ are isomorphisms of vector spaces.
3. *For every $p \in P$, the vertical subspace V_p coincides with the tangent space of the fibre at p and, thus, with* $\ker(\pi_p')$.

Proof 1. This follows from Proposition I/6.2.2/1.

2. By construction, ψ is a surjective vertical morphism of vector bundles. Since Ψ is a free action, Proposition I/6.2.2/3 implies that ψ is injective. Thus, the tangent mapping ψ' is bijective at any point and, consequently, the Inverse Mapping Theorem I/1.5.7 implies that the inverse mapping is smooth.

3. This is an immediate consequence of the Orbit Theorem I/6.2.8. ■

Since, by definition, V is spanned by the Killing vector fields, to prove $V_{\Psi_a(p)} = \Psi_a'(V_p)$ it is enough to study the transport of a Killing vector field under Ψ. One finds

$$\Psi_a' A_*(p) = \left(\mathrm{Ad}\left(a^{-1}\right) A\right)_* (\Psi_a(p)) . \tag{1.3.1}$$

Also note that, by point 2 of Lemma 1.3.1, as a vector bundle, V is trivial.

Now, we can define the notion of connection.

Definition 1.3.2 (*Connection on a principal fibre bundle*) Let (P, G, M, Ψ, π) be a principal fibre bundle. A connection on P is a distribution[9] Γ on P such that

1. $\Gamma_p \oplus V_p = \mathrm{T}_p P$ for all $p \in P$,
2. $\Gamma_{\Psi_a(p)} = \Psi_a'(\Gamma_p)$ for all $p \in P$ and $a \in G$.

Γ_p is called the horizontal subspace at p.

A connection on a principal bundle will be often referred to as a principal connection.

Remark 1.3.3

1. By point 1, every tangent vector $X_p \in \mathrm{T}_p P$ admits a unique decomposition into a horizontal component $\mathrm{hor}\, X_p \in \Gamma_p$ and a vertical component $\mathrm{ver}\, X_p \in V_p$,
$$X_p = \mathrm{hor}\, X_p + \mathrm{ver}\, X_p. \tag{1.3.2}$$

[9] As in Part I, distributions are assumed to be smooth without notice.

Since both Γ and V are smooth, the mappings $\mathrm{hor} : \mathrm{T}P \to \Gamma$ and $\mathrm{ver} : \mathrm{T}P \to V$ are smooth. Thus, if X is a smooth vector field on P, then both $\mathrm{hor}\, X$ and $\mathrm{ver}\, X$ are smooth vector fields, too.

2. For a given connection Γ, the restriction of π' to the horizontal subspace Γ_p yields an isomorphism of Γ_p and $\mathrm{T}_{\pi(p)}M$. Thus, every vector field X on M admits a unique horizontal lift, that is, a vector field X^h on P with values in the horizontal distribution which is π-related to X. It is obtained by applying the inverse of the above isomorphism pointwise to X. By construction, X^h is Ψ-invariant. The proof of smoothness of X^h is left to the reader (Exercise 1.3.1). Conversely, every Ψ-invariant horizontal vector field on P is the horizontal lift of a vector field on M.
3. Every connection on a principal bundle P induces a connection on any bundle associated with P. Indeed, let Γ be a connection on the principal bundle $P(M, G)$ and let $E = P \times_G F$ be an associated bundle. For $f \in F$, we define

$$\iota_f : P \to E, \quad \iota_f(p) = [(p, f)].$$

This mapping has the following properties:

$$\iota_f \circ \Psi_a = \iota_{\sigma_a(f)}, \quad \pi_F \circ \iota_f = \pi. \tag{1.3.3}$$

The horizontal subspace at $e = [(p, f)] \in E$ is defined by

$$\Gamma_e^E := \iota_f'(\Gamma_p). \tag{1.3.4}$$

By the first relation in (1.3.3), the right hand side of this equation does not depend on the choice of the representative (p, f) of e. Since $p \mapsto \Gamma_p$ is a smooth distribution, $e \mapsto \Gamma_e^E$ is smooth, too. We show that this distribution is complementary to the canonical vertical distribution $e \mapsto V_e^E$, where V_e^E denotes the tangent space to the fibre at $e \in E$. By the second equation in (1.3.3), $\iota_f'(V_p)$ is contained in V_e^E and $\pi_F'(\Gamma_e^E) = \pi'(\Gamma_p) = \mathrm{T}_m M$, where $m = \pi_F(e)$. Thus, we have a direct sum decomposition,

$$\mathrm{T}_e E = V_e^E \oplus \Gamma_e^E.$$

The horizontal distribution $e \mapsto \Gamma_e^E$ will be referred to as the connection on E induced by Γ. In particular, the restriction of the tangent mapping π_F' to Γ_e^E defines an isomorphism from Γ_e^E onto $\mathrm{T}_m M$ and, thus, every tangent vector $X \in \mathrm{T}_m M$ admits a unique horizontal lift $X_e^h \in \Gamma_e^E$. By (1.3.4), it is given by

$$X_e^h = \iota_f'(X_p^h), \tag{1.3.5}$$

where X_p^h is the unique horizontal lift of X to the point p of P. ♦

Since, by point 2 of Lemma 1.3.1, $\Psi'_p : \mathfrak{g} \to V_p$ is a vector space isomorphism, with every connection we may associate a $\mathfrak{g}$-valued 1-form on P.

Definition 1.3.4 (*Connection form*) Let (P, G, M, Ψ, π) be a principal bundle and let Γ be a connection on P. The 1-form ω on P with values in $\mathfrak{g}$ defined by

$$\omega_p(X) := (\Psi'_p)^{-1}(\mathrm{ver}\, X), \quad p \in P\,,\; X \in \mathrm{T}_pP, \tag{1.3.6}$$

is called the connection form of Γ.

As an immediate consequence of this definition, we obtain the following formula of the horizontal component of a tangent vector $X \in \mathrm{T}_pP$:

$$\mathrm{hor}\, X = X - \Psi'_p(\omega(X)). \tag{1.3.7}$$

Proposition 1.3.5 *Let (P, G, M, Ψ, π) be a principal bundle and let Γ be a connection on P. Then, the connection form ω of Γ is smooth and has the following properties.*

1. $\ker(\omega_p) = \Gamma_p$ *for all $p \in P$,*
2. $\omega(A_*) = A$ *for all $A \in \mathfrak{g}$,*
3. $\Psi_a^*\omega = \mathrm{Ad}\left(a^{-1}\right) \circ \omega$ *for all $a \in G$.*

Proof By Lemma 1.3.1, we may decompose the mapping $\mathrm{T}P \ni X \mapsto \omega(X) \in \mathfrak{g}$ as follows:

$$\mathrm{T}P \overset{\mathrm{ver}}{\to} V \overset{\psi^{-1}}{\to} P \times \mathfrak{g} \overset{\mathrm{pr}_2}{\to} \mathfrak{g}.$$

This shows that ω is smooth. Assertions 1 and 2 are immediate consequences of the definition of ω. It remains to prove assertion 3. By point 1, it is enough to apply both sides of the equation to a Killing vector field. Using (1.3.1), we obtain

$$\langle \Psi_a^*\omega, A_*\rangle = \langle \omega, \Psi_{a*}A_*\rangle = \left\langle \omega, \left(\mathrm{Ad}\left(a^{-1}\right)A\right)_*\right\rangle = \mathrm{Ad}\left(a^{-1}\right)A = \mathrm{Ad}\left(a^{-1}\right)\langle \omega, A_*\rangle.$$

■

Proposition 1.3.6 *Every $\mathfrak{g}$-valued 1-form ω on P fulfilling the conditions 2 and 3 of Proposition 1.3.5 uniquely defines a connection Γ.*

Proof We put $\Gamma_p := \ker(\omega_p)$. Now, the defining properties of the horizontal distribution $p \mapsto \Gamma_p$ follow directy from the properties of ω (Exercise 1.3.2). ■

Proposition 1.3.7 *Every principal fibre bundle admits a connection.*

Proof Let (P, G, M, Ψ, π) be a principal fibre bundle, let $\{U_i\}_{i\in I}$ be a countable, locally finite covering of M and let $\{f_i\}_{i\in I}$ be a subordinate partition of unity. Choose a system of local trivializations $\{(U_i, \chi_i)\}_{i\in I}$ of P, associated with this covering. At every point $\chi_i^{-1}(m, \mathbb{1})$, $m \in M$, we define a subspace $\Gamma_{\chi_i^{-1}(m,\mathbb{1})}$ of $\mathrm{T}_{\chi_i^{-1}(m,\mathbb{1})}P$ by

$$\Gamma_{\chi_i^{-1}(m,\mathbb{1})} := \left(\chi_i'\right)^{-1} \mathrm{T}_{(m,\mathbb{1})}\left(U_i \times \{\mathbb{1}\}\right). \tag{1.3.8}$$

Clearly, $\Gamma_{\chi_i^{-1}(m,\mathbb{1})}$ is complementary to $V_{\chi_i^{-1}(m,\mathbb{1})}$. If we transport this subspace with Ψ_a', $a \in G$, to the remaining points of the fibre over m, for every $m \in U_i$, then we obtain a connection on the trivial principal G-bundle $\pi^{-1}(U_i)$. Let us denote the corresponding connection form by $\tilde{\omega}_i$. We define the following family of $\mathfrak{g}$-valued 1-forms on P:

$$(\omega_i)_p := \begin{cases} 0 & p \notin \pi^{-1}(U_i) \\ (\pi^* f_i)\, \tilde{\omega}_i & p \in \pi^{-1}(U_i)\,. \end{cases}$$

Since $\{\mathrm{supp}(f_i)\}$ is locally finite, $\omega := \sum_i \omega_i$ is a well-defined smooth 1-form on P with values in $\mathfrak{g}$. It remains to show that ω fulfils conditions 2 and 3 of Proposition 1.3.5.

Condition 2. For $p \in P$ and $A \in \mathfrak{g}$, we have

$$\omega_p(A_*(p)) = \sum_{i\in I} (\omega_i)_p(A_*(p)) = \sum_{i\in I^*} (\omega_i)_p(A_*(p))\,,$$

where $I^* \subset I$ contains exactly those indices for which $\pi(p) \in U_i$. Since every $\tilde{\omega}_i$ is a connection form, for $i \in I^*$, we obtain

$$(\omega_i)_p(A_*(p)) = f_i(\pi(p))(\tilde{\omega}_i)_p(A_*(p)) = f_i(\pi(p))A.$$

Now, $\sum_{i\in I^*} f_i(\pi(p)) = \sum_{i\in I} f_i(\pi(p)) = 1$ implies $\omega_p(A_*(p)) = A$.

Condition 3. It is enough to verify this condition for every ω_i restricted to $\pi^{-1}(U_i)$. Since $\Psi_a^*\left((\pi^* f_i)\, \tilde{\omega}_i\right) = (\pi^* f_i)\left(\Psi_a^* \tilde{\omega}_i\right)$ and since all $\tilde{\omega}_i$ share property 3, the assertion follows. ■

Remark 1.3.8 By the defining properties 2 and 3 of a connection form, cf. Proposition 1.3.5, the difference of two connection forms is a horizontal 1-form of type Ad. Thus, the set of connections of a principal fibre bundle carries the structure of an infinite-dimensional affine space with the translation vector space given by $\Omega^1_{\mathrm{Ad,hor}}(P, \mathfrak{g})$. This space will play a crucial role in gauge theory. ♦

Remark 1.3.9 By Remark 1.3.3/3, a principal connection Γ induces a connection Γ^E on every associated bundle $E = P \times_G F$. If (F, G, σ) is a Lie group representation and, thus, E is a vector bundle, then the canonical vertical subspace V_e^E may be naturally identified with the fibre through $e \in E$. In more detail, since in this case the mapping ι_p, given by (1.2.2), is a vector space isomorphism between F and the fibre $E_{\pi_F(p)}$, the tangent mapping ι_p' is a vector space isomorphism between $\mathrm{T}_f F \cong F$ and V_e^E, where $e = [(p, f)]$. Thus, for any $Z \in V_e^E$, there exists an element $v \in F$ such that $Z = \iota_p'(v)$. Via ι_p, the vector v may be identified with the element $[(p, v)]$ in the fibre $E_{\pi_F(p)}$. Thus, the above mentioned identification is given by

$$V_e^E \to E, \quad Z \mapsto \iota_p \circ (\iota_p')^{-1}(Z)\,.$$

We conclude that in the case of an associated vector bundle E, endowed with an induced connection Γ^E, we have an analogue of the connection form ω:

$$\omega^E : \mathrm{T}E \to E, \quad \omega^E(X) := \iota_p \circ (\iota'_p)^{-1}(X^v), \tag{1.3.9}$$

where $X = X^v + X^h$ is the decomposition of $X \in \mathrm{T}_e E$ with respect to Γ^E. Clearly, ω^E is a vector bundle morphism. It is called the connection mapping induced from ω. Using $\iota_f \circ \Psi_p = \iota_p \circ \sigma_f$, one easily finds the following relation between ω and ω^E (Exercise 1.3.3):

$$\omega^E \circ \iota'_f(X) = \iota_p \circ \sigma'(\omega(X))f, \quad X \in \mathrm{T}_p P. \tag{1.3.10}$$

Here, $\sigma' \equiv \mathrm{d}\sigma : \mathfrak{g} \to \mathrm{End}(F)$ is the representation of the Lie algebra $\mathfrak{g}$ of G induced from σ. The assignment $X \to \sigma'(\omega(X))$ defines a 1-form on P with values in $\mathrm{End}(F)$ which will be denoted by $\sigma'(\omega)$. ♦

It turns out that a connection on $P(M, G)$ is uniquely characterized in terms of its local representatives on the base space M. Let $s\colon U \to \pi^{-1}(U)$ be a local section. The local representative of a connection form ω on P is defined by

$$\mathscr{A} := s^*\omega. \tag{1.3.11}$$

Remark 1.3.10 Let (U, φ) be a local chart on M and let $\{\mathbf{t}_a\}$ be a basis in $\mathfrak{g}$. Then, the collection

$$\{\mathrm{d}\varphi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\varphi^{\mu_k} \otimes \mathbf{t}_a\} \tag{1.3.12}$$

yields a local frame in the bundle of $\mathfrak{g}$-valued k-forms on M. With respect to this frame, the local representative $\mathscr{A}$ takes the form

$$\mathscr{A} = \mathscr{A}^a_\mu \, \mathrm{d}\varphi^\mu \otimes \mathbf{t}_a.$$

♦

We show that ω may be reconstructed from $\mathscr{A}$ locally. For that purpose, recall from the proof of Proposition 1.1.6 that a section s defines an equivariant mapping $\kappa : P \to G$ by

$$\Psi_{\kappa(p)}(s \circ \pi(p)) = p, \quad p \in P. \tag{1.3.13}$$

Proposition 1.3.11 *Let (P, G, M, Ψ, π) be a principal bundle and let ω be a connection form on P. Let $U \subset M$ be open and let $s : U \to \pi^{-1}(U)$ be a local section. Let $\mathscr{A}$ be the local representative defined by (1.3.11). Then, for every $p \in \pi^{-1}(U)$,*

$$\omega_p = \mathrm{Ad}\left(\kappa(p)^{-1}\right)(\pi^*\mathscr{A})_p + (\kappa^*\theta)_p, \tag{1.3.14}$$

with θ denoting the Maurer–Cartan form[10] *on G.*

[10] See Definition I/5.5.11.

Proof Let $X \in \mathrm{T}_p P$ and let $t \mapsto \gamma(t)$ be a curve representing X. Then, using (1.3.13), we calculate

$$
\begin{aligned}
X &= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \gamma(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \Psi\left(s \circ \pi(\gamma(t)), \kappa(\gamma(t))\right) \\
&= \left(\Psi_{\kappa(p)}\right)'_{s(\pi(p))} \circ (s \circ \pi)'_p(X) + \left(\Psi_{s(\pi(p))}\right)'_{\kappa(p)} \circ \kappa'_p\,(X).
\end{aligned}
$$

Denoting the first and the second summand by X^s and X^v, respectively, we get a decomposition $X = X^s + X^v$, where X^s is tangent to the submanifold $\Psi_{\kappa(p)}(s(U))$ and where X^v is vertical. We calculate

$$
\begin{aligned}
\omega_p(X^s) &= \left(\Psi^*_{\kappa(p)}\omega\right)_{s(\pi(p))} \left((s \circ \pi)'_p(X)\right) \\
&= \mathrm{Ad}\left(\kappa(p)^{-1}\right) (s^*\omega)_{\pi(p)}(\pi'(X)) \\
&= \mathrm{Ad}\left(\kappa(p)^{-1}\right) (\pi^*\mathscr{A})_p(X).
\end{aligned}
$$

This yields the first summand in (1.3.14). On the other hand, by the definition of ω,

$$
\omega_p(X^v) = \left(\Psi'_p\right)^{-1} \circ \left(\Psi_{s(\pi(p))}\right)'_{\kappa(p)} \circ \kappa'_p(X).
$$

Using the obvious identity $\Psi_p^{-1} \circ \Psi_{s(\pi(p))} \circ L_{\kappa(p)} = \mathrm{id}_G$, together with

$$
(\kappa^*\theta)_p(X) = L'_{\kappa(p)^{-1}} \circ \kappa'_p(X),
$$

we obtain $\omega_p(X^v) = (\kappa^*\theta)_p(X)$. This proves (1.3.14). ∎

The following corollary is immediate (Exercise 1.3.4).

Corollary 1.3.12 *Let P be a principal G-bundle and let ω be a connection form on P. Let $\{(U_i, \chi_i)\}$ be a system of local trivializations of P with corresponding equivariant mappings $\{\kappa_i\}$, local sections $\{s_i\}$ and transition mappings $\{\rho_{ij}\}$. Let*

$$
\mathscr{A}_i = s_i^*\omega.
$$

Then, for any pair (i, j) such that $U_i \cap U_j \neq \varnothing$, the local representatives $\mathscr{A}_i$ and $\mathscr{A}_j$ are related as follows:

$$
\left(\mathscr{A}_j\right)_m = \mathrm{Ad}\left(\rho_{ij}(m)^{-1}\right) \circ (\mathscr{A}_i)_m + \left(\rho_{ij}^*\theta\right)_m, \quad m \in U_i \cap U_j. \tag{1.3.15}
$$

Conversely, any system of Lie algebra-valued 1*-forms* $\{\mathscr{A}_i\}$ *fulfilling* (1.3.15) *defines a unique connection form* ω *with local representatives* $\{\mathscr{A}_i\}$. ■

Next, let us discuss the transformation properties of connections under principal bundle morphisms.

Proposition 1.3.13 *Let* (ϑ, λ) *be a morphism of the principal bundles* $P_1(M_1, G_1)$ *and* $P_2(M_2, G_2)$ *such that the induced mapping* $\tilde{\vartheta} : M_1 \to M_2$ *is a diffeomorphism. Let* Γ^1 *be a connection on* P_1 *and let* ω_1 *be its connection form.*

1. *There exists a unique connection* Γ^2 *on* P_2 *such that* ϑ' *maps horizontal subspaces of* Γ^1 *to horizontal subspaces of* Γ^2.
2. *The connection form* ω_2 *of* Γ^2 *fulfils* $\vartheta^*\omega_2 = \mathrm{d}\lambda \circ \omega_1$, *where* $\mathrm{d}\lambda : \mathfrak{g}_1 \to \mathfrak{g}_2$ *is the induced homomorphism of Lie algebras. Moreover,* $\vartheta^*\Omega_2 = \mathrm{d}\lambda \circ \Omega_1$.

We call Γ^2 the image of Γ^1 under the morphism (ϑ, λ).[11]

Proof Denote the right group actions and the canonical projections in P_i, $i = 1, 2$, by Ψ^i and π_i, respectively.

1. We define a distribution Γ^2 on P_2 as follows. Since $\tilde{\vartheta}$ is surjective, for a given $p_2 \in P_2$, we can choose a pair $(p_1, a) \in P_1 \times G_2$ such that $p_2 = \Psi^2_a(\vartheta(p_1))$ and define

$$\Gamma^2_{p_2} := \left(\Psi^2_a\right)' \circ \vartheta' \left(\Gamma^1_{p_1}\right),$$

where $\Gamma^1_{p_1}$ is the horizontal subspace of Γ^1 at p_1. By (1.1.3), this definition does not depend on the choice of the pair (p_1, a). We prove that Γ^2 is a connection on P_2. First, we calculate

$$\left(\Psi^2_b\right)' \left(\Gamma^2_{p_2}\right) = \left(\Psi^2_b\right)' \circ \left(\Psi^2_a\right)' \circ \vartheta' \left(\Gamma^1_{p_1}\right) = \left(\Psi^2_{ab}\right)' \circ \vartheta' \left(\Gamma^1_{p_1}\right) = \Gamma^2_{\Psi^2_b(p_2)},$$

because $\Psi^2_b(p_2) = \Psi^2_{ab}(\vartheta(p_1))$. Thus, Γ^2 is G_2-equivariant. To prove that Γ^2 is complementary to the vertical distribution V^2 on P_2, by local triviality of the bundles, it is enough to show that the restriction of $\pi_2' : \mathrm{T}P_2 \to \mathrm{T}M_2$ to Γ^2 yields pointwise isomorphisms of vector spaces. Thus, consider the mapping $\pi_2' : \Gamma^2_{p_2} \to \mathrm{T}_{\pi_2(p_2)}M_2$. By G-equivariance of Γ^2, we may assume $p_2 = \vartheta(p_1)$. Then, from $\tilde{\vartheta} \circ \pi_1 = \pi_2 \circ \vartheta$, we have

$$\tilde{\vartheta}'_{\pi_1(p_1)} \circ (\pi_1)'_{p_1} = (\pi_2)'_{p_2} \circ \vartheta'_{p_1}.$$

Since, by assumption, $\tilde{\vartheta}$ is a diffeomorphism and Γ^1 is a connection, $\tilde{\vartheta}'$ and π_1' are both isomorphisms of vector spaces. Thus, $\pi_2' : \Gamma^2_{p_2} \to \mathrm{T}_{\pi_2(p_2)}M_2$ is an isomorphism, too. We conclude that Γ^2 is a connection. By construction, it is unique.

2. The first assertion is equivalent to

$$(\omega_2)_{\vartheta(p_1)} \left(\vartheta'(X)\right) = \mathrm{d}\lambda \left((\omega_1)_{p_1}(X)\right),$$

[11] It is also common to speak of the push forward or the transport of Γ^1 by the morphism (ϑ, λ).

for any $p_1 \in P_1$ and $X \in \mathrm{T}_{p_1} P_1$. Since ϑ' maps horizontal vectors to horizontal vectors, it is enough to prove this equality for vertical vectors, that is, for values of Killing vector fields. Thus, let A_* be the Killing vector field generated by $A \in \mathfrak{g}_1$. Since, for any $a \in G_1$,

$$\vartheta \circ \Psi^1_{p_1}(a) = \vartheta \circ \Psi^1_a(p_1) = \Psi^2_{\lambda(a)} \circ \vartheta(p_1) = \Psi^2_{\vartheta(p_1)} \circ \lambda(a),$$

we obtain

$$(\omega_2)_{\vartheta(p_1)}\left(\vartheta'(A_*)_{p_1}\right) = (\omega_2)_{\vartheta(p_1)}\left(\left(\Psi^2_{\vartheta(p_1)}\right)' \circ \mathrm{d}\lambda(A)\right) = \mathrm{d}\lambda(A).$$

Now, the assertion follows from the fact that $A = \omega_1(A_*)$. It remains to prove the second statement: for $X, Y \in \mathrm{T}_p P$, we calculate

$$\begin{aligned}
\vartheta^* \tilde{\Omega}(X, Y) &= \mathrm{d}\tilde{\omega}\left(\mathrm{hor}_{\tilde{\omega}} \circ \vartheta'(X), \mathrm{hor}_{\tilde{\omega}} \circ \vartheta'(Y)\right) \\
&= \mathrm{d}\tilde{\omega}\left(\vartheta' \circ \mathrm{hor}_{\omega}(X), \vartheta' \circ \mathrm{hor}_{\omega}(Y)\right) \\
&= \mathrm{d}(\vartheta^* \tilde{\omega})\left(\mathrm{hor}_{\omega}(X), \mathrm{hor}_{\omega}(Y)\right) \\
&= \mathrm{d}(\mathrm{d}\lambda \circ \omega)\left(\mathrm{hor}_{\omega}(X), \mathrm{hor}_{\omega}(Y)\right) \\
&= \mathrm{d}\lambda \circ \mathrm{d}\omega\left(\mathrm{hor}_{\omega}(X), \mathrm{hor}_{\omega}(Y)\right) \\
&= \mathrm{d}\lambda \circ \Omega(X, Y)\,.
\end{aligned}$$

■

Proposition 1.3.13 immediately implies the following.

Corollary 1.3.14 *For a Lie group homomorphism $\lambda : H \to G$ and a principal H-bundle Q, let $P := Q^{[\lambda]}$. Then, any connection Γ^Q on Q induces a unique connection Γ^P on P. The corresponding connection forms are related via*

$$\vartheta^* \omega^P = \mathrm{d}\lambda \circ \omega^Q,$$

where $\vartheta : Q \to P$ is the corresponding bundle morphism. ■

The induced connection Γ^P is often referred to as the λ-extension of Γ^Q.

Since the proof of the following proposition is by arguments similar to those in the proof of Proposition 1.3.13, we leave it to the reader, see Exercise 1.3.5.

Proposition 1.3.15 *Let (ϑ, λ) be a morphism of the principal bundles $P_1(M_1, G_1)$ and $P_2(M_2, G_2)$ such that $\lambda : G_1 \to G_2$ is an isomorphism. Let Γ^2 be a connection on P_2 and let ω_2 be its connection form.*

1. *There exists a unique connection Γ^1 on P_1 such that ϑ' maps horizontal subspaces of Γ^1 to horizontal subspaces of Γ^2.*
2. *The connection form ω_1 of Γ^1 fulfils $\vartheta^* \omega_2 = \mathrm{d}\lambda \circ \omega_1$ and $\vartheta^* \Omega_2 = \mathrm{d}\lambda \circ \Omega_1$.* ■

We call Γ^1 the connection induced by Γ^2 via the morphism (ϑ, λ).

Corollary 1.3.16 *Under the assumptions of Proposition 1.3.15, additionally, assume $G_1 = G_2 = G$ and let λ be the identical automorphism. Then, $\omega_1 = \vartheta^*\omega_2$. This means, in particular:*

1. *The pullback of a connection form under an automorphism of a principal bundle is a connection form.*
2. *For a principal bundle $P(M, G)$ and a mapping $f : N \to M$, every connction in P induces a connection on the pullback bundle f^*P.* ■

Remark 1.3.17

1. Proposition 1.3.13 remains true under the weaker assumptions that $\tilde{\vartheta}$ be a surjective submersion and that M_1 and M_2 have the same dimension. Similarly, in Proposition 1.3.15, it suffices to assume that $\mathrm{d}\lambda$ be an isomorphism of Lie algebras.
2. From the proof of Proposition 1.3.13 we read off the following. Let (ϑ, λ) be a morphism of the principal bundles $P_1(M_1, G_1)$ and $P_2(M_2, G_2)$. For $i = 1, 2$, let ω_i be a connection form on P_i and let Ω_i be its curvature form. If $\vartheta^*\omega_2 = \mathrm{d}\lambda \circ \omega_1$, then $\vartheta^*\Omega_2 = \mathrm{d}\lambda \circ \Omega_1$.
3. Consider the special case of the fibre product bundle $P_1 \times_M P_2 = \Delta^*(P_1 \times P_2)$, cf. Remark 1.1.9/2. Let $(\pi_i, \lambda_i) : P_1 \times_M P_2 \to P_i$, $i = 1, 2$, be the natural principal bundle homomorphisms defined by restriction of the canonical projections $\mathrm{pr}_i : P_1 \times P_2 \to P_i$ to $P_1 \times_M P_2$. The corresponding Lie group homomorphisms $\lambda_i : G_1 \times G_2 \to G_i$ are given by the canonical projections onto the first and the second component, respectively. Let Γ_1 and Γ_2 be connections on P_1 and P_2, respectively, and let ω_1 and ω_2 be the corresponding connection forms. Then,

$$\omega = \mathrm{pr}_1^*\,\omega_1 + \mathrm{pr}_2^*\,\omega_2$$

is obviously a connection form on $P_1 \times P_2$.[12] Now, by Corollary 1.3.16, $\vartheta^*\omega$ is the unique connection on $\Delta^*(P_1 \times P_2) = P_1 \times_M P_2$ induced from ω, where $\vartheta : P_1 \times_M P_2 \to P_1 \times P_2$ is the induced morphism. It is given by

$$\vartheta^*\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2. \tag{1.3.16}$$

♦

We close this section with a number of examples. All of them are related to the Maurer-Cartan form θ of a Lie group G. By Remark I/5.5.12/2, we have $\theta_a = a^{-1}\mathrm{d}a$, $a \in G$. Thus, θ is left invariant and right equivariant under the action of G by left and right translations, respectively. Clearly, the right equivariance property reads

$$R_a^*\theta = \mathrm{Ad}(a^{-1}) \circ \theta.$$

[12] The first summand takes values in the Lie algebra $\mathfrak{g}_1$ of G_1 and the second takes values in the Lie algebra $\mathfrak{g}_2$ of G_2. The embedding mappings $\mathfrak{g}_i \to \mathfrak{g}_1 \oplus \mathfrak{g}_2$ are omitted.

Example 1.3.18 (*Canonical connection of the product bundle*) Consider the product bundle $P = M \times G$, cf. Example 1.1.4/1. Take the connection Γ defined by (1.3.8) with χ being the identical mapping. The connection form corresponding to Γ is given by

$$\omega = \mathrm{pr}_G^* \theta,$$

where $\mathrm{pr}_G : M \times G \to G$ denotes the canonical projection. Details are left to the reader (Exercise 1.3.7). ♦

Example 1.3.19 (*Reductive homogeneous space*) Let G be a Lie group and let $H \subset G$ be a closed subgroup. Then, by Example 1.1.4/3, G carries the structure of a principal H-bundle over the homogeneous space G/H. Assume, additionally, that G/H is reductive, that is, the Lie algebra $\mathfrak{g}$ of G admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

such that $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. Here, $\mathfrak{h}$ denotes the Lie algebra of H. If G is semisimple, then $\mathfrak{m}$ can be chosen to be the orthogonal complement to $\mathfrak{h}$ in the sense of the Killing form (Exercise 1.3.6).

Clearly, the vertical subspace at $a \in G$ is given by $L'_a(\mathfrak{h})$. Since for any $a \in G$, we have $\mathrm{T}_a G = L'_a(\mathfrak{h}) \oplus L'_a(\mathfrak{m})$, the left invariant distribution $a \mapsto \Gamma_a := L'_a(\mathfrak{m})$ on G is complementary to the canonical vertical distribution. Using the reductivity, it is easy to show that Γ is right H-equivariant. Thus, Γ defines a connection on G. The corresponding connection form is given by

$$\omega^0 = \mathrm{pr}_{\mathfrak{h}} \circ \theta, \tag{1.3.17}$$

where $\mathrm{pr}_{\mathfrak{h}}$ is the canonical projection onto the first summand of the above reductive decomposition. Details are left to the reader (Exercise 1.3.7). ♦

Example 1.3.20 (*Canonical connection on the Stiefel bundle*) Recall the Stiefel bundles

$$S_{\mathbb{K}}(k, n) \cong \mathrm{U}_{\mathbb{K}}(n)/\mathrm{U}_{\mathbb{K}}(n-k) \to G_{\mathbb{K}}(k, n) \cong \mathrm{U}_{\mathbb{K}}(n)/(\mathrm{U}_{\mathbb{K}}(n-k) \times \mathrm{U}_{\mathbb{K}}(k))$$

discussed in Example 1.1.24. Denote the Lie algebra of the isometry group $\mathrm{U}_{\mathbb{K}}(i)$ by $\mathfrak{u}_{\mathbb{K}}(i)$, $i = k, n-k, n$. Since $\mathrm{U}_{\mathbb{K}}(n-k)$ and $\mathrm{U}_{\mathbb{K}}(k)$ act in complementary orthogonal subspaces of $\mathbb{K}^n$, the direct sum of their Lie algebras is a Lie subalgebra of $\mathfrak{u}_{\mathbb{K}}(n)$ and we have a direct sum decomposition

$$\mathfrak{u}_{\mathbb{K}}(n) = \mathfrak{u}_{\mathbb{K}}(k) \oplus \mathfrak{m}.$$

Here, $\mathfrak{m} = \mathfrak{u}_{\mathbb{K}}(n-k) \oplus \mathfrak{n}$ and $\mathfrak{n}$ is the orthogonal complement of

$$\mathfrak{u}_{\mathbb{K}}(k) \oplus \mathfrak{u}_{\mathbb{K}}(n-k) \subset \mathfrak{u}_{\mathbb{K}}(n)$$

with respect to the Killing form. The restriction of the adjoint representation $\mathrm{Ad}(\mathrm{U}_{\mathbb{K}}(k))$ acts on $\mathfrak{u}_{\mathbb{K}}(n-k)$ trivially and leaves the subspace $\mathfrak{n}$ invariant. Thus, the above decomposition is reductive. As in Example 1.3.19, we put

$$\omega^c := \mathrm{pr}_{\mathfrak{u}_{\mathbb{K}}(k)} \circ \theta. \tag{1.3.18}$$

Clearly, ω^c is a $\mathfrak{u}_{\mathbb{K}}(k)$-valued 1-form on $U_{\mathbb{K}}(n)$. Since ω^c is invariant under the $\mathrm{U}_{\mathbb{K}}(n-k)$-action on $\mathrm{U}_{\mathbb{K}}(n)$, it descends to a $\mathfrak{u}_{\mathbb{K}}(k)$-valued 1-form on $S_{\mathbb{K}}(k,n)$ which we denote by the same symbol. We claim that ω^c is a connection form. To prove this, we have to check the defining conditions 2 and 3 of Proposition 1.3.5. To check condition 2, note that the Killing vector field of the right $\mathrm{U}_{\mathbb{K}}(k)$-action on $\mathrm{U}_{\mathbb{K}}(n)$ generated by $A \in \mathfrak{u}_{\mathbb{K}}(k)$ coincides with A viewed as a left invariant vector field,

$$(A_*)_a = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction_0} (a \exp(tA)) = aA, \quad a \in \mathrm{U}_{\mathbb{K}}(n).$$

Since the right actions of $\mathrm{U}_{\mathbb{K}}(k)$ and $\mathrm{U}_{\mathbb{K}}(n-k)$ on $\mathrm{U}_{\mathbb{K}}(n)$ commute, the Killing vector field of the right $\mathrm{U}_{\mathbb{K}}(k)$-action on $S_{\mathbb{K}}(k,n)$ generated by $A \in \mathfrak{u}_{\mathbb{K}}(k)$ may be identified with A_*. Now, condition 2 follows from the defining equation of the Maurer-Cartan form, $\theta(A) = A$. Condition 3 follows immediately from the right $\mathrm{U}_{\mathbb{K}}(n)$-equivariance of θ. The connection defined by ω^c is called the canonical or universal[13] connection of the Stiefel bundle. By left invariance of the Maurer-Cartan form, the canonical connection is invariant under left translations of $\mathrm{U}_{\mathbb{K}}(n)$.

We give an explicit description of ω^c in terms of matrix-valued functions: let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis in $\mathbb{K}^n$. If we choose the k-frame $u_0 = (\mathbf{e}_1, \dots, \mathbf{e}_k)$, then the subgroups $\mathrm{U}_{\mathbb{K}}(k)$ and $\mathrm{U}_{\mathbb{K}}(n-k)$ are given in block matrix form by an upper diagonal $(k \times k)$-block and by a lower diagonal $((n-k) \times (n-k))$-block in $\mathrm{U}_{\mathbb{K}}(n)$, respectively. Let $a \in \mathrm{U}_{\mathbb{K}}(n)$ and let $a^i{}_j$ be the corresponding $(n \times n)$-matrix with respect to the standard basis. Since $a^\dagger a = \mathbb{1}$, ω^c is represented by a $(k \times k)$-valued 1-form on $S_{\mathbb{K}}(k,n)$,

$$(\omega^c)^\alpha{}_\beta = (a^\dagger)^\alpha{}_j \, \mathrm{d}a^j{}_\beta,$$

where $\alpha, \beta = 1, \dots k$ and $j = 1, \dots, n$. Denoting by u the matrix-valued function which assigns to the k-frame $u_\alpha = a^j{}_\alpha \mathbf{e}_j$ the $(n \times k)$-matrix $a^j{}_\alpha$, we obtain

$$\omega^c = u^\dagger \mathrm{d}u. \tag{1.3.19}$$

Since $a^\dagger a = \mathbb{1}$, we have $u^\dagger u = \mathbb{1}_k$. ♦

Remark 1.3.21 In the above realization, the horizontal vectors of ω^c at the point $p_0 = \begin{bmatrix} \mathbb{1}_k \\ 0 \end{bmatrix} \in S_{\mathbb{K}}(k,n)$ are given by matrices of the form $\begin{bmatrix} 0 & -T^\dagger \\ T & 0 \end{bmatrix}$, where T is an arbitrary $((n-k) \times n)$-matrix (Exercise 1.3.8). ♦

[13] This name will be explained in Sect. 3.8.

For later purposes, let us consider the following special case.

Example 1.3.22 (*Canonical connection on the Hopf bundle*) As noted in Remark 1.1.25, the Hopf bundles of Examples 1.1.20 and 1.1.22 coincide with the Stiefel bundles $S_{\mathbb{K}}(1,2) \to G_{\mathbb{K}}(1,2)$ with $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$, respectively. First, consider the complex Hopf bundle. In the notation of the above example, we have

$$u = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2, \quad |z_1|^2 + |z_2|^2 = 1,$$

and thus the canonical connection is given by

$$\omega^c = \overline{z_1}\, \mathrm{d}z_1 + \overline{z_2}\, \mathrm{d}z_2. \tag{1.3.20}$$

It takes values in the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$ of $\mathrm{U}(1)$. In complete analogy, for the quaternionic Hopf bundle, we have

$$u = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \in \mathbb{H}^2, \quad |\mathbf{q}_1|^2 + |\mathbf{q}_2|^2 = 1,$$

and the canonical connection is given by

$$\omega^c = \overline{\mathbf{q}_1}\, \mathrm{d}\mathbf{q}_1 + \overline{\mathbf{q}_2}\, \mathrm{d}\mathbf{q}_2. \tag{1.3.21}$$

It takes values in the Lie algebra $\mathfrak{sp}(1)$ of $\mathrm{Sp}(1)$. ♦

Example 1.3.23 In contrast to the complex Hopf bundle, consider the product bundle $P = \mathrm{S}^2 \times \mathrm{U}(1)$ endowed with the canonical connection of Example 1.3.18. In the parameterization $z = e^{i\alpha}$ of $\mathrm{U}(1)$, the canonical connection form is $\omega = \mathrm{d}\alpha$. ♦

Exercises

1.3.1 Prove that the horizontal lift of a vector field, defined in Remark 1.3.3/2, is smooth. Moreover, show the following: if X^h and Y^h are horizontal lifts of X and Y, respectively, then

(a) $X^h + Y^h$ is the horizontal lift of $X + Y$,
(b) for any $f \in C^\infty(M)$, the vector field $(\pi^* f)X^h$ is the horizontal lift of fX.
(c) the horizontal component of $[X^h, Y^h]$ is the horizontal lift of $[X, Y]$.

1.3.2 Complete the proof of Proposition 1.3.6.

1.3.3 Prove formula (1.3.9).

1.3.4 Prove Corollary 1.3.12.

1.3.5 Prove Proposition 1.3.15.
Hint. Since λ is an isomorphism, $\mathrm{d}\lambda$ is an isomorphism of Lie algebras. Use this fact to define the connection Γ^1 via its connection form putting $\omega_1 := (\mathrm{d}\lambda)^{-1} \circ \vartheta^*\omega_2$.

1.3.6 Show the following. If G is a semisimple Lie group and if H is a closed subgroup, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ defines a reductive decomposition. Here, $\mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form.

1.3.7 Complete the proof of the statements made in Examples 1.3.18 and 1.3.19.

1.3.8 Prove the statement of Remark 1.3.21.

1.4 Covariant Exterior Derivative and Curvature

The following notion plays a basic role in the theory of connections.

Definition 1.4.1 (*Covariant exterior derivative*) Let P be a principal bundle and let F be a finite-dimensional vector space. The covariant exterior derivative[14] of an F-valued differential k-form α on P with respect to a connection Γ is the differential $(k+1)$-form with values in F defined by

$$D_\omega\alpha(X_0, \dots, X_k) := \mathrm{d}\alpha(\mathrm{hor}\, X_0, \dots, \mathrm{hor}\, X_k), \quad X_0, \dots, X_k \in \mathfrak{X}(P).$$

By definition, D_ω fulfils the same product rule as the ordinary exterior derivative and $D_\omega\alpha$ is horizontal. Moreover, as will be shown, D_ω preserves the symmetry type of any horizontal form.

We wish to derive an explicit formula for the covariant exterior derivative. For that purpose, we need the following.

Lemma 1.4.2 *Let $P(M, G)$ be a principal bundle with a connection Γ, let A_* be a Killing vector field on P, let $X \in \mathfrak{X}(P)$ be horizontal and let $Y \in \mathfrak{X}(M)$. Then, $[A_*, X]$ is horizontal and $[A_*, Y^h] = 0$.*

Proof For any $p \in P$, we have

$$[A_*, X]_p = (\mathscr{L}_{A_*} X)_p = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \left((\Psi_{\exp(-tA)})_* X\right)_p .$$

Since X is horizontal, $(\Psi_{\exp(-tA)})_* X$ is horizontal for all t. Thus, $[A_*, X]$ is horizontal, too. To prove the second statement, recall that the horizontal lift Y^h is G-invariant, that is, the curve $t \mapsto \left((\Psi_{\exp(-tA)})_* Y^h\right)_p$ is constant and equal to Y^h_p. This yields the assertion. ∎

Recall from Remark 1.3.9 that $\sigma'(\omega)$ is a 1-form on P with values in $\mathrm{End}(F)$.

[14] Or covariant exterior differential.

Proposition 1.4.3 *Let $P(M, G)$ be a principal bundle, let (F, G, σ) be a finite-dimensional representation and let ω be a connection form on P.*

1. *The covariant exterior derivative D_ω of a horizontal F-valued k-form on P of type σ is a horizontal $(k+1)$-form of type σ.*
2. *Let $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$. Then,*

$$D_\omega \tilde{\alpha} = \mathrm{d}\tilde{\alpha} + \sigma'(\omega) \wedge \tilde{\alpha}, \tag{1.4.1}$$

where

$$(\sigma'(\omega) \wedge \tilde{\alpha})_p(X_0, \ldots, X_k) := \sum_{i=0}^{k} (-1)^i \sigma'(\omega_p(X_i))\big(\tilde{\alpha}_p(X_0, \overset{X_i}{\check{\ldots}}, X_k)\big),$$

with $p \in P$ and $X_0, \ldots, X_k \in \mathrm{T}_p P$.

Proof 1. Let $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$. By definition of the covariant exterior derivative, $D_\omega \tilde{\alpha}$ is an F-valued horizontal $(k+1)$-form. For $X_0, \ldots, X_k \in \mathfrak{X}(P)$, we calculate

$$\begin{aligned}
(\Psi_a^* D_\omega \tilde{\alpha})\,(X_0, \ldots, X_k) &= D_\omega \tilde{\alpha}\,(\Psi_{a*} X_0, \ldots, \Psi_{a*} X_k) \\
&= \mathrm{d}\tilde{\alpha}\,(\mathrm{hor}\,\Psi_{a*} X_0, \ldots, \mathrm{hor}\,\Psi_{a*} X_k) \\
&= \mathrm{d}\tilde{\alpha}\,(\Psi_{a*}\,\mathrm{hor}\, X_0, \ldots, \Psi_{a*}\,\mathrm{hor}\, X_k) \\
&= \mathrm{d}(\Psi_a^* \tilde{\alpha})\,(\mathrm{hor}\, X_0, \ldots, \mathrm{hor}\, X_k) \\
&= \mathrm{d}(\sigma_{a^{-1}} \circ \tilde{\alpha})\,(\mathrm{hor}\, X_0, \ldots, \mathrm{hor}\, X_k) \\
&= \sigma_{a^{-1}} \circ D_\omega \tilde{\alpha}\,(X_0, \ldots, X_k)\ .
\end{aligned}$$

This shows that $D_\omega \tilde{\alpha}$ is of type σ.

2. Since each of the vectors $X_0, \ldots, X_k \in \mathrm{T}_p P$ may be decomposed into a vertical and a horizontal part, it is enough to consider the following cases:
(a) Let all vectors X_i be horizontal. Then, $\omega(X_i) = 0$ and formula (1.4.1) follows from Definition 1.4.1.
(b) Let one of the vectors X_i, say X_0, be vertical and let the remaining vectors be horizontal. Then, there exists an element $A \in \mathfrak{g}$ such that $X_0 = \Psi'_p(A)$ and a family of vector fields $Y_1, \ldots Y_k \in \mathfrak{X}(M)$ such that their horizontal lifts Y_i^h at p coincide with the vectors $X_1, \ldots, X_k$. Then,

$$D_\omega \tilde{\alpha}(X_0, \ldots, X_k) = 0, \quad (\sigma'(\omega) \wedge \tilde{\alpha})(X_0, \ldots, X_k) = \sigma'(A)(\tilde{\alpha}(X_1, \ldots, X_k)).$$

Using Proposition I/4.1.6, Lemma 1.4.2, the horizontality of $\tilde{\alpha}$ and the G-invariance of the horizontal lifts Y_i^h, we calculate

$$
\begin{aligned}
(\mathrm{d}\tilde{\alpha})_p(X_0, \ldots, X_k) &= (A_*)_p(\tilde{\alpha}(Y_1^h, \ldots, Y_k^h)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\, \tilde{\alpha}_{\Psi_{\exp(tA)}(p)}(Y_1^h, \ldots, Y_k^h) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\, \tilde{\alpha}_{\Psi_{\exp(tA)}(p)}(\Psi'_{\exp(tA)}(X_1), \ldots, \Psi'_{\exp(tA)}(X_k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} \left(\Psi^*_{\exp(tA)}\tilde{\alpha}\right)_p (X_1, \ldots, X_k) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\, \sigma_{\exp(-tA)}\tilde{\alpha}_p(X_1, \ldots, X_k) \\
&= -\sigma'(A)(\tilde{\alpha}_p(X_1, \ldots, X_k)).
\end{aligned}
$$

Thus, in this case, the right hand side of (1.4.1) also vanishes.
(c) Let at least two of the vectors X_i be vertical and let the remaining vectors be horizontal. Then,

$$
D_\omega\tilde{\alpha}(X_0, \ldots, X_k) = 0, \quad (\sigma'(\omega) \wedge \tilde{\alpha})(X_0, \ldots, X_k) = 0,
$$

and it remains to show that $\mathrm{d}\tilde{\alpha}(X_0, \ldots, X_k) = 0$. Since the commutator of vertical vector fields is vertical, the assertion follows from Proposition I/4.1.6 and the horizontality of $\tilde{\alpha}$. ■

Remark 1.4.4 In particular, the covariant exterior derivative of an equivariant mapping $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$ is given by

$$
D_\omega\tilde{\Phi} = \mathrm{d}\tilde{\Phi} + \sigma'(\omega) \circ \tilde{\Phi}. \tag{1.4.2}
$$

Clearly, this is an immediate consequence of formula (1.4.1). The following independent proof gives some additional insight.

$$
\begin{aligned}
(D_\omega\tilde{\Phi})_p(X) &= (\mathrm{d}\tilde{\Phi})_p(\mathrm{hor}\, X) \\
&= (\mathrm{d}\tilde{\Phi})_p(X - \Psi'_p(\omega(X))) \\
&= (\mathrm{d}\tilde{\Phi})_p(X) - (\Psi'_p(\omega(X)))_p(\tilde{\Phi}) \\
&= (\mathrm{d}\tilde{\Phi})_p(X) - \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} \left(\tilde{\Phi} \circ \Psi_{\exp(t\omega(X))}(p)\right) \\
&= (\mathrm{d}\tilde{\Phi})_p(X) - \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} \left(\sigma_{\exp(-t\omega(X))} \circ \tilde{\Phi}\right)(p) \\
&= (\mathrm{d}\tilde{\Phi})_p(X) + \sigma'(\omega(X)) \circ \tilde{\Phi}(p).
\end{aligned}
$$

♦

Definition 1.4.5 Let P be a principal G-bundle, let $E = P \times_G F$ be associated with P and let ω be a connection on P. An element $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ will be called parallel with respect to ω if

$$D_\omega \tilde{\alpha} = 0. \tag{1.4.3}$$

Next, recall that the ordinary exterior derivative d fulfils $\mathrm{d} \circ \mathrm{d} = 0$. In sharp contrast, $D_\omega \circ D_\omega$ does not vanish in general. This non-vanishing property is closely related to the notion of curvature.

Definition 1.4.6 (*Curvature form*) Let P be a principal bundle and let ω be a connection form on P. The curvature form of ω is defined by

$$\Omega := D_\omega \omega.$$

By definition, Ω is horizontal. Moreover, by point 3 of Proposition 1.3.5,

$$\Psi_a^* \Omega = \mathrm{Ad}\left(a^{-1}\right) \circ \Omega, \quad a \in G. \tag{1.4.4}$$

Thus, the curvature form is a $\mathfrak{g}$-valued horizontal 2-form on P of type Ad.

Remark 1.4.7

1. By definition, we have $\Omega(X, Y) = \mathrm{d}\omega(X, Y)$ for any pair of horizontal vector fields X and Y. Using Proposition I/4.1.6 and the defining equation (1.3.6), we obtain
$$\mathrm{ver}([X, Y])_p = -\Psi_p'(\Omega(X, Y)). \tag{1.4.5}$$
By the Frobenius Theorem, we conclude that the horizontal distribution Γ defining the connection form ω is integrable iff the curvature form Ω vanishes. A connection with vanishing curvature is said to be flat.
2. Since Ω is a horizontal 2-form of type Ad, by Proposition 1.2.12, it may be viewed as a 2-form on M with values in the associated bundle
$$\mathrm{Ad}(P) := P \times_G \mathfrak{g}, \tag{1.4.6}$$
which will be referred to as the adjoint bundle of P. ♦

Remark 1.4.8

1. Below, we will often deal with the exterior product of Lie algebra-valued forms. According to Remark I/4.1.10/2, the exterior product of a k-form α with an l-form β on a manifold M, both with values in a Lie algebra $\mathfrak{g}$, is defined as follows:
$$[\alpha, \beta](X_1, \ldots, X_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \mathrm{sign}(\sigma)\left[\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})\right] \tag{1.4.7}$$

for $X_1, \ldots, X_{k+l} \in \mathfrak{X}(M)$. If the Lie algebra $\mathfrak{g}$ is a subalgebra of the associative algebra $\mathfrak{gl}(n, \mathbb{K})$, then one can work with the associative wedge product $\alpha \wedge \beta$ as well. The latter is defined by (1.4.7) with the Lie product on the right hand side replaced by the associative product inherited from $\mathfrak{gl}(n, \mathbb{K})$. Clearly, then $[\alpha, \beta]$ may be expressed in terms of the associative wedge product (Exercise 1.4.1),

$$[\alpha, \beta] = \alpha \wedge \beta + (-1)^{kl+1} \beta \wedge \alpha. \tag{1.4.8}$$

2. Clearly, point 1 applies, in particular, to horizontal forms on a principal bundle $P(M, G)$ with values in $\mathfrak{g}$. On the other hand, note that the vector space isomorphisms (1.2.2) identifying $\mathfrak{g}$ with the fibres of the adjoint bundle $\mathrm{Ad}(P)$ transport the Lie algebra structure from $\mathfrak{g}$ to the fibres of $\mathrm{Ad}(P)$. Thus, for elements of $\Omega^k(M, \mathrm{Ad}(P))$ we have a natural commutator denoted in the same way. This remark applies, of course, to any vector bundle whose fibres carry the structure of a Lie algebra. ♦

Proposition 1.4.9 (Structure Equation) *Let P be a principal bundle, let ω be a connection form on P and let Ω be its curvature form. Then,*

$$\mathrm{d}\omega = -\frac{1}{2}[\omega, \omega] + \Omega. \tag{1.4.9}$$

Proof We evaluate both sides of (1.4.9) on vector fields $X, Y \in \mathfrak{X}(P)$. By (1.4.7), we have $\frac{1}{2}[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$. Clearly, it is enough to consider the following three cases:

1. X and Y are horizontal. Then, $\omega(X) = \omega(Y) = 0$ and

$$\Omega(X, Y) = \mathrm{d}\omega(\mathrm{hor}\, X, \mathrm{hor}\, Y) = \mathrm{d}\omega(X, Y)\,.$$

2. X is vertical and Y is horizontal. Then, $\omega(Y) = 0$ and $\Omega(X, \cdot) = 0$. Thus, the right hand side of (1.4.9) vanishes. To calculate the left hand side, without loss of generality, we may assume $X = A_*$ for some $A \in \mathfrak{g}$. Then, $\omega(A_*) = A$ and we obtain

$$\mathrm{d}\omega(A_*, Y) = Y(\omega(A_*)) - A_*(\omega(Y)) - \omega([A_*, Y]) = -\omega([A_*, Y]) = 0,$$

because, according to Lemma 1.4.2, $[A_*, Y]$ is horizontal.

3. X and Y are vertical. Then, $\Omega(X, Y) = 0$. Taking $X = A_*$ and $Y = B_*$, for some $A, B \in \mathfrak{g}$, and using[15] $[A_*, B_*] = [A, B]_*$, we calculate

$$\mathrm{d}\omega(A_*, B_*) = -\omega([A_*, B_*]) = -\omega([A, B]_*) = -[A, B] = -[\omega(A_*), \omega(B_*)].$$

■

[15] See Proposition I/6.2.2/2.

Remark 1.4.10

1. By (1.4.8), if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{K})$, then we can rewrite the Structure Equation in terms of the associative wedge product,

$$\mathrm{d}\omega = -\omega \wedge \omega + \Omega .$$

2. Recall the transformation properties $\vartheta^*\omega_2 = \mathrm{d}\lambda \circ \omega_1$ and $\vartheta^*\Omega_2 = \mathrm{d}\lambda \circ \Omega_1$ under some special principal bundle morphisms (ϑ, λ) as proved in Propositions 1.3.13/2 and 1.3.15/2. Note that, by the Structure Equation, the transformation law for the curvature is an immediate consequence of the transformation law for the connection. ♦

As an immediate consequence of the Structure Equation, we obtain the following.

Proposition 1.4.11 (Bianchi Identity) *Let P be a principal bundle and let Ω be the curvature form of a connection form ω on P. Then, Ω is parallel with respect to ω,*

$$D_\omega \Omega = 0. \tag{1.4.10}$$

Proof Clearly, it is enough to show that $\mathrm{d}\Omega(X, Y, Z) = 0$ for arbitrary horizontal vector fields X, Y and Z on P. Using the Structure Equation, we calculate

$$\begin{aligned}\mathrm{d}\Omega(X, Y, Z) &= \frac{1}{2}\mathrm{d}\big([\omega, \omega]\big)(X, Y, Z)\\ &= X([\omega(Y), \omega(Z)]) - Y([\omega(X), \omega(Z)]) + Z([\omega(X), \omega(Y)])\\ &\quad - [\omega([X, Y]), \omega(Z)] + [\omega([X, Z]), \omega(Y)] - [\omega([Y, Z]), \omega(X)]\\ &= 0,\end{aligned}$$

because ω vanishes on horizontal vector fields. ■

Remark 1.4.12 Let $\tilde{\alpha} \in \Omega^k_{\mathrm{Ad,hor}}(P, \mathfrak{g})$. Since $\mathrm{ad}(\omega) \wedge \tilde{\alpha} = [\omega, \tilde{\alpha}]$, Proposition 1.4.3 implies

$$D_\omega \tilde{\alpha} = \mathrm{d}\tilde{\alpha} + [\omega, \tilde{\alpha}].$$

Thus, in particular, $D_\omega \Omega = \mathrm{d}\Omega + [\omega, \Omega]$, and the Bianchi Identity (1.4.10) takes the form

$$\mathrm{d}\Omega + [\omega, \Omega] = 0\,. \tag{1.4.11}$$

♦

Applying D_ω to equation (1.4.1), one finds the following (Exercise 1.4.3).

Proposition 1.4.13 *Let $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$. Then,*

$$D_\omega \circ D_\omega \tilde{\alpha} = \sigma'(\Omega) \wedge \tilde{\alpha}. \tag{1.4.12}$$

■

This yields another geometric interpretation of the curvature form Ω. It measures to which extent $D_\omega \circ D_\omega$ is non-vanishing when acting on a horizontal form. If the connection is flat, then $D_\omega \circ D_\omega = 0$.

We close this section by giving the local description of the above defined geometric objects. First, let us find the local representative of the covariant exterior derivative of $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$. For simplicity, we will omit the index ω in the covariant exterior derivative. By Proposition 1.4.3, $D\tilde{\alpha}$ is an element of $\Omega^{k+1}_{\sigma,\mathrm{hor}}(P, F)$. Thus, we read off its local representative from (1.2.15):

$$(D\tilde{\alpha})^\chi = s^* D\tilde{\alpha}. \tag{1.4.13}$$

Let us calculate the right hand side of (1.4.13) for a 0-form $\tilde{\Phi}$ explicitly. Formula (1.4.2) implies

$$s^*(D\tilde{\Phi}) = \mathrm{d}(s^*\tilde{\Phi}) + s^*(\sigma'(\omega) \circ \tilde{\Phi}).$$

For the second term, we calculate

$$\begin{aligned}
(s^*\sigma'(\omega)\tilde{\Phi})_m(X) &= \left(\sigma'(\omega)\tilde{\Phi}\right)_{s(m)} (s'X) \\
&= \sigma'\left(\omega_{s(m)}(s'X)\right)\ \tilde{\Phi}(s(m)) \\
&= \sigma'\left((s^*\omega)_m(X)\right)\ \left(s^*\tilde{\Phi}\right)(m) \\
&= \sigma'\left(\mathscr{A}_m(X)\right)\ \varphi(m),
\end{aligned}$$

where $\mathscr{A} = s^*\omega$ is the local representative of ω and $X \in \mathrm{T}_m M$. Thus, denoting $(D\tilde{\Phi})^\chi = D\varphi$, we have

$$D\varphi = \mathrm{d}\varphi + \sigma'(\mathscr{A})\varphi. \tag{1.4.14}$$

Here, $\sigma'(\mathscr{A})$ is a 1-form on U with values in $\mathrm{End}(F)$. In the following remark, we analyze formula (1.4.14) further.

Remark 1.4.14 If (U, κ) is a local chart, $\{\mathbf{t}_a\}$ a basis in $\mathfrak{g}$ and $\{\mathbf{e}_\alpha\}$ is a basis in F, we can decompose

$$\varphi(x) = \varphi^\alpha(x)\ \mathbf{e}_\alpha\,, \quad \mathscr{A} = \mathscr{A}^a_\mu\ \mathrm{d}\kappa^\mu \otimes \mathbf{t}_a\,, \quad D\varphi = D_\mu\varphi^\alpha\ \mathrm{d}\kappa^\mu \otimes \mathbf{e}_\alpha.$$

To determine the coefficient funtions $D_\mu\varphi^\alpha$, we compute

$$\begin{aligned}
\sigma'(\mathscr{A})\varphi &= \sigma'\left(\mathscr{A}^a_\mu\, d\kappa^\mu \otimes \mathbf{t}_a\right)\ \varphi^\alpha \mathbf{e}_\alpha \\
&= \left(\mathscr{A}^a_\mu\, \varphi^\alpha\ \mathrm{d}\kappa^\mu\right) \otimes \left(\sigma'(\mathbf{t}_a)\mathbf{e}_\alpha\right) \\
&= \mathscr{A}^a_\mu\, \varphi^\alpha \sigma_{a\alpha}{}^\beta\ \mathrm{d}\kappa^\mu \otimes \mathbf{e}_\beta,
\end{aligned}$$

with ${\sigma_{a\alpha}}^{\beta}$ representing the endomorphism $\sigma'(\mathbf{t}_a)$ in the basis $\{\mathbf{e}_\alpha\}$. Using this and denoting $\mathscr{A}_{\mu}^{\alpha}{}_{\beta} = {\sigma_{a\beta}}^{\alpha}\mathscr{A}_{\mu}^{a}$, we obtain

$$D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + \mathscr{A}_{\mu}^{\alpha}{}_{\beta}\varphi^\beta \,. \tag{1.4.15}$$

♦

Next, let us discuss the local description of the curvature form. If $s\colon U \to \pi^{-1}(U)$, with $U \subset M$ open, is a local section, then we define the local representative of Ω by

$$\mathscr{F} := s^*\Omega \,. \tag{1.4.16}$$

Let $\mathscr{A}$ be the local representative of ω with respect to the section s. Then, the Structure Equation for ω implies

$$\mathscr{F} = \mathrm{d}\mathscr{A} + \frac{1}{2}[\mathscr{A}, \mathscr{A}]. \tag{1.4.17}$$

Remark 1.4.15

1. In complete analogy to Proposition 1.3.11 and Corollary 1.3.12, we have the local reconstruction formula

$$\Omega_p = \mathrm{Ad}\left(\kappa(p)^{-1}\right)(\pi^*(\mathscr{F}))_p, \tag{1.4.18}$$

and the transformation law

$$\left(\mathscr{F}_j\right)_m = \mathrm{Ad}\left(\rho_{ij}(m)^{-1}\right) \circ (\mathscr{F}_i)_m \,, \quad m \in U_i \cap U_j, \tag{1.4.19}$$

(Exercise 1.4.4).

2. With respect to the local frame in the bundle of $\mathfrak{g}$-valued k-forms on P given by (1.3.12), $\mathscr{F}$ reads

$$\mathscr{F} = \frac{1}{2}\mathscr{F}_{\mu\nu}^{a}\, \mathrm{d}\varphi^\mu \wedge \mathrm{d}\varphi^\nu \otimes \mathbf{t}_a,$$

and the Structure Equation takes the form

$$\mathscr{F}_{\mu\nu}^{a} = \partial_\mu \mathscr{A}_{\nu}^{a} - \partial_\nu \mathscr{A}_{\mu}^{a} + {c^a}_{bc}\mathscr{A}_{\mu}^{b}\mathscr{A}_{\nu}^{c}.$$

Here, ${c^a}_{bc}$ are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\mathbf{t}_a\}$. ♦

Exercises

1.4.1 Prove formula (1.4.8).

1.4.2 Calculate the curvature of the canonical connections of the complex and quaternionic Hopf bundles.

1.4.3 Prove Proposition 1.4.13.

1.4.4 Prove the statements of Remark 1.4.15/2.

1.5 The Koszul Calculus

In this section, we show that the notion of covariant exterior derivative with respect to a connection on a principal bundle implies a calculus for covariant derivatives acting as differential operators in the space of sections of any associated vector bundle.[16] This is often referred to as the Koszul calculus.[17]

As above, let $P(M, G)$ be a principal bundle, let (F, G, σ) be a Lie group representation and let $E = P \times_G F$ be the associated vector bundle. Recall that, by Remark 1.3.3/3, a connection Γ on P induces a connection Γ^E on E and the connection form ω of Γ induces a connection mapping $\omega^E : \mathrm{T}E \to E$, given by (1.3.9). Using the isomorphism between $\Omega^k_{\sigma,\mathrm{hor}}(P, F)$ and $\Omega^k(M, E)$ provided by Proposition 1.2.12, we can carry over the notion of covariant exterior derivative to $\Omega^k(M, E)$.

Definition 1.5.1 Let $\alpha \in \Omega^k(M, E)$. The covariant exterior derivative $\mathrm{d}_\omega\alpha$ is defined to be the image of $D_\omega\tilde{\alpha}$ under the isomorphism $\Omega^{k+1}_{\sigma,\mathrm{hor}}(P, F) \to \Omega^{k+1}(M, E)$, that is,

$$\widetilde{\mathrm{d}_\omega\alpha} := D_\omega\tilde{\alpha}. \tag{1.5.1}$$

By definition, for $p \in \pi^{-1}(m)$ and $X_i \in \mathrm{T}_mM$, $Y_i \in \mathrm{T}_pP$ fulfilling $\pi'(Y_i) = X_i$, we have

$$(\mathrm{d}_\omega\alpha)_m(X_1, \ldots, X_{k+1}) = \iota_p \circ (D_\omega\tilde{\alpha})_p(Y_1, \ldots, Y_{k+1}). \tag{1.5.2}$$

Since $\Omega^0(M, E) = \Gamma^\infty(E)$ and $\Omega^1(M, E) = \Gamma^\infty(\mathrm{T}^*M \otimes E)$, d_ω restricted to 0-forms yields a linear operator from $\Gamma^\infty(E)$ to $\Gamma^\infty(\mathrm{T}^*M \otimes E)$.

Definition 1.5.2 The linear operator

$$\nabla^\omega := (\mathrm{d}_\omega)_{\restriction\Omega^0(M,E)} : \ \Gamma^\infty(E) \to \Gamma^\infty(\mathrm{T}^*M \otimes E)$$

is called the covariant derivative on E induced from ω.

By (1.5.2) and the definition of D_ω, for any $m \in M$ and any $\Phi \in \Gamma^\infty(E)$, we have

$$(\nabla^\omega\Phi)_m(X) = \iota_p \circ (D_\omega\tilde{\Phi})_p(Y) = \iota_p \circ (X^h_p(\tilde{\Phi})), \quad p \in \pi^{-1}(m), \tag{1.5.3}$$

[16] By Remark 1.2.9/2, in doing so we exhaust all finite-rank vector bundles.

[17] See [390].

where $Y \in \mathrm{T}_p P$ fulfilling $\pi'(Y) = X$ and X^h is the horizontal lift of X to P.

In the sequel, we assume that a connection has been chosen and, for simplicity, we write ∇ instead of ∇^ω.

Formula (1.5.3) implies a useful expression for the action of ∇ on local frames of E. To derive it, recall that a local trivialization of a principal G-bundle P over M induces a local trivialization of any associated bundle $E = P \times_G F$. Correspondingly, for a chosen basis $\{\mathbf{e}_\alpha\}$ of the typical fibre F, a local section s of P induces a local frame $\{e_\alpha\}$, $\alpha = 1, \dots, p$, of E via

$$e_\alpha(m) = \iota_{s(m)}(\mathbf{e}_\alpha). \tag{1.5.4}$$

Let $\tilde{e}_\alpha : P \to F$ be the equivariant mapping corresponding to e_α. Then,

$$\tilde{e}_\alpha(s(m)) = \mathbf{e}_\alpha. \tag{1.5.5}$$

Proposition 1.5.3 *Let P be a principal G-bundle over M endowed with a connection form ω, let $E = P \times_G F$ be an associated vector bundle and let ∇ be the covariant derivative induced from ω. Let s be a local section of P and let $\{e_\alpha\}$ be a local frame of E induced from s. Then,*

$$\nabla e_\alpha = \mathscr{A}^\beta{}_\alpha \, e_\beta, \tag{1.5.6}$$

where $\mathscr{A} = s^*\omega$ is the local representative of ω and $\mathscr{A}^\beta{}_\alpha$ denotes its matrix with respect to the basis $\{\mathbf{e}_\alpha\}$ of F, cf. Remark 1.4.14.

Proof Consider (1.5.3) for a point $m \in M$ belonging to the domain of s. Since we can take its right hand side at any point in the fibre over m, we calculate it at $s(m)$ and for Y we take the vector $s'(X)$ which is tangent to the section s at $s(m)$. Using (1.4.2), (1.5.4) and (1.5.5), for any $X \in \mathrm{T}_m M$, we calculate

$$\begin{aligned}
(\nabla e_\alpha)_m(X) &= \iota_{s(m)} \left((D\,\tilde{e}_\alpha)_{s(m)}(s'(X)) \right) \\
&= \iota_{s(m)} \left((\mathrm{d}\tilde{e}_\alpha)(s'(X)) + \sigma'\big(\omega(s'(X))\big)\tilde{e}_\alpha(s(m)) \right) \\
&= \iota_{s(m)} \left(\mathrm{d}(s^*\tilde{e}_\alpha)(X) + \sigma'(\mathscr{A}(X))\mathbf{e}_\alpha \right) \\
&= \iota_{s(m)} \left(\mathscr{A}(X)^\beta{}_\alpha \mathbf{e}_\beta \right) \\
&= \left(\mathscr{A}(X)^\beta{}_\alpha e_\beta \right)(m).
\end{aligned}$$

■

Proposition 1.5.4 *For any $f \in C^\infty(M)$ and $\Phi \in \Gamma^\infty(E)$,*

$$\nabla(f\Phi) = \mathrm{d}f \otimes \Phi + f\nabla\Phi. \tag{1.5.7}$$

Proof Using Remark 1.2.13, for $m \in M$, $X \in \mathrm{T}_m M$ and $p \in \pi^{-1}(m)$, we calculate

$$
\begin{aligned}
(\nabla(f\Phi))_m(X) &= \iota_p \circ \mathrm{d}(\widetilde{f\Phi})_p(X^h) \\
&= \iota_p \circ \mathrm{d}(\tilde{f}\tilde{\Phi})_p(X^h) \\
&= \iota_p \circ ((\mathrm{d}\tilde{f})\,\tilde{\Phi} + \tilde{f}\,(\mathrm{d}\tilde{\Phi}))_p(X^h) \\
&= (\mathrm{d}f)_m(X)\Phi(m) + f(m)(\nabla\Phi)_m(X).
\end{aligned}
$$

■

Equation (1.5.7) is called the Leibniz rule for ∇.

Remark 1.5.5

1. Combining Propositions 1.5.3 and 1.5.4 with Remark 1.4.14, for a local section $\varphi = \varphi^\alpha e_\alpha$ of E, decomposed with respect to a local frame e_α, we obtain

$$\nabla\varphi = \mathrm{d}\varphi^\alpha \otimes e_\alpha + \mathscr{A}^\beta{}_\alpha \varphi^\alpha e_\beta. \tag{1.5.8}$$

2. We have the following obvious generalization of Proposition 1.5.4 (Exercise 1.5.1). For $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^l(M)$,

$$\mathrm{d}_\omega(\beta \wedge \alpha) = \mathrm{d}\beta \wedge \alpha + (-1)^l \beta \wedge \mathrm{d}_\omega\alpha. \tag{1.5.9}$$

3. Let E be a $\mathbb{K}$-vector bundle of rank k over M. By point 2 of Remark 1.2.9, E is naturally associated with the bundle $L(E)$ of linear frames, that is, there exists a vector bundle isomorphism $E \cong L(E) \times_{\mathrm{GL}(k,\mathbb{K})} \mathbb{K}^k$. By definition, a connection on E is a $\mathbb{C}$-linear mapping $\nabla : \Gamma^\infty(E) \to \Gamma^\infty(\mathrm{T}^*M \otimes E)$ fulfilling the Leibniz rule (1.5.7). Then, by the above correspondence, connections on E are in one-to-one correspondence with connections on $L(E)$. Thereby, the connection ∇ corresponding to the connection form ω coincides with the covariant derivative defined by ω. Thus, the theory of connections on arbitrary vector bundles boils down to the theory of covariant derivatives in associated vector bundles.
4. By point 3, Proposition 1.5.3 immediately extends to any vector bundle E endowed with a connection. Then, P coincides with the $L(E)$ and σ is the basic representation of $\mathrm{GL}(n, \mathbb{K})$. ♦

The following proposition clarifies the relation of the covariant derivative with the connection mapping, cf. Remark 1.3.9.

Proposition 1.5.6 *For* $X \in \mathfrak{X}(M)$,

$$\nabla\Phi(X) = \omega^E(\Phi'(X)).$$

Proof By the definition of ω^E, we must decompose $\Phi'(X)$ into its vertical and horizontal parts. For that purpose, let $t \mapsto \gamma(t)$ be an integral curve of X through $m \in M$. Then,

$$\Phi' X_m = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 \Phi \circ \gamma(t).$$

Choose a point $p \in \pi^{-1}(m)$ and take the integral curve $t \mapsto \gamma^h(t)$ through p of the horizontal lift X^h of X to P. Then, (1.2.11) implies

$$\Phi \circ \gamma(t) = \iota\left(\gamma^h(t), \tilde{\Phi}(\gamma^h(t))\right)$$

and, thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \Phi \circ \gamma(t) &= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota\left(\gamma^h(t), \tilde{\Phi}(\gamma^h(t))\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota_{\tilde{\Phi}(p)}\left(\gamma^h(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota_p\left(\tilde{\Phi}(\gamma^h(t))\right).\end{aligned}$$

For the first term, using (1.3.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota_{\tilde{\Phi}(p)}\left(\gamma^h(t)\right) = \iota'_{\tilde{\Phi}(p)} X^h_p = X^h_{\Phi(m)}.$$

This is the horizontal component of $\Phi'(X)$ at $\Phi(m)$. The second term reads

$$\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota_p\left(\tilde{\Phi}(\gamma^h(t))\right) = \iota'_p\left(\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \tilde{\Phi}(\gamma^h(t))\right) = \iota_p\left(X^h(\tilde{\Phi})\right).$$

This is the vertical component of $\Phi'(X)$ at $\Phi(m)$. Thus, by (1.3.9) and (1.5.3),

$$\omega^E(\Phi'(X)) = \iota_p \circ X^h(\tilde{\Phi}) = \nabla\Phi(X).$$

∎

Next, recall the notion of parallelity, cf. Definition 1.4.5. By (1.5.3), a section $\Phi \in \Gamma^\infty(E)$ is parallel iff $\nabla_X \Phi = 0$ for all $X \in \mathfrak{X}(M)$. Proposition 1.5.6 implies the following.

Corollary 1.5.7 *A section $\Phi \in \Gamma^\infty(E)$ is parallel with respect to a connection Γ iff $\mathrm{im}(\Phi'_m) \subset \Gamma^E_{\Phi(m)}$ for all $m \in M$.* ∎

In the sequel, it will be often useful to view the covariant derivative as a differential operator acting on sections: for every $X \in \mathfrak{X}(M)$, the covariant derivative induces a mapping

$$\nabla_X \colon \Gamma^\infty(E) \to \Gamma^\infty(E), \quad \nabla_X \Phi := \nabla\Phi(X). \tag{1.5.10}$$

Proposition 1.5.8 *For $X, X_1, X_2 \in \mathfrak{X}(M)$, $\Phi, \Phi_1, \Phi_2 \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$,*

1. $\nabla_{X_1+X_2}\Phi = \nabla_{X_1}\Phi + \nabla_{X_2}\Phi$,
2. $\nabla_X(\Phi_1 + \Phi_2) = \nabla_X(\Phi_1) + \nabla_X(\Phi_2)$,
3. $\nabla_{fX}\Phi = f\,\nabla_X\Phi$,
4. $\nabla_X(f\Phi) = f\,\nabla_X\Phi + X(f)\,\Phi$.

Proof Points 1 and 2 are immediate consequences of the definition of ∇_X. Using Remark 1.2.13, together with $(fX)^h = \tilde{f}X^h$, we get

$$(\nabla\Phi)_m(fX) = \iota_p \circ ((fX)^h(\tilde{\Phi})) = \iota_p \circ (\tilde{f}X^h(\tilde{\Phi})) = f(m)(\nabla\Phi)_m(X),$$

for any $p \in \pi^{-1}(m)$. This proves point 3. Point 4 is an immediate consequence of Proposition 1.5.4. ■

Remark 1.5.9

1. By the locality property 3 of Proposition 1.5.8, for any point $m \in M$, the value of $(\nabla_X\Phi)(m)$ depends only on the value of X at m and on the values of the section $\Phi : M \to E$ along any smooth curve representing X_m. Thus, we obtain a mapping $\nabla : \mathrm{T}M \times \Gamma^\infty(E) \to E$ defined by

$$\nabla_{Y_m}\Phi = (\nabla_X\Phi)(m),$$

where X is an arbitrary extension of the tangent vector $Y_m \in \mathrm{T}_mM$ to a smooth vector field on M. Sometimes, it is useful to view a covariant derivative in this way.
2. The covariant derivative on a vector bundle E over M naturally induces covariant derivatives on all tensor bundles over E: for the dual bundle E^* we define

$$(\nabla_X^{E^*}\Phi^*)(\Phi) := X(\langle\Phi^*, \Phi\rangle) - \langle\Phi^*, \nabla_X^E\Phi\rangle, \tag{1.5.11}$$

where $X \in \mathfrak{X}(M)$, $\Phi \in \Gamma^\infty(E)$ and $\Phi^* \in \Gamma^\infty(E^*)$. Next, we extend ∇_X to any tensor product built from E and E^* by requiring that it be a derivation with respect to the tensor product of sections.
3. Let E_1 and E_2 be vector bundles over M endowed with connections ∇^1 and ∇^2. Then,

$$\nabla(s_1 \otimes s_2) := (\nabla^1 s_1) \otimes s_2 + s_1 \otimes (\nabla^2 s_2), \quad s_i \in \Gamma^\infty(E_i), i = 1, 2, \tag{1.5.12}$$

defines a connection on $E_1 \otimes E_2$ called the tensor product connection.
In particular, let E_1 and E_2 be associated with the principal bundles $P_1(M, G_1)$ and $P_2(M, G_2)$. Then, by Example 1.2.4/3, $E_1 \otimes E_2$ is naturally associated with the fibre product $P_1 \times_M P_2$, cf. Remark 1.1.9/2. If ω_1 and ω_2 are connection forms on P_1 and P_2, respectively, then the latter is endowed with the natural connection form $\vartheta^*\omega$ given by (1.3.16). If ∇^1 and ∇^2 are the covariant derivatives in E_1 and E_2 induced from ω_1 and ω_2, respectively, then the covariant derivative induced from $\vartheta^*\omega$ coincides with the tensor product connection $\nabla^1 \otimes \nabla^2$ (Exercise 1.5.2). ◆

Next, recall that, as a consequence of Proposition 1.4.13, the square of the covariant exterior derivative in general does not vanish and that this non-vanishing is measured by the curvature of the connection under consideration. Let us find the counterpart of this fact within the Koszul calculus. For that purpose, recall that Ω is a horizontal 2-form on P with values in $\mathfrak{g}$. Since σ' is a homomorphism from $\mathfrak{g}$ to $\mathrm{End}(F)$, $\sigma'(\Omega)$ is a horizontal 2-form on P with values in $\mathrm{End}(F)$. Thus, by Proposition 1.2.12, to Ω there corresponds a 2-form on M with values in the endomorphism bundle $\mathrm{End}(E)$:

$$\mathsf{R}^{\nabla}_m(X, Y) := \iota_p \circ \sigma'(\Omega_p(X^h, Y^h)) \circ \iota_p^{-1}, \tag{1.5.13}$$

where $m \in M$, $p \in \pi^{-1}(m)$, $X, Y \in \mathrm{T}_m M$ and X^h and Y^h are the horizontal lifts of X and Y to p, respectively.

Definition 1.5.10 The 2-form R^{∇} is called the curvature endomorphism form associated with Ω.

Proposition 1.5.11 *For any pair of vector fields $X, Y \in \mathfrak{X}(M)$,*

$$\mathsf{R}^{\nabla}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{1.5.14}$$

Proof Let $X, Y \in \mathfrak{X}(M)$ and let X^h, Y^h be their horizontal lifts to P. Let $p \in P$. Using (1.4.5), (1.5.3) and $\mathrm{hor}([X^h, Y^h]) = [X, Y]^h$, we calculate

$$\begin{aligned}
\Psi'_p(\Omega(X^h, Y^h))\tilde{\Phi}(p) &= -\,\mathrm{ver}([X^h, Y^h])_p(\tilde{\Phi}) \\
&= -[X^h, Y^h]_p(\tilde{\Phi}) + \mathrm{hor}([X^h, Y^h])_p(\tilde{\Phi}) \\
&= -[X^h, Y^h]_p(\tilde{\Phi}) + [X, Y]^h_p(\tilde{\Phi}) \\
&= -X^h_p(Y^h(\tilde{\Phi})) + Y^h_p(X^h(\tilde{\Phi})) + [X, Y]^h_p(\tilde{\Phi}) \\
&= -\iota_p^{-1} \circ \left(\nabla_X \nabla_Y \Phi - \nabla_Y \nabla_X \Phi - \nabla_{[X,Y]}\Phi\right)(m).
\end{aligned}$$

Now, the assertion follows from $\Psi'_p(A)(\tilde{\Phi}) = -\sigma'(A)\tilde{\Phi}(p)$ for all $A \in \mathfrak{g}$. ∎

Remark 1.5.12

1. Viewing the covariant derivative as a linear mapping

$$\nabla : \mathfrak{X}(M) \to \mathrm{End}(\Gamma^{\infty}(E)), \quad X \mapsto \nabla_X,$$

   we conclude that this mapping is a Lie algebra homomorphism iff the curvature endomorphism form vanishes.
2. Formula (1.5.14) extends to sections in arbitrary tensor bundles $\mathbb{T}^k_l(E)$ over E, where $\mathsf{R}^{\nabla}(X, Y)$ acts on $\mathbb{T}^k_l(E)$ in the representation induced by σ' (Exercise 1.5.5). ♦

In Sect. 1.3, we have discussed in detail the transport of connections on principal bundles under morphisms fulfilling some additional conditions, cf. Propositions 1.3.13 and 1.3.15 and the associated corollaries. Clearly, the transported connections induce covariant derivatives in the corresponding associated bundles. For later purposes, the pullback connection will be especially important. Thus, we discuss it in some detail.

Let $P(M, G)$ be a principal bundle, let N be a manifold, let $\varphi : N \to M$ be a smooth mapping and let $\varphi^* P$ be the pullback bundle induced by φ. Let (F, G, σ) be a Lie group representation and let $E = P \times_G F$ be the corresponding bundle associated with P. By Example 1.2.4/2, the pullback bundle $\varphi^* E$ is naturally associated with $\varphi^* P$ via the vector bundle isomorphism

$$\varphi^* E \to \varphi^* P \times_G F, \quad (y, [(p, f)]) \mapsto [((y, p), f)],$$

cf. (1.2.6). By point 2 of Corollary 1.3.16, every connection ω on P induces a connection $\vartheta^* \omega$ on the pullback bundle $\varphi^* P$. Here, $\vartheta : \varphi^* P \to P$ is the induced bundle morphism. If Γ^E denotes the connection on E induced from ω, then the connection $\Gamma^{\varphi^* E}$ on $\varphi^* E$ induced from the pullback connection $\vartheta^* \omega$ is given by

$$\Gamma^{\varphi^* E} = (\pi_2')^{-1}(\Gamma^E).$$

Using the obvious identification

$$\mathrm{T}_{(y,e)} \varphi^* E \cong \pi_1'(\mathrm{T}_{(y,e)} \varphi^* E) \oplus \pi_2'(\mathrm{T}_{(y,e)} \varphi^* E),$$

we obtain

$$\mathrm{T}_{(y,e)} \varphi^* E = \left\{ (Y, Z) \in \mathrm{T}_y N \oplus \mathrm{T}_e E : \varphi_y'(Y) = (\pi_F)_e'(Z) \right\}.$$

Thus, the decomposition of $(Y, Z) \in \mathrm{T}_{(y,e)} \varphi^* E$ with respect to $\Gamma^{\varphi^* E}$ is given by

$$(Y, Z) = (0, Z^v) + (Y, Z^h), \tag{1.5.15}$$

with $Z = Z^v + Z^h$ being the decomposition with respect to Γ^E.

Let us analyze the induced covariant derivative $\nabla^{\vartheta^* \omega}$. For that purpose, it is convenient to view the space of sections of $\varphi^* E$ as follows.

Definition 1.5.13 In the above notation, a section of E along φ is a mapping $\phi : N \to E$ fulfilling

$$\pi_F \circ \phi = \varphi.$$

The vector space of sections of E along φ is denoted by $\Gamma_\varphi^\infty(E)$.

Clearly, ϕ is a section of E along φ iff $y \mapsto (y, \phi(y))$ is a section of $\varphi^* E$, that is, $\Gamma^\infty(\varphi^* E)$ is canonically isomorphic to $\Gamma_\varphi^\infty(E)$.

Now, let $\Phi \in \Gamma^\infty(\varphi^* E)$ and let $Y \in \mathfrak{X}(N)$. Representing Φ by a section $\phi \in \Gamma_\varphi^\infty(E)$ and using Proposition 1.5.6, together with (1.2.6), (1.5.15) and (1.3.9), we calculate

$$\begin{aligned}
(\nabla^{\vartheta^*\omega}\Phi)_{(y,e)}(Y) &= \omega^{\varphi^*E}_{(y,e)}\left(\Phi'(Y)\right)\\
&= \omega^{\varphi^*E}_{(y,e)}\left((Y,\phi'(Y)\right)\\
&= \iota_{(y,p)} \circ (\iota'_{(y,p)})^{-1}\left(0,(\phi'(Y))^{v}\right)\\
&= \left(y, \iota_p \circ (\iota'_p)^{-1}(\phi'(Y))^{v}\right)\\
&= \left(y, \omega^E_e(\phi'(Y))\right).
\end{aligned}$$

We see that, associated with $\nabla^{\vartheta^*\omega}_Y$, there is an operator

$$\nabla^{\varphi}_Y : \Gamma^{\infty}_{\varphi}(E) \to \Gamma^{\infty}_{\varphi}(E), \quad \nabla^{\varphi}_Y\phi := \omega^E(\phi'(Y)). \tag{1.5.16}$$

Definition 1.5.14 The operator ∇^{φ} is called the covariant derivative along the mapping φ.

We have

$$\nabla^{\vartheta^*\omega}_Y(\mathrm{id}_N \times \phi) = \mathrm{id}_N \times \nabla^{\varphi}_Y\phi,$$

and, by construction, ∇^{φ}_Y inherits the properties listed in Proposition 1.5.8. Moreover, it fulfils an obvious chain rule: for another mapping $\chi : L \to N$, the composition $\phi \circ \chi$ is a section along $\varphi \circ \chi$ and for $X \in \mathrm{T}L$ we have (Exercise 1.5.3)

$$\nabla^{\varphi\circ\chi}_X(\phi \circ \chi) = \nabla^{\varphi}_{\chi'(X)}\phi. \tag{1.5.17}$$

Exercises

1.5.1 Prove the statement of point 1 of Remark 1.5.5.

1.5.2 Prove the statements of Remarks 1.5.9/2 and 1.5.9/3.

1.5.3 Prove formula (1.5.17).

1.5.4 Using (1.4.12), calculate d^2_{ω} in terms of the curvature endomorphism form.

1.5.5 Prove point 2 of Remark 1.5.12.

1.6 Bundle Reduction

Recall from Sect. 1.1 that a morphism (ϑ, λ) of principal bundles $Q(M, H)$ and $P(M, G)$ is called a λ-reduction or, simply, a reduction of P to H if Q is a subbundle of P fulfilling $\tilde{\vartheta} = \mathrm{id}_M$. In that case, P is called λ-reducible to H and Q is called a λ-reduction of P.

We start with giving two criteria for the reducibility of principal bundles. For the following, recall the description of principal bundles in terms of transition mappings.

Proposition 1.6.1 *Let $P(M, G)$ be a principal bundle and let $\lambda : H \to G$ be an injective Lie group homomorphism. Then, P is λ-reducible iff there exists a covering $\{U_i\}$ of M and an associated 1-cocycle $\{\rho_{ij}\}$ of P with values in* $\mathrm{im}(\lambda)$.

Proof Let $Q(M, H)$ be a λ-reduction of P and let (ϑ, λ) be the corresponding morphism. Let $\{(U_i, \chi_i^Q)\}$ be a bundle atlas of Q and let $\{\tau_{ij}\}$ be the corresponding 1-cocycle. Since every local section s in Q defines a local section in P by $\vartheta \circ s$, each χ_i^Q defines a local trivialization χ_i^P of P over U_i. Let κ_i^Q and κ_i^P be the equivariant mappings corresponding to χ_i^Q and χ_i^P, respectively. One can check that $\lambda \circ \kappa_i^Q = \kappa_i^P \circ \vartheta$. Hence, the transition mappings of the family $\{\chi_i^P\}$ are given by

$$\rho_{ij}(m) = \kappa_i^P(\vartheta(q))\kappa_j^P\big(\vartheta(q)\big)^{-1} = \lambda(\kappa_i^Q(q))\lambda\big(\kappa_j^Q(q)\big)^{-1} = \lambda\big(\kappa_i^Q(q)\kappa_j^Q(q)^{-1}\big)$$

and, thus,

$$\rho_{ij} = \lambda \circ \tau_{ij} \quad \text{for all } i, j. \tag{1.6.1}$$

Conversely, let $\{\rho_{ij}\}$ be the 1-cocycle associated with a bundle atlas $\{(U_i, \chi_i^P)\}$. Assume that it takes values in $\mathrm{im}(\lambda)$. Since λ is injective, the ρ_{ij} define mappings $\tau_{ij} : U_i \cap U_j \to H$ via (1.6.1). Since injective Lie group homomorphisms are immersions, cf. Corollary I/5.3.7, (H, λ) is a Lie subgroup. Since Lie subgroups are initial submanifolds, cf. Proposition I/5.6.4, the τ_{ij} are smooth. Moreover, the cocycle property of $\{\rho_{ij}\}$ implies that of $\{\tau_{ij}\}$. According to Proposition 1.1.10, the 1-cocycle $\{\tau_{ij}\}$ defines a principal H-bundle Q over M. Let $\pi_Q : Q \to M$ be the canonical projection and let $\{(U_i, \chi_i^Q)\}$ be the bundle atlas of Q constructed in the proof of this proposition. For every i, we define a mapping

$$\vartheta_i : \pi_Q^{-1}(U_i) \to \pi_P^{-1}(U_i), \quad \vartheta_i := \big(\chi_i^P\big)^{-1} \circ (\mathrm{id}_{U_i} \times \lambda) \circ \chi_i^Q,$$

where $\pi_P : P \to M$ is the canonical projection of P. By (1.6.1), we have $\vartheta_i = \vartheta_j$ for any pair (i, j) such that $U_i \cap U_j \neq \varnothing$. Thus, the family of mappings $\{\vartheta_i\}$ defines an equivariant mapping $\vartheta : Q \to P$. By construction, (ϑ, λ) is a λ-reduction. ■

The next proposition provides a criterion for reducibility in terms of equivariant mappings.

Proposition 1.6.2 *Let (P, G, M, Ψ, π) be a principal bundle and let (F, G, σ) be a transitive Lie group action. Let $f \in F$ and let $G_f \subset G$ be the stabilizer of f under the action σ. Then, every equivariant mapping $\varphi \in \mathrm{Hom}_G(P, F)$ defines a reduction of P to an embedded principal G_f-subbundle*

$$Q_f = \{p \in P : \varphi(p) = f\}. \tag{1.6.2}$$

Conversely, every such reduction defines an element $\varphi \in \mathrm{Hom}_G(P, F)$.

Proof Let $\varphi \in \mathrm{Hom}_G(P, F)$ and let Q_f be given by (1.6.2). Since σ is transitive and φ is equivariant, φ is a submersion. Hence, by the Level Set Theorem, $Q_f = \varphi^{-1}(f)$

is an embedded submanifold of P for all $f \in F$. Since for every $q \in Q_f$ and every $a \in G_f$,

$$\varphi(\Psi_a(q)) = \sigma_{a^{-1}}(\varphi(q)) = \sigma_{a^{-1}} f = f,$$

Q_f is G_f-invariant. Thus, Ψ induces a free right action of G_f on Q_f, denoted by the same symbol. Since G_f is closed, cf. Proposition I/6.1.5, the induced action is proper, cf. Proposition I/6.3.4. Hence, Q_f is a principal G_f-bundle over the manifold Q_f/G_f. The natural inclusion mappings $Q_f \to P$ and $G_f \to G$ define a principal bundle morphism. Let $\tilde{\vartheta} : Q_f/G_f \to M$ be the corresponding projection. It remains to show that $\tilde{\vartheta}$ is a diffeomorphism. By local triviality, it suffices to show that $\tilde{\vartheta}$ is bijective. For that purpose, we show that Q_f intersects every fibre of P and that the intersections coincide with the G_f-orbits. For the first statement, let $m \in M$ and let $p \in \pi^{-1}(m)$. Since σ acts transitively, there exists an $a \in G$ such that $\varphi(p) = \sigma_a(f)$. Then, $\varphi(\Psi_a(p)) = \sigma_{a^{-1}}(\varphi(p)) = f$, that is, $\Psi_a(p) \in Q_f$. To prove the second statement, let $q_1, q_2 \in Q_f$ and $a \in G$ such that $q_2 = \Psi_a(q_1)$. Then,

$$f = \varphi(q_2) = \varphi(\Psi_a(q_1)) = \sigma_{a^{-1}}(\varphi(q_1)) = \sigma_{a^{-1}}(f),$$

that is, $a \in G_f$. Thus, Q_f is a reduction of P to the subgroup G_f.

Conversely, let there be given a reduction of P to $Q \subset P$ with structure group $G_f \subset G$ and with the morphism given by the natural inclusion mapping. Then, we take the constant mapping $\varphi : Q \to F$, $\varphi(q) := f$, and extend it to a mapping $\varphi : P \to F$ by

$$\varphi(\Psi_a(q)) := \sigma_{a^{-1}}(f), \quad a \in G, q \in Q.$$

This mapping is well defined: if $\Psi_{a_1}(q_1) = \Psi_{a_2}(q_2)$ for $q_1, q_2 \in Q$, then $a_1 a_2^{-1} \in G_f$ and thus

$$\sigma_{a_2^{-1}}(f) = \sigma_{a_1^{-1}} \circ \sigma_{a_1 a_2^{-1}}(f) = \sigma_{a_1^{-1}}(f).$$

By construction, φ is smooth and equivariant. ■

Remark 1.6.3 The bundle reduction Q_f depends on the choice of $f \in F$ as follows: for every $f' \in F$, there exists an $a \in G$ such that $f' = \sigma_a(f)$. Then, for every $q \in Q_f$, we have

$$\varphi(\Psi_{a^{-1}}(q)) = \sigma_a(\varphi(q)) = \sigma_a(f) = f'.$$

Thus, $Q_{f'} = \Psi_{a^{-1}}(Q)$. Moreover, the corresponding structure group is

$$G_{f'} = G_{\sigma_a(f)} = aG_f a^{-1}.$$

♦

The following proposition characterizes the isomorphism classes of bundle reductions to a given structure group G_f.

Proposition 1.6.4 *Let $\varphi \in \mathrm{Hom}_G(P, F)$ and let ϑ be a vertical automorphism of P. If φ defines the bundle reduction Q_f, then $\varphi \circ \vartheta$ defines the bundle reduction $\vartheta^{-1}(Q_f)$. In particular, two reductions to the structure group G_f are equivalent iff the defining equivariant mappings are related by a vertical automorphism.*

Proof The reduced bundle defined by $\varphi \circ \vartheta$ is

$$\{p \in P : \varphi \circ \vartheta(p) = f\} = \vartheta^{-1}\left(\{p \in P : \varphi(p) = f\}\right) = \vartheta^{-1}(Q_f).$$

■

Proposition 1.6.2 implies a useful characterization of reductions of a principal bundle $P(M, G)$ to a given closed subgroup H of G. To formulate it, we consider the natural action of G on the homogeneous space G/H by left translation and build the associated bundle $P \times_G G/H$, cf. Example 1.2.4.

Corollary 1.6.5 *The reductions of a principal G-bundle P to a closed subgroup H of G are in one-to-one correspondence with the smooth sections of the associated bundle $P \times_G G/H$.*

Proof By Proposition 1.2.6, the sections of $P \times_G G/H$ are in one-to-one correspondence with the elements of $\mathrm{Hom}_G(P, F)$. Since G/H is a transitive G-manifold, we can apply Proposition 1.6.2 with $f = [\mathbb{1}]$. ■

The proofs of the following example are left to the reader (Exercise 1.6.1).

Example 1.6.6

1. Let E be a $\mathbb{K}$-vector bundle of rank k, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let $L(E)$ be its frame bundle. Recall from Remark I/2.2.2/3 that E is called orientable iff it admits a family of local trivializations whose transition mappings have positive determinant. Equivalently, E is orientable iff it admits a nowhere vanishing section of $\bigwedge^k E^*$ (the determinant line bundle of E). Thus, an orientation of E may be viewed as a section of the associated bundle

   $$L(E) \times_{\mathrm{GL}(k,\mathbb{K})} \mathrm{GL}(k, \mathbb{K})/\mathrm{GL}_+(k, \mathbb{K}),$$

   where $\mathrm{GL}_+(k, \mathbb{K}) \subset \mathrm{GL}(k, \mathbb{K})$ is the subgroup of elements with positive determinant. Now, Corollary 1.6.5 implies that E is orientable iff $L(E)$ is reducible to $\mathrm{GL}_+(k, \mathbb{K})$.
2. We take up Examples 1.1.15 and 1.1.18. Let E be a $\mathbb{K}$-vector bundle of rank n, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, endowed with a fibre metric.

   (a) Let $\mathbb{K} = \mathbb{R}$. A fibre metric may be viewed as a section of the associated bundle

   $$L(E) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^* \overset{s}{\otimes} (\mathbb{R}^n)^*,$$

where $\overset{s}{\otimes}$ denotes the symmetric tensor product. By the Sylvester Theorem, $\mathrm{GL}(n,\mathbb{R})$ acts transitively on the subspace $S^2_{(k,l)}\mathbb{R}^n \subset (\mathbb{R}^n)^* \overset{s}{\otimes} (\mathbb{R}^n)^*$ consisting of elements of rank $n = k + l$ and signature (k, l) and the stabilizer of the element $\eta = \mathbb{1}_k \oplus (-\mathbb{1}_l)$ is $\mathrm{O}(k, l)$. Thus,

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(k,l) \cong S^2_{(k,l)}\mathbb{R}^n,$$

and E admits a fibre metric with signature (k, l) iff $L(E)$ is reducible to a principal $\mathrm{O}(k, l)$-bundle. Clearly, the latter is the bundle of orthonormal frames $O(E)$.
(b) Let $\mathbb{K} = \mathbb{C}$. A fibre metric may be viewed as a section of the associated bundle

$$L(E) \times_{\mathrm{GL}(n,\mathbb{C})} (\overline{\mathbb{C}^n})^* \overset{s}{\otimes} (\mathbb{C}^n)^*.$$

By the Sylvester Theorem, $\mathrm{GL}(n,\mathbb{C})$ acts transitively on the subset of non-degenerate elements in $(\overline{\mathbb{C}^n})^* \overset{s}{\otimes} (\mathbb{C}^n)^*$ with stabilizer $\mathrm{U}(n)$. Thus, E admits a Hermitean fibre metric iff $L(E)$ is reducible to a principal $\mathrm{U}(n)$-bundle, which then coincides with the bundle of unitary frames $U(E)$. ♦

The following proposition shows that principal bundle reductions do not change the isomorphism class of associated vector bundles.

Proposition 1.6.7 *Let (P, G, M, Ψ, π) be a principal bundle and let (F, G, σ) be a Lie group representation. Let Q be a reduction of P to the structure group H defined by the morphism (ϑ, λ). Let $(F, H, \sigma \circ \lambda)$ be the associated Lie group representation of H. Then, the associated vector bundles $P \times_G F$ and $Q \times_H F$ are isomorphic.*

Proof Denote the H-action on Q by Ψ^Q. The canonical projection of Q is given by $\pi_Q = \pi \circ \vartheta$. Consider the mapping

$$\psi : Q \times_H F \to P \times_G F, \quad \psi([(q,f)]) := [(\vartheta(q), f)].$$

Since, for every $h \in H$,

$$\psi\big([(\Psi^Q_h(q), \sigma_{\lambda(h^{-1})}f)]\big) = [(\Psi_{\lambda(h)}(\vartheta(q)), \sigma_{\lambda(h)^{-1}}f)] = [(\vartheta(q), f)],$$

the mapping ψ is well defined. By construction, ψ is fibre-preserving and the induced mappings ψ_q of the fibres are linear. Finally, since ψ projects to the identical mapping of M and since $\psi_q \circ \iota^Q_q = \iota^P_{\vartheta(q)}$, the mapping ψ is bijective. As a consequence of the Inverse Mapping Theorem, the inverse mapping ψ^{-1} is smooth. ∎

Given an injective Lie group homomorphism $\lambda : H \to G$ and a principal H-bundle Q, we can form the associated principal G-bundle $P = Q^{[\lambda]}$. Then, Q is a λ-reduction of P and P is called a λ-extension of Q. By Corollary 1.3.14, the λ-extension of a connection always exists. The case of a λ-reduction is slightly more involved. Let us assume that G/H is a reductive homogeneous space. Then, in general, a principal bundle reduction induces a decomposition of a connection form into a pair of geometrical objects.

Proposition 1.6.8 *Let $P(M, G)$ be a principal bundle, let $H \subset G$ be a closed subgroup and let $Q(M, H)$ be a reduction of P given by the morphism (ϑ, i_H) with $i_H : H \to G$ being the natural inclusion mapping. Assume that the Lie algebra $\mathfrak{g}$ of G admits a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

with $\mathfrak{h}$ denoting the Lie algebra of H. Let ω be a connection form on P and let $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{m}}$ be its $\mathfrak{h}$- and $\mathfrak{m}$-components, respectively. Then,

1. *$\vartheta^*\omega_{\mathfrak{h}}$ is a connection form on Q,*
2. *$\vartheta^*\omega_{\mathfrak{m}}$ is an $\mathfrak{m}$-valued horizontal 1-form on Q of type $\mathrm{Ad}(H)\mathfrak{m}$.*

Proof Let Ψ and $\tilde{\Psi}$ be the G- and the H-actions on P and Q, respectively. Then,

$$\vartheta \circ \tilde{\Psi}_a(q) = \Psi_a \circ \vartheta(q) \tag{1.6.3}$$

for any $q \in Q$ and $a \in H$.
1. We must check that $\vartheta^*\omega_{\mathfrak{h}}$ has the properties of a connection form. Decomposing $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$ on P and using that ω is a connection form, for any $A \in \mathfrak{h}$, we have

$$A = \omega(A_*) = \omega_{\mathfrak{h}}(A_*) + \omega_{\mathfrak{m}}(A_*),$$

where A_* denotes the Killing vector field on P generated by A. Thus, $\omega_{\mathfrak{m}}(A_*) = 0$, that is, $\omega_{\mathfrak{h}}(A_*) = A$. Let $\tilde{A}_*$ denote the Killing vector field on Q generated by A. Then, by (1.6.3), $\vartheta' \circ \tilde{A}_* = A_* \circ \vartheta$ and thus $\vartheta^*\omega_{\mathfrak{h}}(\tilde{A}_*) = A$. It remains to show H-equivariance. For $a \in H$, we have

$$\Psi_a^*\omega = \mathrm{Ad}(a^{-1}) \circ \omega = \mathrm{Ad}(a^{-1}) \circ \omega_{\mathfrak{h}} + \mathrm{Ad}(a^{-1}) \circ \omega_{\mathfrak{m}}$$

and, on the other hand,

$$\Psi_a^*\omega = \Psi_a^*\omega_{\mathfrak{h}} + \Psi_a^*\omega_{\mathfrak{m}}.$$

By reductivity, $\mathrm{Ad}(a^{-1}) \circ \omega_{\mathfrak{m}}$ takes values in $\mathfrak{m}$. Hence, $\Psi_a^*\omega_{\mathfrak{h}} = \mathrm{Ad}(a^{-1}) \circ \omega_{\mathfrak{h}}$. Then, taking the pullback of this equation under ϑ and using (1.6.3), we obtain the assertion. Moreover, for later use, we note

$$\Psi_a^*\omega_{\mathfrak{m}} = \mathrm{Ad}(a^{-1}) \circ \omega_{\mathfrak{m}}. \tag{1.6.4}$$

2. By (1.6.4), $\vartheta^*\omega_{\mathfrak{m}}$ is an $\mathfrak{m}$-valued 1-form of type $\mathrm{Ad}(H)\mathfrak{m}$. It remains to show that it is horizontal: for any $A \in \mathfrak{h}$, using (1.6.3), we calculate

$$(\vartheta^*\omega_{\mathfrak{m}})_q(\tilde{A}_*) = (\vartheta^*\omega)_q(\tilde{A}_*) - (\vartheta^*\omega_{\mathfrak{h}})_q(\tilde{A}_*).$$

The second term yields $-A$. For the first term, we compute

$$(\vartheta^*\omega)_q(\tilde{A}_*) = \omega_{\vartheta(q)}(\vartheta' \circ \tilde{A}_*(q)) = \omega_{\vartheta(q)}(A_* \circ \vartheta(q)) = A\,.$$

■

We conclude that only in the case when $\vartheta^*\omega$ takes values in $\mathfrak{h}$, its restriction to Q is a connection form on Q. This result suggests the following definition.

Definition 1.6.9 (*Reducible connection*) Let P be a principal G-bundle over a connected manifold M. Let Γ be a connection on P and let ω be its connection form. Let $Q(M, H)$ be a reduction of P given by the morphism (ϑ, i_H). Then, Γ is called reducible to H if $\vartheta^*\omega$ takes values in $\mathfrak{h}$. Γ is called irreducible if P is not reducible to any genuine Lie subgroup of G.

Recall that reductions of a principal G-bundle P to a closed subgroup H of G are in bijective correspondence with smooth sections of the associated bundle $P \times_G G/H$, cf. Corollary 1.6.5. Since orbits of Lie group actions are initial submanifolds, we can carry over Proposition 1.6.2 to the case of a general Lie group action (F, G, σ) by applying it to elements of $\mathrm{Hom}_G(P, F)$ with values in a single orbit of σ.

Proposition 1.6.10 *Let $P(M, G)$ be a principal bundle and let (F, G, σ) be a representation. Let $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$ and assume that it takes values in a single orbit O of σ. Let $Q(M, H)$ be the reduction of P defined by $\tilde{\Phi}$ and some element $f \in O$. Then, a connection Γ on P is reducible to a connection Γ' on Q iff $\tilde{\Phi}$ is parallel with respect to Γ.*

Proof Let ω be the connection form of Γ and let (ϑ, i_H) be the morphism corresponding to the reduction Q. Since $Q = \left\{p \in P : \tilde{\Phi}(p) = f\right\}$, we have $\vartheta^*\tilde{\Phi} = f$ on Q. Thus,

$$\vartheta^*\left(D_\omega\tilde{\Phi}\right) = \mathrm{d}(\vartheta^*\tilde{\Phi}) + \sigma'(\vartheta^*\omega)(\vartheta^*\tilde{\Phi}) = \sigma'(\vartheta^*\omega)(\vartheta^*\tilde{\Phi}).$$

Now, Γ is reducible iff $\vartheta^*\omega$ takes values in the Lie algebra $\mathfrak{h}$ of H, that is, iff $\sigma'(\vartheta^*\omega)(\vartheta^*\tilde{\Phi}) = 0$, that is, iff $\vartheta^*\left(D_\omega\tilde{\Phi}\right) = 0$. By the G-equivariance of Γ the latter is equivalent to $D_\omega\tilde{\Phi} = 0$ on the whole of P. ■

Definition 1.6.11 (*Compatible connection*) Let $Q(M, H) \subset P(M, G)$ be a principal bundle reduction defined by an element $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$ taking values in a single orbit O and by a point $f \in O$. A connection Γ on P will be referred to as compatible with $\tilde{\Phi}$ if it is reducible to Q.

By Proposition 1.6.10, a connection Γ is compatible with $\tilde{\Phi}$ iff

$$D_\omega\tilde{\Phi} = 0. \tag{1.6.5}$$

Here, ω is the connection form of Γ. Note that $\tilde{\Phi}$ takes values in a single orbit O iff the corresponding section Φ of $P \times_G F$ takes values in $P \times_G O$. In terms of Φ, (1.6.5) takes the form

$$\nabla^\omega \Phi = 0. \tag{1.6.6}$$

In the sequel, we will frequently meet compatible connections, in particular in the context of H-structures to be discussed in Chap. 2. Here, we discuss one important class of examples.

Example 1.6.12 (Connection compatible with a fibre metric) We take up point 2 of Example 1.6.6. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let E be a $\mathbb{K}$-vector bundle of rank n over M endowed with a fibre metric h. Recall that h may be viewed as a section of the associated bundle $L(E) \times_{\mathrm{GL}(n,\mathbb{K})} \mathscr{F}$, where $\mathscr{F}$ denotes the space of inner products in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. In the case $\mathbb{K} = \mathbb{C}$, $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\mathscr{F}$, whereas in the case $\mathbb{K} = \mathbb{R}$ it does not. If, in the latter case, we assume that M is connected, then h takes values in a single $\mathrm{GL}(n, \mathbb{R})$-orbit on $\mathscr{F}$. Now, by (1.6.6), a connection form ω on $L(E)$ is compatible with h iff

$$\nabla^\omega \mathsf{h} = 0.$$

Since ∇_X is a derivation of the tensor algebra, this condition takes the following form:

$$X(\mathsf{h}(\Phi_1, \Phi_2)) = \mathsf{h}(\nabla_X \Phi_1, \Phi_2) + \mathsf{h}(\Phi_1, \nabla_X \Phi_2), \tag{1.6.7}$$

for any $\Phi_1, \Phi_2 \in \Gamma^\infty(E)$ and $X \in \mathfrak{X}(M)$. If ω is compatible, then it is reducible to the bundle of h-orthonormal or h-unitary frames of E, respectively. ♦

In the next section, we will show that there exists a smallest reduction of a principal G-bundle P with connection Γ, namely the reduction to the holonomy bundle. We will see that a connection is irreducible iff P coincides with its holonomy bundle.

Exercises

1.6.1 Prove the statements of Example 1.6.6.

1.7 Parallel Transport and Holonomy

From elementary geometry, the reader knows the notion of parallel transport of a vector in an affine space, say, in the 2-plane. Here, we show that this notion generalizes to the abstract theory of connections on fibre bundles.

Definition 1.7.1 Let P be a principal G-bundle over M with canonical projection π, let Γ be a connection on P and let γ and $\tilde{\gamma}$ be smooth curves in M and P, respectively. The curve $\tilde{\gamma}$ is called

1. a lift of γ if $\pi \circ \tilde{\gamma} = \gamma$,
2. horizontal relative to Γ, if all tangent vectors $\dot{\tilde{\gamma}}$ are horizontal relative to Γ.

Proposition 1.7.2 *Let (P, G, M, Ψ, π) be a principal bundle and let Γ be a connection on P. Let I be an open interval containing 0 and let $\gamma : I \to M$ be a smooth curve. Then, for every point $p_0 \in \pi^{-1}(\gamma(0))$, there exists a unique horizontal lift γ^h of γ fulfilling $p_0 = \gamma^h(0)$.*

The proposition generalizes to piecewise smooth curves.

Proof We choose an arbitrary lift $\tilde{\gamma}$ of γ starting at p_0 and seek the horizontal lift of γ in the following form:

$$t \mapsto \gamma^h(t) = \Psi_{g(t)}\tilde{\gamma}(t),$$

see Fig. 1.1. We will prove that there exists a unique curve $t \mapsto g(t)$ in G such that γ^h is horizontal. Since $\Psi_{\gamma^h(t)} = \Psi_{\tilde{\gamma}(t)} \circ L_{g(t)}$, we have

$$\begin{aligned}
\dot{\gamma}^h(t) &= \left(\Psi_{g(t)}\right)'_{\tilde{\gamma}(t)} \left(\dot{\tilde{\gamma}}(t)\right) + \left(\Psi_{\tilde{\gamma}(t)}\right)'_{g(t)} (\dot{g}(t)) \\
&= \left(\Psi_{g(t)}\right)'_{\tilde{\gamma}(t)} \left(\dot{\tilde{\gamma}}(t)\right) + \left(\Psi_{\gamma^h(t)}\right)'_{\gamma^h(t)} \circ \left(L_{g(t)^{-1}}\right)'_{g(t)} (\dot{g}(t)) \,.
\end{aligned}$$

Let ω be the connection form of Γ. The curve γ^h is horizontal iff $\omega(\dot{\gamma}^h(t)) = 0$ for all $t \in I$. Inserting the formula for $\dot{\gamma}^h$, we obtain

$$\omega\left(\left(\Psi_{g(t)}\right)'_{\tilde{\gamma}(t)} \left(\dot{\tilde{\gamma}}(t)\right)\right) + \omega\left(\left(\Psi_{\gamma^h(t)}\right)'_{\gamma^h(t)} \circ \left(L_{g(t)^{-1}}\right)'_{g(t)} (\dot{g}(t))\right) = 0 \,.$$

Now, point 3 of Proposition 1.3.5 implies

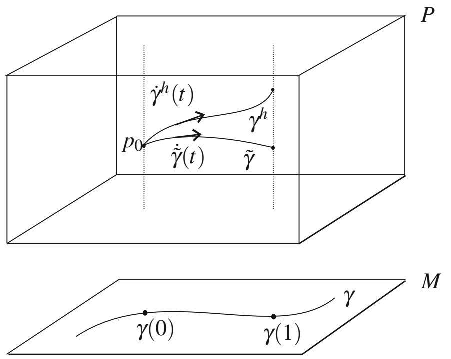

Fig. 1.1 Construction of the horizontal lift γ^h in the proof of Proposition 1.7.2

$$\mathrm{Ad}\left(g(t)^{-1}\right) \circ \omega(\dot{\tilde{\gamma}}(t)) = -\left(L_{g(t)^{-1}}\right)'_{g(t)}(\dot{g}(t)).$$

This is an ordinary first order differential equation for $t \mapsto g(t)$ with the initial condition $g(0) = \mathbb{1}$. Using the standard existence and uniqueness theorem for differential equations of this type, we obtain the assertion.[18] ■

Now, let $I = [0, 1]$. Recall that the concatenation of curves $\gamma, \tau : I \to M$ satisfying $\gamma(1) = \tau(0)$ is defined by

$$\tau \cdot \gamma(t) := \begin{cases} \gamma(2t) & | \ t \leq \frac{1}{2}, \\ \tau(2t-1) & | \ t > \frac{1}{2} \end{cases} \tag{1.7.1}$$

and that the inverse curve is defined by $\gamma^{-1}(t) = \gamma(1-t)$. The proof of the following lemma is left to the reader (Exercise 1.7.2).

Lemma 1.7.3 *Let $\gamma : I \to M$ be a piecewise smooth curve, let $p \in \pi^{-1}(\gamma(0))$ and let γ^h be the horizontal lift of γ through p.*

1. *The horizontal lift of γ through $\Psi_a(p)$ is given by $\Psi_a \circ \gamma^h$.*
2. *If $\tau : I \to M$ is another piecewise smooth curve fulfilling $\tau(0) = \gamma(1)$, then the horizontal lift of $\tau \cdot \gamma$ through p is given by $\tau^h \cdot \gamma^h$, where τ^h is the horizontal lift of τ through the point $\gamma^h(1)$.*
3. *The horizontal lift of γ^{-1} to the point $\gamma^h(1)$ is given by $(\gamma^h)^{-1}$.*

Via the horizontal lift, every piecewise smooth curve $\gamma : I \to M$ defines a mapping

$$\hat{\gamma}_\Gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)),$$

which assigns to $p \in \pi^{-1}(\gamma(0))$ the point $\gamma^h(1)$, where γ^h is the horizontal lift of γ through p.

Definition 1.7.4 The mapping $\hat{\gamma}_\Gamma$ is called the operator of parallel transport along γ with respect to the connection Γ.

By point 1 of Lemma 1.7.3, $\hat{\gamma}_\Gamma$ is equivariant and, thus, an isomorphism of G-manifolds. By point 3, we have

$$\left(\hat{\gamma}_\Gamma\right)^{-1} = \widehat{\left(\gamma^{-1}\right)}_\Gamma.$$

Remark 1.7.5

1. By construction, the parallel transport operator does not depend on the choice of the parameterization of the curve γ. If γ is a smooth curve from m_0 to m_1 and τ is

[18] For a detailed discussion of this theorem for differential equations on Lie groups we refer to [383], see Sect. II/3.

a smooth curve from m_1 to m_2, by point 2 of Lemma 1.7.3, $\hat{\gamma}_\Gamma$ can be composed with $\hat{\tau}_\gamma$ and we have

$$\widehat{(\tau \cdot \gamma)}_\Gamma = \hat{\tau}_\Gamma \circ \hat{\gamma}_\Gamma. \tag{1.7.2}$$

2. For a given horizontal lift γ^h of γ, we obtain

$$\hat{\gamma}_\Gamma = \Psi_{\gamma^h(1)} \circ \left(\Psi_{\gamma^h(0)}\right)^{-1}, \tag{1.7.3}$$

or, more generally,

$$\hat{\gamma}_\Gamma(t) = \Psi_{\gamma^h(t)} \circ \left(\Psi_{\gamma^h(0)}\right)^{-1} : \pi^{-1}(m_0) \to \pi^{-1}(\gamma(t)). \tag{1.7.4}$$

♦

Now, let us consider the important special case of parallel transport along closed curves in M. Let $C(m)$ be the set of piecewise smooth closed curves starting and ending at $m \in M$. The parallel transport along $\gamma \in C(m)$ yields an automorphism of the fibre $\pi^{-1}(m)$. For the trivial curve it coincides with the identity. Thus, the set of parallel transports along elements of $C(m)$ form a subgroup of the group of automorphisms of the fibre $\pi^{-1}(m)$.

Definition 1.7.6 The group of parallel transports along elements of $C(m)$ is called the holonomy group of Γ with base point m. It will be denoted by $\mathscr{H}_m(\Gamma)$.

Let us denote by $C^0(m) \subset C(m)$ the subset of closed curves which are homotopic to the trivial curve. The corresponding subgroup $\mathscr{H}^0_m(\Gamma) \subset \mathscr{H}_m(\Gamma)$ is called the restricted holonomy group of Γ with base point m.

We note that the holonomy groups can be naturally viewed as subgroups of the structure group G: for every $p \in \pi^{-1}(m)$ and $\gamma \in C(m)$, there exists a unique $a \in G$ such that

$$\hat{\gamma}_\Gamma(p) = \Psi_a(p). \tag{1.7.5}$$

For another closed curve $\tau \in C(m)$, let $b \in G$ be the corresponding group element. Then,

$$\hat{\tau}_\Gamma \circ \hat{\gamma}_\Gamma(p) = \hat{\tau}_\Gamma(\Psi_a(p)) = \Psi_a \circ \hat{\tau}_\Gamma(p) = \Psi_a \circ \Psi_b(p) = \Psi_{ba}(p),$$

that is, to $\hat{\tau}_\Gamma \circ \hat{\gamma}_\Gamma$ there corresponds the product ba of elements of G. The subgroup of G defined in this way is called the holonomy group of Γ with base point p. It is denoted by $\mathscr{H}_p(\Gamma)$. Correspondingly, the restricted holonomy group with base point p is denoted by $\mathscr{H}^0_p(\Gamma)$. Obviously, $\mathscr{H}_p(\Gamma)$ and $\mathscr{H}_m(\Gamma)$ are isomorphic as abstract groups.

Remark 1.7.7 Let us define the following equivalence relation on P: two points p_1 and p_2 of P are equivalent iff they can be joined by a horizontal curve of Γ. Then,

$\mathscr{H}_p(\Gamma)$ coincides with the subset of elements $a \in G$ such that $p \in P$ is equivalent to $\Psi_a(p)$. ♦

The following proposition is a simple exercise which we leave to the reader (Exercise 1.7.1).

Proposition 1.7.8 *Let P be a principal G-bundle over M and let Γ be a connection on P.*

1. *The holonomy groups of Γ with base points p and $\Psi_a(p)$, $a \in G$, are conjugate in G,*
$$\mathscr{H}_{\Psi_a(p)}(\Gamma) = a^{-1}\mathscr{H}_p(\Gamma)a.$$
The same is true for the restricted holonomy groups.
2. *If two points in P can be joined by a horizontal curve, then their holonomy groups coincide.* ■

Clearly, if M is connected, then for each pair p_1 and p_2 of points in P, there exists a group element $a \in G$, such that p_1 and $\Psi_a(p_2)$ can be joined by a horizontal curve. In this case, Proposition 1.7.8 implies that all holonomy groups $\mathscr{H}_p(\Gamma)$, $p \in P$, are conjugate in G. Consequently, they are all isomorphic to each other.

Theorem 1.7.9 *Let P be a principal G-bundle over M, let M be connected and let Γ be a connection on P. Then, for every $p \in P$,*

1. *$\mathscr{H}_p^0(\Gamma)$ is a connected Lie subgroup of G,*
2. *$\mathscr{H}_p^0(\Gamma)$ is a normal subgroup of $\mathscr{H}_p(\Gamma)$ and $\mathscr{H}_p(\Gamma)/\mathscr{H}_p^0(\Gamma)$ is countable.*

Our proof is along the lines of Sect. 19.7 of [447]. First, we need the following lemma. Recall that a manifold is said to be C^∞-pathwise connected if any two of its points can be joined by a smooth curve.

Lemma 1.7.10 *Let H be a C^∞-pathwise connected subgroup of a Lie group G. Then, H is a connected Lie group and a Lie subgroup of G.*

Proof Consider the following subset of the Lie algebra of G:

$$\mathfrak{h} := \left\{h'(0) \in \mathrm{T}_{\mathbb{1}}G : h \in C^\infty(\mathbb{R}, G),\ h(\mathbb{R}) \subset H,\ h(0) = \mathbb{1}\right\}. \tag{1.7.6}$$

One can check that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ (Exercise 1.7.3). Let $\tilde{H}$ be the corresponding connected Lie subgroup of G provided by Proposition I/5.6.5. As shown in the proof of this proposition, $\tilde{H}$ is the maximal integral submanifold through $\mathbb{1}$ of the distribution $D^{\mathfrak{h}}$ generated by $\mathfrak{h}$. Now, let $t \mapsto h(t)$ be a smooth curve in G such that $h(\mathbb{R}) \subset H$ and $h(0) = \mathbb{1}$. Clearly,

$$L'_{h(t)^{-1}}h'(t) = \frac{\mathrm{d}}{\mathrm{d}s}_{\restriction s=0} h(t)^{-1}h(t+s) \in \mathfrak{h}.$$

Thus, by left invariance of $D^{\mathfrak{h}}$, $t \mapsto h(t)$ lies in $\tilde{H}$. Since, by assumption, every point in H is connected with $\mathbb{1}$ via such a curve, we conclude $H \subset \tilde{H}$.

To prove $\tilde{H} \subset H$, choose a basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ in $\mathfrak{h}$ and a family of smooth curves $t \mapsto h_i(t)$ in G such that $h_i(\mathbb{R}) \subset H$, $h_i(0) = \mathbb{1}$ and $h_i'(0) = \mathbf{e}_i$. Consider the mapping

$$F : \mathbb{R}^n \to \tilde{H}, \quad F(\mathbf{t}) := h_1(t_1) \ldots h_n(t_n).$$

Clearly, $F'(0)$ maps $\mathbb{R}^n$ bijectively onto $\mathfrak{h}$. Thus, by the Inverse Mapping Theorem, F is a local diffeomorphism mapping an open neighbourhood of the origin in $\mathbb{R}^n$ onto an open neighbourhood of $\mathbb{1}$ in $\tilde{H}$. We conclude that H contains an open neighbourhood of $\mathbb{1}$ in $\tilde{H}$ and, thus, $\tilde{H} \subset H$. ■

Proof of the theorem. We prove that the restricted holonomy group is C^∞-pathwise connected and apply the lemma.

Let $m = \pi(p)$, let $[0, 1] \ni s \mapsto \gamma(s) \in M$ be an element of $C^0(m)$ and let γ^h be its horizontal lift starting at p. Choose a smooth homotopy $\varphi : \mathbb{R}^2 \to M$ such that

$$\varphi(1, s) = \gamma(s), \quad \varphi(0, s) = \varphi(t, 0) = \varphi(t, 1) = m,$$

for all $(t, s) \in [0, 1] \times [0, 1]$. By Corollary 1.3.16, the connection Γ induces a connection Γ^φ on the pullback principal bundle $\varphi^* P$. Let $\vartheta : \varphi^* P \to P$ be the induced morphism projecting to φ. Clearly, for every $t \in [0, 1]$, the preimage under φ of the closed curve $s \mapsto \varphi(t, s)$ is the line segment $s \mapsto (t, s)$ in $\mathbb{R}^2$. Let Φ be the flow of the Γ^φ-horizontal lift of ∂_s. Then,

$$t \mapsto \vartheta \circ \Phi(t, 1)$$

is a smooth curve in P starting at p_0 and ending at $\gamma^h(1)$. Via (1.7.5), it defines a smooth curve in G starting at the unit element $\mathbb{1}$ and ending at the element of $\mathscr{H}^0_{p_0}(\Gamma)$ defined by γ, that is, $\mathscr{H}^0_{p_0}(\Gamma)$ is a C^∞-pathwise connected subgroup of G. Now, Lemma 1.7.10 implies the first assertion.

Let us prove the second assertion. Clearly, for smooth closed curves τ and γ starting at $m_0 \in M$, with γ being null-homotopic, the curve $\tau \cdot \gamma \cdot \tau^{-1}$ is null-homotopic, too. Thus, by (1.7.3), $\mathscr{H}^0_p(\Gamma)$ is a normal subgroup of $\mathscr{H}_p(\Gamma)$. To prove that the quotient group $\mathscr{H}_p(\Gamma)/\mathscr{H}^0_p(\Gamma)$ is countable, we define a homomorphism F from the fundamental group $\pi_1(M, m)$ of M based at m onto $\mathscr{H}_p(\Gamma)/\mathscr{H}^0_p(\Gamma)$ as follows: for a given $\alpha \in \pi_1(M, m)$, let $t \mapsto \gamma(t)$ be a piecewise smooth closed curve representing α. We put $F(\alpha) := [\hat{\gamma}_\Gamma]$. This mapping is well defined: if γ_1 and γ_2 are two representatives of α, then $\gamma_1 \circ \gamma_2^{-1}$ is null-homotopic and thus defines an element of $\mathscr{H}^0_p(\Gamma)$. Clearly, it is surjective. Thus, F is a homomorphism onto $\mathscr{H}_p(\Gamma)/\mathscr{H}^0_p(\Gamma)$, indeed. Now, countability of this quotient follows from the countability of $\pi_1(M, m)$. ■

Remark 1.7.11 By Theorem 1.7.9, $\mathscr{H}_p(\Gamma)$ is a Lie subgroup of G whose connected component of the identity coincides with $\mathscr{H}^0_p(\Gamma)$. In particular, if M is simply connected, then $\mathscr{H}_p(\Gamma)$ is connected. ♦

Proposition 1.7.12 *Let (P, G, M, Ψ, π) be a principal bundle with connected base manifold M and let Γ be a connection on P. Let $p_0 \in P$ and let $P_{p_0}(\Gamma)$ be the subset of points in P which can be joined to p_0 by a horizontal curve of Γ. Then,*

1. *$P_{p_0}(\Gamma)$ is a reduction of P with structure group $\mathscr{H}_{p_0}(\Gamma)$.*
2. *The connection Γ is reducible to a connection on $P_{p_0}(\Gamma)$.*

Proof 1. Since M is connected, the restriction of π to $P_{p_0}(\Gamma)$ is surjective. By Proposition 1.7.8 and Remark 1.7.7, $P_{p_0}(\Gamma)$ is invariant under the right action of the Lie subgroup $\mathscr{H}_{p_0}(\Gamma) \subset G$ and $P_{p_0}(\Gamma)$ intersects the fibres of P in $\mathscr{H}_{p_0}(\Gamma)$-orbits.

Next, we show that $P_{p_0}(\Gamma)$ is a subbundle of P. For that purpose, let $p \in P_{p_0}(\Gamma)$ and let (U, κ) be a local chart at $m = \pi(p)$ such that $\kappa(U)$ is an open ball in $\mathbb{R}^{\dim M}$ and $\kappa(m) = 0$. For any $\tilde{m} \in U$, let $t \mapsto \gamma(t)$ be the unique curve from m to $\tilde{m}$ such that $t \mapsto \kappa \circ \gamma(t)$ is the line segment from 0 to $\kappa(\tilde{m})$. Define

$$s\colon U \to P, \quad s(\tilde{m}) := \hat{\gamma}_\Gamma(p).$$

Clearly, s is a smooth local section fulfilling $s(U) \subset P_{p_0}(\Gamma)$. Now, for every $p \in \pi^{-1}(U)$, there exists a unique element $a \in G$ such that $p = \Psi_a s(\pi(p))$. Then,

$$\tilde{\chi} : \pi^{-1}(U) \to U \times G, \quad \tilde{\chi}(p) := (\pi(p), a)$$

is a bijective mapping which induces a bijective mapping

$$\chi : P_{p_0}(\Gamma) \cap \pi^{-1}(U) \to U \times \mathscr{H}_{p_0}(\Gamma).$$

Constructing, this way, a system of bijective mappings $\{(U_i, \kappa_i)\}$ such that $\{U_i\}$ is a covering of M and requiring that the mappings χ_i be diffeomorphisms, we endow $P_{p_0}(\Gamma)$ with a manifold structure and with a system of local trivializations. To see that $P_{p_0}(\Gamma)$ is a submanifold of P, note that $\{\pi^{-1}(U_i)\}$ is a covering of $P_{p_0}(\Gamma)$ with open subsets of P such that every subset $P_{p_0}(\Gamma) \cap \pi^{-1}(U)$ is a submanifold of P. This follows from the fact that $\mathscr{H}_{p_0}(\Gamma)$ is a submanifold of G and that the χ_i are diffeomorphisms.

We conclude that $P_{p_0}(\Gamma)$ is a reduction of P with structure group $\mathscr{H}_{p_0}(\Gamma)$ with the corresponding morphism (ϑ, λ) given by the natural inclusion mappings $\vartheta : P_{p_0}(\Gamma) \to P$ and $\lambda : \mathscr{H}_{p_0}(\Gamma) \to G$.

2. Let $p \in P_{p_0}(\Gamma)$ and let $X \in \Gamma_p$. Then, there exists a curve γ starting at $\pi(p)$ such that its horizontal lift γ^h starting at p fulfils $X = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 \gamma^h(t)$. Since p can be joined to p_0 by a horizontal curve, we conclude that the image of γ^h is contained in $P_{p_0}(\Gamma)$ and that $X \in \mathrm{T}_p\left(P_{p_0}(\Gamma)\right)$. Thus, for every $p \in P_{p_0}(\Gamma)$, the horizontal subspace Γ_p is tangent to $P_{p_0}(\Gamma)$. This means that the connection Γ is reducible to $P_{p_0}(\Gamma)$: the horizontal subspace at $p \in P_{p_0}(\Gamma)$ of the reduced connection is given by Γ_p. ∎

Definition 1.7.13 The subbundle $P_{p_0}(\Gamma)$ is called the holonomy bundle of Γ with base point p_0.

Note that $P_{p_0}(\Gamma) = P_{p_1}(\Gamma)$ iff p_0 and p_1 may be joined by a horizontal curve. Thus, for any pair (p_0, p_1) of points in P, we have either $P_{p_0}(\Gamma) = P_{p_1}(\Gamma)$ or $P_{p_0}(\Gamma) \cap P_{p_1}(\Gamma) = \varnothing$, that is, P decomposes into the union of disjoint holonomy bundles. One can check that all holonomy bundles of a given connection are isomorphic (Exercise 1.7.4).

Remark 1.7.14 Let $P(M, G)$ be a principal bundle, let $H \subset G$ be a Lie subgroup and let $Q(M, H)$ be a reduction of P given by a morphism (ϑ, i_H) with $i_H : H \to G$ being the natural inclusion mapping. Let Γ be a connection on P which is reducible to a connection $\tilde{\Gamma}$ in Q. Then, by Proposition 1.3.13, $\tilde{\Gamma}$ defines a connection $\hat{\Gamma}$ on P (the image of $\tilde{\Gamma}$ under ϑ). Now, $\hat{\Gamma}$ is either irreducible or not. In the first case, $\vartheta(Q)$ coincides with the holonomy bundle $P_p(\Gamma)$, $p \in \vartheta(Q)$, and $\hat{\Gamma}$ coincides with the reduction of Γ to $P_p(\Gamma)$. In the second case, by Proposition 1.7.12, $\hat{\Gamma}$ is reducible to the holonomy bundle. Thus, in this case, for all $p \in \vartheta(Q)$, we have

$$P_p(\Gamma) \subset \vartheta(Q), \quad \hat{\Gamma}_{\restriction P_p(\Gamma)} = \Gamma_{\restriction P_p(\Gamma)}.$$

Thus, the holonomy bundle is the smallest possible reduction of a principal bundle with connection. In particular, a connection Γ on P is irreducible iff $P = P_p(\Gamma)$ and $G = \mathscr{H}_p(\Gamma)$ for all $p \in P$. ♦

The following classical theorem characterizes the Lie algebra of the holonomy group of a connection in terms of its curvature [18].

Theorem 1.7.15 (Ambrose–Singer) *Let (P, G, M, Ψ, π) be a principal bundle with connected base manifold M and let Γ be a connection on P with connection form ω and curvature form Ω. Let $\mathfrak{g}$ be the Lie algebra of G. Then, for any $p_0 \in P$, the Lie algebra $\mathfrak{h}_{p_0}(\Gamma)$ of the holonomy group $\mathscr{H}_{p_0}(\Gamma)$ coincides with the subspace of $\mathfrak{g}$ generated by elements of the form $\Omega_p(X, Y)$, where $p \in P_{p_0}(\Gamma)$ and $X, Y \in \Gamma_p$.*

Proof By Proposition 1.7.12, without loss of generality we may assume $\mathscr{H}_{p_0}(\Gamma) = G$ and $P_{p_0}(\Gamma) = P$. Let

$$\mathfrak{h} = \operatorname{span}\left\{\Omega_p(X, Y) \in \mathfrak{g} : p \in P_{p_0}(\Gamma),\ X, Y \in \Gamma_p\right\}.$$

We must show that $\mathfrak{h} = \mathfrak{g}$. First, since Ω is a horizontal form of type Ad, the subspace $\mathfrak{h}$ is invariant under the adjoint action of G. Thus, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Next, consider the distribution

$$p \mapsto D_p := \Gamma_p \oplus \Psi'_p(\mathfrak{h}).$$

We show that D is involutive. Since Γ_p is spanned by horizontal vector fields and $\Psi'_p(\mathfrak{h})$ is spanned by Killing vector fields generated by elements of $\mathfrak{h}$, we must consider the following three cases:

(a) Let $A, B \in \mathfrak{h}$ and let A_* and B_* be the corresponding Killing vector fields. Then, $[A, B] \in \mathfrak{h}$ and since $[A_*, B_*] = [A, B]_*$, we have $[A_*, B_*]_p \in \Psi'_p(\mathfrak{h})$.

(b) Let X be a horizontal vector field and let $A \in \mathfrak{h}$. Then, by Lemma 1.4.2, $[X, A_*]$ is a horizontal vector field.
(c) Let X and Y be horizontal vector fields. Then, by (1.4.5),

$$\operatorname{ver}\left([X, Y]_p\right) = -\Psi_p'\left(\Omega_p(X, Y)\right).$$

Now, the Frobenius Theorem I/3.5.12 and Theorem I/3.5.17 yield the existence of a maximal connected integral manifold N through $p_0 \in P$. A point $p \in P$ belongs to N iff there exists a curve γ joining p to p_0 such that $\dot{\gamma}(t) \in D_{\gamma(t)}$ for every t. Since $\Gamma \subset \mathrm{T}N$, we conclude $P_{p_0}(\Gamma) = P \subset N$. Thus, $N = P$ and $D = \mathrm{T}P$ and, consequently,

$$\dim \mathfrak{g} = \dim P - \dim M = \dim N - \dim M = \dim \mathfrak{h},$$

that is, $\mathfrak{g} = \mathfrak{h}$. ■

The proofs of the following statements are left to the reader (Exercise 1.7.5).

Remark 1.7.16

1. As already stated after Definition 1.7.13, P is a disjoint union of holonomy bundles. By the proof of the Ambrose–Singer Theorem, this disjoint union coincides with the foliation defined by the distribution D.
2. If the curvature Ω of Γ vanishes, then $\mathscr{H}_p^0(\Gamma) = \{\mathbb{1}\}$ and each holonomy bundle $P_p(\Gamma)$ is a covering of M. These bundles are all isomorphic and are associated with the universal covering of M, which is a principal bundle with structure group $\pi_1(M)$, cf. Example 1.1.26.
3. If the curvature Ω of Γ vanishes and if, additionally, M is simply connected, then P is isomorphic to the trivial bundle $M \times G$ and the isomorphism maps Γ to the canonical connection on $M \times G$, cf. Example 1.3.18.
4. If G is connected, then Γ is irreducible iff $\mathfrak{h}_{p_0}(\Gamma) = \mathfrak{g}$.
5. Using the Ambrose–Singer Theorem, one can show the following. For every principal fibre bundle $P(M, G)$, with M connected and $\dim M \geq 2$, there exists a connection Γ on P such that $P_p(\Gamma) = P$ for all $p \in P$. For the rather technical proof we refer to [490]. A direct, yet also technical proof can be found in [383], see Chapter II/Theorem 8.2 of Part I. ♦

In the remainder of this section, we show that the concept of parallel transport on a principal G-bundle P carries over to any associated vector bundle $E = P \times_G F$ with (F, G, σ) being a representation of G.

As in the case of principal bundles, the horizontal lift of vectors implies the lift of curves in M to horizontal curves in E. By (1.3.5), the unique lift of a curve γ in M to the horizontal curve γ_E^h in E starting at the point $[(p,f)] \in \pi_F^{-1}(\gamma(0))$ is given by

$$\gamma_E^h(t) = \iota_f(\gamma_P^h(t)), \tag{1.7.7}$$

where γ_P^h is the horizontal lift of γ to P starting at p. The corresponding parallel transport operators along γ will be denoted by

$$\hat{\gamma}_{\Gamma^E}(t) : \pi_F^{-1}(\gamma(0)) \to \pi_F^{-1}(\gamma(t)).$$

As in the case of the principal bundle, for $t = 1$, we simply write $\hat{\gamma}_{\Gamma^E}$. For a given horizontal lift γ_P^h to P, formula (1.7.4) implies

$$\hat{\gamma}_{\Gamma^E}(t) = \iota_{\gamma_P^h(t)} \circ \left(\iota_{\gamma_P^h(0)}\right)^{-1}. \tag{1.7.8}$$

In particular, for closed curves in M, we can define the holonomy group

$$\mathscr{H}_m(\Gamma^E) := \left\{\hat{\gamma}_{\Gamma^E} : \gamma \in C(m)\right\} \subset \mathrm{GL}(\pi_F^{-1}(\gamma(0))) \tag{1.7.9}$$

and, correspondingly, the restricted holonomy group $\mathscr{H}_m^0(\Gamma^E)$. Using (1.7.7), it is easy to show (Exercise 1.7.7) that for any $p \in \pi^{-1}(m)$,

$$\mathscr{H}_m(\Gamma^E) = \iota_p \circ \sigma\left(\mathscr{H}_p(\Gamma)\right) \circ \iota_p^{-1}. \tag{1.7.10}$$

In particular, if σ is injective, then $\mathscr{H}_m(\Gamma^E)$ and $\mathscr{H}_p(\Gamma)$ are isomorphic.

Finally, we relate the concept of parallel transport to the notion of parallelity of sections, cf. Definition 1.4.5 and Corollary 1.5.7. For that purpose, let us denote the subspace of sections of E which are parallel with respect to Γ^E by $\mathscr{P}(E, \Gamma^E)$. The following proposition provides a geometric interpretation of the covariant derivative in terms of parallel transport.

Proposition 1.7.17 *Let Γ^E be a connection on E, let ∇ be its covariant derivative and let $\Phi \in \Gamma^\infty(E)$. Then, for any $m \in M$ and any $X \in \mathfrak{X}(M)$,*

$$\nabla_X \Phi(m) = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \left(\hat{\gamma}_{\Gamma^E}(t)\right)^{-1} \circ \Phi(\gamma(t)), \tag{1.7.11}$$

where $\gamma : I \to M$ is an integral curve of X through $m = \gamma(0)$ and $I \subset \mathbb{R}$ is an open interval containing 0.

Proof Let γ_P^h be the horizontal lift of γ to P starting at $p \in \pi^{-1}(m)$. Let X^h be the horizontal lift of X to P. Then,

$$X_p^h(\tilde{\Phi}) = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \tilde{\Phi} \circ \gamma_P^h(t) = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \left(\iota_{\gamma_P^h(t)}\right)^{-1} \Phi(\gamma(t))$$

and, thus, (1.5.3) and (1.7.8) imply (1.7.11). ∎

Rewriting formula (1.7.11) as

$$\nabla_X \Phi(m) = \lim_{t \to 0} \frac{\left(\hat{\gamma}_{\Gamma^E}(t)\right)^{-1} \circ \Phi(\gamma(t)) - \Phi(\gamma(0))}{t}, \tag{1.7.12}$$

we obtain a geometric interpretation of the covariant derivative. In particular, we note that a section Φ is parallel iff the curve $\Phi \circ \gamma$ in E is horizontal for any integral curve γ of X.

Now, let us consider an arbitrary smooth curve $\gamma : I \to M$. By Definition 1.5.13, a section of E along $\gamma : I \to M$ is a mapping $\phi : I \to E$ fulfilling $\pi_F \circ \phi = \gamma$. Recall that ϕ is a section of E along γ iff $t \mapsto (t, \phi(t))$ is a section of $\gamma^* E$, that is, there is a canonical isomorphism between $\Gamma^\infty(\gamma^* E)$ and the vector space $\Gamma^\infty_\gamma(E)$ of sections of E along γ. Also recall that there is an associated covariant derivative along the mapping γ. According to (1.5.16), it is given by

$$\nabla^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}} : \Gamma^\infty_\gamma(E) \to \Gamma^\infty_\gamma(E), \quad \nabla^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}} \phi = \omega^E \left(\phi' \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \right), \tag{1.7.13}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ is the standard unit vector field on $I \subset \mathbb{R}$ and ω^E is the connection mapping in E. Now, clearly, for any $\Phi \in \Gamma^\infty(E)$,

$$\phi = \Phi \circ \gamma$$

is a section of E along γ and, for this choice of ϕ, formula (1.7.13) takes the form

$$\nabla^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}} (\Phi \circ \gamma) = \omega^E \left(\Phi'(\dot{\gamma}) \right) = \omega^E \left(\frac{\mathrm{d}}{\mathrm{d}t} (\Phi \circ \gamma) \right), \tag{1.7.14}$$

where

$$\dot{\gamma}(t) = \gamma'_t \left(\frac{\mathrm{d}}{\mathrm{d}t}\restriction_t \right)$$

is the tangent vector field of γ, cf. Example I/1.5.5. Thus, a section $\Phi \circ \gamma$ of E along γ is parallel iff

$$\nabla^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}} (\Phi \circ \gamma) = 0. \tag{1.7.15}$$

To summarize, we obtain the following.

Proposition 1.7.18 *The parallel transport operator $\hat{\gamma}_{\Gamma^E} : E_{\gamma(0)} \to E_{\gamma(1)}$ along a curve γ is given by the set of solutions of the differential equation* (1.7.15) *with the initial condition $\Phi(\gamma(0))$ running through the fibre $E_{\gamma(0)}$.* ■

Remark 1.7.19 (Synchronous framing) For a vector bundle $E \to M$ with connection ∇, let ω be the connection form of ∇ in the frame bundle $L(E)$ and let Ω be its curvature. Denote $\dim M = n$ and consider an open ball $B \subset \mathbb{R}^n$ centered at 0. Let $x^1, \ldots, x^n$ be the standard coordinates on B and let

$$X^r := \sum_i x^i \partial_i$$

be the corresponding radial vector field on B. Let (U, κ) be a local chart sending U to B. Via κ, parallel transport along rays $t \mapsto t\mathbf{x}$, $\mathbf{x} \in B$, provides a local trivialization of E over B by identifying the fibres $E_{\mathbf{x}}$ with E_0. The corresponding local frame is said to be synchronous.

Let $\mathbb{A} = \kappa^* \mathscr{A}$ be the local representative of ω on B with respect to a synchronous frame $\{e_\alpha\}$, viewed as a local section of the frame bundle $L(E)$, and let $\mathbb{F} = \kappa^* \mathscr{F}$ be the corresponding representative of Ω. Then, (1.5.6) implies

$$\nabla_{X^r} e_\alpha = \mathbb{A}^\beta{}_\alpha(X^r)\, e_\beta = 0, \tag{1.7.16}$$

that is, $(X^r \lrcorner \mathbb{A}) = 0$. This implies

$$\mathscr{L}_{X^r} \mathbb{A} = X^r \lrcorner \, \mathrm{d}\mathbb{A} = X^r \lrcorner \, \mathbb{F}.$$

We decompose $\mathbb{A} = \mathbb{A}_i \, \mathrm{d}x^i$, $\mathbb{F} = \frac{1}{2}\mathbb{F}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$. Then,

$$\mathscr{L}_{X^r} \mathbb{A} = \mathbb{F}_{ij} x^i \mathrm{d}x^j.$$

On the other hand, by the derivation property of the Lie derivative,

$$\mathscr{L}_{X^r} \mathbb{A} = X^r(\mathbb{A}_i)\mathrm{d}x^i + \mathbb{A}_i \mathrm{d}x^i.$$

Comparing these two formulae, we read off

$$X^r(\mathbb{A}_i) + \mathbb{A}_i = -\mathbb{F}_{ij} x^j \, .$$

This implies

$$\mathbb{A}_i(\mathbf{x}) \sim -\frac{1}{2}\mathbb{F}_{ij}(0) x^j + 0(\|\mathbf{x}\|^2). \tag{1.7.17}$$

In particular, we have $\mathbb{A}(0) = 0$. ♦

Finally, we show that the holonomy group of Γ^E can be used to characterize the set $\mathscr{P}(E, \Gamma^E)$ of sections of E which are parallel with respect to Γ^E. Given $p \in P$, an element of F is called holonomy-invariant if it is invariant under the restriction of the representation σ to the holonomy group $\mathscr{H}_p(\Gamma)$.

Proposition 1.7.20 (Holonomy principle) *If M is connected, then there is a bijective correspondence between $\mathscr{P}(E, \Gamma^E)$ and the space of holonomy-invariant vectors in F.*

Proof Let $m_0 \in M$ and $p_0 \in \pi^{-1}(m_0)$. By Proposition 1.7.12, P reduces together with Γ to the holonomy bundle $P_{p_0}(\Gamma)$ and, by Proposition 1.6.7, we have the following isomorphism of associated vector bundles:

$$E = P \times_G F \cong P_{p_0}(\Gamma) \times_{\mathscr{H}_{p_0}(\Gamma)} F.$$

Thus, it is enough to consider sections of the associated bundle on the right hand side which are parallel in the sense of the reduced connection.

1. Let $f \in F$ be holonomy invariant, that is, $\sigma_h f = f$ for all $h \in \mathscr{H}_{p_0}(\Gamma)$. Define

$$\tilde{\Phi} : P_{p_0}(\Gamma) \to F, \quad \tilde{\Phi}(p) := f. \tag{1.7.18}$$

Since, for all $h \in \mathscr{H}_{p_0}(\Gamma)$, we have

$$\tilde{\Phi}(\Psi_h(p)) = f = \sigma_{h^{-1}} f = \sigma_{h^{-1}} \tilde{\Phi}(p),$$

$\tilde{\Phi}$ is $\mathscr{H}_{p_0}(\Gamma)$-equivariant. Thus, by Proposition 1.2.6, it induces a smooth section Φ of E. Since $\tilde{\Phi}$ is constant on $P_{p_0}(\Gamma)$, we have $X^h(\tilde{\Phi}) = X^h(f) = 0$ for any $X \in \mathfrak{X}(M)$. Then, (1.5.3) implies that Φ is parallel.

2. Conversely, let Φ be a parallel section and let $\tilde{\Phi}$ be the corresponding equivariant mapping. By (1.5.3), we have $X^h(\tilde{\Phi}) = 0$ for every horizontal vector field on $P_{p_0}(\Gamma)$. Thus, $\tilde{\Phi}$ is constant along any horizontal curve in $P_{p_0}(\Gamma)$, that is, $\tilde{\Phi}$ is constant on $P_{p_0}(\Gamma)$. Let $f := \tilde{\Phi}(p) \in F$ be this constant vector. The equivariance of $\tilde{\Phi}$ implies the holonomy invariance of f. ∎

Exercises

1.7.1 Prove Proposition 1.7.1.

1.7.2 Prove Lemma 1.7.3.

1.7.3 Show that $\mathfrak{h}$ defined by (1.7.6) is a Lie subalgebra of $\mathfrak{g}$.

1.7.4 Prove that all holonomy bundles of a given connection are isomorphic.

1.7.5 Prove the statements of Remark 1.7.16.

1.7.6 Within the class of principal bundles defined in Example 1.1.4/3, take

(a) $G = \mathrm{GL}(n, \mathbb{C})$ and $H = \mathrm{SL}(n, \mathbb{C})$. Then, $G/H \cong \mathbb{C}_*$ and the canonical projection is given by the determinant.
(b) $G = \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$. Then, $G/H = \mathrm{S}^2$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. In both cases, decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ reductively and define a connection on each of these bundles by putting $\Gamma_{\mathbb{1}} = \mathfrak{m}$ and $\Gamma_a = L'_a \Gamma_{\mathbb{1}}$, cf. Example 1.3.19. Calculate the holonomy groups of these connections.
Hint. For case (b), use the Ambrose–Singer Theorem.

1.7.7 Prove formula (1.7.10).

1.7.8 Confirm formula (1.7.17).

1.8 Automorphisms

By Definition 1.1.7, if (ϑ, λ) is an automorphism of a principal G-bundle P, then λ is an automorphism of G. In the sequel, we limit our attention to the restricted class of automorphisms fulfilling $\lambda = \mathrm{id}_G$. This class corresponds to the equivariant automorphisms of the G-manifold (P, G, Ψ), cf. Definition I/6.1.1. We denote the group of equivariant automorphisms by $\mathrm{Aut}(P)$. Recall from Remark 1.1.8/2 that an automorphism ϑ of P is called vertical if $\tilde{\vartheta} = \mathrm{id}_M$.

Remark 1.8.1 The vertical automorphisms of P constitute a group which will be denoted by $\mathrm{Aut}_M(P)$. By (1.1.4), the mapping $\mathrm{Aut}(P) \ni \vartheta \mapsto \tilde{\vartheta} \in \mathrm{Diff}(M)$ is a homomorphism of groups and, by definition, $\mathrm{Aut}_M(P)$ coincides with the kernel of this homomorphism. Thus, $\mathrm{Aut}_M(P)$ is a normal subgroup of $\mathrm{Aut}(P)$ and the following sequence is exact,

$$0 \to \mathrm{Aut}_M(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M).$$

In gauge theory, $\mathrm{Aut}_M(P)$ plays the role of the group of local gauge transformations. It can be turned into an infinite-dimensional Hilbert-Lie group, see Chaps. 6 and 8. ♦

We start by giving a characterization of $\mathrm{Aut}_M(P)$ in terms of equivariant mappings which is useful in gauge theory. For a given principal bundle (P, G, M, Ψ, π), consider the set $\mathrm{Hom}_G(P, G)$ of equivariant smooth mappings $u : P \to G$, where G is viewed as a right G-manifold endowed with the G-action by conjugation, that is, $(a, b) \mapsto b^{-1}ab$. Then, equivariance means

$$u(\Psi_a(p)) = a^{-1}\, u(p)\, a, \quad a \in G, p \in P. \tag{1.8.1}$$

We endow $\mathrm{Hom}_G(P, G)$ with a group structure by putting $(uv)(p) := u(p)\, v(p)$ for any $u, v \in \mathrm{Hom}_G(P, G)$. Then,

$$(uv)(\Psi_a(p)) = u(\Psi_a(p))\, v(\Psi_a(p)) = (a^{-1}\, u(p)\, a)(a^{-1}\, v(p)\, a) = a^{-1}\, (uv)(p)\, a,$$

showing that $uv \in \mathrm{Hom}_G(P, G)$. The unit element is given by the constant mapping $p \mapsto \mathbb{1}$ and the inverse of u is given by the mapping $p \mapsto u(p)^{-1}$.

Remark 1.8.2 By Proposition 1.2.6, $\mathrm{Hom}_G(P, G)$ may be identified with the space of sections of the associated bundle $P \times_G G$, with G acting on the typical fibre G by inner automorphisms. ♦

For $u \in \mathrm{Hom}_G(P, G)$, we define

$$\vartheta_u : P \to P, \quad \vartheta_u(p) := \Psi_{u(p)}(p). \tag{1.8.2}$$

Proposition 1.8.3 *For every $u \in \mathrm{Hom}_G(P, G)$, the mapping ϑ_u defined by (1.8.2) is a vertical automorphism of P. The assignment $u \mapsto \vartheta_u$ defines an isomorphism of groups.*

Proof Since Ψ and u are smooth, $\vartheta_u : P \to P$ is smooth as a composition of smooth mappings. Then, $\vartheta_{u^{-1}}$ is also smooth and we have

$$\vartheta_{u^{-1}} \circ \vartheta_u(p) = \Psi_{u\left(\Psi_{u(p)}(p)\right)^{-1}} \circ \Psi_{u(p)}(p) = \Psi_{u(p)^{-1}\,u(p)^{-1}\,u(p)} \circ \Psi_{u(p)}(p) = p,$$

that is, $\vartheta_{u^{-1}} \circ \vartheta_u = \mathrm{id}_P$ and, analogously, $\vartheta_u \circ \vartheta_{u^{-1}} = \mathrm{id}_P$. Thus, ϑ_u is a diffeomorphism. Moreover, by equivariance of u,

$$\vartheta_u \circ \Psi_a(p) = \Psi_{u(\Psi_a(p))} \circ \Psi_a(p)) = \Psi_{a^{-1}\,u(p)\,a} \circ \Psi_a(p)) = \Psi_a \circ \vartheta_u(p),$$

showing that ϑ_u is an automorphism of P. By definition, it is vertical.

To prove the second assertion, we first note that the mapping $u \mapsto \vartheta_u$ is a homomorphism of groups:

$$\vartheta_u \circ \vartheta_v(p) = \Psi_{u\left(\Psi_{v(p)}(p)\right)} \circ \Psi_{v(p)}(p) = \Psi_{u(p)\,v(p)}(p) = \vartheta_{uv}(p).$$

Since the G-action Ψ is free, the mapping $u \mapsto \vartheta_u$ is injective. It is also surjective. Indeed, let $\vartheta \in \mathrm{Aut}_M(P)$. Since $\vartheta(p)$ and p belong to the same fibre, there exists a unique element $u(p) \in G$ such that $\vartheta(p) = \Psi_{u(p)}(p)$. This yields a smooth mapping

$$u : P \to G, \quad u(p) = \Psi_p^{-1} \circ \vartheta(p).$$

Finally, we must show the equivariance of u. On the one hand, we have

$$\vartheta(\Psi_a(p)) = \Psi_{u(\Psi_a(p))} \circ \Psi_a(p) = \Psi_{a\,u(\Psi_a(p))}(p)$$

and, on the other hand, by (1.1.3),

$$\vartheta(\Psi_a(p)) = \Psi_a(\vartheta(p)) = \Psi_a \circ \Psi_{u(p)}(p) = \Psi_{u(p)\,a}(p).$$

This yields $a\,u(\Psi_a(p)) = u(p)\,a$, that is, $u(\Psi_a(p)) = a^{-1}\,u(p)\,a$. ■

Next, we show that a vertical automorphism of P induces a vertical automorphism in every associated bundle $(P \times_G F, M, \pi_F)$.

Proposition 1.8.4 *Let ϑ be a vertical automorphism of P. Then, the mapping*

$$\hat{\vartheta} : P \times_G F \to P \times_G F, \quad \hat{\vartheta}([(p, f)]) := [(\vartheta(p), f)],$$

is a vertical automorphism of $P \times_G F$*. If* ϑ *is given by* $u \in \mathrm{Hom}_G(P, G)$*, then*

$$\hat{\vartheta}_u([(p, f)]) = \big[\big(p, \sigma_{u(p)}(f)\big)\big]. \tag{1.8.3}$$

Proof The first assertion follows from Proposition 1.2.8/3. To prove (1.8.3), we calculate

$$\hat{\vartheta}_u([(p, f)]) = \big[(\vartheta_u(p), f)\big] = \big[\big(\Psi_{u(p)}(p), f\big)\big] = \big[\big(p, \sigma_{u(p)}(f)\big)\big].$$

■

Corollary 1.8.5 *If* (F, G, σ) *is a Lie group representation, then* $\hat{\vartheta}$ *is a vertical automorphism of vector bundles.*

Proof Let $m \in M$ and let $p \in \pi^{-1}(m)$. Formula (1.8.3) implies

$$(\hat{\vartheta}_u)\restriction_{\pi_F^{-1}(m)} = \iota_p \circ \sigma_{u(p)^{-1}} \circ \iota_p^{-1}. \tag{1.8.4}$$

According to Proposition 1.2.8, the diffeomorphism ι_p is a linear mapping. Thus, (1.8.4) defines an endomorphism of the fibre $\pi_F^{-1}(m)$. ■

Remark 1.8.6 From the proof of Proposition 1.8.4 we read off the following formula for the local representative of $\hat{\vartheta}$:

$$\xi \circ \hat{\vartheta} \circ \xi^{-1}(m, f) = (m, \sigma_{\rho(m)} f). \tag{1.8.5}$$

Here, $\rho = u \circ s$ denotes the local representative of $u \in \mathrm{Hom}_G(P, G)$. ♦

We know from Corollary 1.3.16 that the image $\vartheta'(\Gamma)$ and the preimage $(\vartheta^{-1})'(\Gamma)$ of a connection Γ under an automorphism ϑ of P are both connections. In particular, the image of a horizontal curve under ϑ is horizontal with respect to $\vartheta'(\Gamma)$ and formula (1.7.4) immediately implies the following transformation law for the parallel transport operator:

$$\hat{\gamma}_{\vartheta'(\Gamma)} = \vartheta \circ \hat{\gamma}_\Gamma \circ \vartheta^{-1}. \tag{1.8.6}$$

Proposition 1.8.7 *Let* $P(M, G)$ *be a principal bundle and let* Γ *be a connection on* P*. Then, for* $\vartheta \in \mathrm{Aut}_M(P)$ *corresponding to* $u \in \mathrm{Hom}_G(P, G)$*, one has the following transformation laws:*

1. *If* ω *is the connection form of* Γ*, then* $\vartheta^*\omega$ *is the connection form of* $(\vartheta^{-1})'(\Gamma)$ *and*
$$(\vartheta^*\omega)_p = \mathrm{Ad}(u(p)^{-1}) \circ \omega_p + (u^*\theta)_p, \tag{1.8.7}$$
with θ *denoting the Maurer-Cartan form on* G*.*
2. *If* Ω *is the curvature form of* ω*, then* $\vartheta^*\Omega$ *is the curvature form of* $\vartheta^*\omega$ *and*
$$(\vartheta^*\Omega)_p = \mathrm{Ad}(u(p)^{-1}) \circ \Omega_p. \tag{1.8.8}$$

3. *For the operator of covariant exterior derivative, one has*

$$D_{\vartheta^*\omega} = \vartheta^* \circ D_\omega \circ (\vartheta^*)^{-1}. \tag{1.8.9}$$

Proof 1. To prove (1.8.7), we must calculate $\vartheta'(X)$ for $X \in \mathrm{T}_pP$. Let γ be a curve representing X. Then, using (1.8.2), we have

$$\begin{aligned}
\vartheta'_p(X) &= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \vartheta(\gamma(t)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \Psi\big(\gamma(t), u(\gamma(t))\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \Psi\big(\gamma(t), u(p)\big) + \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \Psi\big(p, u(\gamma(t))\big) \\
&= \Psi'_{u(p)}(X) + \Psi'_p \circ u'_p(X).
\end{aligned}$$

The second term describes a vertical vector in $\vartheta(p)$. Thus, we may write it in the form $\Psi'_{\vartheta(p)}(A)$ with $A \in \mathfrak{g}$. Explicitly, since $u'_p(X) \in \mathrm{T}_{u(p)}G$, we can write

$$\Psi'_p \circ u'_p(X) = \Psi'_p \circ L'_{u(p)} \circ L'_{u(p)^{-1}} \circ u'_p(X).$$

Using

$$L'_{u(p)^{-1}} \circ u'_p(Y) = \theta(u'_p(X)) = (u^*\theta)_p(X)$$

and $\Psi_p \circ L_{u(p)} = \Psi_{\vartheta(p)}$, we obtain

$$\vartheta'_p(X) = \Psi'_{u(p)}(X) + \Psi'_{\vartheta(p)}\big(u^*\theta(X)\big). \tag{1.8.10}$$

Using this equation, together with the equivariance of ω, we obtain (1.8.7).

2. Using the Structure Equation, we obtain

$$\vartheta^*\Omega = \vartheta^*(\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]) = \mathrm{d}(\vartheta^*\omega) + \frac{1}{2}[\vartheta^*\omega, \vartheta^*\omega],$$

that is, $\vartheta^*\Omega$ is the curvature form of $\vartheta^*\omega$, indeed. To prove (1.8.8), we must calculate $\vartheta^*\Omega(X, Y)$ for any $X, Y \in \mathrm{T}_pP$. For that purpose, we use the decomposition (1.8.10) for both tangent vectors. Since Ω is horizontal, only the first terms of this decomposition contribute. Then, using the equivariance of Ω, one immediately obtains (1.8.8).

3. Using the horizontality of the covariant exterior derivative and the fact that $\vartheta^*\omega$ is the connection form of $(\vartheta^{-1})'(\Gamma)$, we obtain

$$\vartheta' \circ \mathrm{hor}^{\vartheta^*\omega} = \mathrm{hor}^\omega \circ \vartheta'$$

and thus

$$(D_{\vartheta^*\omega}(\vartheta^*\tilde{\alpha}))(X_0, \dots, X_k) = (\vartheta^* D_\omega \tilde{\alpha})(X_0, \dots, X_k),$$

for any $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ and $X_i \in \mathrm{T}_p P$. This yields the assertion. ∎

Remark 1.8.8

1. Sometimes, we will use the following short-hand notation for the above transformation laws:

$$\vartheta^*\omega = \mathrm{Ad}(u^{-1}) \circ \omega + u^*\theta, \quad \vartheta^*\Omega = \mathrm{Ad}(u^{-1}) \circ \Omega.$$

   In matrix notation, we have $u^*\theta = u^{-1}\mathrm{d}u$, cf. Remark I/5.5.12/2. Then,

$$\vartheta^*\omega = u^{-1}\omega u + u^{-1}\mathrm{d}u, \quad \vartheta^*\Omega = u^{-1}\Omega u.$$

2. Using the local representative $\rho = u \circ s$, introduced in Remark 1.8.6, from (1.8.7) and (1.8.8) we read off the following transformation laws for the local representatives of ω and Ω, cf. formulae (1.3.11) and (1.4.16):

$$\mathscr{A}' = \mathrm{Ad}(\rho^{-1}) \circ \mathscr{A} + \rho^*\theta, \quad \mathscr{F}' = \mathrm{Ad}(\rho^{-1}) \circ \mathscr{F}. \tag{1.8.11}$$

♦

1.9 Invariant Connections

In this section, we consider the following geometrical setting. Let there be given a principal bundle (P, G, M, Ψ, π) and let the base manifold M be endowed with a left Lie group action (M, K, δ). Assume that both K and G are compact[19] connected Lie groups. By a lift of the K-action to P we mean a homomorphism $\Delta : K \to \mathrm{Aut}(P)$ projecting to δ, that is, $\pi \circ \Delta_k = \delta_k \circ \pi$ for any $k \in K$. The following natural problems arise:

(a) Classify the lifts of the K-actions.
(b) Classify the connections on P which are invariant under a lifted K-action.

In pure mathematics, these problems are a natural part of fibre bundle theory. We will cite a number of relevant contributions later on. In physics, these questions are closely related to model building in the spirit of Kaluza–Klein theories, see Sects. 7.8 and 7.9. We also refer to Chap. 6 for various applications. Here, we address the above problems under the following additional assumptions.

(a) We assume that the K-action δ have only one orbit type. In the sequel, such an action will be referred to as a simple K-action.

[19]It will become clear below for which statements the compactness assumption is necessary. In particular, under this assumption the K-action δ is proper.

(b) Since $\varDelta_k$ is an automorphism of P for every $k \in K$, the actions $\varPsi$ and $\varDelta$ commute and thus they induce a left action,

$$\rho : (K \times G) \times P \to P, \quad \rho_{(k,g)}(p) := \varDelta_k \circ \varPsi_{g^{-1}}(p). \tag{1.9.1}$$

We assume that this action be simple, too.

Let us denote the orbit type of δ by $[H]$ and let us consider a representative H of the conjugacy class $[H]$. In the notation of Sect. 6.6. of Part I, let $N_K(H)$ be the normalizer of H in K and let $\varGamma_H = N_K(H)/H$. Recall that $\varGamma_H$ acts on K/H naturally from the left,

$$\varGamma_H \times K/H \to K/H, \quad (aH, kH) \mapsto ka^{-1}H. \tag{1.9.2}$$

By Proposition I/6.6.1, the subset $M_H \subset M$ of isotropy type H is a principal $\varGamma_H$-bundle over the orbit space $\hat{M} \equiv M/K$ with right $\varGamma_H$-action $(a, m) \mapsto \delta_{a^{-1}}(m)$ and δ induces a K-equivariant diffeomorphism

$$M_H \times_{\varGamma_H} K/H \to M, \quad [(m, [k])] \mapsto \delta_k(m), \tag{1.9.3}$$

with the K-action on $M_H \times_{\varGamma_H} K/H$ given by left translation on K/H.

Now, let $(P, G, M, \varPsi, \pi)$ be a principal bundle, with G compact connected, and let $\varDelta : K \to \operatorname{Aut}(P)$ be a lift of δ. Then, for every isotropy group H, the submanifold $\pi^{-1}(M_H) \subset P$ is a principal G-bundle over M_H and the restriction of $\varDelta$ to H defines a homomorphism from H to $\operatorname{Aut}_{M_H}(\pi^{-1}(M_H))$. By Proposition 1.8.3, the latter induces a mapping $\lambda : H \times \pi^{-1}(M_H) \to G$, given by

$$\varDelta_h(p) = \varPsi_{\lambda(h,p)}(p), \quad \pi(p) \in M_H, \ h \in H. \tag{1.9.4}$$

For every $h \in H$, the induced mapping $\lambda_h : \pi^{-1}(M_H) \to G$ is G-equivariant,

$$\lambda_h(\varPsi_g(p)) = g^{-1}\lambda_h(p)g, \tag{1.9.5}$$

and, for every $p \in \pi^{-1}(M_H)$, the induced mapping $\lambda_p : H \to G$ is a homomorphism of Lie groups. By (1.9.5), a change of the point in a given fibre of $\pi^{-1}(M_H)$ results in a conjugate homomorphism, that is,

$$\lambda_{\varPsi_g(p)} = g^{-1}\lambda_p g. \tag{1.9.6}$$

By assumption, the left action ρ of $K \times G$ on P given by (1.9.1) is simple. Let us calculate the isotropy group $(K \times G)_p$ for a chosen point $p \in \pi^{-1}(M_H)$. From $\varDelta_k \circ \varPsi_{g^{-1}}(p) = p$ we read off $\delta_k(\pi(p)) = \pi(p)$, that is, $k \in H$. Thus, using (1.9.4), we obtain

$$(K \times G)_p = \left\{(h, \lambda_p(h)) \in K \times G : h \in H\right\}. \tag{1.9.7}$$

Now, let us choose an isotropy subgroup I and let us consider the subset

$$P_I \subset \pi^{-1}(M_H) \subset P$$

of isotropy type I. By definition of P_I, the restriction of the mapping $p \mapsto \lambda(h, p)$ to P_I is constant. In the sequel, it will be denoted by λ_0. Denoting

$$\Gamma_I = N_{K\times G}(I)/I$$

and, again using Proposition I/6.6.1, we conclude that P_I is a principal Γ_I-bundle over the orbit space $P/(K \times G) = \hat{M}$ with right Γ_I-action

$$\Psi^I : \Gamma_I \times P_I \to P_I, \quad (a, p) \mapsto \Psi^I(a, p) := \rho_{a^{-1}}(p), \tag{1.9.8}$$

and that ρ induces a $(K \times G)$-equivariant diffeomorphism

$$P_I \times_{\Gamma_I} (K \times G)/I \to P, \quad [(p, [(k, g)])] \mapsto \rho_{(k,g)}(p). \tag{1.9.9}$$

Thus, a principal G-bundle P admitting a simple lift of a simple K-action has the form

$$P = P_I \times_{\Gamma_I} (K \times G)/I, \tag{1.9.10}$$

where I is a chosen isotropy group of the induced action ρ.

Remark 1.9.1 If we take another representative $I' = aIa^{-1}$, $a \in K \times G$, of the orbit type $[I]$, then the isotropy submanifold P_I gets translated by a, that is, $P_{I'} = \rho_a(P_I)$. Thus, P is uniquely characterized by an equivalence class $[(I, P_I)]$. ♦

The following remark shows that, depending on the context, formula (1.9.10) may be interpreted in various ways.

Remark 1.9.2

1. By (1.9.7), the action of I on $K \times G$ may be identified with the action of H given by

$$H \times (K \times G) \to K \times G, \quad (h, (k, g)) \mapsto (kh, g\lambda_0(h)). \tag{1.9.11}$$

   Thus, we can write

$$(K \times G)/I = K \times_H G, \tag{1.9.12}$$

   where K is viewed as a principal H-bundle over K/H.
2. Consider $K \times G$ as a principal $N_{K\times G}(I)$-bundle over $(K \times G)/N_{K\times G}(I)$. Since $I \subset N_{K\times G}(I)$ is a normal subgroup, by Corollary I/6.5.3/1, the right action of $N_{K\times G}(I)$ on $K \times G$ descends to a free proper action of $\Gamma_I = N_{K\times G}(I)/I$ on $(K \times G)/I$ and $\mathrm{id}_{K\times G}$ induces a diffeomorphism between $(K \times G)/N_{K\times G}(I)$ and $\big((K \times G)/I\big)/\Gamma_I$. Thus, $(K \times G)/I$ may be viewed as a principal Γ_I-bundle over

$(K \times G)/N_{K\times G}(I)$ and the isomorphism (1.9.9) may be rewritten as follows:

$$P \cong (K \times G)/I \times_{\Gamma_I} P_I. \tag{1.9.13}$$

♦

Next, we will show that from the above data, we can construct a principal G-bundle admitting the lift of a simple K-action. For that purpose, we must gain some insight into the structure of $\Gamma_I = N_{K\times G}(I)/I$ and of P_I, respectively. First, note that $(k, g) \in N_{K\times G}(I)$ iff

$$k \in N_K(H), \quad g\lambda_0(h)g^{-1} = \lambda_0(khk^{-1}) \quad \text{for all } h \in H. \tag{1.9.14}$$

Next, consider the centralizer $C_G(\lambda_0(H))$. By (1.9.14), we have

$$N_{K\times G}(I) \cap (\{\mathbb{1}_K\} \times G) = \{\mathbb{1}_K\} \times C_G(\lambda_0(H)) \equiv Z. \tag{1.9.15}$$

Since $Z \cap I = \{\mathbb{1}_K \times \mathbb{1}_G\}$, we may view Z as a (normal) subgroup of Γ_I. Thus, Z acts freely on P_I and, by (1.9.15), transitively on each intersection of P_I with a fibre of P. We conclude that P_I carries the structure of a principal Z-bundle over

$$M_I := \pi(P_I) \cong P_I/Z,$$

with the right action of Z given by restriction of Ψ to $Z \times P_I \subset G \times P$. Clearly, $M_I \subset M_H$ and thus

$$\Gamma_I/Z \subset \Gamma_H. \tag{1.9.16}$$

To summarize, we have a sequence of principal bundles

$$P_I \xrightarrow{\pi_{M_I}} M_I \xrightarrow{\pi_{\hat{M}}} \hat{M}, \tag{1.9.17}$$

with structure groups Z and Γ_I/Z, respectively. Let us denote the Lie algebras of, respectively,

$$K,\ H,\ N_K(H),\ \Gamma_H,\ G,\ I,\ N_{K\times G}(I),\ \Gamma_I,\ Z \quad \text{by} \quad \mathfrak{k},\ \mathfrak{h},\ \mathfrak{n}_H,\ \hat{\mathfrak{n}}_H,\ \mathfrak{g},\ \mathfrak{i},\ \mathfrak{n}_I,\ \hat{\mathfrak{n}}_I,\ \mathfrak{z}.$$

Lemma 1.9.3 *The Lie algebra $\hat{\mathfrak{n}}_I$ of Γ_I is the direct sum of two ideals,*

$$\hat{\mathfrak{n}}_I = \hat{\mathfrak{n}}_H \oplus \mathfrak{z}. \tag{1.9.18}$$

Proof Let $[(A, B)] \in \hat{\mathfrak{n}}_I$. Then, by (1.9.14), $A \in \mathfrak{n}_H$. Since K is compact, we can decompose

$$\mathfrak{n}_H = \mathfrak{h} \oplus \mathfrak{h}^\perp \tag{1.9.19}$$

with respect to some Ad-invariant scalar product in $\mathfrak{k}$. By invariance and since $\mathfrak{h}$ is an ideal, $\mathfrak{h}^\perp$ is an ideal and thus (1.9.19) is a decomposition into a direct sum of Lie algebras. Clearly, we may choose $A \in \mathfrak{h}^\perp$. Then, $[A, X] = 0$ for any $X \in \mathfrak{h}$. Now, for any $B \in \mathfrak{z}$, the second equation of (1.9.14) implies

$$[B, \lambda_0'(X)] = \lambda_0'([A, X]) = 0.$$

■

From (1.9.18) we read off that the connected components of the identity of $\varGamma_I/Z$ and $\varGamma_H$ coincide,

$$(\varGamma_I/Z)_0 = (\varGamma_H)_0, \tag{1.9.20}$$

and, using (1.9.16), we conclude that $\varGamma_I/Z$ is the union of a number of connected components of $\varGamma_H$. In particular, $\varGamma_H/(\varGamma_I/Z)$ is a discrete group.

Lemma 1.9.4 *The manifold M_I is a reduction of the principal $\varGamma_H$-bundle $M_H \to \hat{M}$ to the closed subgroup $\varGamma_I/Z$ and we have the following isomorphism of associated bundles:*

$$M_H \times_{\varGamma_H} K/H \cong M_I \times_{\varGamma_I/Z} K/H. \tag{1.9.21}$$

Proof Note that $M_H/(\varGamma_I/Z)$ may be viewed as a section of the associated bundle $M_H \times_{\varGamma_I/Z} \varGamma_H/(\varGamma_I/Z)$. By Corollary 1.6.5, this section defines a reduction of M_H to the closed subgroup $\varGamma_I/Z$. Thus, M_I is a reduction of M_H to $\varGamma_I/Z$. The isomorphism (1.9.21) follows from Proposition 1.6.7. ■

We conclude from (1.9.21) that, via the K-equivariant diffeomorphism (1.9.3), we may identify M with $M_I \times_{\varGamma_I/Z} K/H$. Now, we can prove the announced converse statement.

Proposition 1.9.5 *Let K and G be compact connected Lie groups and let (M, K, δ) be a simple Lie group action. Let $H \subset K$ be an isotropy subgroup of δ and let $\lambda_0 : H \to G$ be a Lie group homomorphism. Let $(\hat{P}, \varGamma_I, \hat{M}, \hat{\Psi}, \hat{\pi})$ be a principal bundle, where $I = \{(h, \lambda_0(h)) \in K \times G : h \in H\}$ and $\varGamma_I = N_{K\times G}(I)/I$. Then, the bundle*

$$P = \hat{P} \times_{\varGamma_I} (K \times G)/I \tag{1.9.22}$$

associated with $\hat{P}$ carries the structure of a principal G-bundle over M, where G acts by inverse left translation on the factor G. The natural K-action $\varDelta$ on P given by left translation on $(K \times G)/I$ yields a group homomorphism $\varDelta : K \to \mathrm{Aut}(P)$ and projects onto δ.

Proof First, we show that P carries the structure of a principal G-bundle over M. The right G-action is defined by

$$G \times P \to P, \quad \Psi(a, [(\hat{p}, [(k, g)])]) := [(\hat{p}, [(k, a^{-1}g)])].$$

This action is obviously free. The canonical bundle projection is defined as the projection onto the orbit space of this action, $\pi : P \to P/G$. We must show that the orbit space is diffeomorphic to M. Since G is compact, the action Ψ is proper and thus, as explained in Sect. 6.5 of Part I, the Tubular Neighbourhood Theorem I/6.4.3 implies that $(P, G, P/G, \Psi, \pi)$ is a principal bundle. Next, we have the sequence (1.9.17) with P_I replaced by $\hat{P}$ and, thus, $\hat{P}/Z = M_I$. Using this and the fact that Z is normal in Γ_I, we obtain

$$P/G = \hat{P} \times_{\Gamma_I} K/H = \hat{P}/Z \times_{\Gamma_I/Z} K/H = M_I \times_{\Gamma_I/Z} K/H.$$

Thus, using (1.9.21) and (1.9.3), we obtain $P/G = M$. Finally, P can be endowed with the natural left K-action

$$\Delta : K \times P \to P, \quad \Delta(l, [(\hat{p}, [(k, g)])]) := [(\hat{p}, [(lk, g)])],$$

which obviously commutes with the G-action and which projects onto δ. ■

Remark 1.9.6 As in Remark 1.9.1, we may pass to another subgroup $I' = aIa^{-1}$, $a \in K \times G$. Correspondingly, $\Gamma_{I'}$ is isomorphic to Γ_I. Choosing a principal $\Gamma_{I'}$-bundle $\hat{P}'$ which is vertically isomorphic to $\hat{P}$, the construction yields a principal bundle P' isomorphic to P. ♦

Next, we discuss two important special cases. First, we consider the classical case of a transitive K-action, see [383, 647].

Remark 1.9.7

1. If δ is transitive, then ρ is also transitive. Thus, in this case, $\hat{M}$ and, therefore, also P_I/Γ_I is the one-point space and, by (1.9.12), formula (1.9.22) reduces to

$$P = K \times_H G, \tag{1.9.23}$$

where K is viewed as a principal H-bundle over K/H. Thus, in the transitive case, principal G-bundles admitting a lift of a K-action are completely characterized by Lie group homomorphisms $\lambda_0 : H \to G$.
2. The bundle P given by (1.9.23) is trivial iff λ_0 extends to a smooth mapping $\tilde{\lambda}_0 : K \to G$ fulfilling $\tilde{\lambda}_0(kh) = \tilde{\lambda}_0(k)\lambda_0(h)$ for $k \in K$ and $h \in H$ (Exercise 1.9.2).
3. If the action of K is free, then λ_0 is the trivial homomorphism and thus P is a trivial bundle. This means that a principal bundle over a Lie group K admits a lift of the natural action of K on itself by left translation iff it is trivial.
4. The triples $(K, P(M, G), \Delta)$, where K is a Lie group, $P(M, G)$ is a principal G-bundle over a homogeneous K-space M and Δ is an action of K on P by automorphisms which projects to the transitive K-action on M, form a category, called the category of homogeneous principal bundles. Correspondingly, one may consider the category of homogeneous principal bundles with base point $p \in P$. As a consequence of Proposition 1.9.5, in the latter category, every object is isomorphic to $(K, P(K/H, G), \Delta)$ with base point $(\mathbb{1}_K, \mathbb{1}_G)$, see [634] for details.

♦

Before we proceed to a more general case, we give an example illustrating that a lift does not always exist, see [632]. For a discussion of the lifting problem we refer to [89, 256, 266, 486, 502].

Example 1.9.8 Put $K = \mathrm{SO}(3)$, $G = \mathrm{U}(1)$ and $M = \mathrm{S}^2$, endowed with the natural action of K. Then, $H = \mathrm{SO}(2) \cong \mathrm{U}(1)$ and we must consider homomorphisms $\lambda : \mathrm{U}(1) \to \mathrm{U}(1)$. It is well known that such homomorphisms are labelled by the integers, that is, they are of the form $\lambda_n(z) = z^n$, with $z \in \mathrm{U}(1)$ and $n \in \mathbb{Z}$. Thus, for $n > 0$, we obtain

$$P_n = \mathrm{SO}(3) \times_{\mathrm{U}(1)} \mathrm{U}(1) \cong \mathrm{SO}(3)/\mathbb{Z}_n \cong \mathrm{SU}(2)/\mathbb{Z}_{2n}.$$

These are the even 3-dimensional lens spaces. In particular, for $n = 1$, the bundle manifold is $\mathrm{SO}(3)$. For $n = 0$ the bundle manifold is $\mathrm{S}^2 \times \mathrm{U}(1)$. We conclude that the complex Hopf bundle $\mathrm{S}^3(\mathrm{S}^2, \mathrm{U}(1))$ does not admit a lift of the natural $\mathrm{SO}(3)$-action on S^2. ♦

The following case was considered in various versions in [284, 285, 539, 546].

Remark 1.9.9 Assume that the principal Γ_I/Z-bundle[20] $M_I \to \hat{M}$ is trivial. Then, we may choose a global section $s : \hat{M} \to M_I$. Let us denote $\tilde{M} := s(\hat{M})$ and

$$\tilde{P} := \pi^{-1}(\tilde{M}) \cap P_I = (\pi_{M_I})^{-1}(\tilde{M}).$$

By construction, $\tilde{P}$ is a subbundle of $P_I(\hat{M}, \Gamma_I)$ carrying the structure of a principal Z-bundle. In particular, since $\tilde{M}$ and $\hat{M}$ may be identified via the section s, this yields a reduction of P_I to the structure group Z. Thus, by Proposition 1.6.7,

$$P_I \times_{\Gamma_I} (K \times G)/I = \tilde{P} \times_Z (K \times G)/I.$$

Since $Z = \{\mathbb{1}_K\} \times C_G(\lambda_0(H)) \subset \Gamma_I$, the action of I on $K \times G$ commutes with the action of Z on this product. Thus,

$$\tilde{P} \times_Z (K \times G)/I \cong K \times_H \left(\tilde{P} \times_{C_G(\lambda_0(H))} G\right),$$

where H acts on $\tilde{P} \times_{C_G(\lambda_0(H))} G$ by right translation on the factor G via λ_0. Viewing the twisted product $\tilde{P} \times_{C_G(\lambda_0(H))} G$ as a bundle associated with the principal $C_G(\lambda_0(H))$-bundle $G(C_G(\lambda_0(H)), G/C_G(\lambda_0(H)))$, we finally obtain

$$P \cong K \times_H \left(G \times_{C_G(\lambda_0(H))} \tilde{P}\right), \tag{1.9.24}$$

with the right H-action on $K \times \left(G \times_{C_G(\lambda_0(H))} \tilde{P}\right)$ induced by (1.9.11),

[20] And thus also the principal Γ_H-bundle $M_H \to \hat{M}$.

$$(h, (k, [(g, \tilde{p})]) \mapsto (kh, [(g\lambda_0(h), \tilde{p})]), \quad h \in H,$$

cf. [539]. The diffeomorphism (1.9.24) is induced from (1.9.9) in an obvious way:

$$[(k, [(g, \tilde{p})])] \mapsto \Delta_k \circ \Psi_{g^{-1}}(\tilde{p}).$$

By Remark 1.9.6, passing from λ_0 to a conjugate homomorphism yields an isomorphic principal G-bundle P. ♦

Next, we will use the above results to classify G-invariant connections in the present context.

Definition 1.9.10 Let $P(M, G)$ be a principal bundle and let $\Delta : K \to \mathrm{Aut}(P)$ be a group homomorphism. A connection form ω on P is called K-invariant if for all $k \in K$

$$\Delta_k^* \omega = \omega.$$

The following result yields the classification of invariant connections for the case of simple group actions. It belongs to Jadczyk and Pilch [345]. To formulate it, we need a reductive decomposition

$$\mathfrak{k} = \mathfrak{n}_H \oplus \mathfrak{p}, \tag{1.9.25}$$

whose existence is guaranteed by the compactness of K. Let $L(\mathfrak{p}, \mathfrak{g})$ be the space of linear mappings from $\mathfrak{p}$ to $\mathfrak{g}$. Note that $L(\mathfrak{p}, \mathfrak{g})$ is endowed with a natural $N_{K\times G}(I)$-action given by

$$N_{K\times G}(I) \times L(\mathfrak{p}, \mathfrak{g}) \to L(\mathfrak{p}, \mathfrak{g}), \quad ([(k, g)], F) \mapsto \mathrm{Ad}(g) \circ F \circ \mathrm{Ad}(k^{-1}),$$

and that this action descends to a Γ_I-action on the subspace $L(\mathfrak{p}, \mathfrak{g})^H \subset L(\mathfrak{p}, \mathfrak{g})$ of H-invariant elements, that is, linear mappings fulfilling

$$F = \mathrm{Ad}(\lambda_0(h)) \circ F \circ \mathrm{Ad}(h^{-1}), \quad h \in H. \tag{1.9.26}$$

Theorem 1.9.11 *Let (M, K, δ) be a simple Lie group action, let (P, G, M, Ψ, π) be a principal bundle admitting a lift $\Delta : K \to \mathrm{Aut}(P)$ of the K-action. Then, there is a one-to-one correspondence between K-invariant connection forms ω on P and pairs $(\hat{\omega}, \hat{\Phi})$, where $\hat{\omega}$ is a $\mathfrak{z}$-valued 1-form of type* Ad *on $P_I(\hat{M}, \Gamma_I)$ fulfilling*

$$\hat{\omega}_p(A_*) = A, \quad A \in \mathfrak{z},\ p \in P_I, \tag{1.9.27}$$

and $\hat{\Phi} : P_I \to L(\mathfrak{p}, \mathfrak{g})^H$ is a Γ_I-equivariant mapping.

Proof 1. Let ω be K-invariant. According to Remark 1.9.2/2, we may view P as a bundle associated with the principal Γ_I-bundle

$$(K \times G)/I \to (K \times G)/N_{K\times G}(I). \tag{1.9.28}$$

Since G is compact, we may decompose $\mathfrak{g}$ into $\mathfrak{z}$ and its orthogonal complement

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^{\perp}. \tag{1.9.29}$$

Clearly, this decomposition is reductive. Using the decompositions (1.9.25) and (1.9.29), together with $\mathfrak{n}_H = \mathfrak{h} \oplus \hat{\mathfrak{n}}_H$, we obtain

$$\mathrm{T}_{[\mathbb{1}]}((K \times G)/I) = (\hat{\mathfrak{n}}_H \oplus \mathfrak{z}) \oplus (\mathfrak{p} \oplus \mathfrak{z}^{\perp}). \tag{1.9.30}$$

Since the decompositions (1.9.25) and (1.9.29) are reductive, this decomposition is reductive, too. Using (1.9.18), we obtain

$$\mathrm{T}_{[\mathbb{1}]}\big((K \times G)/N_{K\times G}(I)\big) = \mathfrak{p} \oplus \mathfrak{z}^{\perp}. \tag{1.9.31}$$

By Example 1.3.19, $\mathfrak{p} \oplus \mathfrak{z}^{\perp}$ defines a connection on the principal bundle (1.9.28), which in turn induces a connection on the associated bundle $(K \times G)/I \times_{\Gamma_I} P_I$. By (1.9.9), the corresponding splitting of the tangent bundle $\mathrm{T}P$ is pointwise given by

$$\mathrm{T}_{\rho_{(k,g)}(p)}P = \rho'_{(k,g)}\left\{\mathrm{T}_p P_I \oplus \rho'_p(\mathfrak{p} \oplus \mathfrak{z}^{\perp})\right\}, \tag{1.9.32}$$

where $(k, g) \in N_{K\times G}(I)$ and $p \in P_I$ (Exercise 1.9.1). The first summand in (1.9.32) is vertical and the second one is the horizontal subspace of the induced connection. With respect to this splitting, every 1-form α on P may be decomposed into its vertical and horizontal parts,

$$\alpha = \alpha^v + \alpha^h,$$

and the horizontal part may be further decomposed as

$$\alpha^h = \alpha^{\mathfrak{p}} + \alpha^{\mathfrak{z}^{\perp}}.$$

We define

$$\hat{\omega} := (\omega^v)_{\restriction P_I}. \tag{1.9.33}$$

Using the K-invariance of ω and (1.9.4), on P_I we obtain

$$\omega_p = (\Delta_h^*\omega)_p = (\Psi^*_{\lambda_0(h)}\omega)_p = \mathrm{Ad}(\lambda_0(h)^{-1})\omega_p,$$

for every $p \in P_I$ and every $h \in H$. Thus, $\hat{\omega}$ takes values in $\mathfrak{z}$. By point 3 of Proposition 1.3.5 and, again, by K-invariance of ω,

$$\rho^*_{(k,g)}\hat{\omega} = \mathrm{Ad}(g) \circ \hat{\omega}, \quad (k, g) \in N_{K\times G}(I).$$

Since $\hat{\omega}$ is $\mathfrak{z}$-valued and $\mathfrak{z} \subset \hat{\mathfrak{n}}_I = \hat{\mathfrak{n}}_H \oplus \mathfrak{z}$, we may rewrite this relation as follows:

$$(\Psi^I)^*_a \hat{\omega} = \mathrm{Ad}(a^{-1}) \circ \hat{\omega}, \quad a \in \Gamma_I, \tag{1.9.34}$$

showing that $\hat{\omega}$ is of type Ad. Finally, formula (1.9.27) is an immediate consequence of point 2 of Proposition 1.3.5. Next, we define

$$\Phi : P_I \to (\mathfrak{p} \oplus \mathfrak{z}^\perp)^* \otimes \mathfrak{g}, \qquad \Phi(p) := \rho_p^*\big((\omega^h)_{\restriction P_I}\big), \quad p \in P_I,$$

where $\rho_p : N_{K\times G}(I) \to P_I$ is defined by restriction. Since

$$\rho'_p(\mathfrak{p} \oplus \mathfrak{z}^\perp) = \Delta'_p(\mathfrak{p}) \oplus \Psi'_p(\mathfrak{z}^\perp),$$

the two horizontal components are

$$\hat{\Phi} : P_I \to \mathfrak{p}^* \otimes \mathfrak{g} = L(\mathfrak{p}, \mathfrak{g}), \quad \hat{\Phi}(p) := \Delta_p^*\big((\omega^{\mathfrak{p}})_{\restriction P_I}\big), \tag{1.9.35}$$

and

$$\check{\Phi} : P_I \to (\mathfrak{z}^\perp)^* \otimes \mathfrak{g} = L(\mathfrak{z}^\perp, \mathfrak{g}), \quad \check{\Phi}(p) := \Psi_p^*\big((\omega^{\mathfrak{z}^\perp})_{\restriction P_I}\big).$$

Here, as above, Δ_p and Ψ_p stand for the appropriate restrictions. We show that $\hat{\Phi}$ is Γ_I-equivariant and that $\check{\Phi}$ is constant and equal to the identical mapping on $\mathfrak{z}^\perp$. Using the G-equivariance and the K-invariance of ω, together with

$$\rho'_{\rho_{(k,g)}(p)}(A, B) = \rho'_{(k,g)} \circ \rho'_p\big(\mathrm{Ad}(k^{-1})A, \mathrm{Ad}(g^{-1})B\big),$$

where $(k, g) \in N_{K\times G}(I)$ and $(A, B) \in \mathfrak{p} \oplus \mathfrak{z}^\perp$, we calculate

$$\begin{aligned}
\Phi(\rho_{(k,g)}(p))(A, B) &= \omega^h_{\rho_{(k,g)}(p)}\big(\rho'_{\rho_{(k,g)}(p)}(A, B)\big)\\
&= \omega_{\rho_{(k,g)}(p)}\big(\Delta'_k \circ \Psi'_{g^{-1}} \circ \rho'_p(\mathrm{Ad}(k^{-1})A, \mathrm{Ad}(g^{-1})B)\big)\\
&= \mathrm{Ad}(g) \circ \omega_p\big(\rho'_p(\mathrm{Ad}(k^{-1})A, \mathrm{Ad}(g^{-1})B\big)\\
&= \mathrm{Ad}(g) \circ \Phi(p)\big(\mathrm{Ad}(k^{-1})A, \mathrm{Ad}(g^{-1})B\big).
\end{aligned}$$

Thus,

$$\hat{\Phi}(\rho_{(k,g)}(p)) = \mathrm{Ad}(g) \circ \hat{\Phi}(p) \circ \mathrm{Ad}(k^{-1}), \tag{1.9.36}$$

showing the $N_{K\times G}(I)$-equivariance of $\hat{\Phi}$, and

$$\check{\Phi}(\rho_{(k,g)}(p))(B) = \mathrm{Ad}(g) \circ \omega_p^{\mathfrak{z}^\perp}(\Psi'_p(Ad(g^{-1})B)) = B.$$

Finally, the H-invariance of $\hat{\Phi}(p)$ follows immediately: for $(h, \lambda_0(h)) \in I$ we have $\rho_{(h,\lambda_0(h))}(p) = p$ for all $h \in H$, and thus (1.9.36) implies

$$\hat{\Phi}(p) = \mathrm{Ad}(\lambda_0(h)) \circ \hat{\Phi}(p) \circ \mathrm{Ad}(h^{-1}).$$

Thus, $\hat{\varPhi}$ is $\varGamma_I$-equivariant.

2. Conversely, let $(\hat{\omega}, \hat{\varPhi})$ be a pair of objects defined on the principal bundle $P_I(\hat{M}, \varGamma_I)$ with the desired properties. Using the $(K \times G)$-equivariant diffeomorphism

$$P \cong P_I \times_{\varGamma_I} (K \times G)/I$$

given by (1.9.9), we extend $\hat{\varPhi}$ by the constant mapping $\check{\varPhi} : P_I \to \mathrm{id}_{\mathfrak{z}^\perp}$ to a mapping

$$\varPhi : P_I \to (\mathfrak{p} \oplus \mathfrak{z}^\perp)^* \otimes \mathfrak{g}, \quad \varPhi := \hat{\varPhi} + \check{\varPhi},$$

and use (1.9.32) to define

$$\omega_p(Z) := \hat{\omega}_p(X) + \varPhi(p)(A, B),$$

with $p \in P_I$ and $Z = X + \rho'_p(A, B) \in \mathrm{T}_p P$. Finally, we extend ω to P via ρ. The proof that this yields a well-defined K-invariant connection form on P is left to the reader (Exercise 1.9.3). ∎

Remark 1.9.12

1. Combining Theorem 1.9.11 with Proposition 1.9.5, one finds that pairs (P, ω), where P is a principal bundle over M admitting a lift of the K-action and ω is a K-invariant connection, are in bijective correspondence with triples $(\hat{P}, \hat{\omega}, \hat{\varPhi})$, where $\hat{P}$ is a principal $\varGamma_I$-bundle, $\hat{\omega}$ is a $\mathfrak{z}$-valued 1-form of type Ad on $\hat{P}$ fulfilling (1.9.27) and $\hat{\varPhi} : \hat{P} \to L(\mathfrak{p}, \mathfrak{g})^H$ is a $\varGamma_I$-equivariant mapping.
2. Note that $\hat{\omega}$ is is not a connection form on $P_I(\hat{M}, \varGamma_I)$, because point 2 of Proposition 1.3.5 need not be fulfilled for elements $A \in \hat{\mathfrak{n}}_I$. On the other hand, owing to the fact that $Z \subset \varGamma_I$, formula (1.9.34) holds for any $a \in Z$. Together with (1.9.27), this implies that $\hat{\omega}$ is a connection form on P_I viewed as a principal Z-bundle over M_I.
3. Let μ be a connection form on the principal $\varGamma_I/Z$-bundle $\pi_{\hat{M}} : M_I \to \hat{M}$. Define

$$\hat{\tau} := \hat{\omega} - \hat{\omega} \circ \pi^*_{M_I}\mu + \pi^*_{M_I}\mu, \tag{1.9.37}$$

where

$$\hat{\omega} \circ \pi^*_{M_I}\mu : \mathrm{T}_p P_I \to \hat{\mathfrak{n}}_I, \quad \hat{\omega} \circ \pi^*_{M_I}\mu(X) := \hat{\omega}((\mu(\pi'_{M_I}(X))_*).$$

Here, $\mu(\pi'_{M_I}(X))$ is viewed as an element of $\hat{\mathfrak{n}}_I$ via (1.9.18). It is easy to see that $\hat{\tau}$ is a connection form on $P_I(\hat{M}, \varGamma_I)$ (Exercise 1.9.4). This shows that any connection form on $M_I(\hat{M}, \varGamma_I/Z)$ completes the connection form $\hat{\omega}$ on $P_I(M_I, Z)$ to a connection form on $P_I(\hat{M}, \varGamma_I)$.
4. The H-invariance condition (1.9.26) for $\hat{\varPhi}(p)$, $p \in P_I$, may be rewritten as

$$\hat{\varPhi}(p) \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda_0(h)) \circ \hat{\varPhi}(p), \quad h \in H. \tag{1.9.38}$$

In this form, it means that the linear mapping $\hat{\Phi}(p)$ is an operator intertwining the restrictions of the adjoint representations of K and G to H acting on $\mathfrak{m}$ and to $\lambda_0(H)$ acting on $\mathfrak{g}$, respectively. Under the canonical identification $L(\mathfrak{p}, \mathfrak{g}) = \mathfrak{p}^* \otimes \mathfrak{g}$, condition (1.9.26) takes the form

$$\left(\mathrm{Ad}^*(h) \otimes \mathrm{Ad}(\lambda_0(h))\right) \hat{\Phi}(p) = \hat{\Phi}(p).$$

♦

Let us apply Theorem 1.9.11 to the two special cases treated before. First, let us consider the case of a transitive K-action addressed in Remark 1.9.7. By this remark, principal G-bundles admitting a lift of the K-action are completely characterized by Lie group homomorphisms $\lambda_0 : H \to G$ and have the following structure:

$$P = K \times_H G.$$

In this case, $\hat{M}$ is the one-point space and thus the principal Γ_I-bundle P_I coincides with the principal Z-bundle $\Gamma_I \to \Gamma_I/Z$. Consequently, by (1.9.34) and (1.9.27), $\hat{\omega}$ is a Γ_I-invariant connection form on this bundle and, therefore, by (1.9.18), it is given by a linear mapping $\hat{\phi} : \hat{\mathfrak{n}}_H \to \mathfrak{z}$. Since $\mathrm{Ad}(H)$ acts trivially on $\hat{\mathfrak{n}}_H$, this mapping is H-invariant. To summarize, if we denote

$$\mathfrak{m} = \hat{\mathfrak{n}}_H \oplus \mathfrak{p}, \tag{1.9.39}$$

then $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and we may merge $\hat{\phi}$ and $\hat{\Phi}$ to an H-equivariant mapping $\tilde{\Phi} : \mathfrak{m} \to \mathfrak{g}$, that is, a mapping fulfilling

$$\tilde{\Phi} \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda_0(h)) \circ \tilde{\Phi}, \quad h \in H. \tag{1.9.40}$$

This way, we get the following classical result of Wang [647].

Corollary 1.9.13 (Wang) *If the K-action is transitive, then K-invariant connections on P are in one-to-one correspondence with H-equivariant linear mappings $\tilde{\Phi} : \mathfrak{m} \to \mathfrak{g}$.* ■

Some details of the proofs of the statements contained in the following remark are left to the reader (Exercise 1.9.5).

Remark 1.9.14

1. For later purposes, we give an explicit reconstruction formula for the K-invariant connections described by Corollary 1.9.13. Choose $p_0 = [(\mathbb{1}_K, \mathbb{1}_G)] \in K \times_H G$. Then, any tangent vector $Z_{p_0} \in \mathrm{T}_{p_0}(K \times_H G)$ may be written as

$$Z_{p_0} = [(A, B)], \quad A \in \mathfrak{k},\ B \in \mathfrak{g},$$

and, for any $p \in K \times_H G$, there exist elements $k \in K$ and $g \in G$ such that

$$p_0 = \Delta_k \circ \Psi_g(p).$$

We define

$$\omega_p(Z) = \mathrm{Ad}(g)\left(\lambda_0'(A_{\mathfrak{h}}) + \tilde{\Phi}(A_{\mathfrak{m}}) + B\right), \tag{1.9.41}$$

where $A_{\mathfrak{h}} \in \mathfrak{h}$ and $A_{\mathfrak{m}} \in \mathfrak{m}$ are the components of A with respect to the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. It is easy to show that ω is a (correctly defined) K-invariant connection form on $K \times_H G$, indeed.

2. Clearly, among the invariant connections labeled by $\tilde{\Phi}$ there is a distinguished element, defined by

$$\tilde{\Phi} = 0. \tag{1.9.42}$$

By (1.9.41), it is given by

$$\omega_p(Z) = \mathrm{Ad}(g)\left(\lambda_0'(A_{\mathfrak{h}}) + B\right), \tag{1.9.43}$$

that is, it is uniquely determined by the homomorphism λ_0. Therefore, it is called the canonical invariant connection on P.

3. In the transitive case, the compactness assumptions on K and G may be dropped. Then, in general, there is no reductive decomposition (1.9.25) and the (slightly more general) classification reads as follows: K-invariant connection forms are in one-to-one correspondence with H-invariant linear mappings $\Lambda : \mathfrak{k} \to \mathfrak{g}$ fulfilling $\Lambda(A) = \lambda_0'(A)$ for any $A \in \mathfrak{h}$.
4. Using the Structure Equation, it is easy to calculate the curvature Ω of a K-invariant connection form. Clearly, it suffices to calculate Ω on Killing vector fields of K. This yields

$$\Omega(A_*, A'_*) = [\Lambda(A), \Lambda(A')] - \Lambda([A, A']), \quad A, A' \in \mathfrak{k}.$$

Thus, a K-invariant connection is flat iff Λ is a Lie algebra homomorphism. ♦

Application of Theorem 1.9.11 to the case addressed in Remark 1.9.9 yields the following, see [546].

Corollary 1.9.15 *If the principal Γ_I/Z-bundle $M_I \to \hat{M}$ is trivial, then K-invariant connections on P are in one-to-one correspondence with pairs $(\tilde{\omega}, \tilde{\Phi})$, where*

1. *$\tilde{\omega}$ is a connection form on the principal $C_G(\lambda_0(H))$-bundle $\tilde{P}$ over $\tilde{M}$,*
2. *$\tilde{\Phi} : \tilde{P} \to L(\mathfrak{m}, \mathfrak{g})^H$ is a $C_G(\lambda_0(H))$-equivariant mapping.*

Proof Since Γ_I acts freely on P_I and Z is a normal subgroup of Γ_I we have the following diffeomorphism:

$$\varphi : \tilde{P} \times_Z \Gamma_I \to P_I \quad \varphi([(\tilde{p}, a)]) := \rho_a(\tilde{p}). \tag{1.9.44}$$

Using this identification and (1.9.18), we get a splitting of the tangent bundle,

$$\mathrm{T}_{\rho_a(\tilde{p})}P_I = \rho'_a\left\{\mathrm{T}_{\tilde{p}}\tilde{P} \oplus \varDelta'_{\tilde{p}}(\hat{\mathfrak{n}}_H)\right\}. \tag{1.9.45}$$

Decomposing $\hat{\omega}$ with respect to this splitting yields a pair $(\tilde{\omega}, \hat{\phi})$, where $\tilde{\omega}$ is a connection form on $\tilde{P}$ and $\hat{\phi}$ is a mapping given by

$$\hat{\phi} : \tilde{P} \to (\hat{\mathfrak{n}}_H)^* \otimes \mathfrak{z}, \quad \hat{\phi}(\tilde{p})(A) = \hat{\omega}(\varDelta'_{\tilde{p}}(A)), \quad A \in \hat{\mathfrak{n}}_H. \tag{1.9.46}$$

Finally, as above, merging $\hat{\phi}$ with $\hat{\varPhi}$ we get a mapping

$$\tilde{\varPhi} : \tilde{P} \to (\mathfrak{m})^* \otimes \mathfrak{g},$$

fulfilling

$$\tilde{\varPhi}(\tilde{p}) \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda_0(h)) \circ \tilde{\varPhi}(\tilde{p}), \quad h \in H. \tag{1.9.47}$$

Conversely, given a pair $(\tilde{\omega}, \tilde{\varPhi})$, one first reconstructs the pair $(\hat{\omega}, \hat{\varPhi})$ and then, using Theorem 1.9.11, the invariant connection ω. ■

Remark 1.9.16

1. By construction, see (1.9.35) and (1.9.46), $\tilde{\varPhi}$ is given by

$$\tilde{\varPhi}(\tilde{p})(A) = \omega_{\tilde{p}}(\varDelta'_{\tilde{p}}(A)) = \varDelta^*_{\tilde{p}}(\omega)(A), \quad A \in \mathfrak{m}. \tag{1.9.48}$$

2. Comparing with point 3 of Remark 1.9.12, in this case, the connection form μ is simply given by the section $s : \hat{M} \to M_I$, cf. Example 1.3.18. ♦

To conclude this section, we discuss two simple examples of the above type which are relevant in physics.

Example 1.9.17 (*Rotational invariance*) Consider the defining representation of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ or, equivalently, the adjoint representation of $\mathrm{SU}(2)$ under the identification $\mathbb{R}^3 \cong \mathfrak{su}(2)$.[21] If we remove the origin, we have $\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}_+ \times \mathrm{S}^2$ and thus we deal with the situation described by Remark 1.9.9 and Corollary 1.9.15, with

$$G = \mathrm{SU}(2), \quad K = \mathrm{SU}(2), \quad H = \mathrm{U}(1), \quad \tilde{M} = \mathbb{R}_+.$$

Let us classify the K-invariant $\mathrm{SU}(2)$-connections over $\mathbb{R}^3 \setminus \{0\}$.

(a) Principal $\mathrm{SU}(2)$-bundles over $\tilde{M}$ admitting a lift of the adjoint representation of $\mathrm{SU}(2)$ are labeled by conjugacy classes of homomorphisms $\lambda : \mathrm{U}(1) \to \mathrm{SU}(2)$. Clearly, with $\mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$, for every integer n, the mapping

$$\lambda_n(z) = \mathrm{diag}(z^n, z^{-n})$$

[21] See Examples 5.2.8 and 5.4.7 of Part I.

is a homomorphism. Since any unitary matrix is diagonalizable via conjugation by unitary matrices, all other homomorphisms are conjugate to some λ_n. Thus, the conjugacy classes of homomorphisms classifying the admissible principal SU(2)-bundles are labeled by $n \in \mathbb{Z}$. The principal SU(2)-bundles admitting a lift of δ are given by (1.9.24),

$$P \cong K \times_H \left(G \times_{C_G(\lambda_n(H))} \tilde{P} \right),$$

where $\tilde{P}$ is necessarily trivial, that is, $\tilde{P} = \mathbb{R}_+ \times C_G(\lambda_n(H))$. Thus, P can be naturally identified as follows

$$P \cong \mathbb{R}_+ \times (K \times_H G). \tag{1.9.49}$$

(b) Let us apply Corollary 1.9.15. By direct inspection, we see that the centralizer $C_{\mathrm{SU}(2)}(\lambda_n(\mathrm{U}(1)))$ is U(1) for $n \neq 0$ and SU(2) for $n = 0$. Consequently, $\tilde{\omega}$ is $\mathfrak{u}(1)$-valued for $n \neq 0$ and $\mathfrak{su}(2)$-valued for $n = 0$. By (1.9.49), $\tilde{\omega}$ may be globally represented by a $\mathfrak{z}$-valued 1-form $\tilde{\mathbb{A}}$ on $\mathbb{R}_+$. Let us analyze the mapping $\tilde{\Phi}$. Again by (1.9.49), it is a function on $\mathbb{R}_+$ with values in the K-invariant connections on $K \times_H G$, cf. point 1 of Remark 1.9.14. The latter are given by (1.9.41). Since $\tilde{\Phi}$ takes values in $L(\mathfrak{m}, \mathfrak{su}(2))^{\mathrm{U}(1)}$, where $\mathfrak{m}$ is defined by the orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{m}$, we must analyze the U(1)-invariance condition

$$\tilde{\Phi} \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda_n(h)) \circ \tilde{\Phi}, \quad h \in \mathrm{U}(1).$$

Here we interpret $\tilde{\Phi}$ as an intertwiner of the representations $\mathrm{Ad}(\mathrm{U}(1))_{\restriction \mathfrak{m}}$ and $\mathrm{Ad}(\lambda_n(\mathrm{U}(1)))$. For that purpose, it is convenient to pass to the complexification of the Lie algebras under consideration and to use the standard representation theory of complex simple Lie algebras.[22] Correspondingly, we extend $\tilde{\Phi}$ by linearity to the complexified spaces. Let $\mathbf{h}$ be a Cartan element and let $\mathbf{e}_-$, $\mathbf{e}_+$ be root vectors for the complexification of $\mathfrak{k} = \mathfrak{su}(2)$. Clearly, $\mathfrak{u}(1)$ is spanned by $\mathbf{h}$ and $\mathfrak{m}$ is spanned by the root vectors. By direct inspection, we see that $\mathfrak{m}$ decomposes into irreducible components of $\mathrm{Ad}(\mathrm{U}(1))\mathfrak{m}$ as

$$\mathfrak{m} = \mathbb{C}\mathbf{e}_+ \oplus \mathbb{C}\mathbf{e}_-, \quad \mathrm{ad}(\mathbf{h})_{\restriction \mathbb{C}\mathbf{e}_\pm} = \pm 2\,.$$

In physics notation, this is summarized in the formula

$$\underline{2} = (2) + (-2). \tag{1.9.50}$$

If we denote the Cartan element and the root vectors for $\mathfrak{g} = \mathfrak{su}(2)$ by $\mathbf{H}$, $\mathbf{E}_-$, $\mathbf{E}_+$, respectively, then we have $\lambda_n'(\mathbf{h}) = n\mathbf{H}$. Consequently, the decomposition of $\mathfrak{g}$ into irreducible components reads, in physics notation,

$$\underline{3} = (0) + (2n) + (-2n). \tag{1.9.51}$$

[22] See [329, 344].

Comparing (1.9.50) with (1.9.51) we see that for $n \neq \pm 1$, the decompositions do not contain equivalent representations, that is, the intertwining operator $\tilde{\Phi}$ vanishes. In that case, the corresponding invariant connection on $K \times_H G$ is the canonical one given by (1.9.43). Since $\lambda'_n(\mathbf{h}) = n\mathbf{H}$, for $n = 0$, this connection degenerates to a 'pure gauge'. For $n = \pm 1$, we get a nontrivial solution. For every $r \in \mathbb{R}_+$, it is given by

$$\tilde{\Phi}(\mathbf{e}_-) = c_- \mathbf{E}_-, \quad \tilde{\Phi}(\mathbf{e}_+) = c_+ \mathbf{E}_+ \quad c_\pm \in \mathbb{C}. \tag{1.9.52}$$

Finally, returning to the original mapping $\tilde{\Phi}$ by restricting the above intertwiner to the real vector space $\mathfrak{m}$ implies $c_+ = \bar{c}_-$. Thus, $\tilde{\Phi}$ is labeled by two $\mathbb{R}$-valued functions on $\mathbb{R}_+$. The corresponding invariant connections are given by (1.9.41). ♦

Example 1.9.18 (*Translational invariance*) Consider the orthogonal decomposition of the Euclidean space

$$\mathbb{R}^4 = \mathbb{R}\mathbf{e}_0 \oplus \mathbb{R}^3$$

and write pr_i, $i = 1, 2$, for the canonical projections onto the first and the second component. For $\mathbf{x} \in \mathbb{R}^4$, denote $\mathrm{pr}_1(\mathbf{x}) = x^0$ and $\mathrm{pr}_2(\mathbf{x}) = \tilde{\mathbf{x}}$. In this notation, the action of the Abelian group $\mathbb{R}$ by translations[23] on the first factor is given by

$$\delta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4, \quad \delta(s, (x^0, \tilde{\mathbf{x}})) = (x^0 + s, \tilde{\mathbf{x}}).$$

For a given Lie group G, let us classify the $\mathbb{R}$-invariant connections over $\mathbb{R}^4$.

(a) Principal G-bundles $\pi : P \to \mathbb{R}^4$ admitting a lift of δ are given by (1.9.24). Here, $K = \mathbb{R}$ and $H = \{0\}$. Thus, λ_0 must be the trivial homomorphism sending 0 to $\mathbb{1}_G$. Consequently, $C_G(\lambda_0(H)) = G$ and we obtain

$$P \cong \mathbb{R} \times \tilde{P}, \quad \tilde{P} = \pi^{-1}(\mathbb{R}^3),$$

with $\tilde{\pi} : \tilde{P} \to \mathbb{R}^3$ being a (trivial) principal G-bundle. Under this isomorphism, the lift Δ of δ to automorphisms of P is given by translations on the first factor, $\Delta(s, (x^0, \tilde{p})) = (x^0 + s, \tilde{p})$.

(b) According to Corollary 1.9.15, $\mathbb{R}$-invariant connections ω on P are given by pairs $(\tilde{\omega}, \tilde{\Phi})$, where $\tilde{\omega}$ is a connection form on $\tilde{P}$ and $\tilde{\Phi}$ is an equivariant mapping from $\tilde{P}$ to $L(\mathbb{R}\mathbf{e}_0, \mathfrak{g}) \cong (\mathbb{R}\mathbf{e}_0)^* \otimes \mathfrak{g}$. Thus,

$$\tilde{\Phi}(\tilde{p}) = \tilde{\phi}(\tilde{p}) \otimes \mathbf{e}_0^*, \quad \tilde{p} \in \tilde{P},$$

where $\mathbf{e}_0^*$ is the basis in $(\mathbb{R}\mathbf{e}_0)^*$ dual to $\mathbf{e}_0$ and $\tilde{\phi} \in \mathrm{Hom}_G(\tilde{P}, \mathfrak{g})$. Given $(\tilde{\omega}, \tilde{\Phi})$, let us reconstruct ω: pulling back $\tilde{\phi}$ and $\mathbf{e}_0^*$ with the natural projections $P \to \tilde{P}$ and

[23] Although the group $\mathbb{R}$ is not compact, the action under consideration is proper and, since $\mathbb{R}$ is Abelian, the standard scalar product is trivially Ad-invariant. As a consequence, the above theory applies.

$P \to \mathbb{R}\mathbf{e}_0$, respectively, we obtain from $\tilde{\Phi}$ a horizontal 1-form $\tilde{\tau}$ of type Ad on P. Extending $\tilde{\omega}$ via the $\mathbb{R}$-action to P, we obtain

$$\omega = \tilde{\omega} + \tilde{\tau}. \tag{1.9.53}$$

♦

Exercises

1.9.1 Prove formula (1.9.32).

1.9.2 Prove the statements of Remark 1.9.7/2 and 1.9.7/3.

1.9.3 Complete point 2 of the proof of Theorem 1.9.11.

1.9.4 Prove the statement of Remark 1.9.12/3.

1.9.5 Work out the details in Remark 1.9.14.

Chapter 2
Linear Connections and Riemannian Geometry

In Sects. 2.1 and 2.2, we present the general theory of linear connections together with the reduction theory of the underlying frame bundle to some Lie subgroup of the general linear group. These reductions are usually referred to as H-structures.[1] They lead to a unified view on possible geometric structures manifolds may be endowed with. Using this framework, we discuss almost complex, pseudo-Riemannian, conformal, almost Hermitean and almost symplectic structures including a discussion of the corresponding compatible connections. Thus, from the perspective of H-structures, Riemannian geometry is an important special example. In Sects. 2.3 and 2.5, we continue to study H-structures by investigating torsion-free compatible connections. We ask which holonomy groups may occur for such connections. This fundamental question has been first systematically studied by Berger. In this delicate analysis, the central object to be studied is the curvature mapping of the connection under consideration. In Sect. 2.3, we study the class of connections which are not locally symmetric with emphasis on the metric case, where the H-structure defines a pseudo-Riemannian manifold. For that case, we formulate the classification result of Berger without giving a proof. We also comment on the classification in the non-metric case. In Sect. 2.5, we study the case of locally symmetric connections. This leads us to the theory of symmetric spaces. We present the basics of this theory in a fairly consistent manner including a number of important classes of examples. Next, in Sect. 2.6, we extend our discussion of compatible connections to vector bundles with emphasis on Hermitean bundles and holomorphic structures. In Sect. 2.7, we present the basics of Hodge Theory[2] including a detailed study of Weitzenboeck-type formulae. Finally, in Sect. 2.8, we discuss properties of Riemannian manifolds which are special in dimension four.

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[1] Also called G-structures in the older literature.

[2] But, the proof of the Hodge Decomposition Theorem is postponed to Chap. 5.

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2.1 Linear Connections

Let M be an n-dimensional differentiable manifold and let $L(M)$ be its bundle of linear frames, cf. Example 1.1.14. Recall that a linear frame at $m \in M$ is an ordered basis $u = (u_1, \dots, u_n)$ in $\mathrm{T}_m M$ and that $\pi : L(M) \to M$, $\pi(u) = m$, is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. The free right action of GL(n, $\mathbb{R}$) on $L(M)$ is given by

$$L(M) \times \mathrm{GL}(n, \mathbb{R}) \to L(M) , \quad (u, a) \mapsto ua . \tag{2.1.1}$$

Here, $ua = (u_i a^i{}_1, \dots, u_i a^i{}_n)$.

In the sequel, the basic representation of $\mathrm{GL}(n, \mathbb{R})$ given by matrix multiplication of elements of $\mathbb{R}^n$ from the left will be denoted by σ_n^0. Thus, $\sigma_n^0(a)\mathbf{x} = a\mathbf{x}$.

Definition 2.1.1 A principal connection Γ on the frame bundle $L(M)$ will be referred to as a linear connection on M.[3]

Given a linear connection on M, it induces connections on all tensor bundles over M. To see this, it is enough to show that all tensor bundles over M are vector bundles associated with $L(M)$. For the proof, take the basic representation σ_n^0 of $\mathrm{GL}(n, \mathbb{R})$ and the corresponding associated bundle $E := L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$. Define

$$\varphi \colon E \to \mathrm{T}M , \quad \varphi([(u, \mathbf{x})]) := x^i u_i , \tag{2.1.2}$$

where x^i are the components of $\mathbf{x} \in \mathbb{R}^n$ in the standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$. It is easy to show that φ is an isomorphism of vector bundles (Exercise 2.1.1). Thus,

$$\mathrm{T}M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n . \tag{2.1.3}$$

Via the dual of the basic representation, this induces an isomorphism

$$\mathrm{T}^*M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^* \tag{2.1.4}$$

and, thus,

$$\mathbb{T}^k_l M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{T}^k_l \mathbb{R}^n . \tag{2.1.5}$$

Remark 2.1.2 Often, a frame $u \in L(M)$ will be viewed as an isomorphism

$$u : \mathbb{R}^n \to \mathrm{T}_{\pi(u)} M , \quad u(\mathbf{x}) := x^i u_i .$$

By (2.1.2), we have

$$\varphi \circ \iota_u = u . \tag{2.1.6}$$

♦

[3] As in the general theory, Γ is a horizontal distribution on $L(M)$. Below, it will become clear why it is reasonable to speak of a connection on the base manifold M.

Now we can start discussing the theory of linear connections. First, we exhibit a structure which distinguishes frame bundles from general principal fibre bundles.

Definition 2.1.3 The differential form $\theta \in \Omega^1(L(M), \mathbb{R}^n)$ defined by

$$\theta(X) := u^{-1}(\pi'(X)), \quad X \in \mathrm{T}_u L(M), \tag{2.1.7}$$

is called the canonical $\mathbb{R}^n$-valued 1-form on $L(M)$, or, the soldering form.

Proposition 2.1.4 *The soldering form θ is a horizontal 1-form of type σ_n^0,*

$$\Psi_a^*\theta = a^{-1} \circ \theta, \quad a \in \mathrm{GL}(n, \mathbb{R}).$$

Proof By definition, θ is horizontal. Let $u \in L(M)$ and $a \in \mathrm{GL}(n, \mathbb{R})$. If we view u as a mapping $\mathbb{R}^n \to \mathrm{T}_{\pi(u)}M$, then to $\Psi_a(u)$ there corresponds the mapping

$$u \circ a : \mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} \mathrm{T}_{\pi(u)}M.$$

Thus, for any $X \in \mathrm{T}_u L(M)$,

$$\begin{aligned}
(\Psi_a^*\theta)_u(X) &= \theta_{\Psi_a(u)}(\Psi_a' X) \\
&= (\Psi_a(u))^{-1}(\pi' \circ \Psi_a'(X)) \\
&= (u \circ a)^{-1}(\pi'(X)) \\
&= a^{-1}\theta_u(X).
\end{aligned}$$

■

Remark 2.1.5 By Proposition 1.2.12, via the isomorphism (2.1.2), to θ there corresponds a unique 1-form $\hat{\theta} \in \Omega^1(M, \mathrm{T}M)$ given by

$$\hat{\theta}_m(X) = u \circ \theta(X^*) = u \circ u^{-1} \circ \pi'(X^*) = X,$$

where $\pi(u) = m$, $X \in \mathrm{T}_m M$ and $X^* \in \mathrm{T}_u L(M)$ fulfilling $\pi'(X^*) = X$. Thus, $\hat{\theta}(X) = X$. That is why $\hat{\theta}$ is usually called the tautological 1-form. ♦

Now, let Γ be a linear connection on M and let ω be its connection form on $L(M)$. Then, any $\mathbf{x} \in \mathbb{R}^n$ defines a Γ-horizontal vector field $B(\mathbf{x})$ on $L(M)$ by assigning to $u \in L(M)$ the unique Γ-horizontal lift of $u(\mathbf{x}) \in \mathrm{T}_{\pi(u)}M$ to the point u.

Definition 2.1.6 The vector field $B(\mathbf{x})$ is called the horizontal standard vector field defined by $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2.1.7 *For any $\mathbf{x} \in \mathbb{R}^n$, the horizontal standard vector field fulfils*

1. $\theta(B(\mathbf{x})) = \mathbf{x}$,
2. $\Psi_{a*}B(\mathbf{x}) = B(a^{-1}\mathbf{x})$, $a \in \mathrm{GL}(n, \mathbb{R})$,
3. *if* $\mathbf{x} \neq 0$, *then* $B(\mathbf{x})$ *vanishes nowhere.*

Proof 1. We calculate

$$\theta_u(B(\mathbf{x})) = u^{-1}(\pi'(B(\mathbf{x})_u)) = u^{-1}(u(\mathbf{x})) = \mathbf{x}\,.$$

2. By Proposition 2.1.4 and point 1, we have

$$\theta(\Psi_{a*}B(\mathbf{x})) = \Psi_a^*\theta(B(\mathbf{x})) = a^{-1}\theta(B(\mathbf{x})) = a^{-1}\mathbf{x}\,,$$

and, thus, $\pi'(\Psi_{a*}B(\mathbf{x})) = u(a^{-1}\mathbf{x})$. Since $\Psi_{a*}B(\mathbf{x})$ is horizontal, the assertion follows from the uniqueness of the horizontal lift.

3. Clearly, $B(\mathbf{x})_u = 0$ iff $u(\mathbf{x}) = 0$ and, thus, iff $\mathbf{x} = 0$, because $u : \mathbb{R}^n \to \mathrm{T}_{\pi(u)}M$ is a vector space isomorphism. ■

Remark 2.1.8 Let $\{\mathbf{e}_i\}$ be the standard basis in $\mathbb{R}^n$. Then, the horizontal standard vector fields $B_i = B(\mathbf{e}_i)$ span the horizontal distribution defined by Γ. Moreover, $B(\mathbf{x})$ is uniquely determined by the conditions

$$\theta(B(\mathbf{x})) = \mathbf{x}\,, \quad \omega(B(\mathbf{x})) = 0\,. \tag{2.1.8}$$

♦

Lemma 2.1.9 *Let A_* be the Killing vector field on $L(M)$ generated by $A \in \mathfrak{gl}(n,\mathbb{R})$ and let $\mathbf{x} \in \mathbb{R}^n$. Then,*

$$[A_*, B(\mathbf{x})] = B(A\mathbf{x})\,. \tag{2.1.9}$$

Proof Let $a_t = \exp(tA)$. Using point 2 of Proposition 2.1.7, we obtain

$$[A_*, B(\mathbf{x})]_u = (\mathscr{L}_{A_*}B(\mathbf{x}))_u = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 \Big(\big(\Psi_{a_t^{-1}}\big)_* B(\mathbf{x})\Big)_u = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 B(a_t\mathbf{x})_u = B(A\mathbf{x})_u\,.$$

■

Definition 2.1.10 Let Γ be a linear connection on M and let ω be its connection form. The 2-form $\Theta \in \Omega^2(L(M),\mathbb{R}^n)$ defined by

$$\Theta := D_\omega\theta \tag{2.1.10}$$

is called the torsion form of Γ.

Clearly, Θ is a horizontal 2-form of type σ_n^0. The Structure Equation (1.4.9) for the curvature of a linear connection is supplemented by a structure equation involving the torsion form.

Proposition 2.1.11 (Structure Equations) *Let ω, Ω and Θ be, respectively, the connection, curvature and torsion forms of a linear connection Γ on M. Then, for any $X, Y \in \mathrm{T}_uL(M)$,*

$$\mathrm{d}\omega(X, Y) = -[\omega(X), \omega(Y)] + \Omega(X, Y)\,, \tag{2.1.11}$$
$$\mathrm{d}\theta(X, Y) = -\big(\omega(X)\theta(Y) - \omega(Y)\theta(X)\big) + \Theta(X, Y)\,. \tag{2.1.12}$$

Proof Equation (2.1.11) coincides with the Structure Equation (1.4.9) of the general theory. Since θ is a horizontal form, (2.1.12) follows immediately from formula (1.4.1), with σ being the basic representation. ∎

Remark 2.1.12 Using

$$\omega \wedge \theta(X, Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X)\,,$$

the Structure Equations may be rewritten as follows:

$$\mathrm{d}\omega = -\omega \wedge \omega + \Omega\,, \quad \mathrm{d}\theta = -\omega \wedge \theta + \Theta\,. \tag{2.1.13}$$

If we decompose the above forms with respect to the standard bases $\{\mathbf{e}_i\}$ in $\mathbb{R}^n$ and $\{E^i{}_j\}$ in $\mathfrak{gl}(n, \mathbb{R})$,

$$\theta = \theta^i \mathbf{e}_i\,, \quad \Theta = \Theta^i \mathbf{e}_i\,, \quad \omega = \omega^i{}_j E^j{}_i\,, \quad \Omega = \Omega^i{}_j E^j{}_i\,, \tag{2.1.14}$$

then we obtain the Structure Equations in the form

$$\mathrm{d}\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \Omega^i{}_j\,, \quad \mathrm{d}\theta^i = -\omega^i{}_j \wedge \theta^j + \Theta^i\,. \tag{2.1.15}$$

♦

The Bianchi identity for the curvature has a counterpart for the torsion.

Proposition 2.1.13 (Bianchi Identities) *Let ω, Ω and Θ be, respectively, the connection, curvature and torsion forms of a linear connection Γ on M. Then,*

$$D_\omega \Omega = 0\,, \tag{2.1.16}$$
$$D_\omega \Theta = \Omega \wedge \theta\,. \tag{2.1.17}$$

Proof Equation (2.1.16) coincides with the Bianchi Identity (1.4.10) of the general theory. Equation (2.1.17) is an immediate consequence of Proposition 1.4.12, with $\sigma = \sigma_n^0$. ∎

Alternatively, (2.1.17) may be checked by direct inspection. It is obtained by differentiating the first of the two equations in (2.1.15) and by using both of these equations thereafter (Exercise 2.1.5).

Remark 2.1.14

1. The 1-forms ω and θ may be combined to the joint object

$$\omega + \theta \in \Omega^1(L(M), \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n)\,.$$

Clearly, $\mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n$ is the Lie algebra of the affine group on $\mathbb{R}^n$. Its commutation relations are obtained by supplementing the commutation relations of $\mathfrak{gl}(n, \mathbb{R})$ by

$$[A, \mathbf{x}] = -[\mathbf{x}, A] = A\mathbf{x}\,, \quad [\mathbf{x}, \mathbf{y}] = 0\,, \quad A \in \mathfrak{gl}(n, \mathbb{R})\,, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\,.$$

Accordingly, we may pass from the bundle $L(M)$ of linear frames to the bundle $A(M)$ of affine frames. Clearly, $\omega + \theta$ defines a connection form on $A(M)$ which is called the affine connection form induced by ω. This explains why linear connection and affine connection are often used as synonyms in the literature. Obviously,

$$D_{\omega+\theta}(\omega + \theta) = \mathrm{d}(\omega + \theta) + \frac{1}{2}[\omega + \theta, \omega + \theta] = \Omega + \Theta\,,$$

that is, curvature and torsion constitute a joint object on $A(M)$, namely the curvature of $\omega + \theta$.

2. Let $\{\mathbf{e}_i\}$ and $\{E^j{}_i\}$ be the standard bases of $\mathbb{R}^n$ and $\mathfrak{gl}(n, \mathbb{R})$, respectively. Let B_i be the horizontal standard vector field with respect to a chosen connection Γ generated by $\mathbf{e}_i$ and let $E^j{}_{i*}$ be the Killing vector field generated by $E^j{}_i$. Since the $E^j{}_{i*}$ span the vertical subspace $V_u \subset \mathrm{T}_u L(M)$, for every $u \in L(M)$, and since the $\{B_i\}$ span the (complementary) Γ-horizontal subspace Γ_u, these $n^2 + n$ vector fields provide a global frame in the tangent bundle $\mathrm{T}L(M)$ which is, therefore, trivial. One says that the manifold $L(M)$ admits a global parallelism given by the vector fields B_i, $E^j{}_{i*}$. Moreover, the vector fields B_i, $E^j{}_{i*}$ are dual to the 1-forms θ^i, $\omega^i{}_j$,

$$\begin{aligned} \theta^k(B_i) &= \delta^k{}_i\,, & \theta^k(E^j{}_{i*}) &= 0\,, \\ \omega^k{}_l(B_i) &= 0\,, & \omega^k{}_l(E^j{}_{i*}) &= \delta^k{}_i \delta^j{}_l\,. \end{aligned} \tag{2.1.18}$$

Thus, $\mathrm{T}^*L(M)$ is trivial, too, and the 1-forms $\theta^i, \omega^i{}_j$ provide a global frame of $\mathrm{T}^*L(M)$, or, in more abstract terms, the affine connection $\omega + \theta$ induces an absolute parallelism on $A(M)$. As a consequence, every horizontal k-form α on $L(M)$ may be expanded with respect to the 1-forms θ^i,

$$\alpha = \frac{1}{k!}\alpha_{i_1 \ldots i_k}\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\,. \tag{2.1.19}$$

In particular,

$$\Omega^i{}_j = \tfrac{1}{2}\Omega^i_{klj}\,\theta^k \wedge \theta^l\,, \quad \Theta^i = \tfrac{1}{2}\Theta^i_{jk}\,\theta^j \wedge \theta^k\,. \tag{2.1.20}$$

♦

Since both Ω and Θ are horizontal 2-forms on $L(M)$ of type Ad, respectively, they uniquely correspond to 2-forms on M with values in certain associated vector bundles. By Proposition 1.2.12 and by the isomorphism (2.1.3), to $\Theta \in \Omega^2(L(M), \mathbb{R}^n)$

there corresponds an element $\mathsf{T} \in \Omega^2(M, \mathrm{T}M)$ defined by

$$\mathsf{T}_m(X, Y) = u(\Theta_u(X^*, Y^*)) , \tag{2.1.21}$$

where $X, Y \in \mathrm{T}_m M$, $\pi(u) = m$ and $X^*, Y^* \in \mathrm{T}_u L(M)$ fulfilling $\pi'(X^*) = X$ and $\pi'(Y^*) = Y$.[4] By Remark 1.4.7, to Ω there corresponds a 2-form on M with values in the adjoint bundle $\mathrm{Ad}(L(M))$. Since the differential of the basic representation σ_n^0 identifies $\mathfrak{gl}(n, \mathbb{R})$ naturally with $\mathrm{End}(\mathbb{R}^n)$, this 2-form may be identified with the curvature endomorphism form $\mathsf{R} \in \Omega^2(M, \mathrm{End}(\mathrm{T}M))$,

$$\mathsf{R}_m(X, Y) = u \circ \Omega_u(X^*, Y^*) \circ u^{-1} , \tag{2.1.22}$$

cf. (1.5.13). Since R takes values in $\mathrm{End}(\mathrm{T}M)$, we may apply it to any tangent vector $Z \in \mathrm{T}_m M$:

$$\mathsf{R}_m(X, Y)Z = u\big(\Omega_u(X^*, Y^*)(u^{-1}Z)\big) . \tag{2.1.23}$$

Definition 2.1.15 Let Γ be a linear connection on $L(M)$ and let Θ and Ω be its curvature and torsion forms. The 2-forms T and R defined by (2.1.21) and (2.1.22) are called the torsion tensor field associated with Θ and the curvature tensor field associated with Ω, respectively.

Remark 2.1.16 Since, for any $u \in L(M)$, the assignment $\mathbb{R}^n \to \Gamma_u$, $\mathbf{x} \mapsto B(\mathbf{x})$, is an isomorphism of vector spaces, we have an induced isomorphism

$$b(u) : \bigwedge^2 \mathbb{R}^n \to \bigwedge^2 \Gamma_u , \quad b(u)(\mathbf{x} \wedge \mathbf{y}) = B(\mathbf{x})_u \wedge B(\mathbf{y})_u .$$

Using this, we get yet another presentation of curvature and torsion, which will turn out to be useful. We define mappings

$$\mathscr{R} : L(M) \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathfrak{gl}(n, \mathbb{R}) , \quad \mathscr{T} : L(M) \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

by

$$\mathscr{R}(u) := \Omega_u \circ b(u) , \quad \mathscr{T}(u) := \Theta_u \circ b(u) . \tag{2.1.24}$$

In the sequel, $\mathscr{R}$ and $\mathscr{T}$ will be referred to as the curvature and the torsion mappings, respectively. Using that Ω and Θ are horizontal forms of type Ad and σ_n^0, respectively, together with (1.2.3), one finds:

$$\mathscr{R}(\Psi_a(u))(\mathbf{x}, \mathbf{y}) = \mathrm{Ad}(a^{-1}) \circ \big(\mathscr{R}(u)(a\mathbf{x}, a\mathbf{y})\big) , \tag{2.1.25}$$

$$\mathscr{T}(\Psi_a(u))(\mathbf{x}, \mathbf{y}) = a^{-1} \circ \big(\mathscr{T}(u)(a\mathbf{x}, a\mathbf{y})\big) . \tag{2.1.26}$$

By Proposition 1.2.6, to $\mathscr{R}$ and $\mathscr{T}$, there correspond unique sections of the associated bundles

[4] Clearly, for X^* and Y^* we may take the horizontal lifts of X and Y with respect to Γ.

$$L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \left(\bigwedge^2(\mathbb{R}^n)^* \otimes \mathfrak{gl}(n,\mathbb{R})\right), \quad L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \left(\bigwedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n\right),$$

respectively. By (2.1.24), they are given by

$$m \mapsto u \circ \mathscr{R}(u) \circ u^{-1} = \mathsf{R}_u \circ \bigwedge^2 u, \quad m \mapsto u \circ \mathscr{T}(u) = \mathsf{T}_u \circ \bigwedge^2 u, \tag{2.1.27}$$

where $\bigwedge^2 u : \mathbb{R}^n \wedge \mathbb{R}^n \to \mathrm{T}_{\pi(u)}M \wedge \mathrm{T}_{\pi(u)}M$ and $m = \pi(u)$. ♦

Next, we discuss the covariant derivative of tensor fields and apply the Koszul calculus developed in Sect. 1.5 to the case under consideration. By Definition 1.5.2, the covariant derivative

$$\nabla^\omega = (\mathrm{d}_\omega)_{\restriction \Omega^0(M,E)} : \ \Gamma^\infty(E) \to \Gamma^\infty(\mathrm{T}^*M \otimes E)$$

on an associated bundle $E = P \times_G F$, induced from a connection form ω, is given by

$$(\nabla^\omega \Phi)_m(X) = \iota_p \circ (D_\omega \tilde{\Phi})_p(X^*), \tag{2.1.28}$$

with $\pi(p) = m$ and $X^* \in \mathrm{T}_p P$ fulfilling $\pi'(X^*) = X$. Applying this to a section Y of $TM \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$, that is, to a vector field on M, we read off

$$(\nabla^\omega Y)_m(X) = u \circ (D_\omega \tilde{Y})_u(X^*), \quad \pi(u) = m, \tag{2.1.29}$$

where $\tilde{Y} \in \mathrm{Hom}_{\mathrm{GL}(n,\mathbb{R})}(L(M), \mathbb{R}^n)$ is given by $Y(m) = u \circ \tilde{Y}(u)$. According to (1.5.10), we have an associated operator

$$\nabla^\omega_X \colon \Gamma^\infty(\mathrm{T}M) \to \Gamma^\infty(\mathrm{T}M), \quad \nabla^\omega_X Y := (\nabla^\omega Y)(X). \tag{2.1.30}$$

In the sequel, we assume that a connection has been chosen and, for simplicity, we write ∇ instead of ∇^ω.

Remark 2.1.17

1. By (1.5.3), formula (2.1.29) may be rewritten as $(\nabla_X Y)(m) = u(X^*_u(\tilde{Y}))$, where X^* is the horizontal lift of X. Thus, using

$$\theta_u(Y^*) = u^{-1} \circ \pi'(Y^*) = u^{-1} Y_m = \tilde{Y}_u,$$

we obtain

$$(\nabla_X Y)(m) = u(X^*_u(\theta(Y^*))). \tag{2.1.31}$$

2. Clearly, the covariant derivative ∇_X given by (2.1.30) has all the properties listed in Proposition 1.5.8. Moreover, it induces covariant derivatives in all tensor bundles over M. A general formula is easily derived from (1.4.2) by taking for σ the tensor

product representation of p copies of σ_n^0 and q copies of its dual, cf. Exercise 2.1.2. If not otherwise stated, by ∇ we mean the covariant derivative in $\mathrm{T}M$. ♦

The proof of the following proposition is left to the reader (Exercise 2.1.3). It provides an axiomatic characterization of the covariant derivative of a tensor field.

Proposition 2.1.18 *Let Γ be a linear connection on a manifold M and let ∇ be its covariant derivative in* $\mathrm{T}M$. *Then, the covariant derivative*

$$\nabla_X : \Gamma^\infty(\mathrm{T}^r_s M) \to \Gamma^\infty(\mathrm{T}^r_s M)\,,$$

acting on tensor fields of type (r, s) is uniquely determined by the following properties.

1. $\nabla_X f = X(f)$ *, for $f \in C^\infty(M)$.*
2. ∇_X *is a derivation of the tensor algebra.*
3. ∇_X *commutes with any contraction.*

We express the curvature and torsion tensor fields in terms of the covariant derivative.

Proposition 2.1.19 *Let ∇ be the covariant derivative of a linear connection Γ on M. Then, the curvature and the torsion tensor fields of Γ are given by*

$$\mathsf{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,, \tag{2.1.32}$$

$$\mathsf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\,. \tag{2.1.33}$$

Proof Formula (2.1.32) follows from Proposition 1.5.11 as a special case. To prove formula (2.1.33), let X^*, Y^* be the horizontal lifts of X and Y. Then, $\Theta(X^*, Y^*) = \mathrm{d}\theta(X^*, Y^*)$. Using this, together with (2.1.31) and $\pi'([X^*, Y^*]) = [X, Y]$, we obtain

$$\begin{aligned}\mathsf{T}(X, Y)(m) &= u(\Theta_u(X^*, Y^*))\\ &= u(X^*_u(\theta(Y^*)) - Y^*_u(\theta(X^*)) - \theta_u([X^*, Y^*]))\\ &= (\nabla_X Y - \nabla_Y X - [X, Y])(m)\,.\end{aligned}$$

■

Finally, we carry over the concept of parallel transport and holonomy as developed in Sect. 1.7 to the case of linear connections on M. In this way, for a given connection, we obtain the operation of parallel transport along curves in M both for the frame bundle $L(M)$ and for any associated tensor bundle $\mathrm{T}^r_s M$. Correspondingly, we obtain holonomy groups in all associated tensor bundles. As in the general theory, there is a deep relation between holonomy and curvature, provided by the Ambrose-Singer Theorem 1.7.15. This has tremendous consequences for the structure theory of (pseudo-)Riemannian manifolds, see Sect. 2.3.

Clearly, comparing with the general theory, the situation here is special in so far as the parallel transport operators apply to geometric objects living on the base manifold M. Related to this fact, there is a special class of curves which we discuss next. Applying the theory to the tangent bundle, for any curve $\gamma : I \to M$, we obtain a unique parallel transport of tangent vectors along γ. In the sequel, let $I \subset \mathbb{R}$ denote an open interval containing 0. Let $\dot{\gamma}$ be the tangent vector field of γ. By Example I/1.5.5, it is given by

$$\dot{\gamma}(t) = \gamma'_t\left(\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_t}\right),$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the unit vector field on I. Applying the notions developed in Sect. 1.7, a vector field X on M is parallel (with respect to a connection Γ) along a curve γ if

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} X = 0\,. \tag{2.1.34}$$

Here, ∇^{γ} is the covariant derivative along the mapping γ and X must be viewed as a section of $\mathrm{T}M$ along γ.[5] In particular, since $\dot{\gamma}$ is certainly a section of $\mathrm{T}M$ along γ, we may consider the equation

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} \dot{\gamma} = 0 \tag{2.1.35}$$

and we may ask whether it admits solutions.

Definition 2.1.20 Let Γ be a linear connection. A curve $\gamma : I \to M$, $t \mapsto \gamma(t)$, is called a geodesic with respect to Γ if it fulfils equation (2.1.35).

The following proposition is left as an exercise to the reader (Exercise 2.1.4).

Proposition 2.1.21 *If a curve $\gamma : I \to M$ is a geodesic, then for any $\alpha, \beta \in \mathbb{R}$ the curve $t \mapsto \gamma(\alpha \cdot t + \beta)$ is a geodesic, too.* ■

Proposition 2.1.22 *Let Γ be a linear connection on M. Then, the projection under $\pi : L(M) \to M$ of any integral curve of a horizontal standard vector field is a geodesic. Conversely, every geodesic is obtained in this way.*

Proof Let $\mathbf{x} \in \mathbb{R}^n$. By definition, $B(\mathbf{x})_u$ is the unique Γ-horizontal lift of $u(\mathbf{x}) \in \mathrm{T}_{\pi(u)}M$ to $u \in L(M)$. Let $t \mapsto \tilde{\gamma}(t)$ be an integral curve of $B(\mathbf{x})$. Define $\gamma := \pi \circ \tilde{\gamma}$. Then, using the natural identification (2.1.2) and omitting φ,

$$\dot{\gamma}(t) = \pi' \circ \dot{\tilde{\gamma}}(t) = \pi'(B(\mathbf{x})_{\tilde{\gamma}(t)}) = \tilde{\gamma}(t)(\mathbf{x}) = \iota_{\mathbf{x}}(\tilde{\gamma}(t))\,,$$

where $\tilde{\gamma}(t) : \mathbb{R}^n \to \mathrm{T}_{\gamma(t)}M$ as usual. Thus, by (1.7.13) and (1.3.4), we have

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} \dot{\gamma} = \omega^E\left(\iota'_{\mathbf{x}}(\dot{\tilde{\gamma}}(t))\right) = 0\,.$$

Conversely, let $\gamma : I \to M$ be a geodesic. Let $u_0 \in L(M)$ be such that $\pi(u_0) = \gamma(0)$ and let $\mathbf{x} := u_0^{-1}(\dot{\gamma}(0)) \in \mathbb{R}^n$. Let $t \mapsto \tilde{\gamma}(t)$ be the horizontal lift of γ through u_0.

[5] That is, more precisely, we should write $X \circ \gamma$ instead of X.

If $\mathbf{x} = 0$, we are done. Thus, let $\mathbf{x} \neq 0$. Then, there exists a curve $t \to \sigma(t)$ in $L(M)$ such that $\dot{\gamma}(t) = \sigma(t)(\mathbf{x})$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t) = \iota'_{\mathbf{x}}\dot{\sigma}(t) \,.$$

Since γ is a geodesic, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t) \in \Gamma^{\mathrm{T}M} \subset \mathrm{T}(\mathrm{T}M)$, this formula implies that $t \mapsto \sigma(t)$ is horizontal in $L(M)$. Since $\sigma(0) = u_0$ and $\pi \circ \sigma = \gamma$, uniqueness of the horizontal lift implies $\sigma = \tilde{\gamma}$. Thus, $\dot{\gamma}(t) = \tilde{\gamma}(t)(\mathbf{x})$ and, since $\tilde{\gamma}$ is horizontal,

$$\theta(\dot{\tilde{\gamma}}(t)) = \tilde{\gamma}(t)^{-1}(\pi'(\dot{\tilde{\gamma}}(t))) = \tilde{\gamma}(t)^{-1}(\dot{\gamma}(t)) = \mathbf{x} \,.$$

Thus, $t \mapsto \tilde{\gamma}(t)$ is an integral curve of $B(\mathbf{x})$. ■

Corollary 2.1.23 *Let Γ be a connection on M. For every $m \in M$ and every $X \in \mathrm{T}_m M$, there exists a unique geodesic $\gamma : I \to M$ with initial conditions (m, X), that is, $\gamma(0) = m$ and $\dot{\gamma}(0) = X$.* ■

We say that a linear connection Γ on M is complete if every geodesic of Γ may be extended to $I = \mathbb{R}$. Then, we have another corollary following immediately from Proposition 2.1.22.

Corollary 2.1.24 *A linear connection on M is complete iff every horizontal standard vector field on $L(M)$ is complete.* ■

If M is endowed with a complete linear connection Γ, we may define the following mapping. For every $m \in M$ and every $X \in \mathrm{T}_m M$, we take the unique geodesic γ with initial conditions $(\gamma(0) = m, \dot{\gamma}(0) = X)$ and put

$$\exp : \mathrm{T}M \to M \,, \quad \exp(X) := \gamma(1) \,. \tag{2.1.36}$$

This mapping is called the exponential mapping of Γ.

Remark 2.1.25 If Γ is not complete, then exp may still be defined. In this case, one defines exp on a neighbourhood of the zero section in $\mathrm{T}M$. This way, one obtains a smooth mapping which, for every $m \in M$, yields a local diffeomorphism from a neighbourhood of the origin in $\mathrm{T}_m M$ onto a neighbourhood U_m of m in M, see Fig. 2.1. For details, we refer to Propositions 8.1 and 8.2 in Chap. III of [381]. ♦

In the remainder of this section, we describe the above structures locally. Thus, let

$$m \mapsto \mathfrak{e}(m) = (e_1(m), \ldots, e_n(m))$$

be a local section of $L(M)$, that is, a local frame of $\mathrm{T}M$, and let

$$m \mapsto \vartheta(m) = (\vartheta^i(m), \ldots \vartheta^n(m))$$

be its dual coframe. Recall that $\mathfrak{e}(m)(\mathbf{e}_i) = e_i(m)$ for the standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$.

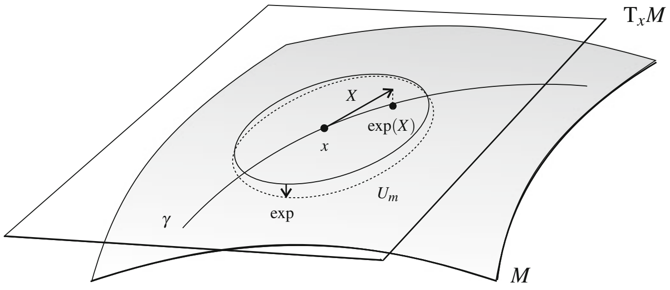

Fig. 2.1 Exponential mapping

Lemma 2.1.26 *For any local frame* $\mathfrak{e}$,

$$\mathfrak{e}^*\theta = \vartheta^i \otimes \mathbf{e}_i \,. \tag{2.1.37}$$

Proof For any $X \in \mathrm{T}_m M$, we calculate

$$(\mathfrak{e}^*\theta)_m(X) = \theta_{\mathfrak{e}(m)}(\mathfrak{e}'(X)) = (\mathfrak{e}(m))^{-1}(\pi' \circ \mathfrak{e}'(X)) = (\mathfrak{e}(m))^{-1}(X) \,.$$

Thus, decomposing $X = X^i e_i(m)$ and using $\mathfrak{e}(m)(\mathbf{e}_i) = e_i(m)$, we obtain

$$(\mathfrak{e}^*\theta)_m(X) = X^i(m)\mathbf{e}_i = \vartheta^i_m(X)\mathbf{e}_i \,.$$ ■

Thus, for the components of θ with respect to the decomposition (2.1.14),

$$\mathfrak{e}^*\theta^i = \vartheta^i \,. \tag{2.1.38}$$

Next, the local representative $\mathscr{A} = \mathfrak{e}^*\omega$ of a linear connection Γ with connection form ω is a 1-form on M with values in $\mathfrak{gl}(n, \mathbb{R})$. Thus, it may be written as

$$\mathscr{A} = \mathscr{A}^i{}_k E^k{}_i = \Gamma^i{}_{jk}\, \vartheta^j \otimes E^k{}_i \,. \tag{2.1.39}$$

The coefficient functions $\Gamma^i{}_{jk}$ are called the Christoffel symbols of Γ in the local frame $\mathfrak{e}$.

Remark 2.1.27 Consider a change $\mathfrak{e} \to \mathfrak{e}'$ of the local frame.[6] Using (1.3.15), we obtain the following induced transformation formula for the Christoffel symbols (Exercise 2.1.6)

$$\Gamma'^l{}_{mn} = \Gamma^i{}_{jk}\, \rho^j{}_m \rho^k{}_n (\rho^{-1})^l{}_i + \rho^j{}_m \big(\partial_j\, \rho^i{}_n\big)(\rho^{-1})^l{}_i \,. \tag{2.1.40}$$

♦

[6] We emphasize the passive interpretation here, but formula (2.1.40) may also be interpreted actively.

Let us calculate the local representatives of curvature and torsion. For that purpose, we take the pullback of (2.1.20) under $\mathfrak{e}$,

$$\mathfrak{e}^*\Omega^i{}_j = \tfrac{1}{2}\left(\mathfrak{e}^*\Omega^i_{klj}\right)\vartheta^k \wedge \vartheta^l\,, \quad \mathfrak{e}^*\Theta^i = \tfrac{1}{2}\left(\mathfrak{e}^*\Theta^i_{jk}\right)\vartheta^j \wedge \vartheta^k\,, \tag{2.1.41}$$

and denote the local coefficient functions as follows:

$$\mathsf{R}^i{}_{klj} = \mathfrak{e}^*\Omega^i_{klj}\,, \quad \mathsf{T}^i{}_{jk} = \mathfrak{e}^*\Theta^i_{jk}\,.$$

To calculate them, we use the Structure Equations in the form given by (2.1.15). Taking the pullback of the first equation yields

$$\tfrac{1}{2}\mathsf{R}^i{}_{klj}\,\vartheta^k \wedge \vartheta^l = \mathrm{d}\mathscr{A}^i{}_j + \mathscr{A}^i{}_k \wedge \mathscr{A}^k{}_j\,.$$

Inserting (2.1.39) into this equation, we obtain (Exercise 2.1.7)

$$\mathsf{R}^i{}_{jkl} = e_j(\Gamma^i{}_{kl}) - e_k(\Gamma^i{}_{jl}) + \Gamma^m{}_{kl}\Gamma^i{}_{jm} - \Gamma^m{}_{jl}\Gamma^i{}_{km} - C^m{}_{jk}\Gamma^i{}_{ml}\,, \tag{2.1.42}$$

where the $C^i{}_{jk}$ are the structure functions of the local frame $\mathfrak{e}$ defined by

$$[e_j, e_k] = C^i{}_{jk}e_i\,. \tag{2.1.43}$$

In the same way, taking the pullback of the second equation in (2.1.15), we read off

$$\mathsf{T}^i{}_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj} - C^i{}_{jk}\,. \tag{2.1.44}$$

Next, by Proposition 1.5.3, the local version of the Koszul calculus is based upon the following formula. For a local frame $\mathfrak{e}$, we have

$$\nabla e_j = \Gamma^k{}_{ij}\vartheta^i \otimes e_k\,. \tag{2.1.45}$$

Correspondingly,

$$\nabla_{e_i} e_j = \Gamma^k{}_{ij}\, e_k\,. \tag{2.1.46}$$

Next, acting with ∇_{e_i} on the pairing $\vartheta^j(e_k) = \delta^j{}_k$ and using that the covariant derivative is a derivation of the tensor algebra, we obtain

$$\nabla_{e_i}\vartheta^j = -\Gamma^j{}_{ik}\,\vartheta^k\,. \tag{2.1.47}$$

Thus,

$$\nabla\vartheta^j = -\Gamma^j{}_{ik}\,\vartheta^i \otimes \vartheta^k\,. \tag{2.1.48}$$

Now, decomposing an arbitrary tensor field with respect to a local frame $\mathfrak{e}$ and its dual coframe ϑ and using (2.1.46) and (2.1.47), together with the properties of the covariant derivative, one can derive a local formula for the covariant derivative of any tensor field, see Exercise 2.1.7. In particular, for a vector field X and a 1-form α we obtain

$$\nabla_{e_i} X = \big(e_i(X^k) + \Gamma^k{}_{ij} X^j\big) e_k\,, \tag{2.1.49}$$
$$\nabla_{e_i} \alpha = \big(e_i(\alpha_j) - \Gamma^k{}_{ij} \alpha_k\big) \vartheta^j\,. \tag{2.1.50}$$

Using (1.5.8), we get $\nabla X = \vartheta^i \otimes \nabla_{e_i} X$ and $\nabla \alpha = \vartheta^i \otimes \nabla_{e_i} \alpha$. Clearly, the covariant derivative of any tensor field t may also be decomposed in this way,

$$\nabla t = \vartheta^i \otimes \nabla_{e_i} t\,, \tag{2.1.51}$$

in accordance with the fact that $\nabla t \in \Omega^1(M, \mathbb{T}^k_l(M))$.

Remark 2.1.28 By point 2 of Remark 1.2.15, it is clear that the local representatives of Ω and R, as well as the local representatives of Θ and T, coincide. Thus,

$$\mathsf{R}\left(e_j, e_k\right) e_l = \mathsf{R}^i{}_{jkl} e_i\,, \quad \mathsf{T}\left(e_j, e_k\right) = \mathsf{T}^i{}_{jk} e_i\,. \tag{2.1.52}$$

This can also be checked by direct inspection, inserting (2.1.46) into (2.1.32) and (2.1.33) and comparing with (2.1.42) and (2.1.44) (Exercise 2.1.8). ♦

Remark 2.1.29 (*Holonomic frame*) Let (U, κ) be a local chart of M and let x^i be the corresponding local coordinates. Then, $\{\partial_j\}$ is a local frame of $\mathrm{T}M$, called the induced holonomic frame of $\mathrm{T}M$ and $\{\mathrm{d}x^j\}$ is the dual coframe of T^*M. The name holonomic refers to the fact that $[\partial_i, \partial_j] = 0$, that is, the structure functions of a holonomic frame vanish. In such a frame, the formulae (2.1.39), (2.1.42), (2.1.44) and (2.1.45) take the following form:

$$\mathscr{A} = \Gamma^i{}_{jk}\, \mathrm{d}x^j \otimes E^k{}_i\,, \tag{2.1.53}$$
$$\mathsf{R}^i_{jkl} = \partial_j\, \Gamma^i{}_{kl} - \partial_k\, \Gamma^i{}_{jl} + \Gamma^m{}_{kl} \Gamma^i{}_{jm} - \Gamma^m{}_{jl} \Gamma^i{}_{km}\,, \tag{2.1.54}$$
$$\mathsf{T}^i_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj}\,, \tag{2.1.55}$$
$$\nabla \partial_j = \Gamma^k{}_{ij}\, \mathrm{d}x^i \otimes \partial_k\,. \tag{2.1.56}$$

The change from one holonomic frame to another one is described by the Jacobi matrix of the coordinate transformation. Thus, here, the transition function is

$$x \mapsto \rho(x) = \left(\frac{\partial x^i}{\partial x'^l}\right)$$

and the transformation formula (2.1.40) reads

$$\Gamma'^l{}_{mn} = \Gamma^i{}_{jk} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \cdot \frac{\partial x'^l}{\partial x^i} + \frac{\partial^2 x^i}{\partial x'^m \partial x'^n} \frac{\partial x'^l}{\partial x^i}\,. \tag{2.1.57}$$

♦

It remains to analyze Eqs. (2.1.34) and (2.1.35) in local coordinates. Then, γ is given by $t \mapsto x^i(t)$ and, correspondingly, $X = X^i \partial_i$ and $\dot{\gamma} = \dot{x}^i \partial_i$. Using points 3 and 4 of Proposition 1.5.8 we calculate:

$$\nabla_{\dot{\gamma}} X = \nabla_{\dot{x}^i \partial_i} \left(X^j \partial_j \right) = \left(\dot{x}^i X^j \Gamma^k{}_{ij} + \partial_i \left(X^k \right) \dot{x}^i \right) \partial_k \, ,$$

that is, Eq. (2.1.34) reads

$$\frac{dX^k}{dt} + \Gamma^k{}_{ij} \dot{x}^i X^j = 0 \, . \tag{2.1.58}$$

This is a system of first order ordinary differential equations, which according to standard theorems admits unique local solutions depending smoothly on the initial values $(t_0, X(t_0))$. The solution $t \mapsto X(t)$ provides the parallel transport

$$\hat{\gamma}_{\Gamma^{\mathrm{T}M}}(t) \colon \mathrm{T}_{\gamma(t_0)} M \to \mathrm{T}_{\gamma(t)} M \, . \tag{2.1.59}$$

Inserting $X^i = \dot{x}^i$ into (2.1.58), we obtain the local form of the geodesic equation:

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} + \Gamma^k{}_{ij} \dot{x}^i \dot{x}^j = 0 \, . \tag{2.1.60}$$

This is a system of second order ordinary differential equations, which admits unique local solutions depending smoothly on the initial conditions $(t_0, x^i(t_0), \dot{x}^i(t_0))$.

Remark 2.1.30

1. Consider the exponential mapping of a linear connection Γ on M, cf. equation (2.1.36) and Remark 2.1.25. Via the exponential mapping, any frame $u : \mathbb{R}^n \to \mathrm{T}_m M$ at $m \in M$ provides a local chart on $\mathrm{T}_m M$:

$$\varphi := \exp \circ u : \mathbb{R}^n \to U_m \, .$$

   This is a local diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ onto a neighbourhood $U_m \subset M$ of m. Taking $\kappa := \varphi^{-1}$ we obtain a local chart (U, κ) centered at m which will be referred to as a local geodesic chart. The local coordinates x^i of that chart mapping will be called normal coordinates at m. In normal coordinates, any geodesic takes the form $x^i(t) = a^i \cdot t$. Thus, at m, we obviously have $\Gamma^k{}_{ij} + \Gamma^k{}_{ji} = 0$. That is, for vanishing torsion, the Christoffel symbols vanish at m (Exercise 2.1.9).
2. The parallel transport of a tangent vector along a closed curve yields a geometric interpretation of curvature. Note that this is in accordance with the Ambrose-Singer Theorem 1.7.15. We have (Exercise 2.1.9)

$$\triangle X^i = -\frac{1}{2} \mathsf{R}^i{}_{jkl} X^l \cdot f^{jk} \, , \tag{2.1.61}$$

   where f^{jk} is a bivector field characterizing the plane enclosed by γ.

3. The quantity

$$a^i := \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i{}_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t}$$

is the natural generalization of the notion of acceleration of a point particle to curved space. For $a^i = 0$, the particle moves on a geodesic. This occurs if the particle is not acted upon by additional (non-gravitational) external forces. ◆

Exercises

2.1.1 Prove that the mapping φ defined by (2.1.2) is an isomorphism of vector bundles.

2.1.2 Derive from (1.4.2) a formula for the covariant derivative of a tensor field t of type (r, s) by taking for σ the tensor product representation of s copies of σ_n^0 and r copies of its dual.

2.1.3 Prove Proposition 2.1.18.

2.1.4 Prove Proposition 2.1.21.

2.1.5 Prove equation (2.1.17) by a direct calculation using the Structure Equations.

2.1.6 Prove formula (2.1.40).

2.1.7 Prove the local formulae (2.1.42), (2.1.44), (2.1.49) and (2.1.50). Derive a local formula for the covariant derivative of an arbitrary tensor field t, cf. Exercise 2.1.2. Conclude that, in particular, in local coordinates the covariant derivative of t is given by

$$\nabla_{\partial_k} t^{i_1 \ldots i_s}_{j_1 \ldots j_r} = \partial_k \, t^{i_1 \ldots i_s}_{j_1 \ldots j_r} + \sum_l \Gamma^{i_l}_{km} t^{i_1 \ldots i_l = m \ldots i_s}_{j_1 \ldots j_r} - \sum_l \Gamma^m_{kj_l} t^{i_1 \ldots i_s}_{j_1 \ldots j_l = m \ldots j_r} \,.$$

2.1.8 Prove the statement of Remark 2.1.28.

2.1.9 Prove the statements of points 1 and 2 of Remark 2.1.30.

2.2 *H*-Structures and Compatible Connections

In the sequel, we will meet reductions of the frame bundle $L(M)$ to various Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$. The following concept allows for a unified treatment of all of them.

Definition 2.2.1 (*H-structure*) Let M be a smooth manifold.

1. A reduction P of the frame bundle $L(M)$ to a Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ is called an H-structure on M.
2. An H-structure P is called integrable if for every point $m \in M$ there exists a local chart (U, κ) with local coordinates x^j such that the induced holonomic frame $\{\partial_j\}$ is a local section of P. Such local coordinates are called admissible.
3. Let $\varphi : M \to M$ be a diffeomorphism. If $\varphi' : \mathrm{T}M \to \mathrm{T}M$ leaves P invariant, then φ is called an automorphism of the H-structure.

Clearly, the automorphisms of an H-structure form a group. By Corollary 1.6.5, reductions of $L(M)$ to a Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ are in one-to-one correspondence with smooth sections of the associated bundle

$$L(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathrm{GL}(n, \mathbb{R})/H)\,, \tag{2.2.1}$$

or, equivalently, with elements of $\mathrm{Hom}_{\mathrm{GL}(n,\mathbb{R})}(L(M), \mathrm{GL}(n, \mathbb{R})/H)$. Thus, the existence of an H-structure on a manifold M is a topological problem which can be dealt with by applying methods of obstruction theory. In particular, if $\mathrm{GL}(n, \mathbb{R})/H$ is contractible, then an H-structure certainly exists. Note that, geometrically, an H-structure should be viewed as a bundle of distinguished frames on M.

Recall from Definition 1.6.11 the general notion of compatible connection.

Definition 2.2.2 A linear connection on M is called compatible with the H-structure P if it is reducible to P.

Next, recall Proposition 1.6.10 characterizing the reducibility of connections on principal bundles in terms of G-homomorphisms.

Proposition 2.2.3 *Let P be an H-structure on M and let*

$$\tilde{\Phi} : L(M) \to \mathrm{GL}(n, \mathbb{R})/H$$

be the $\mathrm{GL}(n, \mathbb{R})$-equivariant mapping defining P. Assume that $\mathrm{GL}(n, \mathbb{R})/H$ embeds into a $\mathrm{GL}(n, \mathbb{R})$-module F. Then, a linear connection ω on $L(M)$ is compatible with the H-structure P iff $\tilde{\Phi}$ is parallel with respect to ω, that is, iff

$$D_\omega \tilde{\Phi} = 0\,.$$

Proof By the proof of Proposition 1.6.2, $P = \Big\{u \in L(M) : \tilde{\Phi}(u) = [\mathbb{1}]\Big\}$. Thus, the restriction of $D_\omega \tilde{\Phi} = 0$ to P reads

$$\sigma'(\omega)[\mathbb{1}] = 0\,,$$

which holds iff ω restricted to P takes values in the Lie algebra of H. This is equivalent to being reducible to P. ∎

Clearly, for a given H-structure P we may restrict the soldering form θ of $L(M)$ to P and, thus, for any connection ω on P we have a torsion 2-form Θ on P defined by (2.1.10). One says that ω is torsion-free if Θ vanishes.

Proposition 2.2.4 *If P is an integrable H-structure on M, then it admits a torsion-free connection.*

Proof Let $\pi : P \to M$ be the canonical projection. Let s be an integrable local section of P over $U \subset M$. Taking the tangent bundle of the graph of s and extending it using the right H-action to a distribution on P, we obtain a connection on $\pi^{-1}(U) \subset P$. Then, integrability implies $s^* \mathrm{d}\theta = 0$ (Exercise 2.2.1) and, thus, vanishing of the torsion. Next, we patch together these local connections to a connection on P using a partition of unity. Since torsion is additive this yields the assertion. ∎

Since any other connection ω' on P differs from ω by a horizontal 1-form α on P with values in the Lie algebra $\mathfrak{h}$ of H,

$$\Theta' = \Theta + \alpha \wedge \theta \,.$$

By Remark 2.1.16, Θ and α may be identified with H-equivariant functions

$$\mathscr{T} : P \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,, \quad \tilde{\alpha} : P \to (\mathbb{R}^n)^* \otimes \mathfrak{h} \,,$$

respectively. Since $H \subset \mathrm{GL}(n, \mathbb{R})$, we have a natural inclusion

$$\iota_{\mathfrak{h}} : \mathfrak{h} \to \mathrm{End}(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,.$$

Thus, under the above identification, $\alpha \wedge \theta$ is a function on P with values in $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We claim that it coincides with the image of $\tilde{\alpha}$ under the mapping

$$\delta : (\mathbb{R}^n)^* \otimes \mathfrak{h} \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,, \quad \delta := (a \otimes \mathrm{id}_{\mathbb{R}^n}) \circ (\mathrm{id}_{(\mathbb{R}^n)^*} \otimes \iota_{\mathfrak{h}}) \,, \tag{2.2.2}$$

where $a : (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \to \bigwedge^2 (\mathbb{R}^n)^*$ is the anti-symmetrization mapping. Indeed, using $\tilde{\alpha}(u)(\mathbf{x}) = \alpha\big(B(\mathbf{x})\big)$, we calculate

$$(\alpha \wedge \theta)_u\big(B(\mathbf{x}), B(\mathbf{y})\big) = \big(\tilde{\alpha}(u)(\mathbf{x})\big)\mathbf{y} - \big(\tilde{\alpha}(u)(\mathbf{y})\big)\mathbf{x} = \big(\delta \circ \tilde{\alpha}(u)\big)(\mathbf{x}, \mathbf{y}) \,.$$

As a result,

$$\mathscr{T}' = \mathscr{T} + \delta(\tilde{\alpha}) \,. \tag{2.2.3}$$

Let

$$\mathrm{pr} : \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \mathrm{coker}(\delta) = \Big(\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n\Big) / \mathrm{im}(\delta)$$

be the natural projection.[7] Then, the mapping

$$\tau : P \to \operatorname{coker}(\delta)\,, \quad \tau(u) := \operatorname{pr}(\mathscr{T}(u))\,, \tag{2.2.4}$$

does not depend on the choice of the connection. This motivates the following definition.

Definition 2.2.5 The mapping τ is called the intrinsic torsion of the H-structure P. Moreover, P is called torsion-free if τ vanishes.

Clearly, τ yields the obstruction to the existence of a torsion-free connection on P.

Proposition 2.2.6 *Let P be an H-structure. Then, the following hold.*

1. *If ω and ω' are torsion-free connections on P and $\omega' = \omega + \alpha$, then $\tilde{\alpha}(u) \in \ker\delta$ for every $u \in P$. In particular, if $\ker(\delta) = 0$, then P admits at most one torsion-free connection.*
2. *P has a torsion-free connection iff it is torsion-free.*

Proof The first assertion follows immediately from (2.2.3). For the second one, if P has a torsion-free connection, then it is clearly torsion-free. We prove the converse: let ω be a connection with (non-vanishing) torsion Θ. By assumption, $\tau = 0$. Thus, $\mathscr{T}(u) \in \operatorname{im}(\delta)$ for every $u \in P$. That is, there exists an equivariant mapping $\tilde{\alpha} : P \to (\mathbb{R}^n)^* \otimes \mathfrak{h}$ such that $\mathscr{T} = \delta(\tilde{\alpha})$. Let α be the unique horizontal 1-form on P corresponding to $\tilde{\alpha}$. Then, $\omega' = \omega - \alpha$ is a torsion-free connection. ■

In particular, as an immediate consequence, we obtain

Corollary 2.2.7 *If δ is bijective, then P admits a unique torsion-free connection.* ■

Next, let us discuss a number of relevant examples.

Example 2.2.8 (*Orientation*) We take $H = \mathrm{GL}_+(n, \mathbb{R})$. Then, $\mathrm{GL}(n, \mathbb{R})/H \cong \mathbb{Z}_2$. According to Example 1.6.6, a section of the associated bundle (2.2.1) exists iff the manifold is orientable, that is, iff the first Stiefel-Whitney class[8] of M vanishes. In this case, the H-structure consists of those frames which are compatible with a chosen orientation. Note that this H-structure is integrable. Also note that automorphisms of this H-structure are exactly the orientation-preserving diffeomorphisms of M. ♦

Example 2.2.9 (*Volume form*) We consider $H = \mathrm{SL}(n, \mathbb{R})$. The basic representation of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ induces the following $\mathrm{GL}(n, \mathbb{R})$-action on $\bigwedge^n(\mathbb{R}^n)^*$:

$$\mathrm{GL}(n, \mathbb{R}) \times \textstyle\bigwedge^n(\mathbb{R}^n)^* \to \bigwedge^n(\mathbb{R}^n)^*\,, \quad (a, \mathsf{v}) \mapsto \det(a) \cdot \mathsf{v}\,.$$

[7] The mapping δ and its cokernel have an interpretation in terms of Spencer cohomology of $\mathfrak{h}$ which we suppress here. For details, see e.g. [569].

[8] See Sect. 4.2.

Restricted to $\bigwedge^n(\mathbb{R}^n)^* \setminus \{0\}$, this action is transitive and has the common stabilizer $\mathrm{SL}(n,\mathbb{R})$. Thus,

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R}) \cong \bigwedge^n(\mathbb{R}^n)^* \setminus \{0\}\,.$$

Via the natural isomorphism $\bigwedge^n \mathrm{T}^*M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \bigwedge^n(\mathbb{R}^n)^*$, the sections of the associated bundle (2.2.1) are in one-to-one correspondence with volume forms on M. The $\mathrm{SL}(n,\mathbb{R})$-structure corresponding to a given volume form v consists of those frames u fulfilling

$$\mathsf{v} = \mathsf{v}_0 \circ \bigwedge^n u\,,$$

where v_0 is the canonical volume form on $\mathbb{R}^n$. Since $\mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R})$ is homotopy equivalent to $\mathrm{GL}(n,\mathbb{R})/\mathrm{GL}_+(n,\mathbb{R})$, M admits an $\mathrm{SL}(n,\mathbb{R})$-structure iff M is orientable. Moreover, it is easy to show that any $\mathrm{SL}(n,\mathbb{R})$-structure is integrable (Exercise 2.2.2). Finally, note that the automorphisms of this H-structure are the volume-preserving diffeomorphisms of M. ♦

Example 2.2.10 (*Almost complex structure*) Take $H = \mathrm{GL}(n,\mathbb{C})$ canonically embedded in $\mathrm{GL}(2n,\mathbb{R})$ via

$$a + ib \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a, b \in \mathrm{GL}(n,\mathbb{R})\,, \tag{2.2.5}$$

and consider the canonical complex structure on $\mathbb{R}^{2n}$ given by

$$\mathsf{J}_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{2.2.6}$$

Since $\mathrm{End}(\mathbb{R}^{2n}) \cong (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$, the basic representation of $\mathrm{GL}(2n,\mathbb{R})$ induces a $\mathrm{GL}(2n,\mathbb{R})$-module structure on $\mathrm{End}(\mathbb{R}^{2n})$ given by

$$\mathrm{GL}(2n,\mathbb{R}) \times \mathrm{End}(\mathbb{R}^{2n}) \to \mathrm{End}(\mathbb{R}^{2n})\,, \quad (g, A) \mapsto g^{-1}Ag\,.$$

Since $\mathrm{End}(\mathbb{R}^{2n})$ is the Lie algebra of $\mathrm{GL}(2n,\mathbb{R})$, this is merely the adjoint representation. Now, by Proposition I/7.1.2, the induced action of $\mathrm{GL}(2n,\mathbb{R})$ on the subset of endomorphisms fulfilling $A^2 = -\mathrm{id}$ is transitive and the stabilizer of J_0 is

$$H_{\mathsf{J}_0} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathrm{GL}(n,\mathbb{R}) \right\} = \mathrm{GL}(n,\mathbb{C})\,. \tag{2.2.7}$$

Thus,

$$\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}) \cong \left\{ A \in \mathrm{End}(\mathbb{R}^{2n}) : A^2 = -\mathrm{id} \right\}.$$

Thus, by (2.2.1), $\mathrm{GL}(n,\mathbb{C})$-structures are in one-to-one correspondence with sections J of $\mathrm{End}(\mathrm{T}M)$ fulfilling $\mathsf{J}_m^2 = -\mathrm{id}$ for every $m \in M$. A $\mathrm{GL}(n,\mathbb{C})$-structure will be referred to as an almost complex structure on M and (M, J) will be called an almost complex manifold. Since $\mathrm{End}(\mathbb{R}^{2n}) \cong (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$, J may be viewed as a tensor field on M of type $(1, 1)$. The $\mathrm{GL}(n, \mathbb{C})$-structure defined by J will be denoted by $C(M, \mathsf{J})$ and will be referred to as the bundle of complex linear frames. Note that it consists of frames fulfilling

$$u \circ \mathsf{J}_0 = \mathsf{J}_m \circ u\,, \tag{2.2.8}$$

where $u : \mathbb{R}^{2n} \to \mathrm{T}_m M$ as usual. It is easy to show that every almost complex manifold is orientable (Exercise 2.2.4). For a discussion of the obstructions to the existence of almost complex structures we refer to [431].

Next, let us discuss integrability. By (2.2.8), an almost complex structure (M, J) is integrable if M has the structure of a complex manifold such that for any system of admissible local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ we have

$$\mathsf{J}\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}\,, \quad \mathsf{J}\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}\,.$$

Then, $z^k := x^k + i y^k$ provide M with a local chart of complex coordinates. Conversely, we have

Proposition 2.2.11 *Viewed as a real C^∞-manifold, every complex manifold M carries a natural induced integrable almost complex structure.*

Proof Let $\{(U_i, \kappa_i)\}$ be a holomorphic atlas of M consisting of charts $\kappa_i : U_i \to \mathbb{C}^n$. For every i, we define an associated mapping $\tilde{\kappa}_i : U_i \to \mathbb{R}^{2n}$ given by

$$\tilde{\kappa}_i(m) := (\mathrm{Re}(\kappa_1(m)), \ldots, \mathrm{Re}(\kappa_n(m)), \mathrm{Im}(\kappa_1(m)), \ldots, \mathrm{Im}(\kappa_n(m)))\ ,$$

which clearly provides a C^∞-chart on U_i. Thus, $\{(U_i, \tilde{\kappa}_i)\}$ endows M with the structure of a real C^∞-manifold. Next, consider $\mathbb{R}^{2n}$ with the global coordinates $x^1, \ldots, x^n, y^1, \ldots y^n$. Then,

$$\mathsf{J}\left(\frac{\partial}{\partial x^k}\right) := \frac{\partial}{\partial y^k}\,, \quad \mathsf{J}\left(\frac{\partial}{\partial y^k}\right) := -\frac{\partial}{\partial x^k}\,,$$

clearly defines a complex structure on $\mathbb{R}^{2n}$. We transport this complex structure to M, viewed as a real manifold, via the local charts $\tilde{\kappa}_i$. The almost complex structure defined in this way is independent of the choice of the atlas, because the transition mappings are holomorphic and a mapping of an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$ leaves an almost complex structure on $\mathbb{C}^n$ invariant iff it is holomorphic (Exercise 2.2.3). By construction, the above almost complex structure is integrable. Indeed,

$$(\mathbf{x}, \mathbf{y}) \mapsto \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right)$$

provides a local section of the $\mathrm{GL}(n, \mathbb{C})$-structure defined by J. ∎

To summarize, an almost complex structure is integrable iff it is induced from a complex structure. The following notion provides a criterion for integrability.

Definition 2.2.12 Let (M, J) be an almost complex manifold. The Nijenhuis tensor of (M, J) is the tensor field $N \in \Gamma^\infty(\mathrm{T}^1_2(M))$ defined by

$$N(X, Y) := [\mathsf{J}X, \mathsf{J}Y] - [X, Y] - \mathsf{J}([X, \mathsf{J}Y]) - \mathsf{J}([\mathsf{J}X, Y]) , \quad X, Y \in \mathfrak{X}(M) .$$

The following deep theorem holds, see [485].

Theorem 2.2.13 (Newlander–Nirenberg) *An almost complex structure* J *is integrable iff the Nijenhuis tensor of* J *vanishes.* ■

Next, we show that J implies a natural splitting of tensor bundles over M. In particular, this will imply a variety of equivalent criteria for integrability. From now on, let $\mathrm{T} = \mathbb{R}^{2n}$ denote the basic $\mathrm{GL}(2n, \mathbb{R})$-module, let T^* be the dual (contragredient) module and let $\mathrm{T}_\mathbb{C}$ and $\mathrm{T}^*_\mathbb{C}$ be the complexifications of T and T^*, respectively. We extend J_0 to a $\mathbb{C}$-linear mapping of $\mathrm{T}_\mathbb{C}$ and decompose $\mathrm{T}_\mathbb{C}$ into eigenspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ corresponding to the eigenvalues i and $-i$ of J_0:

$$\mathrm{T}_\mathbb{C} = \mathrm{T}^{1,0} \oplus \mathrm{T}^{0,1} . \tag{2.2.9}$$

Then,

$$\mathrm{T}^{1,0} = \{X - i\mathsf{J}_0 X : X \in \mathrm{T}\} , \quad \mathrm{T}^{0,1} = \{X + i\mathsf{J}_0 X : X \in \mathrm{T}\} . \tag{2.2.10}$$

On the other hand, recall from Sect. 7.5 of Part I that J_0 endows T with the structure of a complex vector space, denoted by V, via

$$(a + ib)X := aX + b\mathsf{J}_0 X , \quad a, b \in \mathbb{R} , \ X \in \mathrm{T} . \tag{2.2.11}$$

Clearly, $V \cong \mathbb{C}^n$ carries the basic $\mathrm{GL}(n, \mathbb{C})$-module structure. Let ι be the natural embedding of V into $\mathrm{T}_\mathbb{C}$. Via this mapping, a chosen basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ in V induces a basis $(\mathbf{e}_1, \mathsf{J}_0\mathbf{e}_1, \dots, \mathbf{e}_n, \mathsf{J}_0\mathbf{e}_n)$ in $\mathrm{T}_\mathbb{C}$. By (2.2.11), for $Z = (X^k + iY^k)\mathbf{e}_k$ we have

$$\iota(Z) = X^k\mathbf{e}_k + Y^k\mathsf{J}_0\mathbf{e}_k . \tag{2.2.12}$$

Note that ι is not complex linear. Next, let $\mathrm{pr}^{1,0} : \mathrm{T}_\mathbb{C} \to \mathrm{T}^{1,0}$ and $\mathrm{pr}^{0,1} : \mathrm{T}_\mathbb{C} \to \mathrm{T}^{0,1}$ be the canonical projections. Then,

$$\mathrm{pr}^{1,0} \circ \iota : V \to \mathrm{T}^{1,0} , \quad \mathrm{pr}^{0,1} \circ \iota : V \to \mathrm{T}^{0,1}_m , \tag{2.2.13}$$

are $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear vector space isomorphisms, respectively (Exercise 2.2.6). Next, recall the embedding $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$ given by (2.2.5). It extends to $\mathrm{T}_\mathbb{C}$ by

$$\rho : \mathrm{GL}(n,\mathbb{C}) \times \mathrm{T}_{\mathbb{C}} \to \mathrm{T}_{\mathbb{C}}\,, \quad \rho(g)\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} aX - bY \\ bX + aY \end{bmatrix}.$$

One easily checks (Exercise 2.2.6) that for any $Z \in V$,

$$\mathrm{pr}^{1,0} \circ \rho(g) \circ \iota(Z) = (a + ib)Z\,, \quad \mathrm{pr}^{0,1} \circ \rho(g) \circ \iota(Z) = (a - ib)\overline{Z}\,. \tag{2.2.14}$$

On the other hand, the subspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ are invariant under the $\mathrm{GL}(n,\mathbb{C})$-action and, by (2.2.5), they carry the basic representation of $\mathrm{GL}(n,\mathbb{C})$ and its conjugate, respectively. It follows that V and $\mathrm{T}^{1,0}$ are isomorphic as $\mathrm{GL}(n,\mathbb{C})$-modules.

Next, note that, by duality, the decomposition (2.2.9) implies a decomposition

$$\mathrm{T}^*{}_{\mathbb{C}} = \mathrm{T}^{*1,0} \oplus \mathrm{T}^{*0,1}\,, \tag{2.2.15}$$

where $\mathrm{T}^{*1,0}$ and $\mathrm{T}^{*0,1}$ are the annihilators of $\mathrm{T}^{0,1}$ and $\mathrm{T}^{1,0}$, respectively. Thus, they carry the dual of the basic and the basic representation of $\mathrm{GL}(n,\mathbb{C})$, respectively. This decomposition induces the following decompositions:

$$\bigwedge{}^k \mathrm{T}^*{}_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge{}^{p,q}\,, \quad \bigwedge{}^{p,q} = \bigwedge{}^p \mathrm{T}^{*1,0} \otimes \bigwedge{}^q \mathrm{T}^{*0,1}\,. \tag{2.2.16}$$

Clearly, in analogy to (2.2.9) and (2.2.15), J induces decompositions

$$\mathrm{T}_{\mathbb{C}}M = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M\,, \quad \mathrm{T}^*{}_{\mathbb{C}}M = \mathrm{T}^{*1,0}M \oplus \mathrm{T}^{*0,1}M\,. \tag{2.2.17}$$

Note that, as a complex vector bundle, $\mathrm{T}M$ is $\mathbb{C}$-linearly isomorphic to $\mathrm{T}^{1,0}M$ via (2.2.13). Corresponding to (2.2.16), we have

$$\bigwedge{}^k \mathrm{T}^*{}_{\mathbb{C}}M = \bigoplus_{p+q=k} \bigwedge{}^{p,q}M\,, \quad \bigwedge{}^{p,q}M = \bigwedge{}^p \mathrm{T}^{*1,0}M \otimes \bigwedge{}^q \mathrm{T}^{*0,1}M\,. \tag{2.2.18}$$

The spaces of sections of $\bigwedge^k \mathrm{T}^*{}_{\mathbb{C}}M$ and $\bigwedge^{p,q}M$ will be denoted by $\Omega^k_{\mathbb{C}}(M)$ and by $\Omega^{p,q}(M)$, respectively. Elements of $\Omega^{p,q}(M)$ are called differential forms of type (p,q). Let us denote the projection to $\Omega^{p,q}(M)$ by $\Pi^{p,q}$. Extending the exterior differential $\mathbb{C}$-linearly, we may define mappings $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ via

$$\partial := \Pi^{p+1,q} \circ \mathrm{d}\,, \quad \overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}\,. \tag{2.2.19}$$

Proposition 2.2.14 *For an almost complex manifold, the following conditions are equivalent:*

1. $N(X,Y) = 0$ *for all* $X, Y \in \mathfrak{X}(M)$.
2. $\mathrm{T}^{1,0}M$ *is involutive.*
3. $\mathrm{d}\big(\Omega^{1,0}(M)\big) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$.

4. *For any* $\alpha \in \Omega^k_{\mathbb{C}}(M)$*, we have* $\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$.

Proof Recall that, as a real vector space, $\mathrm{T}_{\mathbb{C}}$ decomposes as $\mathrm{T}_{\mathbb{C}} = \mathrm{T} + \mathrm{i}\mathrm{T}$. Correspondingly, we have real linear projections $\mathrm{Re}, \mathrm{Im} : \mathrm{T}_{\mathbb{C}} \to \mathrm{T}$ defined by $W = \mathrm{Re}(W) + \mathrm{i}\mathrm{Im}(W)$ for all $W \in \mathrm{T}_{\mathbb{C}}$. Now, for any $X, Y \in \mathfrak{X}(M)$, we calculate

$$\begin{aligned} N(X,Y) &= [\mathsf{J}X, \mathsf{J}Y] - [X,Y] - \mathsf{J}([X,\mathsf{J}Y]) - \mathsf{J}([\mathsf{J}X,Y]) \\ &= -\mathrm{Re}\big([X - i\mathsf{J}X, Y - i\mathsf{J}Y] + i\mathsf{J}[X - i\mathsf{J}X, Y - i\mathsf{J}Y]\big) \\ &= -8\mathrm{Re}\big([X^{1,0}, Y^{1,0}]^{0,1}\big)\,. \end{aligned}$$

Since for elements $W \in \mathrm{T}^{0,1}$ we have $\mathrm{Im}(W) = \mathsf{J}(\mathrm{Re}(W))$, points 1 and 2 are equivalent. For $\alpha \in \Omega^{1,0}(M)$ and $X, Y \in \Gamma^\infty(\mathrm{T}^{1,0}M)$,

$$\mathrm{d}\overline{\alpha}(X,Y) = X(\overline{\alpha}(Y)) - Y(\overline{\alpha}(X)) - \overline{\alpha}([X,Y]) = -\overline{\alpha}([X,Y])\,,$$

where $\overline{\alpha} \in \Omega^{0,1}(M)$ defined by $\overline{\alpha}(W) = \alpha(\overline{W})$ with $\overline{W}$ denoting the conjugation in $\mathrm{T}_{\mathbb{C}}$. This implies the equivalence of points 2 and 3. Clearly, point 4 implies point 3. Thus, it remains to prove the converse. We note that $\mathrm{d} = \partial + \overline{\partial}$ holds iff $\mathrm{d}\alpha \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ for any $\alpha \in \Omega^{p,q}(M)$. Locally,

$$\alpha = f\vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_p} \wedge \varphi^{j_1} \wedge \ldots \wedge \varphi^{j_q}\,, \quad \vartheta^k \in \Omega^{1,0}(M)\,,\ \varphi^l \in \Omega^{0,1}(M)\,.$$

We have $\mathrm{d}f \in \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, $\mathrm{d}\vartheta^k \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$. Since $\overline{\Omega^{1,0}(M)} = \Omega^{0,1}(M)$, point 3 implies $\mathrm{d}\varphi^l \in \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$ and the assertion follows. ■

Corollary 2.2.15 *If an almost complex structure* J *is integrable, then*

$$\partial^2 = 0\,, \quad \overline{\partial}^2 = 0\,, \quad \overline{\partial} \circ \partial + \partial \circ \overline{\partial} = 0\,. \tag{2.2.20}$$

Conversely, if $\overline{\partial}^2 = 0$*, then* J *is integrable.*[9]

Proof The first assertion is an immediate consequence of $\mathrm{d}^2 = 0$. The second assertion is left to the reader, see Exercise 2.2.7. ■

Let z^k be local coordinates on a complex manifold M. Then, any $\alpha \in \Omega^*_{\mathbb{C}}(M)$ locally reads[10] $\alpha = \alpha_{IJ}\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$ and

$$\partial\alpha = \frac{\partial\alpha_{IJ}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J\,, \quad \overline{\partial}\alpha = \frac{\partial\alpha_{IJ}}{\partial\overline{z}^k}\mathrm{d}\overline{z}^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J\,.$$

[9] Using the operator $\overline{\partial}$, one can build a cohomology theory for complex manifolds, called the Dolbeault cohomology, see [336].

[10] We use the notation of Sect. 4.1 of Part I.

Finally, we note that a diffeomorphism $\varphi : M \to M$ is an automorphism of an almost complex structure J iff $\varphi' \circ \mathsf{J} = \mathsf{J} \circ \varphi'$. If J is integrable, then this means that φ is holomorphic. ♦

The following example is closely related to Example 1.6.6.

Example 2.2.16 (*Pseudo-Riemannian metric*) Denote the vector space of symmetric covariant tensors of second rank on $\mathbb{R}^n$ by $S^2\mathbb{R}^n$. Endow $\mathbb{R}^n$ with a pseudo-Euclidean metric $\eta \in S^2\mathbb{R}^n$ with signature (k, l) where $n = k + l$. The basic representation of $\mathrm{GL}(n, \mathbb{R})$ induces a $\mathrm{GL}(n, \mathbb{R})$-module structure on $S^2\mathbb{R}^n$ given by

$$\sigma : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Aut}(S^2\mathbb{R}^n)\,, \quad \sigma(a) := (a^{-1})^T \otimes (a^{-1})^T\,. \tag{2.2.21}$$

As already noted under point 2 of Example 1.6.6, by the Sylvester Theorem, $\mathrm{GL}(n, \mathbb{R})$ acts transitively on the subspace $S^2_{(k,l)}\mathbb{R}^n \subset S^2\mathbb{R}^n$ consisting of elements with fixed signature, and the stabilizer of η is $\mathrm{O}(k, l)$, that is,

$$\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(k, l) \cong S^2_{(k,l)}\mathbb{R}^n\,.$$

Thus, by (2.2.1), $\mathrm{O}(k, l)$-structures are in one-to-one correspondence with pseudo-Riemannian metrics g on M and the $\mathrm{O}(k, l)$-structure corresponding to g coincides with the bundle $O(M)$ of frames which are orthonormal with respect to g. If (M, g) is oriented, then $O(M)$ further reduces to a principal $\mathrm{SO}(k, l)$-bundle, denoted by $O_+(M)$. Note that $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ is contractible. Thus, an $\mathrm{O}(n)$-structure, that is, a Riemannian metric, always exists. On the contrary, for an arbitrary signature, $\mathrm{O}(k, l)$-structures may not exist. E.g. the obstruction to the existence of a Lorentz-structure[11] on a 4-dimensional oriented manifold is given by the Euler class of the tangent bundle. Thus, for a non-compact M, there is no obstruction. Below, we will see that associated with a pseudo-Riemannian structure, there is a unique torsion-free connection. Then, point 1 of Remark 1.4.7 implies that an $\mathrm{O}(k, l)$-structure is integrable iff the curvature of this connection vanishes. Equivalently, a pseudo-Riemannian structure is integrable iff it is locally flat, that is, if for every point of M there exists a neighbourhood on which g is given by the Euclidean metric.

Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of an $\mathrm{O}(k, l)$-structure iff φ is an isometry of the corresponding pseudo-Riemannian metric g, that is, $\varphi^*\mathsf{g} = \mathsf{g}$. It can be shown, see Theorem 3.4 in Chap. VI of [381], that the group of isometries carries a Lie group structure with respect to the compact-open topology. This Lie group will be denoted by $I(M)$. ♦

Example 2.2.17 (*Conformal structure*) For $n \geq 3$, consider the Lie subgroup

$$\mathrm{CO}(n) := \left\{a \in \mathrm{GL}(n, \mathbb{R}) : a^{\mathrm{T}}a = c\mathbb{1}\,,\ c \in \mathbb{R}\,,\ c > 0\right\}\,.$$

Clearly, $\mathrm{CO}(n) = \mathbb{R}_+ \times \mathrm{O}(n)$. By the previous example, $\mathrm{GL}(n, \mathbb{R})$ acts transitively on the space $S^2_{(k,l)}\mathbb{R}^n$. Thus, it also acts transitively on the set of conformal

[11] A pseudo-Riemannian structure with signature $(+, -, -, -)$.

equivalence classes of elements of $S^2_{(k,l)}\mathbb{R}^n$ defined by the relation $\eta \sim \eta'$ iff $\eta' = c\eta$ for some positive real number c. Clearly, the stabilizer of an element $[\eta]$ is $\mathrm{CO}(n)$. Thus, $\mathrm{CO}(n)$-structures are in one-to-one correspondence with conformal equivalence classes $[\mathsf{g}]$ of metrics on M, with the equivalence defined as follows: two metrics g_1 and g_2 are conformally equivalent iff they differ by a positive function. The $\mathrm{CO}(n)$-structure corresponding to class $[\mathsf{g}]$ is denoted by $CO(M)$ and is referred to as the bundle of conformal frames.

Since $\mathrm{CO}(n) = \mathbb{R}_+ \times \mathrm{O}(n)$, the representation theory of $\mathrm{CO}(n)$ is essentially obtained as an extension of the representation theory of the orthogonal group $\mathrm{O}(n)$. The irreducible representations of $\mathbb{R}_+$ on $\mathbb{R}$ are labeled by real numbers $r \in \mathbb{R}$ and are given by

$$\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (t, x) \mapsto t^r x .$$

The number r is called the conformal weight of the representation under consideration. Let us denote the corresponding representation space by L^r (a copy of $\mathbb{R}$). Then, a typical CO-module is a tensor product of an $\mathrm{O}(n)$-module with L^r. Note that, with respect to the conformal structure $[\mathsf{g}]$, the tangent and the cotangent bundles can no longer be identified, because they correspond to representations containing the factors L^r and L^{-r}, respectively. Clearly, on the level of vector bundles over M, the additional factors L^r corresponds to building the tensor product with an associated line bundle characterized by r.

In close relation to the previous example, one can show that a conformal structure is integrable iff it is locally conformally flat, that is, iff for every point of M there exists a neighbourhood on which the metric is given by $\mathsf{g} = f^2\mathsf{g}_0$, where g_0 is the (flat) Euclidean metric and f is a nowhere vanishing function on that neighbourhood. If this condition holds globally, then one says that (M, g) is conformally flat or, equivalently, that $(M, [\mathsf{g}])$ is flat.

A diffeomorphism $\varphi : M \to M$ is an automorphism of a $\mathrm{CO}(n)$-structure iff there exists a nowhere vanishing function $f \in C^\infty(M)$ such that $\varphi^*\mathsf{g} = f^2\mathsf{g}$, where g is some representative of this structure. The automorphism group of a conformal structure $(M, [\mathsf{g}])$ is called the conformal group of $(M, [\mathsf{g}])$. It will be denoted by $\mathrm{C}(M, [\mathsf{g}])$. The following classical theorem may be found in [381].[12]

Theorem 2.2.18 *Let* (M, g) *be a connected n-dimensional Riemannian manifold with* $n \geq 3$. *Then, its conformal group* $\mathrm{C}(M, [\mathsf{g}])$ *is a Lie group of dimension at most* $\frac{1}{2}(n+1)(n+2)$. ■

For a systematic study of conformal geometry, we refer to [61, 119, 382, 492, 686, 608]. ♦

Example 2.2.19 (*Almost Hermitean structure*) Recall from Example I/7.5.5 that, in the standard embedding (2.2.6) of $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$, we have

$$\mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{GL}(n, C) . \tag{2.2.22}$$

[12] The authors of [381] outline a proof based upon results of Eisenhardt [183] and Palais [499].

Explicitly,

$$\mathrm{U}(n) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : aa^{\mathrm{T}} + bb^{\mathrm{T}} = \mathbb{1}\,,\ ab^{\mathrm{T}} - ba^{\mathrm{T}} = 0\,,\ a, b \in \mathrm{GL}(n, \mathbb{R}) \right\}. \tag{2.2.23}$$

This shows that we may combine an almost complex structure $C(M)$ with the SO($2n$)-structure $O_+(M)$ of a $2n$-dimensional (oriented) Riemannian manifold by intersecting them. On the algebraic level, $\mathsf{J}_0^{\mathrm{T}} \eta \mathsf{J}_0 = \eta$. Thus, if we assume that J is an isometry, that is,

$$\mathsf{g}(\mathsf{J}X, \mathsf{J}Y) = \mathsf{g}(X, Y)\,, \quad X, Y \in \mathfrak{X}(M)\,, \tag{2.2.24}$$

then the intersection

$$U(M) := C(M) \cap O_+(M) \tag{2.2.25}$$

is a U(n)-structure.[13] It is called the bundle of unitary frames. If (2.2.24) is fulfilled, we say that g is a Hermitean metric with respect to J. The triple $(M, \mathsf{g}, \mathsf{J})$ is called an almost Hermitean manifold. If, additionally, J is integrable, then $(M, \mathsf{g}, \mathsf{J})$ is called a Hermitean manifold. Note that

$$\beta(X, Y) := \mathsf{g}(X, \mathsf{J}Y) \tag{2.2.26}$$

is a non-degenerate 2-form on M. Thus, β^n is a nowhere vanishing $2n$-form, that is, an orientation of M. This shows that every almost Hermitean manifold is endowed with a canonical volume form. Existence and integrability criteria of almost Hermitean structures are obtained from Examples 2.2.10 and 2.2.16 above. Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of a U(n)-structure iff it is an automorphism of the GL($n, \mathbb{C}$)- and of the SO($2n$)-structure.

We give an equivalent description of an almost Hermitean manifold $(M, \mathsf{g}, \mathsf{J})$. Viewing its tangent bundle TM as a complex vector bundle, each of its fibres carries a Hermitean scalar product, given by[14]

$$\mathsf{h}(X, Y) := \mathsf{g}(X, Y) + i\mathsf{g}(X, \mathsf{J}Y)\,. \tag{2.2.27}$$

Equivalently, by (2.2.26),

$$\mathsf{h}(X, Y) = \mathsf{g}(X, Y) + i\beta(X, Y) = \beta(\mathsf{J}X, Y) + i\beta(X, Y)\,. \tag{2.2.28}$$

Note that h is linear in the first and anti-linear in the second entry (Exercise 2.2.8). Thus, $(\mathrm{T}M, \mathsf{h})$ is a Hermitean vector bundle, cf. Definition 1.1.16. As usual, let $\tilde{\mathsf{h}}$, $\tilde{\mathsf{g}}$ and $\tilde{\mathsf{J}}$ be the equivariant mappings corresponding to h, g and J, respectively.

[13] It suffices to assume that $C(M)$ and $O_+(M)$ have a nonempty intersection over every point of M.

[14] See Sect. 7.5 of Part I. Note that we have changed conventions in order to be compatible with the standard literature.

Restricted to $U(M)$, $\tilde{\mathsf{g}}$ and $\tilde{\mathsf{J}}$ coincide with the Euclidean metric η and the standard complex structure J_0, respectively. Let h_0 be the Hermitean form defined by η and J_0 via (2.2.27). Since η is $\mathrm{SO}(2n)$-invariant and since J_0 commutes with the $\mathrm{U}(n)$-action, h_0 is $\mathrm{U}(n)$-invariant. This yields the following.

Proposition 2.2.20 *Relative to a given almost complex structure* J *on* M*,* $\mathrm{U}(n)$*-structures on* M *are in one-to-one correspondence with Hermitean fibre metrics on* $\mathrm{T}M$.[15] ∎

Finally, we give a characterization of the above objects in terms of the decompositions (2.2.9), (2.2.15) and (2.2.16). Here, T may be viewed as the basic $\mathrm{SO}(2n)$-module and, by (2.2.22), the subspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ carry the basic representation of $\mathrm{U}(n)$ and its conjugate, respectively. Thus, V and $\mathrm{T}^{1,0}$ are isomorphic as $\mathrm{U}(n)$-modules. For $k = 2$, the decomposition (2.2.16) takes the form

$$\bigwedge^2 \mathrm{T}^*{}_{\mathbb{C}} = \bigwedge^{2,0} \oplus \bigwedge^{1,1} \oplus \bigwedge^{0,2} . \tag{2.2.29}$$

By standard representation theory, the adjoint representation of $\mathrm{U}(n)$ is given by the tensor product of the basic representation and its dual. Thus, after intersecting with the real exterior product $\bigwedge^2 \mathrm{T}^*$, formula (2.2.29) corresponds to the decomposition $\mathfrak{o}(2n) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp$, where

$$\mathfrak{u}(n) = \bigwedge^{1,1} \cap \bigwedge^2 \mathrm{T}^* , \quad \mathfrak{u}(n)^\perp = \Big(\bigwedge^{2,0} \oplus \bigwedge^{0,2}\Big) \cap \bigwedge^2 \mathrm{T}^* . \tag{2.2.30}$$

For a given basis $(\mathbf{e}_1, \mathsf{J}\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathsf{J}\mathbf{e}_n)$ of T, let $(\vartheta^1, \varphi^1, \ldots, \vartheta^n, \varphi^n)$ be the dual basis in T^*. Clearly, the latter yields the bases

$$\{\vartheta^k \wedge \vartheta^l\} , \quad \{\vartheta^k \wedge \varphi^l\} , \quad \{\varphi^k \wedge \varphi^l\} , \quad k < l , \quad k, l = 1, \ldots n ,$$

in, respectively, $\bigwedge^{2,0}$, $\bigwedge^{1,1}$ and $\bigwedge^{0,2}$. In particular, for $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ we may choose the standard basis in $V \cong \mathbb{C}^n$. Since $\tilde{\mathsf{h}}$ takes values in the space of bilinear forms on $\mathrm{T}_{\mathbb{C}}$, we obtain (Exercise 2.2.9)

$$\tilde{\mathsf{h}}(u) = \sum_{k=1}^{n} (\vartheta^k \otimes \vartheta^k + \varphi^k \otimes \varphi^k) - i \sum_{k=1}^{n} \vartheta^k \wedge \varphi^k , \tag{2.2.31}$$

for any $u \in U(M)$. From (2.2.28), we read off

$$\tilde{\mathsf{g}}(u) = \sum_{k=1}^{n} (\vartheta^k \otimes \vartheta^k + \varphi^k \otimes \varphi^k) , \quad \tilde{\beta}(u) = -\sum_{k=1}^{n} \vartheta^k \wedge \varphi^k . \tag{2.2.32}$$

To summarize, for $u \in U(M)$,

[15] Clearly, this is consistent with Example 1.1.18, where we considered the orthonormal frame bundle of an arbitrary vector bundle carrying a fibre metric.

$$\tilde{\mathsf{h}}(u) \in \bigwedge^{1,1}, \quad \tilde{\mathsf{g}}(u) \in \left(\bigwedge^{2,0} \oplus \bigwedge^{0,2}\right) \cap S^2\mathrm{T}^*, \quad \tilde{\beta}(u) \in \bigwedge^{1,1} \cap \bigwedge^2\mathrm{T}^*. \tag{2.2.33}$$

Note that $\beta(u) \in \mathfrak{u}(n)$ is $\mathrm{U}(n)$-invariant. Thus, it spans a 1-dimensional invariant subspace in $\mathfrak{u}(n)$ and gives rise to the decomposition $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus i\mathbb{R}$. ♦

Example 2.2.21 (*Almost symplectic structure*) Consider $H = \mathrm{Sp}(n, \mathbb{R})$. Recall that this is the group of linear transformations of $\mathbb{R}^{2n}$ leaving the standard symplectic form (2.2.6) invariant.[16] Thus, $\mathrm{Sp}(n, \mathbb{R})$-structures are in one-to-one correspondence with 2-forms on M of maximal rank. Such structures are called almost symplectic. By the previous example, each almost Hermitean structure defines such a 2-form β. By Proposition I/7.5.3,

$$\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{Sp}(n, \mathbb{R}),$$

and, thus, each pair built from the triple $(\mathsf{g}, \mathsf{J}, \beta)$ yields the same $\mathrm{U}(n)$-structure. Moreover, since $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ contain $\mathrm{U}(n)$ as their maximal compact subgroup, M admits an almost symplectic structure iff it admits an almost complex structure. Clearly, by the Darboux Theorem I/8.1.5, an almost symplectic structure is integrable iff $\mathrm{d}\beta = 0$. Then (M, β) is called a symplectic manifold. A Hermitean manifold $(M, \mathsf{g}, \mathsf{J})$ such that the 2-form β defined by (2.2.26) is closed is called Kähler. For the discussion of existence, see Remark I/8.1.4.

Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of an $\mathrm{Sp}(n, \mathbb{R})$-structure iff $\varphi^*\beta = \beta$. If (M, β) is symplectic, then φ is called a symplectomorphism. For the study of the group of symplectomorphisms see Sect. 8.8 in Part I. ♦

In the remainder of this section, we discuss compatible connections.

Example 2.2.22 (*Metric connection*) By Example 2.2.16, pseudo-Riemannian manifolds are in one-to-one correspondence with $\mathrm{O}(k, l)$-structures. Thus, let (M, g) be a pseudo-Riemannian manifold and let $O(M)$ be its $\mathrm{O}(k, l)$-structure. In terms of the corresponding equivariant mapping $\tilde{\mathsf{g}}$,

$$O(M) = \{u \in L(M) \colon \tilde{\mathsf{g}}(u) = \eta\}, \tag{2.2.34}$$

where η is the standard inner product on $\mathbb{R}^n$ with signature (k, l). By Proposition 2.2.3, a linear connection ω on M is compatible with the $\mathrm{O}(k, l)$-structure iff g is parallel with respect to ω. A linear connection fulfilling this condition is called metric. By (2.2.21), the metricity condition $D\tilde{\mathsf{g}} = \mathrm{d}\tilde{\mathsf{g}} + \sigma'(\omega)\tilde{\mathsf{g}} = 0$ reads

$$\mathrm{d}\tilde{\mathsf{g}} - \left(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \omega^{\mathrm{T}}\right)(\tilde{\mathsf{g}}) = 0. \tag{2.2.35}$$

More explicitly, decomposing ω with respect to the basis $\{E^j{}_i\}$ in $\mathfrak{gl}(n, \mathbb{R})$ and $\tilde{\mathsf{g}}$ with respect to the basis in $S^2\mathbb{R}^n$ induced from the standard basis of $\mathbb{R}^n$, (2.2.35) takes the form

[16] Note the double role of J_0.

$$\mathrm{d}\tilde{\mathsf{g}}_{jk} - \tilde{\mathsf{g}}_{jl}\omega^l{}_k - \tilde{\mathsf{g}}_{kl}\omega^l{}_j = 0\,. \tag{2.2.36}$$

But, on $O(M)$ we have $\tilde{\mathsf{g}}_{kl} = \eta_{kl}$ and, thus, $\mathrm{d}\tilde{\mathsf{g}}_{jk} = 0$. This shows that ω is metric iff its reduction to $O(M)$ fulfils

$$\omega_{jk} + \omega_{kj} = 0\,,$$

that is, iff this reduction takes values in the Lie algebra $\mathfrak{o}(k, l)$, indeed. Equivalently, the metricity condition is given by $\nabla\mathsf{g} = 0$. Since ∇_X is a derivation of the tensor algebra, the latter is equivalent to

$$X(\mathsf{g}(Y, Z)) = \mathsf{g}(\nabla_X Y, Z) + \mathsf{g}(Y, \nabla_X Z)\,, \tag{2.2.37}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Remark 2.2.23 Let (V, q) be a quadratic vector space over $\mathbb{K}$. Assume that $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and that q is non-degenerate. Recall from Example I/5.2.6 that the Lie algebra $\mathfrak{o}(V, \mathsf{q})$ of the orthogonal group $\mathrm{O}(V, \mathsf{q})$ coincides with those endomorphisms of V which are anti-symmetric with respect to the symmetric bilinear form η of q. In the context of Clifford algebras, see Sect. 5.2, we will see that the following canonical isomorphism of Lie algebras holds:

$$\kappa : \mathfrak{o}(V, \mathsf{q}) \to \bigwedge^2 V\,, \quad \kappa(A) = \frac{1}{4} A(\mathbf{e}_i) \wedge \eta^{-1}(\vartheta^i)\,, \tag{2.2.38}$$

where $\{\mathbf{e}_i\}$ is a q-orthogonal basis in V and $\{\vartheta^j\}$ is the dual basis. Denoting $A_{kl} = \mathsf{g}(\mathbf{e}_k, A\mathbf{e}_l)$, we obtain

$$\kappa(A) = \frac{1}{4}\eta^{ij} A(\mathbf{e}_i) \wedge \mathbf{e}_j = \frac{1}{4} A^{ij}\mathbf{e}_i \wedge \mathbf{e}_j\,. \tag{2.2.39}$$

♦

Proposition 2.2.24 *Any* $\mathrm{O}(k, l)$*-structure has a unique torsion-free connection.*

Proof By Corollary 2.2.7, it is enough to show that the mapping δ given by (2.2.2) is bijective. In the case under consideration, $\mathfrak{h} = \mathfrak{o}(k, l) \cong \bigwedge^2\mathbb{R}^n \cong \bigwedge^2(\mathbb{R}^n)^*$. Thus,

$$\delta : (\mathbb{R}^n)^* \otimes \bigwedge^2(\mathbb{R}^n)^* \to \bigwedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n\,.$$

Let $\alpha \in (\mathbb{R}^n)^* \otimes \bigwedge^2(\mathbb{R}^n)^*$ and let α_{ijk} be the components of α in the basis induced from the standard basis of $\mathbb{R}^n$. Then, $\alpha_{ijk} = -\alpha_{ikj}$ and the components of $\delta(\alpha)$ are given by $\frac{1}{2}(\alpha_{ijk} - \alpha_{jik})$. Assume $\delta(\alpha) = 0$. Then,

$$\alpha_{ijk} = \alpha_{jik} = -\alpha_{jki} = -\alpha_{kji} = \alpha_{kij} = \alpha_{ikj} = -\alpha_{ijk}\,,$$

that is $\ker(\delta) = 0$. Now, bijectivity follows from dimension counting. ■

A classical proof of Proposition 2.2.24 is obtained by using (2.2.37) and (2.1.33),

$$X(\mathsf{g}(Y,Z)) = \mathsf{g}(\nabla_X Y, Z) + \mathsf{g}(Y, \nabla_X Z)\,, \quad 0 = \nabla_X Y - \nabla_Y X - [X,Y]\,.$$

Then, by direct inspection (Exercise 2.2.10),

$$\begin{aligned} 2\mathsf{g}(\nabla_X Y, Z) &= X(\mathsf{g}(Y,Z)) + Y(\mathsf{g}(X,Z)) - Z(\mathsf{g}(X,Y)) \\ &\quad + \mathsf{g}([X,Y],Z) + \mathsf{g}([Z,X],Y) + \mathsf{g}([Z,Y],X)\,. \end{aligned} \tag{2.2.40}$$

One easily checks that this equation defines a torsion-free connection. In the sequel, the unique torsion-free connection defined by g will be called the Levi-Civita connection.

Finally, we derive local formulae for the Levi-Civita connection. In contrast to general linear connections, here we have two natural types of local frames:

(a) local holonomic frames $\{\partial_j\}$ induced from arbitrary local charts (U_j, κ_j),
(b) local frames $\{e_j\}$ which are orthonormal with respect to g.

Since the formulae (2.1.42), (2.1.44), (1.5.8) and (2.1.46)–(2.1.50) hold true for any local frame, they clearly apply here. Let $\mathfrak{e}$ be an arbitrary local frame. By (2.2.40),

$$2\mathsf{g}(\nabla_{e_i} e_j, e_k) = e_i(\mathsf{g}_{jk}) + e_j(\mathsf{g}_{ik}) - e_k(\mathsf{g}_{ij}) + C^l{}_{ij}\mathsf{g}_{lk} + C^l{}_{ki}\mathsf{g}_{lj} + C^l{}_{kj}\mathsf{g}_{li}\,,$$

where $\mathsf{g}_{ij} = \mathsf{g}(e_i, e_j)$. Thus,

$$\begin{aligned} \Gamma^m{}_{ij} &= \frac{1}{2}\mathsf{g}^{mk}\left(e_i(\mathsf{g}_{jk}) + e_j(\mathsf{g}_{ik}) - e_k(\mathsf{g}_{ij})\right) \\ &\quad + \frac{1}{2}\left(C^m{}_{ij} + \mathsf{g}^{km}\mathsf{g}_{lj}C^l{}_{ki} + \mathsf{g}^{km}\mathsf{g}_{li}C^l{}_{kj}\right)\,. \end{aligned} \tag{2.2.41}$$

For the case (a), we have $\mathsf{g}_{ij} = \mathsf{g}(\partial_i, \partial_j)$ and $C^i_{jk} = 0$. Thus,

$$\Gamma^m{}_{ij} = \frac{1}{2}\,\mathsf{g}^{mk}(\mathsf{g}_{jk,i} + \mathsf{g}_{ki,j} - \mathsf{g}_{ji,k})\,, \quad \Gamma^m{}_{ij} = \Gamma^m{}_{ji}\,. \tag{2.2.42}$$

For the case (b), we have $\mathsf{g}_{ij} = \eta_{ij}$ and, therefore,

$$\Gamma^m{}_{ij} = \frac{1}{2}\left(C^m{}_{ij} + \eta^{km}\eta_{lj}C^l{}_{ki} + \eta^{km}\eta_{li}C^l{}_{kj}\right)\,. \tag{2.2.43}$$

Thus, $\Gamma_{kij} = \eta_{km}\Gamma^m{}_{ij} = \frac{1}{2}(C_{kij} + C_{jki} + C_{ikj})$ and, consequently, for case (b) we have

$$\Gamma_{kij} = -\Gamma_{jik}\,, \quad \Gamma^k{}_{ik} = 0\,. \tag{2.2.44}$$

Using (2.1.46) and (2.2.43), we obtain

$$\mathrm{d}\vartheta^i(e_j, e_k) = -\vartheta^i([e_j, e_k]) = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

and, thus,

$$\mathrm{d}\vartheta^i = -\Gamma^i{}_{jk}\vartheta^j \wedge \vartheta^k . \tag{2.2.45}$$

Comparing with (2.1.46), we read off

$$\mathrm{d}\vartheta^i = \vartheta^j \wedge \nabla_{e_j}\vartheta^i . \tag{2.2.46}$$

This implies the following useful formula (Exercise 2.2.11). For any $\alpha \in \Omega^k(M)$,

$$\mathrm{d}\alpha = \vartheta^j \wedge \nabla_{e_j}\alpha . \tag{2.2.47}$$

Since the operator d is intrinsically defined, this formula does not depend on the choice of the frame. It can be rewritten as

$$\mathrm{d}\alpha(e_0, \ldots, e_k) = \sum_j (-1)^j \left(\nabla_{e_j}\alpha\right)\left(e_0, \overset{j}{\curlyvee}, e_k\right) . \tag{2.2.48}$$

By the locality property of ∇ and by the multilinearity of α, we conclude

$$\mathrm{d}\alpha(X_0, \ldots, X_k) = \sum_j (-1)^j \left(\nabla_{X_j}\alpha\right)\left(X_0, \overset{j}{\curlyvee}, X_k\right) , \tag{2.2.49}$$

for any set of vector fields $X_0, \ldots, X_k$ on M. ♦

Example 2.2.25 (*Almost complex connection*) By Example 2.2.10, $\mathrm{GL}(n, \mathbb{C})$-structures on a manifold M are in one-to-one correspondence with sections J of $\mathrm{End}(\mathrm{T}M)$ fulfilling $\mathsf{J}_m^2 = -\,\mathrm{id}$ for every $m \in M$. By Proposition 2.2.3, a linear connection ω on M is compatible with a $\mathrm{GL}(n, \mathbb{C})$-structure iff J is parallel with respect to ω. A linear connection fulfilling this condition is called almost complex. Recall that the obstruction to integrability of an almost complex structure is given by the Nijenhuis tensor N.

Proposition 2.2.26 *An almost complex manifold* (M, J) *admits a torsion-free almost complex linear connection iff* J *is integrable.*

Proof We show that the intrinsic torsion vanishes iff J is integrable. Here, the mapping (2.2.2) takes the form

$$\delta : (\mathbb{R}^{2n})^* \otimes \mathfrak{gl}(n, \mathbb{C}) \to \bigwedge^2 (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n} .$$

We pass to the complexifications of both the domain and the target space of δ and use the decompositions (2.2.9), (2.2.15) and (2.2.29), together with the embedding (2.2.5). Then, the target space reads

$$
\begin{aligned}
\left(\bigwedge^2 \mathrm{T}^*_{\mathbb{C}}\right) \otimes \mathrm{T}_{\mathbb{C}} &= \left(\bigwedge^{2,0} \oplus \bigwedge^{1,1} \oplus \bigwedge^{0,2}\right) \otimes \left(\mathrm{T}^{1,0} \oplus \mathrm{T}^{0,1}\right) \\
&= \left(\bigwedge^{2,0} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{1,1} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{0,2} \otimes \mathrm{T}^{1,0}\right) \\
&\quad \oplus \left(\bigwedge^{2,0} \otimes \mathrm{T}^{0,1}\right) \oplus \left(\bigwedge^{1,1} \otimes \mathrm{T}^{0,1}\right) \oplus \left(\bigwedge^{0,2} \otimes \mathrm{T}^{0,1}\right),
\end{aligned}
$$

and for the image of δ we get

$$
\operatorname{im}(\delta) = \left(\left(\bigwedge^{1,1} \oplus \bigwedge^{0,2}\right) \otimes \mathrm{T}^{0,1}\right) \oplus \left(\left(\bigwedge^{2,0} \oplus \bigwedge^{1,1}\right) \otimes \mathrm{T}^{1,0}\right). \tag{2.2.50}
$$

The latter is obtained by a straightforward calculation, see Exercise 2.2.5. Thus, the intrinsic torsion takes values in

$$
\operatorname{coker}(\delta) = \left(\bigwedge^{0,2} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{2,0} \otimes \mathrm{T}^{0,1}\right).
$$

We give the argument for the first component. Let $\mathfrak{e} = (e_1, \ldots, e_n)$ be a holomorphic frame and let $(\vartheta^1, \ldots, \vartheta^n)$ be the dual coframe. Taking the pullback under $\mathfrak{e}$ of the Structure Equation for the torsion, cf. (2.1.15), we obtain

$$
\mathrm{T}^i = \mathrm{d}\vartheta^i + \mathscr{A}^i{}_j \wedge \vartheta^j .
$$

Evaluating the $(1,0)$-component of this equation on $X_1, X_2 \in \Gamma^\infty(\mathrm{T}^{0,1}M)$, we obtain

$$
\mathrm{T}^i(X_1, X_2) = -\vartheta^i([X_1, X_2]).
$$

We get the same equation for the $(0,1)$-component evaluated on a pair of vector fields of type $(1,0)$. Thus, the intrinsic torsion vanishes iff $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$ are involutive. Now, point 2 of Proposition 2.2.14 yields the assertion. ■

By the above proof and point 1 of Proposition 2.2.14, the Nijenhuis tensor measures the torsion of an almost complex linear connection, see also Theorem 3.4 in Chap. IX of [381] for a classical proof. ♦

Example 2.2.27 (*Unitary connection*) Here, we take up Example 2.2.19. Thus, let $U(M)$ be a $\mathrm{U}(n)$-structure and let $(M, \mathsf{g}, \mathsf{J})$ be the corresponding $2n$-dimensional almost Hermitean manifold. Clearly, by Proposition 2.2.3, a linear connection ω on M is compatible with the $\mathrm{U}(n)$-structure iff both g and J are parallel with respect to ω. Such a connection will be called unitary.

Assume that there exists a torsion-free unitary connection ω on M. Since $U(M) = C(M) \cap O_+(M)$ and since the Levi-Civita connection of g is the unique torsion-free connection on $O_+(M)$, ω is necessarily obtained as a reduction of the Levi-Civita connection to $U(M)$. Thus, if it exists, it is necessarily unique.

Proposition 2.2.28 *Let* $U(M)$ *be a* $\mathrm{U}(n)$*-structure, let* $(M, \mathsf{g}, \mathsf{J})$ *be the corresponding almost Hermitean manifold and let* β *be the almost symplectic form defined* *by the pair* (g, J). *Then, the Levi-Civita connection* ω *of* g *is compatible with the* $\mathrm{U}(n)$*-structure iff* J *is integrable and* β *is symplectic.*

Proof Assume that ω is $\mathrm{U}(n)$-compatible. Then, both g and J are ω-parallel and, by Proposition 2.2.26, since ω is torsion-free and since J is parallel, J is integrable. Moreover, the parallelity of g and J imply the parallelity of β. Then, (2.2.49) yields $\mathrm{d}\beta = 0$. The converse statement follows immediately from the identity

$$2\mathsf{g}((\nabla_X \mathsf{J})Y, Z) = \mathrm{d}\beta(X, \mathsf{J}Y, \mathsf{J}Z) - \mathrm{d}\beta(X, Y, Z) + \mathsf{g}(N(Y, Z), \mathsf{J}X)\,, \qquad (2.2.51)$$

where ∇ is the covariant derivative of ω and $X, Y, Z \in \mathfrak{X}(M)$, see Exercise 2.2.12. ■

Thus, ω is compatible with the $\mathrm{U}(n)$-structure iff $(M, \mathsf{g}, \mathsf{J})$ is Kähler. For a detailed description of Kähler structures in terms of local coordinates we refer to Sects. 4 and 5 of Chap. IX in [381].

Finally, by the discussion in Example 2.2.19, we obtain a characterization of unitary connections in terms of the Hermitean fibre metric h defined by g and J.

Proposition 2.2.29 *A linear connection* ω *on a Hermitean manifold* $(M, \mathsf{g}, \mathsf{J})$ *is unitary iff the Hermitean fibre metric* h *defined by* g *and* J *is parallel with respect to* ω.

According to (2.2.33), $\tilde{\mathsf{h}}(u) \in \bigwedge^{1,1}$. Explicitly, the $\mathrm{U}(n)$-module structure of $\bigwedge^{1,1}$ is given by

$$\sigma : \mathrm{U}(n) \to \mathrm{Aut}\left(\bigwedge\nolimits^{1,1}\right)\,, \quad \sigma(g) = \left(g^{-1}\right)^{\mathrm{T}} \otimes \overline{\left(g^{-1}\right)^{\mathrm{T}}}\,. \qquad (2.2.52)$$

Thus, the metricity condition $D\,\tilde{\mathsf{h}} = \mathrm{d}\tilde{\mathsf{h}} + \sigma'(\omega)\tilde{\mathsf{h}} = 0$ restricted to $U(M)$ implies

$$\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \overline{\omega^{\mathrm{T}}} = 0\,. \qquad (2.2.53)$$

Analyzing (2.2.53) in the standard basis as in Example 2.2.22, we obtain $\omega^{\dagger} + \omega = 0$, that is, ω takes values in the Lie algebra $\mathfrak{u}(n)$, indeed. ♦

Exercises

2.2.1 Show that integrability of a section s in an H-structure P implies $s^*\mathrm{d}\theta = 0$.

2.2.2 Prove that any $\mathrm{SL}(n, \mathbb{R})$-structure is integrable.

2.2.3 Prove that a mapping of an open subset of $\mathbb{C}^n$ to $\mathbb{C}^m$ is compatible with the natural almost complex structures iff it is holomorphic.

2.2.4 Prove that every almost complex manifold is orientable.

2.2.5 Prove formula (2.2.50). *Hint.* Let $\xi \in (\mathbb{R}^{2n})^*$ and $a \in \mathfrak{gl}(n, \mathbb{C}) \cong (\mathbb{C}^n)^* \otimes \mathbb{C}^n$. To calculate $\delta(\xi \otimes a)$, decompose both elements with respect to bases adapted to the decompositions (2.2.9) and (2.2.15) and calculate the image explicitly using (2.2.5).[17]

2.2.6 Prove that the mappings $\mathrm{pr}^{1,0} \circ \iota$ and $\mathrm{pr}^{0,1} \circ \iota$, defined by (2.2.13), are $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear, respectively. Show that (2.2.14) holds.

2.2.7 Prove the second assertion in Corollary 2.2.15. *Hint.* Use point 2 of Proposition 2.2.14.

2.2.8 Prove that h defined by (2.2.27) is linear in the first and anti-linear in the second entry.

2.2.9 Prove formula (2.2.31).

2.2.10 Give an alternative proof of Proposition 2.2.24 by using (2.2.37) and (2.1.33).

2.2.11 Prove formula (2.2.47).

2.2.12 Prove formula (2.2.51). *Hint.* Prove that $\mathsf{g}((\nabla_X \mathsf{J})Y, Z) = \mathsf{g}(\nabla_X(\mathsf{J}Y), Z) + \mathsf{g}(\nabla_X Y, \mathsf{J}Z)$ and rewrite the terms on the right hand side according to (2.2.40). Use formula I/4.1.6. Alternatively, the proof can be found in [381], see Proposition 4.2 in Chap. IX.

2.2.13 Prove that for $H = \mathrm{Sp}(n, \mathbb{R})$, the cokernel of the mapping (2.2.2) is isomorphic to $\bigwedge^3(\mathbb{R}^{2n})^*$. Show that the corresponding intrinsic torsion coincides with the exterior derivative of the almost symplectic form, cf. Example 2.2.21.

2.3 Curvature and Holonomy

In this section, we continue the discussion of connections compatible with H-structures. Here, we consider exclusively torsion-free connections and ask which holonomy groups may occur for such a connection. This question has first been studied systematically by Berger, see [68, 69].

At this point, the reader may wish to recall the basic notions from the general holonomy theory as presented in Sect. 1.7. For a linear connection Γ in $L(M)$, let $P_{u_0}(\Gamma)$ be the holonomy bundle of Γ with base point $u_0 \in L(M)$. By Proposition 1.7.12, Γ is reducible to $P_{u_0}(\Gamma)$ and thus, for any $u \in P_{u_0}(\Gamma)$, the curvature Ω of Γ takes values in the Lie algebra $\mathfrak{h}_{u_0}(\Gamma)$ of the holonomy group $\mathscr{H}_{u_0}(\Gamma) \subset \mathrm{GL}(n, \mathbb{R})$. By the Ambrose-Singer Theorem 1.7.15, we have

$$\mathfrak{h}_{u_0}(\Gamma) = \mathrm{span}\left\{\Omega_u(X, Y) : u \in P_{u_0}(\Gamma)\,,\ \ X, Y \in \Gamma_u\right\}. \tag{2.3.1}$$

It is the condition of torsion-freeness which makes the above question nontrivial. If we drop this assumption, then any closed Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ may occur

[17] Cf. also Example 2.2.19.

as the holonomy group of a linear connection on some n-dimensional manifold M, see [283]. However, in general, such a connection will have a nontrivial torsion. By the Bianchi identity (2.1.17), vanishing of the torsion implies

$$\Omega \wedge \theta = 0\,, \tag{2.3.2}$$

and, by the Ambrose-Singer Theorem, this yields a nontrivial restriction on the holonomy. Now, let $P \subset L(M)$ be an H-structure on an n-dimensional manifold M, let ω be an H-compatible connection and let Ω be its curvature. For simplicity, let us denote $\mathbb{R}^n \equiv V$. By Remark 2.1.16, we may represent Ω equivalently by the curvature mapping

$$\mathscr{R} : P \to \bigwedge^2 V^* \otimes \mathfrak{h} \tag{2.3.3}$$

fulfilling the equivariance condition (2.1.25) with respect to the natural representation $\sigma : H \to \mathrm{Aut}\left(\bigwedge^2 V^* \otimes \mathfrak{h}\right)$ given by

$$\sigma_a((\xi \wedge \tau) \otimes A) := \left((a^{-1})^{\mathrm{T}}\xi \wedge (a^{-1})^{\mathrm{T}}\tau\right) \otimes \mathrm{Ad}(a)A\,. \tag{2.3.4}$$

Since the exterior products of the components θ^i of θ span the spaces of horizontal forms, (2.3.2) implies that $\mathscr{R}$ takes values in the kernel $\mathfrak{K}(\mathfrak{h})$ of the mapping

$$\delta : \bigwedge^2 V^* \otimes \mathfrak{h} \to \bigwedge^3 V^* \otimes V\,, \quad \delta = (a \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \iota_{\mathfrak{h}})\,, \tag{2.3.5}$$

where a is the anti-symmetrization mapping, cf. (2.2.2). Clearly,

$$\mathfrak{K}(\mathfrak{h}) = \left\{F \in \bigwedge^2 V^* \otimes \mathfrak{h} : F(\mathbf{x},\mathbf{y})\mathbf{z} + F(\mathbf{y},\mathbf{z})\mathbf{x} + F(\mathbf{z},\mathbf{x})\mathbf{y} = 0\,, \quad \mathbf{x},\mathbf{y},\mathbf{z} \in V\right\}.$$

The space $\mathfrak{K}(\mathfrak{h})$ is called the space of curvature mappings.

Lemma 2.3.1 *The subspace*

$$\underline{\mathfrak{h}} := \mathrm{span}\,\{F(\mathbf{x},\mathbf{y}) \in \mathfrak{h} : F \in \mathfrak{K}(\mathfrak{h})\,, \quad \mathbf{x},\mathbf{y} \in V\} \tag{2.3.6}$$

is an ideal of $\mathfrak{h}$.

Proof Let $F(\mathbf{x},\mathbf{y}) \in \underline{\mathfrak{h}}$ and let $A \in \mathfrak{h} \subset \mathrm{End}(V)$. Then, we may write

$$[F(\mathbf{x},\mathbf{y}), A] = \tilde{F}(\mathbf{x},\mathbf{y}) - F(A\mathbf{x},\mathbf{y}) - F(\mathbf{x}, A\mathbf{y})\,,$$

where

$$\tilde{F}(\mathbf{x},\mathbf{y}) = [F(\mathbf{x},\mathbf{y}), A] + F(A\mathbf{x},\mathbf{y}) + F(\mathbf{x}, A\mathbf{y})\,.$$

One checks by direct inspection that $\tilde{F} \in \mathfrak{K}(\mathfrak{h})$. ■

Note that $\tilde{F}$ corresponds exactly to the action of A on F obtained by differentiating the equivariance condition (2.1.25).[18] Thus, by the Ambrose-Singer Theorem, for the Lie algebra $\mathfrak{h}_{u_0}(\Gamma)$ of the holonomy group of a torsion-free connection Γ, we have

$$\underline{\mathfrak{h}_{u_0}(\Gamma)} = \mathfrak{h}_{u_0}(\Gamma)\,.$$

We conclude that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R})$ can occur as the Lie algebra of the holonomy group of a torsion-free connection only if it coincides with the ideal $\underline{\mathfrak{h}}$. This is commonly referred to as the first criterion of Berger. It yields a necessary condition for a Lie subalgebra to be the holonomy Lie algebra of a torsion-free connection.

Next, let us analyze the Bianchi identity (2.1.16) in terms of $\mathscr{R}$. The covariant derivative $D\mathscr{R} = \mathrm{d}\mathscr{R} + \sigma'(\omega)\mathscr{R}$ is a horizontal 1-form on P with values in $\mathfrak{K}(\mathfrak{h})$.

Definition 2.3.2 A torsion-free connection fulfilling $D\mathscr{R} = 0$ is called locally symmetric.

Decomposing $D\mathscr{R}$ with respect to the horizontal frame $\{\theta^i\}$, we obtain a function $D\mathscr{R} : P \to V^* \otimes \mathfrak{K}(\mathfrak{h})$. Using the fact that the commutators of horizontal standard vector fields corresponding to a torsion-free connection are vertical (Exercise 2.3.1), we calculate

$$\begin{aligned} D\Omega(B(\mathbf{x}), B(\mathbf{y}), B(\mathbf{z})) &= \mathrm{d}\Omega(B(\mathbf{x}), B(\mathbf{y}), B(\mathbf{z})) \\ &= B(\mathbf{x})\big(\Omega(B(\mathbf{y}), B(\mathbf{z})\big) - \Omega([B(\mathbf{x}), B(\mathbf{y})], B(\mathbf{z})) + \text{cycl.} \\ &= \mathrm{d}\mathscr{R}(B(\mathbf{x}))(\mathbf{y} \wedge \mathbf{z}) + \text{cycl.} \\ &= D\mathscr{R}(\mathbf{x})(\mathbf{y} \wedge \mathbf{z}) + \text{cycl.}\ . \end{aligned}$$

Thus, by the Bianchi identity $D\Omega = 0$, we conclude that the function $D\mathscr{R}$ takes values in the kernel of the mapping

$$\delta' : V^* \otimes \mathfrak{K}(\mathfrak{h}) \to \bigwedge^3 V^* \otimes \mathfrak{h}\,, \tag{2.3.7}$$

defined as the composition

$$V^* \otimes \mathfrak{K}(\mathfrak{h}) \to V^* \otimes \bigwedge^2 V^* \otimes \mathfrak{h} \to \bigwedge^3 V^* \otimes \mathfrak{h}$$

of the inclusion and the anti-symmetrization mappings. Clearly, the kernel of δ' is

$$\mathfrak{K}^1(\mathfrak{h}) := \left\{\Phi \in V^* \otimes \mathfrak{K}(\mathfrak{h}) : \Phi(\mathbf{x})(\mathbf{y}, \mathbf{z}) + \Phi(\mathbf{y})(\mathbf{z}, \mathbf{x}) + \Phi(\mathbf{z})(\mathbf{x}, \mathbf{y}) = 0\,,\ \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V\right\}.$$

Thus, if $\mathfrak{h}$ is the holonomy Lie algebra of a torsion-free linear connection that is not locally symmetric, then necessarily $\mathfrak{K}^1(\mathfrak{h}) \neq 0$. This is usually referred to as the second Berger criterion.

[18] Clearly, this is the action of the Killing vector field generated by A.

Definition 2.3.3 A Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ is called a Berger algebra if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Berger algebra is called symmetric if $\mathfrak{K}^1(\mathfrak{h}) = 0$ and non-symmetric otherwise. Correspondingly, a Lie subgroup $H \subset \mathrm{Aut}(V)$ is referred to as a (symmetric or non-symmetric) Berger group if its Lie algebra is a (symmetric or non-symmetric) Berger algebra.

By the above discussion, we have the following.

Proposition 2.3.4 (Berger) *Let $\mathfrak{h} \subset \mathrm{End}(V)$ be a Lie subalgebra. Then,*

1. *If $\mathfrak{h}$ is the Lie algebra of the holonomy group of a torsion-free connection on some manifold, then $\mathfrak{h}$ is a Berger algebra.*
2. *If $\mathfrak{K}^1(\mathfrak{h}) = 0$, then any torsion-free connection on a manifold whose holonomy Lie algebra is contained in $\mathfrak{h}$ must be locally symmetric.*

Based upon these criteria, Berger started to tackle the above classification problem. It is natural to distinguish between the following two classes:

(a) Lie subalgebras $\mathfrak{h}$ lying in some $\mathfrak{o}(\eta)$, where η is some non-degenerate bilinear form on V. In this case, the associated H-structure defines a pseudo-Riemannian manifold. Therefore, this is called the metric case.
(b) Lie subalgebras which are not contained in any orthogonal Lie algebra. This is called the non-metric case.

Within this general analysis, Berger obtained a list of candidates for Lie subalgebras of type (a) and also an (incomplete) list for type (b).[19] These lists where refined and completed by the work of Alekseevski [14], Bryant [108, 109], Chi [132], Merkulov and Schwachhöfer [569]. The final full classification of irreducible holonomies of torsion-free affine connections was obtained by Merkulov and Schwachhöfer [441]. For an exhaustive discussion, we refer to the reviews of Bryant [110] and Schwachhöfer [570] and to the textbooks of Besse [76], Joyce [353] and Salamon [555]. In [110], the reader can find the complete classification list (divided into four parts) together with a lot of information on methods for proving that a given group in the list really occurs as a holonomy. It turns out that every such group is realized at least locally.[20]

In the remainder of this section, we exclusively consider the metric case. That is, we consider (pseudo-)Riemannian manifolds (M, g), endowed with their unique torsion-free metric connection (the Levi-Civita connection). Under this assumption, the frame bundle reduces to the orthonormal frame bundle $O(M)$ and the whole theory may be described in terms of objects living on $O(M)$. Consequently, in the case under consideration, the holonomy group is a subgroup of the structure group $\mathrm{O}(k, l)$. If the Levi-Civita connection is locally symmetric, we call (M, g) locally symmetric.

[19] The list provided by Theorem 2.3.19 below is included in type (a).

[20] The appropriate method working for three of the above mentioned four tables is to describe torsion-free connections with a given holonomy as solutions to an exterior differential system and to apply Cartan's existence theorem.

Definition 2.3.5 Let (M, g) be a pseudo-Riemannian manifold. The curvature mapping

$$\mathscr{R} : O(M) \to \bigwedge^2 V^* \otimes \mathfrak{o}(k, l)$$

of the Levi-Civita connection of g is called the Riemann curvature mapping. Correspondingly, the curvature tensor R is called the Riemann curvature of (M, g).

Comparing with the general case, $\mathscr{R}$ has some additional properties coming from the fact that we may use the metric η to identify V with V^*. In particular, $\mathfrak{o}(k, l) \cong \bigwedge^2 V^*$, and thus

$$\mathscr{R}(u) \in \bigwedge^2 V^* \otimes \bigwedge^2 V^* , \tag{2.3.8}$$

for every $u \in O(M)$.

Proposition 2.3.6 *The Riemann curvature mapping $\mathscr{R}$ of a pseudo-Riemannian manifold has the following algebraic properties. For any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in V$,*

$$\mathscr{R}(\mathbf{x}, \mathbf{y}) = -\mathscr{R}(\mathbf{y}, \mathbf{x}) , \tag{2.3.9}$$

$$\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) = -\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{w}, \mathbf{z}) , \tag{2.3.10}$$

$$\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) = \eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{x}, \mathbf{y}) , \tag{2.3.11}$$

$$\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z} + \mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{x} + \mathscr{R}(\mathbf{z}, \mathbf{x})\mathbf{y} = 0 . \tag{2.3.12}$$

Proof Formulae (2.3.9) and (2.3.10) follow immediately from (2.3.8) and formula (2.3.12) is a direct consequence of the fact that $\mathscr{R}$ takes values in the kernel $\mathfrak{K}(\mathfrak{h})$ of the mapping (2.3.5). It remains to prove (2.3.11). For that purpose, we write down the following four versions of (2.3.12).

$$\begin{aligned}
0 &= \eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) + \eta(\mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{x}, \mathbf{w}) + \eta(\mathscr{R}(\mathbf{z}, \mathbf{x})\mathbf{y}, \mathbf{w}) ,\\
0 &= \eta(\mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{w}, \mathbf{x}) + \eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{y}, \mathbf{x}) + \eta(\mathscr{R}(\mathbf{w}, \mathbf{y})\mathbf{z}, \mathbf{x}) ,\\
0 &= -\eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{x}, \mathbf{y}) - \eta(\mathscr{R}(\mathbf{w}, \mathbf{x})\mathbf{z}, \mathbf{y}) - \eta(\mathscr{R}(\mathbf{x}, \mathbf{z})\mathbf{w}, \mathbf{y}) ,\\
0 &= -\eta(\mathscr{R}(\mathbf{w}, \mathbf{x})\mathbf{y}, \mathbf{z}) - \eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{w}, \mathbf{z}) - \eta(\mathscr{R}(\mathbf{y}, \mathbf{w})\mathbf{x}, \mathbf{z}) .
\end{aligned}$$

Summation of these equations and using (2.3.9) and (2.3.10) yields the assertion. ∎

Remark 2.3.7

1. By Proposition 2.3.6,

$$\mathscr{R} : O(M) \to S^2\left(\bigwedge^2 V^*\right) , \tag{2.3.13}$$

where $S^2\left(\bigwedge^2 V^*\right) = \bigwedge^2 V^* \overset{s}{\otimes} \bigwedge^2 V^*$ is the symmetrized tensor product. By (2.1.25), $\mathscr{R}$ has the following equivariance property, see Exercise 2.3.2,

$$\mathscr{R}(\Psi_a(u))(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \mathscr{R}(u)(a\mathbf{x}, a\mathbf{y}, a\mathbf{u}, a\mathbf{v}) , \tag{2.3.14}$$

for $a \in \mathrm{O}(k,l)$ and $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in V$.

2. By (2.1.27), the Riemann curvature R fulfils identities corresponding to (2.3.9)–(2.3.12) with $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in V$ replaced by $X, Y, Z, W \in \mathrm{T}_m M$ and η replaced by g. Thus, in particular, $\mathsf{R} \in \Gamma^\infty\big(S^2(\bigwedge^2 \mathrm{T}^* M)\big)$. For a local frame $\{e_i\}$, using (2.1.52) we write
$$\mathsf{R}_{ijkl} \equiv \mathsf{g}(\mathsf{R}(e_i, e_j)e_k, e_l) = \mathsf{R}^m{}_{ijk}\, \mathsf{g}_{ml}\,.$$

In this notation, the algebraic properties (2.3.9)–(2.3.12) read
$$\mathsf{R}_{ijkl} = -\mathsf{R}_{jikl}\,, \quad \mathsf{R}_{ijkl} = -\mathsf{R}_{ijlk}\,, \quad \mathsf{R}_{ijkl} = \mathsf{R}_{klij}\,, \tag{2.3.15}$$
$$\mathsf{R}_{ijkl} + \mathsf{R}_{jkil} + \mathsf{R}_{kijl} = 0 \tag{2.3.16}$$

♦

Using the above properties, the space of Riemann curvature mappings $\mathfrak{K}(\mathfrak{o}(k,l))$ may be characterized as follows. By standard representation theory of the group $\mathrm{O}(k,l)$, for $n \geq 4$, one obtains the following decompositions into $\mathrm{O}(k,l)$-irreducible modules [76, 555]:
$$\textstyle\bigwedge^3 V^* \otimes V^* = \bigwedge^2 V^* \oplus \bigwedge^4 V^* \oplus U\,, \tag{2.3.17}$$
$$S^2\left(\textstyle\bigwedge^2 V^*\right) = \mathbb{R} \oplus \Sigma_0^2 \oplus \textstyle\bigwedge^4 V^* \oplus W\,, \tag{2.3.18}$$

where Σ_0^2 stands for the space of traceless endomorphisms of $\mathbb{R}^n$ (viewed as symmetric 2-tensors) and where U and W are orthogonal complements. By dimension counting, U and W are not isomorphic.

Proposition 2.3.8 *The space of Riemann curvature mappings is given by*
$$\mathfrak{K}(\mathfrak{o}(k,l)) = \ker\varphi \,\cap\, S^2\left(\textstyle\bigwedge^2 V^*\right)\,, \tag{2.3.19}$$

where
$$\varphi : \textstyle\bigwedge^2 V^* \otimes \bigwedge^2 V^* \to \bigwedge^4 V^*\,, \quad \varphi(\xi \otimes \tau) := \xi \wedge \tau\,. \tag{2.3.20}$$

Proof Under the identifications $\mathfrak{o}(k,l) \cong \bigwedge^2 V^*$ and $V \cong V^*$, $\mathfrak{K}(\mathfrak{o}(k,l))$ coincides with the kernel of the mapping $\chi : \bigwedge^2 V^* \otimes \bigwedge^2 V^* \to \bigwedge^3 V^* \otimes V^*$ given by
$$\chi(\alpha \otimes (\zeta \wedge \tau)) := (\alpha \wedge \zeta) \otimes \tau - (\alpha \wedge \tau) \otimes \zeta\,.$$

Now, consider the decompositions (2.3.17) and (2.3.18). Viewing χ as an $\mathrm{O}(k,l)$-intertwining mapping and using Schur's Lemma, together with the fact that χ is surjective, we conclude that χ must be zero on the irreducible subspaces $\mathbb{R}$, Σ_0^2 and W. By dimension counting, these subspaces span the kernel of χ. Moreover, restricted to $S^2(\bigwedge^2 V^*)$, χ maps onto $\bigwedge^4 V^*$ and coincides with φ. ■

Combining (2.3.19) and (2.3.18), for $n \geq 4$, we obtain[21]

$$\mathfrak{K}(\mathfrak{o}(k,l)) = \mathbb{R} \oplus \Sigma_0^2 \oplus W\,. \tag{2.3.21}$$

This yields a decomposition of the Riemann curvature into its irreducible components with respect to the action of $\mathrm{O}(k,l)$. The component Σ_0^2 corresponds to the contraction to $V^* \otimes V^*$ defined by taking the trace of the mapping $\mathbf{z} \mapsto \mathscr{R}(\mathbf{z},\mathbf{x})\mathbf{y}$ and restricting it to $S^2(V^*)$.

Definition 2.3.9 (*Ricci tensor*) Let (M, g) be a pseudo-Riemannian manifold and let $\mathscr{R}$ be its Riemann curvature mapping. The mapping

$$\widetilde{\mathsf{Ric}} : O(M) \to S^2(V^*)\,, \quad \widetilde{\mathsf{Ric}}(u)(\mathbf{x},\mathbf{y}) := \operatorname{tr}\{\mathbf{z} \mapsto \mathscr{R}(u)(\mathbf{z},\mathbf{x})\mathbf{y}\} \tag{2.3.22}$$

is called the Ricci curvature mapping. Correspondingly,

$$\mathsf{Ric} : \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}\,, \quad \mathsf{Ric}(X,Y) := \operatorname{tr}\{Z \mapsto \mathsf{R}(Z,X)Y\} \tag{2.3.23}$$

is called the Ricci tensor of (M, g).

Note that Ric is of the same geometric type as the metric. Thus, viewing it as a mapping $\mathrm{T}_m M \to \mathrm{T}_m^* M$ and using $\mathsf{g}^{-1} : \mathrm{T}_m^* M \to \mathrm{T}_m M$, we can define a scalar on M.

Definition 2.3.10 (*Scalar curvature*) Let (M, g) be a pseudo-Riemannian manifold and let Ric be its Ricci tensor. The function

$$\mathsf{Sc} : M \to \mathbb{R}\,, \quad \mathsf{Sc}(m) := \operatorname{tr}(\mathsf{g}^{-1} \circ \mathsf{Ric})(m) \tag{2.3.24}$$

is called the scalar curvature of (M, g). The corresponding equivariant function $\widetilde{\mathsf{Sc}} : O(M) \to \mathbb{R}$ is called the scalar curvature mapping.

The scalar curvature corresponds to the first component in the decomposition (2.3.21). The component corresponding to the third summand is called the Weyl tensor. In Sect. 2.8, the above decomposition will be discussed in detail for the case $n = 4$.

Remark 2.3.11 Denoting $\mathsf{R}_{ij} = \mathsf{Ric}(e_i, e_j)$, we obtain the following local expressions for the Ricci tensor and the scalar curvature,

$$\mathsf{R}_{ij} = \mathsf{g}^{kl}\,\mathsf{R}_{kijl}\,, \quad \mathsf{Sc} = \mathsf{g}^{ij}\mathsf{R}_{ij}\,. \tag{2.3.25}$$

In particular, for a holonomic frame, we obtain

$$\mathsf{R}_{ij} = \partial_i\,\Gamma^l{}_{jl} - \partial_j\,\Gamma^l{}_{il} + \Gamma^l{}_{jm}\Gamma^m{}_{il} - \Gamma^l{}_{im}\Gamma^m{}_{jl}\,. \tag{2.3.26}$$

[21] For $k + l = 3$, one obtains $\mathfrak{K}(\mathfrak{o}(k,l)) = \mathbb{R} \oplus \Sigma_0^2$. For $k + l = 4$, this result belongs to Singer and Thorpe [592].

For an orthonormal local frame, we have $\mathsf{R}_{ij} = \eta^{kl}\,\mathsf{R}_{kijl}$. This yields the following useful formula

$$\mathsf{Ric}(X,Y) = \eta^{kl}\mathsf{g}(\mathsf{R}(e_k,X)Y,e_l)\,, \quad X,Y \in \mathfrak{X}(M)\,. \tag{2.3.27}$$

♦

There is an important special class of Riemannian manifolds characterized by the fact that their curvature has a trivial Σ_0^2-component in the decomposition (2.3.21).

Definition 2.3.12 (*Einstein manifold*) A (pseudo-)Riemannian manifold (M,g) is called Einstein if its Ricci tensor is a constant multiple of the metric at each point of M.

Note that for an n-dimensional Einstein space (M,g) we have

$$\mathsf{Ric} = \frac{\mathsf{Sc}}{n}\,\mathsf{g}\,, \tag{2.3.28}$$

where Sc is constant. In Sect. 2.5, we will see a large class of Einstein manifolds.

In the next step, we show which impact the above additional structures have on the analysis of the Berger criteria in the metric case. For a chosen orthonormal frame $u_0 \in P_{u_0}(\Gamma)$, let us consider the holonomy bundle $P_{u_0}(\Gamma) \subset O(M)$. Let us denote

$$H = \mathscr{H}_{u_0}(\Gamma)\,, \quad \mathfrak{h} = \mathfrak{h}_{u_0}(\Gamma)\,.$$

On $P_{u_0}(\Gamma)$, the curvature takes values in $\mathfrak{h} \subset \mathfrak{o}(k,l) \cong \bigwedge^2(V^*)$. This fact, together with (2.3.19), implies the following.

Proposition 2.3.13 *For any point $u \in P_{u_0}(\Gamma)$, the Riemann curvature $\mathscr{R}(u)$ belongs to the space*

$$\mathfrak{K}(\mathfrak{h}) = \ker\varphi \,\cap\, S^2(\mathfrak{h})\,. \tag{2.3.29}$$

■

It turns out that for many subgroups $H \subset \mathrm{O}(k,l)$, the restriction of φ to $S^2(\mathfrak{h})$ is injective. This implies $\mathfrak{K}(\mathfrak{h}) = 0$ and, thus, $\underline{\mathfrak{h}} = 0$. Then, the first Berger criterion implies that, in this case, H cannot occur as a holonomy group.

In the same way, the covariant derivative $D\mathscr{R}$ may be dealt with. By the above discussion, we have the following.

Proposition 2.3.14 *For any point $u \in P_{u_0}(\Gamma)$, the covariant derivative of $\mathscr{R}(u)$ takes values in*

$$\mathfrak{K}^1(\mathfrak{h}) = \ker\delta' \,\cap\, \big(V^* \otimes \mathfrak{K}(\mathfrak{h})\big)\,, \tag{2.3.30}$$

where $\delta' : V^* \otimes \mathfrak{K}^1(\mathfrak{o}(k,l)) \to \bigwedge^3 V^* \times \mathfrak{o}(k,l)$, cf. formula (2.3.7). ■

As already mentioned above, the condition $\mathfrak{K}^1(\mathfrak{h}) = 0$ distinguishes a special class of possible candidates. By Proposition 2.3.4, in this case the Riemannian manifold is necessarily locally symmetric. We exclude this class of spaces for a while, postponing its presentation to Sect. 2.5.

Finally, we show that we may limit our attention to the case where the representation of the holonomy group H on $V \equiv \mathbb{R}^n$ is irreducible. We consider the Riemannian metric case and comment on the pseudo-Riemannian case at the end. Under this assumption, the holonomy group is a subgroup of $\mathrm{O}(n)$. Let us assume, on the contrary, that the representation of H is reducible, that is, there exists a proper subspace $W \subset V$ invariant under H. Since we assume that η be definite, there exists an invariant orthogonal complement $W^\perp \subset V$. Proceeding further in this manner, we obtain an invariant orthogonal decomposition

$$V = W_0 \oplus W_1 \oplus \ldots \oplus W_k \, , \tag{2.3.31}$$

with W_0 carrying the trivial representation[22] (acting as the identity) and W_k carrying nontrivial irreducible representations of H for all $k \geq 1$. The following theorem belongs to de Rham [150]. It simplifies the holonomy classification problem essentially.

Theorem 2.3.15 (de Rham Splitting Theorem) *Let (M, g) be a Riemannian manifold. If the holonomy group H acts reducibly on $\mathbb{R}^n$, then the restricted holonomy group*[23] *H^0 of (M, g) is isomorphic to a product,*

$$H^0 = \{e\} \times H_1 \times \ldots \times H_k \, ,$$

and M is locally isomorphic to a product of Riemannian manifolds,

$$M_0 \times M_1 \times \ldots \times M_k \, ,$$

with M_0 being flat.

Proof By the above discussion, $\mathscr{R} : O(M) \to \bigwedge^2 V^* \otimes \mathfrak{o}(n)$ and $\mathscr{R}(u)(\mathbf{x}, \mathbf{y})$ takes values in $\mathfrak{h} \equiv \mathfrak{h}_{u_0}(\Gamma)$, for any $u \in P_{u_0}(\Gamma)$ and any $\mathbf{x}, \mathbf{y} \in V$. Since the decomposition (2.3.31) is invariant, we have

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y})_{\restriction W_0} = 0 \, , \quad \mathscr{R}(u)(\mathbf{x}, \mathbf{y})_{\restriction W_i} \subset W_i \, , \tag{2.3.32}$$

for $1 \leq i \leq k$. We decompose $\mathbf{x} = \sum \mathbf{x}_i$ and $\mathbf{y} = \sum \mathbf{y}_i$ with respect to (2.3.31) and insert this decomposition into $\mathscr{R}(u)(\mathbf{x}, \mathbf{y})$. This yields

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y}) = \sum_i \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i) + \sum_{i \neq j} \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_j) \, .$$

---

[22] Clearly, W_0 may be zero.

[23] By Theorem 1.7.9, this is the identity connected component of H.

By (2.3.12) and (2.3.32), we have $\mathscr{R}(u)(W_i, W_j)W_k = 0$ for i, j and k pairwise distinct. Next, consider the case $i = j \neq k$. Then, again by (2.3.12),

$$\mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)\mathbf{z}_k = 0\,, \quad \mathscr{R}(u)(\mathbf{y}_i, \mathbf{z}_k)\mathbf{x}_i = -\mathscr{R}(u)(\mathbf{z}_k, \mathbf{x}_i)\mathbf{y}_i\,.$$

The first of these equations implies $\mathscr{R}(u)(W_i, W_i)W_k = 0$ for $i \neq k$. Using (2.3.11), from the second equation we obtain

$$\eta\big(\mathscr{R}(u)(\mathbf{z}_k, \mathbf{x}_i)\mathbf{y}_i, \mathbf{x}_i\big) = \eta\big(\mathscr{R}(u)(\mathbf{z}_k, \mathbf{y}_i)\mathbf{x}_i, \mathbf{x}_i\big) = \eta\big(\mathscr{R}(u)(\mathbf{x}_i, \mathbf{x}_i)\mathbf{z}_k, \mathbf{y}_i\big)\,,$$

and the anti-symmetry of $\mathscr{R}$ implies $\mathscr{R}(u)(W_k, W_i)W_i = 0$ for $i \neq k$. We conclude

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y}) = \sum_i \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)\,.$$

Now, according to the equivariance of $\mathscr{R}$, as u ranges over $\pi^{-1}(m) \cap P_{u_0}(\varGamma)$ and $\mathbf{x}, \mathbf{y}$ over V, for every i, the mappings $\mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)$ span an ideal $\mathfrak{h}_i(m) \subset \mathrm{End}(W_i)$ of $\mathfrak{h}$. Finally, varying m yields ideals $\mathfrak{h}_i$ and, by (2.3.1), the decomposition

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k\,.$$

This proves the first assertion. To prove the second assertion, first note that the splitting (2.3.31) induces a splitting of the horizontal distribution $\varGamma$ on $P_{u_0}(\varGamma)$,

$$\varGamma = \varGamma_1 \oplus \ldots \oplus \varGamma_k\,, \quad \varGamma_i := \varGamma \cap \theta^{-1}(W_i)\,.$$

By H-equivariance, this splitting induces a family of distributions $D_i = \pi'(\varGamma_i)$ on M such that

$$\mathrm{T}M = D_1 \oplus \ldots \oplus D_k\,.$$

Moreover, corresponding to (2.3.31), let us decompose

$$\theta = \theta_1 + \ldots + \theta_k\,, \quad \omega = \omega_1 + \ldots + \omega_k\,, \quad \varOmega = \varOmega_1 + \ldots + \varOmega_k\,,$$

with $\theta_i \in \varOmega^1(P_{u_0}(\varGamma)) \otimes W_i$ and $\omega_i\,,\ \varOmega_i \in \varOmega^*(P_{u_0}(\varGamma)) \otimes \mathfrak{h}_i$. We define the distributions

$$\hat{\varGamma}_i := \varGamma_i \oplus V_i$$

on $P_{u_0}(\varGamma)$, with V_i being the vertical distribution spanned by the Killing vector fields generated from elements of $\mathfrak{h}_i$. Clearly, $\varGamma_i$ is spanned by the horizontal standard vector fields generated by any basis of W_i. Thus, $\hat{\varGamma}_i$ annihilates both θ_j, ω_j, and $\varOmega_j$ for any $j \neq i$ and, by point 2 of Remark 2.1.14 and (1.4.5), for every i the distribution $\hat{\varGamma}_i$ is involutive. Consequently, by the Frobenius Theorem, it is integrable and, for every i, we have

$$\mathrm{d}\theta_i + \omega_i \wedge \theta_i = 0\,, \quad \Omega_i = \mathrm{d}\omega_i + \frac{1}{2}[\omega_i, \omega_i]\,. \tag{2.3.33}$$

Let $P_i \subset P_{u_0}(\Gamma)$ be an integral manifold of $\hat{\Gamma}_i$. Integrability of $\hat{\Gamma}_i$ clearly induces integrability of D_i and the integral manifolds U_i of D_i fulfil $U_i = \pi(P_i) \subset M$. Moreover, for every i, the restriction $\pi_i : P_i \to U_i$ of π defines a principal H_i-bundle and, by (2.3.33), ω_i is a torsion-free connection on P_i with restricted holonomy group H_i.

To summarize, for every $m \in M$, there exists a neighbourhood $U \cong U_1 \times \ldots \times U_k$ of m in M, with the U_i being integral manifolds of D_i, and the Levi-Civita connection restricted to U being a product of the Levi-Civita connections on the components U_i. ■

Definition 2.3.16 A Riemannian manifold (M, g) which is locally isomorphic to a product of Riemannian manifolds is called locally reducible. It is called irreducible if it is not locally reducible.

Clearly, by Theorem 2.3.15, if (M, g) is irreducible, then the restricted holonomy group necessarily acts irreducibly. Under additional assumptions, de Rham [150] was able to prove the following global version of Theorem 2.3.15.

Theorem 2.3.17 (Global de Rham Splitting Theorem) *Let (M, g) be a geodesically complete simply connected Riemannian manifold and assume that the holonomy group*[24] *of the Levi-Civita connection acts reducibly. Then, (M, g) is the direct product of geodesically complete simply connected irreducible Riemannian manifolds (M_i, g_i),*

$$(M, \mathsf{g}) = (M_0, \mathsf{g}_0) \times (M_1, \mathsf{g}_1) \times \ldots \times (M_k, \mathsf{g}_k)\,.$$

Here, (M_0, g_0) is a Euclidean vector space whose dimension is possibly zero. ■

Remark 2.3.18 Both versions of the de Rham Splitting Theorem have been extended to the case of an indefinite metric by Wu [682, 683]. ♦

Summarizing our discussion, for finding the possible holonomy groups of a Riemannian manifold (M, g), it is reasonable to make the following assumptions:

(a) M is simply connected. This ensures that the holonomy group is connected and that it coincides with the restricted holonomy group.
(b) (M, g) is irreducible. This implies that the holonomy group acts irreducibly.
(c) (M, g) is not locally symmetric. This requires $\mathfrak{K}^1(\mathfrak{h}) \neq 0$.

Under these assumptions, for the Riemannian case, Berger obtained the following.

Theorem 2.3.19 (Berger) *Let (M, g) be an n-dimensional simply connected irreducible Riemannian manifold which is not locally symmetric. Then, its holonomy group H belongs to one of the following classes:*

[24] By Remark 1.7.11, if M is simply connected, then the holonomy group and the restricted holonomy group coincide.

1. $H = \mathrm{SO}(n),\ n \geq 2$, *(generic Riemannian manifold)*
2. $H = \mathrm{U}(m),\ n = 2m \geq 4$, *(generic Kähler manifold)*
3. $H = \mathrm{SU}(m),\ n = 2m \geq 4$, *(special Kähler manifold)*
4. $H = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1),\ n = 4m \geq 8$, *(quaternionic Kähler manifold)*
5. $H = \mathrm{Sp}(m),\ n = 4m \geq 8$, *(Hyper-Kähler manifold)*
6. $H = G_2,\ n = 7$, *(special holonomy)*
7. $H = \mathrm{Spin}(7),\ n = 8$, *(special holonomy).* ■

For the proof, which is beyond the scope of this book, we refer to [68, 69, 555].

Remark 2.3.20

1. An elegant proof of Theorem 2.3.19 is obtained from the following result of Simons [591]: if M is irreducible, then either the holonomy group H acts transitively on S^{n-1} or its identity component acts trivially on the space of curvature tensors $\mathfrak{K}(\mathfrak{h})$. Then, Theorem 2.3.19 is obtained by using the classification of simple Lie algebras and their representations.
2. According to Examples 2.2.22 and 2.2.27, it was clear from the beginning that the groups $\mathrm{SO}(n)$ and $\mathrm{U}(n)$ must occur in the above list. For a detailed discussion of examples for all the groups occuring in Theorem 2.3.19, we refer to [555]. ♦

Exercises

2.3.1 Show that the commutators of horizontal standard vector fields corresponding to a torsion-free connection are vertical.

2.3.2 Confirm the equivariance property (2.3.14). *Hint:* Under the identification $\mathfrak{o}(n) \cong (\mathbb{R}^n)^* \wedge (\mathbb{R}^n)^*$, the adjoint representation is mapped onto the second exterior power of the dual of the basic representation.

2.3.3 Show that, in terms of the Riemann curvature R, the Bianchi identity (2.1.16) reads

$$(\nabla_X \mathsf{R})(Y, Z) + (\nabla_Y \mathsf{R})(Z, X) + (\nabla_Z \mathsf{R})(X, Y) = 0\,. \tag{2.3.34}$$

2.4 Sectional Curvature

In this section, we discuss a generalization of the classical Gaussian curvature of surfaces in $\mathbb{R}^3$. It reduces the study of the Riemann curvature to the study of real valued functions. Let (M, g) be a pseudo-Riemannian manifold. Let $\Sigma_m \subset \mathrm{T}_m M$ be a 2-dimensional subspace such that $\mathsf{g}_{\restriction \Sigma_m}$ is non-degenerate. Let $\{X, Y\}$ be an arbitrary basis of Σ_m. We put

$$\mathsf{K}(\Sigma_m) := \frac{\langle \mathsf{R}(X, Y)Y, X\rangle}{\parallel X \parallel^2 \parallel Y \parallel^2 - \langle X, Y\rangle^2}\,, \tag{2.4.1}$$

where $\| \cdot \|^2$ and $\langle \cdot, \cdot \rangle$ are the quadratic form and the bilinear form, respectively, induced from g. It can be easily shown that $\mathsf{K}(\Sigma_p)$ is well defined, that is,

(a) the right hand side of (2.4.1) does not depend on the choice of the basis. This is a simple consequence of the symmetry properties of R given by point 2 of Remark 2.3.7 and is, thus, left to the reader (Exercise 2.4.1).
(b) Σ_m is non-degenerate iff $\| X \|^2 \| Y \|^2 - \langle X, Y \rangle^2 \neq 0$, (Exercise 2.4.2).

Note that K may be viewed as a mapping from the Graßmann manifold $G_2(\mathrm{T}_m M)$ to $\mathbb{R}$. Let $G_2^0(\mathrm{T}_m M) \subset G_2(\mathrm{T}_m M)$ be the subset of non-degenerate subspaces.

Definition 2.4.1 The mapping $\mathsf{K} : G_2^0(\mathrm{T}_m M) \to \mathbb{R}$ given by (2.4.1) is called the sectional curvature of the pseudo-Riemannian manifold at $m \in M$.

Clearly, in the Riemannian case, every 2-dimensional subspace of $\mathrm{T}_m M$ is non-degenerate.

Proposition 2.4.2 *The curvature tensor* R *is completely determined by the sectional curvature. If the mapping* K *is constant, that is,* $\mathsf{K}(\Sigma_m) = k(m)$ *for every* $\Sigma_m \in G_2^0(\mathrm{T}_m M)$*, then*

$$\mathsf{R}_m(X, Y)Z = k(m)\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big) . \tag{2.4.2}$$

Conversely, if (2.4.2) *is fulfilled, then all non-degenerate planes have sectional curvature* $k(m)$.

Proof The proof of the first assertion is the consequence of the following simple polarization argument. Denote $\alpha(X, Y) := \langle \mathsf{R}(X, Y)X, Y\rangle$, for any $X, Y \in \mathrm{T}_m M$. Then, by direct inspection,

$$\begin{aligned} -6\langle \mathsf{R}(X, Y)Z, W\rangle = {} & \alpha(X + W, Y + Z) - \alpha(X + W, Y) - \alpha(X + W, Z) \\ & - \alpha(X, Y + Z) - \alpha(W, Y + Z) + \alpha(X, Z) + \alpha(W, Y) \\ & - \alpha(Y + W, X + Z) + \alpha(Y + W, X) + \alpha(Y + W, Z) \\ & + \alpha(Y, X + Z) + \alpha(W, X + Z) - \alpha(Y, Z) - \alpha(W, X) , \end{aligned}$$

showing that R is determined by α and, thus, by K. We prove the second statement. For that purpose, denote

$$\mathsf{R}_0(X, Y)Z := \langle Y, Z\rangle X - \langle X, Z\rangle Y .$$

Note that R_0 shares the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R.[25] Assume that $\mathsf{K}(\Sigma_m) = k(m)$ for all non-degenerate planes. If X, Y span a non-degenerate plane, then by (2.4.1),

$$\langle \mathsf{R}(X, Y)Y, X\rangle = k(m)\big(\langle Y, Y\rangle X - \langle X, Y\rangle Y\big) = \langle k(m)\mathsf{R}_0(X, Y)Y, X\rangle .$$

[25]It also shares the symmetry property (2.3.11), but this is not needed here.

Thus, the tensor $\hat{\mathsf{R}} := \mathsf{R} - k(m)\mathsf{R}_0$ has the above symmetry properties and fulfils

$$\langle \hat{\mathsf{R}}(X, Y)Y, X\rangle = 0\,. \tag{2.4.3}$$

If X and Y span a degenerate plane, we can choose sequences $X_n \to X$ and $Y_n \to Y$ of tangent vectors such that X_n and Y_n span non-degenerate planes for each n.[26] Then, $\langle \hat{\mathsf{R}}(X_n, Y_n)Y_n, X_n\rangle = 0$ for all n and, thus, (2.4.3) holds for degenerate planes as well. Finally, note that this equation is also true for pairs X, Y which are linearly dependent. We conclude that (2.4.3) holds for all $X, Y \in \mathrm{T}_m M$. Now, the assertion is a consequence of the following simple algebraic fact (Exercise 2.4.3): If

$$\tilde{\mathsf{R}} : \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$$

is a quadrilinear mapping sharing the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R, then $\langle \tilde{\mathsf{R}}(X, Y)Y, X\rangle = 0$ implies $\tilde{\mathsf{R}} = 0$.

The converse statement is trivial. ■

Proposition 2.4.2 leads us to an important class of pseudo-Riemannian manifolds.

Definition 2.4.3 If $\mathsf{K}(\Sigma_m) = k(m)$ for every $\Sigma_m \in G_2^0(\mathrm{T}_m M)$, then we say that (M, g) is a space of constant curvature at m. Let k be a real number. We say that (M, g) is a space of constant curvature k if $\mathsf{K}(\Sigma_m) = k$ at every point $m \in M$.

Remark 2.4.4

1. By the proof of Proposition 2.4.2, for a space of constant curvature, we have

$$\mathsf{R}(X, Y)Z = k\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big)\,, \quad k \in \mathbb{R}\,. \tag{2.4.4}$$

2. By a theorem of Schur, see Theorem 2.2. in Chap. V of [381], if (M, g) is a space of constant curvature at every point of M and $\dim M \geq 3$, then M is a space of constant curvature, that is, the mapping $m \to k(m)$ is constant.
3. It is not hard to construct models of spaces of constant curvature. The simplest Riemannian example is the n-sphere of radius r embedded in the standard way in $\mathbb{R}^{n+1}$. This is a space of constant curvature equal to $\frac{1}{r^2}$. The simplest pseudo-Riemannian model is the pseudo-Euclidean space $(\mathbb{R}^n_s, \mathsf{g}^n_s)$ with the signature $(n - s, s)$. It is easy to show that this is a space of constant curvature equal to 0. In Sect. 2.5, we will see a large class of spaces of constant curvature. For an exhaustive presentation of this subject we refer to [676].
4. In the indefinite case, there is a lot of subtleties and there is quite a number of classical papers on that subject, see [63, 145, 257, 395, 490] and further references therein. ♦

[26] By property (b) above, in any fixed basis of $\mathrm{T}_m M$, $\parallel X \parallel^2 \parallel Y \parallel^2 - \langle X, Y\rangle^2$ is a polynomial in the components of X and Y whose zero set does not contain any open subset.

Exercises

2.4.1 Prove that (2.4.1) does not depend on the choice of the basis.

2.4.2 Show that the restriction of a pseudo-Riemannian metric to a 2-dimensional subspace $\Sigma_m \subset \mathrm{T}_m M$ is non-degenerate iff $\parallel X \parallel^2 \parallel Y \parallel^2 - \langle X, Y\rangle^2 \neq 0$.

2.4.3 Prove the following. If $\tilde{\mathsf{R}} : \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$ is a quadrilinear mapping sharing the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R, then $\langle \tilde{\mathsf{R}}(X, Y)Y, X\rangle = 0$ implies $\tilde{\mathsf{R}} = 0$.

2.5 Symmetric Spaces

In this section, we take up the discussion from Sect. 2.3. We analyze the special case $\mathfrak{K}^1(\mathfrak{h}) = 0$, that is, we analyze the condition

$$D\mathscr{R} = 0\,, \tag{2.5.1}$$

defining locally symmetric manifolds, cf. Definition 2.3.2. Thus, we give up assumption (c) prior to Theorem 2.3.19, but we keep on assuming the following.

(a) M is simply connected, which ensures that the holonomy group H is connected and that it coincides with the restricted holonomy group.
(b) (M, g) is irreducible, which implies that H acts irreducibly.

Moreover, as above, we limit our attention to the Riemannian metric case, that is, $H \subset \mathrm{O}(n)$ is a compact Lie subgroup acting irreducibly on $V \equiv \mathbb{R}^n$. Then, by the Holonomy Principle, cf. Proposition 1.7.20, the space of parallel sections of

$$E = O(M) \times_{\mathrm{O}(n)} S^2\left(\bigwedge\nolimits^2 V^*\right)$$

is in one-to-one correspondence with the space of holonomy-invariant vectors in $S^2\left(\bigwedge^2 V^*\right)$ as follows. Any $\mathscr{R}$ satisfying (2.5.1) is constant on $P_{u_0}(\Gamma)$ and, restricted to $P_{u_0}(\Gamma)$, it takes values in $\mathfrak{K}(\mathfrak{h})$ given by (2.3.29). Thus, the Holonomy Principle assigns to $\mathscr{R}$ the H-invariant element

$$F := \mathscr{R}(u) \in \mathfrak{K}(\mathfrak{h}), \quad u \in P_{u_0}(\Gamma)\,. \tag{2.5.2}$$

Lemma 2.5.1 *Let $H \subset \mathrm{O}(n)$ be a closed subgroup and let $F \in \mathfrak{K}(\mathfrak{h})$ be an H-invariant element. Then, $\mathfrak{g} = \mathfrak{h} \oplus V$ carries the structure of a Lie algebra given by*

$$[A, \mathbf{x}] = -[\mathbf{x}, A] = A\mathbf{x}\,, \quad [\mathbf{x}, \mathbf{y}] = -F(\mathbf{x}, \mathbf{y})\,, \quad A \in \mathfrak{h}\,,\ \mathbf{x}, \mathbf{y} \in V\,.$$

Proof Bilinearity and anti-symmetry are obvious. We prove that the Jacobi identity holds. For that purpose, we have to consider three cases:
(a) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Since $F(\mathbf{x}, \mathbf{y}) \in \mathfrak{h} \subset \mathrm{End}(V)$, the definition of $\mathfrak{K}(\mathfrak{h})$ implies

$$[[\mathbf{x}, \mathbf{y}], \mathbf{z}] + [[\mathbf{y}, \mathbf{z}], \mathbf{x}] + [[\mathbf{z}, \mathbf{x}], \mathbf{y}] = 0\,.$$

(b) Let $\mathbf{x}, \mathbf{y} \in V$. By the H-invariance of F, cf. (2.1.25), we have

$$F(\mathbf{x}, \mathbf{y}) = \mathrm{Ad}(a^{-1}) \circ F(a\mathbf{x}, a\mathbf{y})\,, \quad a \in H \subset \mathrm{O}(n)\,.$$

Differentiating this equation, we obtain

$$[F(\mathbf{x}, \mathbf{y}), A] + F(A\mathbf{x}, \mathbf{y}) + F(\mathbf{x}, A\mathbf{y}) = 0$$

for any $A \in \mathfrak{h}$. This implies

$$[[\mathbf{x}, \mathbf{y}], A] + [[\mathbf{y}, A], \mathbf{x}] + [[A, \mathbf{x}], \mathbf{y}] = 0\,.$$

(c) Let $\mathbf{x} \in V$ and $A, B \in \mathfrak{h}$. Then, by definition of the Lie bracket of $\mathfrak{h} \subset \mathrm{End}(V)$,

$$[A, B](\mathbf{x}) = A(B\mathbf{x}) - B(A\mathbf{x})\,.$$

This proves the third case. ■

To make contact with the standard notation, we denote $V = \mathfrak{m}$. Then,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{2.5.3}$$

and the commutation relations of $\mathfrak{g}$ fulfil:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,. \tag{2.5.4}$$

Moreover, by the Ambrose-Singer Theorem,

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}\,. \tag{2.5.5}$$

Associated with the decomposition (2.5.3), there is a linear mapping

$$\lambda : \mathfrak{g} \to \mathfrak{g}\,, \quad \lambda(A, \mathbf{x}) := (A, -\mathbf{x})\,, \quad A \in \mathfrak{h}\,, \mathbf{x} \in \mathfrak{m}\,. \tag{2.5.6}$$

By (2.5.4), λ is an involutive Lie algebra homomorphism (Exercise 2.5.1). Conversely, we have the following.

Lemma 2.5.2 *Any involutive Lie algebra homomorphism λ of a Lie algebra $\mathfrak{g}$ induces a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ fulfilling* (2.5.4).

Proof Since $\lambda^2 = \mathrm{id}$, λ is diagonalizable and we may decompose $\mathfrak{g}$ into the eigenspaces $\mathfrak{h}$ and $\mathfrak{m}$ of λ corresponding to the eigenvalues $+1$ and -1, respectively. Now, the first relation in (2.5.4) is obvious. To check the remaining two, we calculate

$$\lambda([A, \mathbf{x}]) = [\lambda(A), \lambda(\mathbf{x})] = -[A, \mathbf{x}], \quad A \in \mathfrak{h}, \mathbf{x} \in \mathfrak{m},$$

that is, $[A, \mathbf{x}] \in \mathfrak{m}$. Similarly, $\lambda([\mathbf{x}, \mathbf{y}]) = [\mathbf{x}, \mathbf{y}] \in \mathfrak{h}$ for any $\mathbf{x}, \mathbf{y} \in \mathfrak{m}$. ■

Definition 2.5.3 Let $\mathfrak{g}$ be a Lie algebra and let λ be an involutive automorphism of $\mathfrak{g}$. Then, the pair $(\mathfrak{g}, \lambda)$ is called a symmetric Lie algebra. In addition,

1. if the set of fixed points $\mathfrak{h}$ of λ is a compactly embedded Lie subalgebra[27] of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda)$ is called an orthogonal symmetric Lie algebra,
2. if $\mathfrak{h} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda)$ is called effective.
3. if $(\mathfrak{g}, \lambda)$ is effective and $\mathrm{ad}([\mathfrak{m}, \mathfrak{m}])$ acts irreducibly on $\mathfrak{m}$, then $(\mathfrak{g}, \lambda)$ is called irreducible.

Proposition 2.5.4 *The Lie algebra $\mathfrak{g}$ constructed in Lemma 2.5.1, endowed with the involutive automorphism λ given by (2.5.6), is an irreducible orthogonal symmetric Lie algebra.*

Proof By construction, $(\mathfrak{g}, \lambda)$ is symmetric. Since, by assumption, $H \subset \mathrm{O}(n)$ is a compact Lie subgroup acting faithfully on $\mathbb{R}^n$, $\mathrm{ad}(\mathfrak{h})$ is compact and, thus, $(\mathfrak{g}, \lambda)$ is orthogonal. Suppose $A \in \mathfrak{h} \cap \mathfrak{z}$. Then,

$$A\mathbf{x} = [A, \mathbf{x}] = 0$$

for every $\mathbf{x} \in \mathfrak{m}$ and, thus, $A = 0$. Thus, $(\mathfrak{g}, \lambda)$ is effective. Finally, by assumption, H acts irreducibly on $\mathfrak{m}$. Thus, $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, too. This, together with (2.5.5) implies that $(\mathfrak{g}, \lambda)$ is irreducible. ■

In the sequel, the pair $(\mathfrak{g}, \lambda)$ constructed above will be called the canonical symmetric Lie algebra associated with the locally symmetric Riemannian manifold we started with. The decomposition (2.5.3) will be called the canonical decomposition of $(\mathfrak{g}, \lambda)$.

The following proposition characterizes irreducible symmetric Lie algebras.

Proposition 2.5.5 *Let $(\mathfrak{g}, \lambda)$ be an irreducible symmetric Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition induced by λ. Then, one of the following cases occurs:*

1. *$\mathfrak{g}$ is a simple Lie algebra.*
2. *$\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$ with $\tilde{\mathfrak{g}}$ simple, fulfilling $\mathfrak{h} = \{(A, A) : A \in \tilde{\mathfrak{g}}\}$ and $\lambda(A, B) = (B, A)$ for any $A, B \in \tilde{\mathfrak{g}}$.*
3. *$[\mathfrak{m}, \mathfrak{m}] = 0$.*

For the proof we refer the reader to [381].[28]

[27] That is, the group of transformations of $\mathfrak{g}$ generated by $\mathrm{ad}(\mathfrak{h})$ is compact.

[28] Cf. Proposition 7.5 in Vol. 2, Chap. XI of [381].

Remark 2.5.6

1. Assume that either point 1 or point 2 of Proposition 2.5.5 holds. Then, since $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is an ideal in $\mathfrak{g}$, we have $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$. Thus, an effective symmetric Lie algebra is irreducible iff $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$, that is, iff $\mathfrak{g}$ is of the form described either by point 1 or by point 2. In particular, if $(\mathfrak{g}, \lambda)$ is irreducible, then $\mathfrak{g}$ is semisimple.
2. Conversely, if $(\mathfrak{g}, \lambda)$ is an orthogonal symmetric Lie algebra and $\mathfrak{g}$ is simple, then $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, see Proposition 7.4 in Vol. 2, Chap. XI of [381]. ♦

Proposition 2.5.5 and property (2.5.5) imply that the canonical symmetric Lie algebra $(\mathfrak{g}, \lambda)$ is semisimple. Consequently, by Proposition I/5.4.10, the Killing form

$$\mathsf{k} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \mathsf{k}(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)),$$

of $\mathfrak{g}$ is non-degenerate. Moreover, the relations (2.5.4) imply that the decomposition (2.5.3) is orthogonal with respect to k (Exercise 2.5.2). Equivalently, k is λ-invariant. This implies that the restrictions $\mathsf{k}^{\mathfrak{h}}$ and $\mathsf{k}^{\mathfrak{m}}$ of k to $\mathfrak{h}$ and $\mathfrak{m}$, respectively, are both non-degenerate and λ-invariant, too. Moreover, they have the following properties:

(a) By Corollary I/5.5.8, $\mathsf{k}^{\mathfrak{h}}$ is negative semidefinite and, since $(\mathfrak{g}, \lambda)$ is effective, it is negative definite.
(b) Since $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$ and since both $\mathsf{k}^{\mathfrak{m}}$ and the scalar product η on $\mathfrak{m}$ induced from the metric g are $\mathrm{ad}(\mathfrak{h})$-invariant, by Schur's Lemma, they must be proportional to each other,

$$\eta(\mathbf{x}, \mathbf{z}) = -c\, \mathsf{k}^{\mathfrak{m}}(\mathbf{x}, \mathbf{z}), \quad \mathbf{x}, \mathbf{z} \in \mathfrak{m}, \; c \in \mathbb{R}, c \neq 0. \tag{2.5.7}$$

Thus, since η is positive definite, $\mathsf{k}^{\mathfrak{m}}$ is either positive or negative definite.

Definition 2.5.7 An effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \lambda)$ with $\mathfrak{g}$ semisimple is said to be of compact or of non-compact type, if the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$ is, respectively, negative definite or positive definite.

Remark 2.5.8 Combining Proposition 2.5.5 with Propositions 7.4 and 7.5 in in Vol. 2, Chap. XI of [381], one can show that any irreducible orthogonal symmetric Lie algebra is either of compact or of non-compact type. ♦

Next, we show that, given an irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}, \lambda)$, one can construct a special type of homogeneous Riemannian manifold.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition induced from λ. Let $\tilde{G}$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{H}$ be the connected Lie subgroup corresponding to $\mathfrak{h}$. Then, the space of left cosets $M := \tilde{G}/\tilde{H}$ is a simply connected manifold endowed with the natural left $\tilde{G}$-action given by left translations. Let

$$\tilde{Z} = \left\{ g \in \tilde{G} : g(m) = m \ \text{ for all } \ m \in M \right\}$$

be the kernel of this action. Since, by assumption, $(\mathfrak{g}, \lambda)$ is effective, $\tilde{Z}$ must be discrete. Thus, M is an almost effective $\tilde{G}$-manifold. We pass to an effective action by setting $G := \tilde{G}/\tilde{Z}$ and $H := \tilde{H}/\tilde{Z}$. Then, $M = G/H$, G and H are connected, and we have the natural left effective action

$$\delta : G \times G/H \to G/H\,, \quad (a, [g]) \mapsto \delta_a([g]) := [ag]\,.$$

By point 4 of Example 1.1.4, the natural projection $\pi : G \to M$ endows G with the structure of a principal H-bundle P and the tangent mapping π' identifies $\mathfrak{m}$ and $\mathrm{T}_{[\mathbb{1}]}M$ as vector spaces. Under this identification, the isotropy representation

$$H \to \mathrm{Aut}(\mathrm{T}_{[\mathbb{1}]}M)\,, \quad h \mapsto (\delta_h)'_{\mathbb{1}}\,,$$

is given by $\mathrm{Ad}(H)$ acting on $\mathfrak{m}$, cf. point 1 of Remark I/6.2.10. Correspondingly,

$$G \times_{\mathrm{Ad}(H)} \mathfrak{m} \to \mathrm{T}M\,, \quad [(a, \mathbf{x})] \mapsto [\mathrm{L}'_a(\mathbf{x})], \tag{2.5.8}$$

is an isomorphism. Since $(\mathfrak{g}, \lambda)$ is orthogonal and irreducible, there exists an $\mathrm{Ad}(H)$-invariant scalar product η on $\mathfrak{m}$ which is unique up to a positive factor. Clearly, η induces an H-invariant scalar product on $\mathrm{T}_{[\mathbb{1}]}M$ which, using the left G-action δ, can be extended to a G-invariant Riemannian metric g on M. To summarize, we have constructed a simply connected transitive and effective G-manifold (M, g) with G acting by isometries.

Consider the bundle of orthonormal frames $O(M)$ of (M, g). Note that any η-orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathfrak{m}$ induces via π' a g-orthonormal frame $(e_1, \ldots, e_n)$ at $[\mathbb{1}] \in M$ and, thus, an injective bundle morphism

$$\vartheta : P \to O(M)\,, \quad \vartheta(a) := (\delta'_a(e_1), \ldots, \delta'_a(e_n))\,, \tag{2.5.9}$$

projecting onto the identical diffeomorphism of M. The corresponding Lie group homomorphism $\tau : H \to O(n) \subset \mathrm{GL}(n, \mathbb{R}) \cong \mathrm{Aut}(\mathrm{T}_m M)$ is given by the adjoint action of H on $\mathfrak{m} \cong \mathrm{T}_{[\mathbb{1}]}M$. To summarize, P is a subbundle of $O(M)$.

Now, decompose the Maurer–Cartan form $\theta^G \in \Omega^1(G, \mathfrak{g})$ with respect to (2.5.3):

$$\theta^G = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}\,.$$

By Example 1.3.19, $\theta_{\mathfrak{h}}$ coincides with the canonical G-invariant connection[29] ω^c on P. Recall that the corresponding horizontal distribution is generated by $\mathfrak{m}$, that is, by left invariant vector fields $a \mapsto (\mathbf{x}_*)_a = L'_a(\mathbf{x})$ with $\mathbf{x} \in \mathfrak{m}$.

Lemma 2.5.9 *Under the morphism (ϑ, τ), $\theta_{\mathfrak{m}}$ corresponds to the soldering form θ on $O(M)$, that is, $\vartheta^*\theta = \theta_{\mathfrak{m}}$.*

---

[29] Note that this is a special case of the canonical invariant connection defined in point 2 of Remark 1.9.14. It is obtained by setting $G = H$ and $\lambda = \mathrm{id}$ there.

Proof By $\mathfrak{m}$-valuedness of $\theta_{\mathfrak{m}}$ and horizontality of θ, both $\vartheta^*\theta$ and $\theta_{\mathfrak{m}}$ vanish on the left invariant vector fields generated by elements of $\mathfrak{h}$. Thus, let $\mathbf{x}_*$ be generated by $\mathbf{x} \in \mathfrak{m}$. Then, clearly $\theta_{\mathfrak{m}}(\mathbf{x}_*) = \mathbf{x}$. On the other hand,

$$(\vartheta^*\theta)_g(\mathbf{x}_*) = \vartheta(g)^{-1}(\rho' \circ \vartheta'(\mathbf{x}_*)) = \vartheta(g)^{-1}(\pi'(\mathbf{x}_*)) = \vartheta(g)^{-1}(\delta'_g \circ \pi'(\mathbf{x})) = \mathbf{x}\,,$$

where $\rho : O(M) \to M$ is the canonical projection. ■

Proposition 2.5.10 *The Riemannian manifold (M, g) has the following properties:*

1. *Under the morphism (ϑ, τ), the Levi-Civita connection ω^0 of (M, g) corresponds to the canonical connection ω^c, that is, $\vartheta^*\omega^0 = \omega^c$.*
2. *The Riemann curvature of (M, g) is constant and given by the linear mapping*

$$F : \bigwedge\nolimits^2\mathfrak{m} \to \mathfrak{h}\,, \quad F(\mathbf{x}, \mathbf{y}) = -[\mathbf{x}, \mathbf{y}]\,. \tag{2.5.10}$$

3. *The holonomy group based at $\vartheta(\mathbb{1})$ of ω^0 is H and the holonomy bundle coincides with P.*
4. *The Riemann curvature of (M, g) is parallel, that is, (M, g) is locally symmetric.*
5. *For any $\mathbf{x} \in \mathfrak{m}$, $t \mapsto \pi(L_g \exp(t\mathbf{x}))$ is a geodesic through $[g] \in M$. Conversely, every geodesic through $[g]$ is of this form. In particular, M is geodesically complete.*

Proof 1. We decompose the commutator $[\theta^G, \theta^G] \in \Omega^2(G, \mathfrak{g})$ with respect to (2.5.3). By (2.5.4),

$$[\theta^G, \theta^G]_{\mathfrak{h}} = [\theta_{\mathfrak{h}}, \theta_{\mathfrak{h}}] + [\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}}]\,, \quad [\theta^G, \theta^G]_{\mathfrak{m}} = 2[\theta_{\mathfrak{h}}, \theta_{\mathfrak{m}}]\,. \tag{2.5.11}$$

Since the Levi-Civita connection is uniquely characterized by its covariant derivative D_{ω^0} on $\mathrm{T}M$, it is enough to show that the covariant derivative D_{ω^c} induced by ω^c via the isomorphism (2.5.8) coincides with D_{ω^0}. This is done by showing that the extension of ω^c to $O(M)$ is metric and torsionless. By Proposition 1.2.6, we may view any vector field X on M as an H-equivariant mapping $\tilde{X} : G \to \mathfrak{m}$ and, thus,

$$D_{\omega^c}\tilde{X} = \mathrm{d}\tilde{X} + \mathrm{ad}(\omega^c) \circ \tilde{X} = \mathrm{d}\tilde{X} + [\theta_{\mathfrak{h}}, \tilde{X}]\,,$$

cf. Eq. (1.4.2). Let η be the (unique up to a positive factor) $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{m}$. By $\mathrm{Ad}(H)$-invariance, we obtain

$$\eta(D_{\omega^c}\tilde{X}, \tilde{Y}) + \eta(\tilde{X}, D_{\omega^c}\tilde{Y}) = \mathrm{d}(\eta(\tilde{X}, \tilde{Y}))\,.$$

This shows that the extension of ω^c to $O(M)$ is metric. It remains to show that this extension is torsionless: restricting the Maurer–Cartan equation to $\mathfrak{m}$ and using (2.5.11) we get

$$D_{\omega^c}\theta_{\mathfrak{m}} = \mathrm{d}\theta_{\mathfrak{m}} + [\theta_{\mathfrak{h}}, \theta_{\mathfrak{m}}] = 0\,.$$

But, by Lemma 2.5.9, $\vartheta^*\theta = \theta_{\mathfrak{m}}$ and, thus, $\vartheta^*\Theta = 0$. By uniqueness of the Levi-Civita connection, the assertion follows.

2. By the Structure Equation, the curvature form of ω^c is given by[30]

$$\Omega^c = -\frac{1}{2}[\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}}] .$$

By point 1, $\vartheta^*\Omega^0 = \Omega^c$. These two facts immediately imply (2.5.10).

3. By point 2 and by the Ambrose-Singer Theorem, the Lie algebra of the holonomy group of ω^0 is $[\mathfrak{m}, \mathfrak{m}]$. By point 1 of Remark 2.5.6, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ and, thus, the Lie algebra of the holonomy group of ω^0 coincides with $\mathfrak{h}$. Since, by construction, M is simply connected, the holonomy group of ω^0 is connected and coincides with the restricted holonomy group. On the other hand, since H is connected, too, we obtain the assertion. It follows that P coincides with the holonomy bundle of ω^0.

4. Since the curvature is constant on P and, thus, H-invariant, the Holonomy Principle 1.7.20 implies the assertion.

5. By Proposition 2.1.22, the geodesics of (M, g) are given by the projections of integral curves of horizontal standard vector fields on $L(M)$. Since they are horizontal, these curves may be chosen to lie in P. The restriction of $B(\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^n$, to P is given by the left-invariant vector field generated by $\mathbf{x} = y^i\mathbf{e}_i \in \mathfrak{m}$, where $\{\mathbf{e}_i\}$ is a basis in $\mathfrak{m}$. Thus, here, the geodesics are given as projections of (global) one-parameter subgroups $t \mapsto \exp(t\mathbf{x})$ and their left translates by arbitrary group elements $g \in G$. ■

By point 3 of Proposition 2.5.10, the irreducibility of $(\mathfrak{g}, \lambda)$ implies that (M, g) is irreducible. Together with points 4 and 5, this yields the following.

Corollary 2.5.11 (M, g) *is a complete irreducible locally symmetric Riemannian manifold.* ■

Next, we show that the involutive automorphism λ induces a special symmetry for any point $m \in M$. Since any automorphism of a Lie algebra is the differential of a unique automorphism of the corresponding simply connected Lie group,[31] λ induces a unique automorphism σ of $\tilde{G}$. By (2.5.6), it fulfils $\sigma(\tilde{H}) = \tilde{H}$. Thus, σ descends to an involutive diffeomorphism $s : M \to M$. By construction,

$$s'_{[\mathbb{1}]} : \mathrm{T}_{[\mathbb{1}]}M \to \mathrm{T}_{[\mathbb{1}]}M , \quad s'_{[\mathbb{1}]}(X) = -X . \tag{2.5.12}$$

Thus, under the identification $\mathrm{T}_{[\mathbb{1}]}M = \mathfrak{m}$, we have $s'_{[\mathbb{1}]} = \lambda_{\restriction\mathfrak{m}}$.

Lemma 2.5.12 *The origin* $[\mathbb{1}]$ *of* M *is an isolated fixed point of* s. *Moreover,* s *is an isometry of the Riemannian metric* g.

---

[30]Since ω^c is a G-invariant connection, this is a special case of point 4 of Remark 1.9.14.

[31]For a proof, see e.g. Theorem 3.27 in [652].

Proof The proof of the first assertion is left to the reader (Exercise 2.5.4). To prove the second statement, we have to show that the mapping

$$s'_m : \mathrm{T}_m M \to \mathrm{T}_m M$$

is isometric. For the point $m = [\mathbb{1}]$, this follows immediately from (2.5.12), because at the origin g coincides with η and the latter is λ-invariant. To prove the invariance for an arbitrary point $m = [g]$, note that for any $g, h \in G$,

$$s(\delta_g[h]) = s([gh]) = [\sigma(g)\sigma(h)] = \delta_{\sigma(g)}[\sigma(h)] = \delta_{\sigma(g)} s([h]) ,$$

that is, $s \circ \delta_g = \delta_{\sigma(g)} \circ s$. Differentiation of this identity yields

$$s'_{[g]} \circ (\delta_g)'_{[\mathbb{1}]} = (\delta_{\sigma(g)})'_{[\mathbb{1}]} \circ s'_{[\mathbb{1}]} .$$

By construction, g is G-invariant and, thus, $(\delta_g)'_{[\mathbb{1}]}$ and $(\delta_{\sigma(g)})'_{[\mathbb{1}]}$ leave g invariant. This yields the assertion. ■

Remark 2.5.13 For every $g \in \tilde{Z}$, we have $\big(\sigma(g)\big)(m) = s \circ g \circ s(m) = s^2(m) = m$. Hence, $\sigma(\tilde{Z}) = \tilde{Z}$ and σ descends to an automorphism of G, denoted by the same symbol. One has $\sigma(H) = H$. ♦

Next, for any $m = [g] \in M$, we define[32]

$$s_m : M \to M , \quad s_m := \delta_g \circ s \circ \delta_{g^{-1}} . \tag{2.5.13}$$

Differentiating (2.5.13), we obtain $s'_m = \delta'_g \circ s'_{[\mathbb{1}]} \circ \delta'_{g^{-1}}$ for any $m = [g] \in M$. Thus, by Lemma 2.5.12, by formula (2.5.12) and by the G-invariance of g, for any $m \in M$, s_m is an involutive isometry of g fulfilling (Exercise 2.5.5)

$$s_m(m) = m , \quad (s_m)'_m = -\,\mathrm{id} . \tag{2.5.14}$$

The following remark yields a geometric interpretation of the symmetry s_m.

Remark 2.5.14 Let $t \to \gamma(t)$ be a geodesic of (M, g) with $\gamma(0) = m$. Since an isometry transforms geodesics to geodesics, $t \mapsto \tau(t) := s_m(\gamma(t))$ is a geodesic, too. By (2.5.14), its tangent vector at $t = 0$ satisfies

$$\dot{\tau}(0) = (s_m)'_m\, \dot{\gamma}(0) = -\dot{\gamma}(0) . \tag{2.5.15}$$

Now, the uniqueness property of geodesics, see Corollary 2.1.23, implies $\tau(t) = \gamma(-t)$. Thus, for any $m \in M$,

[32] Clearly, this definition does not depend on the choice of the representative.

$$s_m(\gamma(t)) = \gamma(-t)\,, \tag{2.5.16}$$

that is, s_m reverses the geodesics through m. ♦

Definition 2.5.15 (*Riemannian globally symmetric space*) A Riemannian manifold (M, g) is called globally symmetric if for each $m \in M$ there exists an involutive isometry $s_m : M \to M$ such that m is an isolated fixed point of s_m. The mapping s_m is called the symmetry of (M, g) at m.

Taking into account that, in the above construction of (M, g), the scalar product on $\mathfrak{m}$ is unique up to a positive constant and that a change of this constant implies a conformal transformation of g, we obtain the following.

Proposition 2.5.16 *To any irreducible*[33] *orthogonal symmetric Lie algebra* $(\mathfrak{g}, \lambda)$ *there corresponds a unique homothetic equivalence class* $(M, [\mathsf{g}])$ *of simply connected irreducible Riemannian globally symmetric spaces.* ■

It should be clear that the locally symmetric Riemannian manifold we started with and the Riemannian globally symmetric space constructed here are deeply related. Indeed, let (M, g) be a locally symmetric space. Let $(\mathfrak{g}, \lambda)$ be its canonical symmetric Lie algebra with canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $\eta \in S^2(\mathfrak{m}^*)$ be the scalar product on $\mathfrak{m}$ defined by g and let $F \in \mathfrak{K}(\mathfrak{h}) \subset \bigwedge^2 \mathfrak{m}^* \otimes \mathfrak{h}$ be the Riemann curvature of (M, g). Let G/H be the Riemannian globally symmetric space constructed from $(\mathfrak{g}, \lambda)$. Then, for any chosen point $m \in M$, via

$$\mathrm{T}_m M \cong \mathfrak{m} \cong \mathrm{T}_{[\mathbb{1}]} G/H$$

we obtain an isometric isomorphism between $\mathrm{T}_m M$ and $\mathrm{T}_{[\mathbb{1}]} G/H$ and, by point 2 of Proposition 2.5.10, M and G/H have the same Riemann curvature given by the mapping F. By standard arguments,[34] this implies the following.

Corollary 2.5.17 *Every point of a locally symmetric space* (M, g) *admits a neighbourhood isometric to a neighbourhood of the origin of the Riemannian globally symmetric space constructed from the canonical symmetric Lie algebra of* (M, g).

Note, however, that not every locally symmetric space is a Riemannian globally symmetric space. It is even not necessarily homogeneous. As an example,[35] let M be a compact Riemann surface with genus ≥ 2, equipped with a Riemannian metric of constant curvature equal to -1. Then, the isometry group of M is finite and, thus, M is not homogeneous and, consequently, also not globally symmetric.

As an immediate consequence of the existence of the symmetries s_m, we obtain

Proposition 2.5.18 *Any Riemannian globally symmetric space* (M, g) *is complete.*

[33] Remember that irreducibility includes effectiveness, cf. Definition 2.5.3.

[34] See Theorem 7.4 in Chap. VI of [381].

[35] This example is taken from [73].

Proof Consider any geodesic $t \mapsto \gamma(t)$ defined on the interval $[0, t_0[$. Apply the symmetry $s_{\gamma(t_0-\varepsilon)}$ to γ with some ε fulfilling $0 < \varepsilon < \frac{t_0}{2}$. By (2.5.16), this operation extends the domain of γ to $[0, 2t_0 - 2\varepsilon[$. Continuing this procedure, we obtain completeness of (M, g). ■

Next, given a Riemannian globally symmetric space (M, g), for every geodesic $t \mapsto \gamma(t)$ we consider the family of isometries

$$T_t^\gamma := s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}\,, \tag{2.5.17}$$

called the transvections along γ. The following properties are immediate consequences of (2.5.15) and (2.5.16) and are, therefore, left to the reader (Exercise 2.5.3).

Proposition 2.5.19 *Let (M, g) be a Riemannian globally symmetric space and let $t \mapsto \gamma(t)$ be a geodesic. Then,*

1. *T_t^γ acts on γ by translations, that is, $T_t^\gamma(\gamma(s)) = \gamma(t + s)$.*
2. *$(T_t^\gamma)'_{\gamma(s)}$ acts by parallel translation from $\gamma(s)$ to $\gamma(t + s)$ along γ, that is, for any parallel vector field X along γ,*

$$(T_t^\gamma)'_{\gamma(s)}(X(\gamma(s)) = X(\gamma(t + s))\,.$$

3. *$\{T_t^\gamma\}_{t\in\mathbb{R}}$ is a 1-parameter group of isometries, that is, $T_{t+s}^\gamma = T_t^\gamma \circ T_s^\gamma$.* ■

Recall from Example 2.2.16 that the isometry group $I(M)$ of a Riemannian manifold M is a Lie group. Let us denote its identity component by $I_0(M)$. By point 3 of Proposition 2.5.19, for any geodesic γ, the transvections T_t^γ form a subgroup (called the transvection group) of $I_0(M)$. On the other hand, by a classical theorem of Hopf and Rinow,[36] any two points of a complete Riemannian manifold may be joined by a geodesic. Using these two facts, we obtain the following.

Corollary 2.5.20 *Let (M, g) be a Riemannian globally symmetric space. Then,*

1. *Geodesics in M are images of 1-parameter groups of isometries.*
2. *The identity component $I_0(M)$ acts transitively on M.* ■

Proposition 2.5.21 *Let (M, g) be an irreducible Riemannian globally symmetric space and let G be a Lie group acting transitively and isometrically on M. If G acts effectively, then G coincides with $I_0(M)$.*

Proof Clearly, $I_0(M)$ is the largest connected group of isometries of (M, g). Denote $G' = I_0(M)$ and let $\mathfrak{g}'$ be its Lie algebra. Conjugation by s defines an automorphism σ' of G' which clearly restricts to the automorphism σ of G, cf. Remark 2.5.13. The canonical decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ necessarily fulfil $\mathfrak{m}' = \mathfrak{m}$. Here $\mathfrak{h}'$ is the Lie algebra of the stabilizer of the chosen point on M under G'. Thus, by Remark 2.5.6,

[36] See, e.g. [352].

$$\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}', \mathfrak{m}'] = \mathfrak{h}' .$$

This implies $\mathfrak{g}' = \mathfrak{g}$ and, thus, $G' = G$. ■

Thus, in the construction leading to Proposition 2.5.16, the Lie group G actually coincides with $I_0(M)$. Now, we are able to prove the converse of Proposition 2.5.16.

Proposition 2.5.22 *To any simply connected irreducible Riemannian globally symmetric space there corresponds a unique irreducible orthogonal symmetric Lie algebra.*

Proof Let (M, g) be a simply connected irreducible Riemannian globally symmetric space. By Corollary 2.5.20, $G = I_0(M)$ acts transitively and effectively on M. Let H be the isotropy group of this Lie group action at a chosen point $o \in M$. By the homotopy sequence of the fibration $H \to G \to G/H$, the simply-connectedness of G/H and the connectedness of G imply that H is connected. Moreover, by Theorem 3.4 in Chap. VI of [381], the isotropy subgroup $I(M)_m$ at any point $m \in M$ is compact. Hence, $H = G \cap I(M)_o$ is compact, too. Thus, $M = G/H$ and, by standard arguments, $\pi : G \to M$ is a submersion. In particular, $\pi' : \mathrm{T}_{\mathbb{1}} G \to \mathrm{T}_o M$ is an H-equivariant surjective linear mapping whose kernel coincides with $\mathrm{T}_{\mathbb{1}} H$.

Let s be the symmetry at o. Since s is an involutive diffeomorphism, the mapping $g \mapsto \sigma(g) := s \circ g \circ s^{-1}$ defines an involutive automorphism of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Clearly, $\lambda := \sigma'$ is an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{m}$ be the eigenspace of λ corresponding to the eigenvalue -1. By (2.5.14), $\pi'(\mathfrak{m}) = \mathrm{T}_o M$. We prove that $\mathfrak{h}$ is the eigenspace of λ corresponding to the eigenvalue $+1$: let

$$G^\sigma := \{g \in G : \sigma(g) = g\}$$

be the fixed point set of σ. By (2.5.14), s'_o commutes with the isotropy representation of H at o and, thus, H is contained in G^σ. Conversely, if $g \in G^\sigma$, then it commutes with s and, thus, for any 1-parameter subgroup $t \mapsto g_t$ of G^σ,

$$s \circ g_t(o) = g_t \circ s(o) = g_t(o) ,$$

that is, the orbit $g_t(o)$ is left invariant pointwise by s. Now, by Lemma 2.5.12, o is an isolated fixed point. Thus, $g_t(o)$ must coincide with o. But, $g_t(o) = o$ implies that the 1-parameter subgroup $t \mapsto g_t$ is contained in H. Since a connected Lie group is generated by its 1-parameter subgroups, we have $(G^\sigma)^0 \subset H$. Thus,

$$(G^\sigma)^0 \subset H \subset G^\sigma .$$

This relation implies that $\mathfrak{h}$ coincides with the $(+1)$-eigenspace of λ, indeed. To summarize, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is canonical with respect to λ, that is, $(\mathfrak{g}, \lambda)$ is a symmetric Lie algebra. Since H is compact, $\mathrm{ad}(\mathfrak{h})$ is a compactly embedded Lie subalgebra of $\mathfrak{g}$, that is, $(\mathfrak{g}, \lambda)$ is orthogonal. It remains to prove that $(\mathfrak{g}, \lambda)$ is irreducible. Since g is G-invariant, we are in the situation described by Proposition

2.5.10. By this proposition, H coincides with the holonomy group of the Levi-Civita connection of g. Thus, the irreducibility of (M, g), together with the effectiveness of the action of G on M, implies the irreducibility of $(\mathfrak{g}, \lambda)$. ■

Remark 2.5.23 In the course of the above proof, we have found the following structure: a triple (G, H, σ) fulfilling

1. G is a connected Lie group and H is a closed subgroup,
2. σ is an involutive automorphism of G such that $(G^\sigma)^0 \subset H \subset G^\sigma$,
3. $\mathrm{Ad}(H)$ is compact,

is called a Riemannian symmetric pair. This notion clearly constitutes a link between symmetric spaces and symmetric Lie algebras. ♦

Combining Proposition 2.5.16 with Proposition 2.5.22, we obtain the following.

Theorem 2.5.24 *The homothetic equivalence classes of simply connected irreducible Riemannian globally symmetric spaces are in one-to-one correspondence with the irreducible orthogonal symmetric Lie algebras.* ■

This theorem reduces the classification of symmetric spaces of the above type to the classification of irreducible symmetric Lie algebras of compact or of non-compact type. According to a beautiful duality,[37] the problem further reduces to the classification of irreducible symmetric Lie algebras of the non-compact type. The latter can be shown to be in one-to-one correspondence with the real simple Lie algebras of non-compact type. If the complexification of such a Lie algebra is simple as a complex Lie algebra, then M is said to be of type III, otherwise M is said to be of type IV. The corresponding compact irreducible symmetric spaces are obtained by duality and are referred to as of type I and II, respectively. The complete list of simply connected irreducible symmetric spaces with symmetry group being a classical Lie group is given in Tables 2.1 and 2.2.[38] Here, $\mathrm{SO}_0(p, q)$ denotes the identity component of $\mathrm{SO}(p, q)$ and $\mathrm{SO}^*(2n)$ is the subgroup of $\mathrm{SO}(2n, \mathbb{C})$ satisfying

$$g^{\mathrm{T}} \mathsf{J}_0 \overline{g} = \mathsf{J}_0 \,, \quad g^{\mathrm{T}} g = \mathbb{1}_{2n} \,.$$

For the corresponding list with exceptional Lie groups we refer to the textbook of Helgason [293]. As already mentioned, there the reader may find an exhaustive presentation of the whole subject.

Remark 2.5.25 Note that in our considerations, we have excluded the class of symmetric Lie algebras fulfilling $[\mathfrak{m}, \mathfrak{m}] = 0$, cf. case 3 in Proposition 2.5.5. Symmetric Lie algebras with this property are said to be of Euclidean type. By point 2 of Proposition 2.5.10, they are necessarily flat. One can show that if G/H is simply connected,

[37] See Sect. 8 of Chap. XI in [381] or Sect. 2 of Chap. V in [293].

[38] By definition, the rank is the dimension of some maximal Abelian subspace of $\mathfrak{m}$. Any two maximal Abelian subspaces of $\mathfrak{m}$ are $\mathrm{Ad}(H)$-conjugate.

Table 2.1 Classical symmetric spaces of types I and III

Type I	Type III	Dimension	Rank
$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$	$(n-1)(n+2)/2$	$n-1$
$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathrm{SL}(n,\mathbb{H})/\mathrm{Sp}(n)$	$(n-1)(2n+1)$	$n-1$
$\mathrm{SU}(p+q)/S(\mathrm{U}(p)\times\mathrm{U}(q))$	$\mathrm{SU}(p,q)/S(\mathrm{U}(p)\times\mathrm{U}(q))$	$2pq$	$\min(p,q)$
$\mathrm{SO}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$	$\mathrm{SO}_0(p,q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$	pq	$\min(p,q)$
$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$n(n-1)$	$[n/2]$
$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	$n(n+1)$	n
$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$	$\mathrm{Sp}(p,q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$	$4pq$	$\min(p,q)$

Table 2.2 Classical symmetric spaces of types II and IV. For type II, see Proposition X.1.2 and Sect. IV.6 in [293]

Type II	Type IV	Dimension	Rank
$\mathrm{SU}(n+1)$	$\mathrm{SL}(n+1,\mathbb{C})/\mathrm{SU}(n+1)$	$n(n+2)$	n
$\mathrm{Spin}(2n+1)$	$\mathrm{SO}(2n+1,\mathbb{C})/\mathrm{SO}(2n+1)$	$n(2n+1)$	n
$\mathrm{Sp}(n)$	$\mathrm{Sp}(n,\mathbb{C})/\mathrm{Sp}(n)$	$n(2n+1)$	n
$\mathrm{Spin}(2n)$	$\mathrm{SO}(2n,\mathbb{C})/\mathrm{SO}(2n)$	$n(2n-1)$	n

then a symmetric space of this type is isometric to some Euclidean space $\mathbb{R}^n$. Clearly, $\mathbb{R}^n$ itself provides the simplest example, with the symmetry at the origin given by $s:\mathbf{x}\to-\mathbf{x}$. ♦

Next, we show that Riemannian symmetric spaces provide Riemannian manifolds of certain types met before. Recall that if $(\mathfrak{g},\lambda)$ is irreducible, then $\mathfrak{g}$ is necessarily semisimple and thus, the Killing form k is non-degenerate. As already noted, this implies

$$\eta(\mathbf{x},\mathbf{z}) = -c\,\mathsf{k}^{\mathfrak{m}}(\mathbf{x},\mathbf{z})\,,\quad \mathbf{x},\mathbf{z}\in\mathfrak{m}\,, \tag{2.5.18}$$

for some $c\in\mathbb{R}$, $c\neq 0$, cf. (2.5.7). Recall from point 2 of Proposition 2.5.10 that the curvature mapping $\mathscr{R}$ is given by the mapping F, cf. formula (2.5.10). Substituting $\mathbf{x}=F(\mathbf{u},\mathbf{v})\mathbf{w}$ into (2.5.18) and using the $\mathrm{ad}(\mathfrak{h})$-invariance of k, we obtain

$$\eta(F(\mathbf{u},\mathbf{v})\mathbf{w},\mathbf{z}) = c\,\mathsf{k}^{\mathfrak{m}}([[\mathbf{u},\mathbf{v}],\mathbf{w}],\mathbf{z}) = c\,\mathsf{k}^{\mathfrak{h}}([\mathbf{u},\mathbf{v}],[\mathbf{w},\mathbf{z}])\,. \tag{2.5.19}$$

Setting $\mathbf{x}=\mathbf{u}=\mathbf{z}$ and $\mathbf{y}=\mathbf{v}=\mathbf{w}$ in (2.5.19), we immediately obtain the following formula for the sectional curvature:

$$\eta(F(\mathbf{x},\mathbf{y})\mathbf{y},\mathbf{x}) = -c\,\mathsf{k}^{\mathfrak{h}}([\mathbf{x},\mathbf{y}],[\mathbf{x},\mathbf{y}])\,. \tag{2.5.20}$$

This yields useful formulae for the Ricci tensor and for the scalar curvature. For any orthonormal basis $\{\mathbf{e}_i\}$ of $\mathfrak{m}$,

$$\mathsf{Ric}(\mathbf{e}_i, \mathbf{e}_j) = -\sum_k \eta([[\mathbf{e}_k, \mathbf{e}_i], \mathbf{e}_j], \mathbf{e}_k)\,, \quad \mathsf{Sc} = -\sum_{k,l} \eta([[\mathbf{e}_k, \mathbf{e}_l], \mathbf{e}_l], \mathbf{e}_k)\,. \tag{2.5.21}$$

Proposition 2.5.26 *Let (M, g) be an irreducible Riemannian globally symmetric space and let $(\mathfrak{g}, \lambda)$ be the corresponding irreducible orthogonal symmetric Lie algebra.*

1. *If $(\mathfrak{g}, \lambda)$ is of compact type, then (M, g) is a compact Einstein manifold with non-negative sectional curvature and positive definite Ricci tensor.*
2. *If $(\mathfrak{g}, \lambda)$ is of non-compact type, then (M, g) is a simply connected Einstein manifold with non-positive sectional curvature and negative definite Ricci tensor. Moreover, M is diffeomorphic to a Euclidean space.*

Proof Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the canonical decomposition. By Theorem 2.5.24, $G = I_0(M)$ acts transitively and effectively on M and g is G-invariant. Since $k^{\mathfrak{h}}$ is negative definite, the statements about the sectional curvature K follow immediately from (2.5.20). Since the Ricci tensor Ric is a symmetric $\mathrm{ad}(\mathfrak{h})$-invariant bilinear form on $\mathfrak{m}$ and since $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, Ric must be proportional to the metric, that is, (M, g) is an Einstein space.

1. Let $(\mathfrak{g}, \lambda)$ be of compact type. Then, K is non-negative and, thus, Ric is semi-positive definite. Since M is Einstein, Ric is either positive definite or zero. But if Ric is zero, then (2.3.27) implies that K must also be zero, which contradicts the non-degeneracy of k and, thus, the irreducibility of $(\mathfrak{g}, \lambda)$. Finally, since $\mathsf{k}^{\mathfrak{m}}$ is negative definite, k is negative definite and, since $\mathfrak{g}$ is semisimple, G is compact. Thus, M is compact.

2. Let $(\mathfrak{g}, \lambda)$ be of non-compact type. Then, by similar arguments, M is Einstein with negative definite Ricci tensor. The remaining statement follows from Theorem 8.3 in Chap. VIII of [381]. ■

In the remainder of this section, we present the symmetric space structure of a few of the types in Table 2.1 explicitly. By Theorem 2.5.24, it is enough to exhibit the corresponding symmetric Lie algebra structure. For a much more detailed discussion of examples we refer to Chap. XI of [381] and to [692]. We leave it to the reader to check the statements below (Exercise 2.5.6).

Example 2.5.27

1. Consider type I in lines 3, 4, and 7 of Table 2.1. Lines 3 and 7 correspond to the Graßmann manifolds

$$G_{\mathbb{K}}(k, n) \cong \mathrm{U}_{\mathbb{K}}(n)/(\mathrm{U}_{\mathbb{K}}(n-k) \times \mathrm{U}_{\mathbb{K}}(k))\,, \quad \mathbb{K} = \mathbb{C}, \mathbb{H}\,,$$

and line 4 corresponds to the Graßmann manifold of oriented subspaces of $\mathbb{R}^{p+q}$.[39] The corresponding symmetric Lie algebra is given by

$$\mathfrak{u}_{\mathbb{K}}(p+q) = (\mathfrak{u}_{\mathbb{K}}(p) \oplus \mathfrak{u}_{\mathbb{K}}(q)) \oplus \mathfrak{m}\,,$$

where

$$\mathfrak{u}_{\mathbb{K}}(p) \oplus \mathfrak{u}_{\mathbb{K}}(q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathfrak{u}_{\mathbb{K}}(p+q) : A \in \mathfrak{u}_{\mathbb{K}}(p)\,,\ B \in \mathfrak{u}_{\mathbb{K}}(q) \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -X^{\dagger} \\ X & 0 \end{bmatrix} \in \mathfrak{u}_{\mathbb{K}}(p+q) : X \in L(\mathbb{K}^p, \mathbb{K}^q) \right\}\,.$$

The action of $\mathrm{Ad}(H)$ on $\mathfrak{m}$ is given by

$$X \mapsto hXk^{-1}\,, \quad h \in \mathrm{U}_{\mathbb{K}}(q)\,,\ k \in \mathrm{U}_{\mathbb{K}}(p)\,,$$

and the involutive automorphism λ acts via

$$\begin{bmatrix} A & -X^{\dagger} \\ X & B \end{bmatrix} \mapsto \begin{bmatrix} A & X^{\dagger} \\ -X & B \end{bmatrix}.$$

The corresponding involutive automorphism σ is given by conjugation with

$$\mathbb{1}_{p,q} = \begin{bmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{bmatrix}. \tag{2.5.22}$$

2. Consider the special case $p = n$ and $q = 1$ for type I in line 4 of Table 2.1:

$$\mathrm{S}^n = S_{\mathbb{R}}(1, n+1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)\,.$$

The underlying symmetric Lie algebra is given by

$$\mathfrak{o}(n+1) = \mathfrak{o}(n) \oplus \mathfrak{m}\,, \tag{2.5.23}$$

where

$$\mathfrak{o}(n) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \in \mathfrak{o}(n+1) : A \in \mathfrak{o}(n) \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -\mathbf{x}^{\mathrm{T}} \\ \mathbf{x} & 0 \end{bmatrix} \in \mathfrak{o}(n+1) : \mathbf{x} \in \mathbb{R}^n \right\}\,.$$

Then, $\mathrm{Ad}(\mathrm{SO}(n))$ gets identified with the basic representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ and, under the identification $\mathfrak{m} \cong \mathbb{R}^n$, the Euclidean scalar product on $\mathbb{R}^n$ yields

[39] Cf. Example I/7.5.6.

a scalar product on $\mathfrak{m}$ which coincides with the restriction of the Killing form on $\mathfrak{o}(n+1)$ to $\mathfrak{m}$ up to the factor $-2(n-1)$. The involutive automorphisms are read off from the previous point.

3. Consider type I in line 5 of Table 2.1. One easily shows that $\mathrm{SO}(2n)/\mathrm{U}(n)$ is the space of orthogonal complex structures on the $2n$-dimensional Euclidean space.[40] Here we decompose[41]

$$\mathfrak{o}(2n) = \mathfrak{u}(n) \oplus \mathfrak{m}\,,$$

with

$$\mathfrak{u}(n) = \left\{ \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \in \mathfrak{o}(2n) : X, Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ Y = Y^{\mathrm{T}} \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \in \mathfrak{o}(2n) : X, Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ Y = -Y^{\mathrm{T}} \right\}\,.$$

The involutive automorphism $\lambda : \mathfrak{o}(2n) \to \mathfrak{o}(2n)$ corresponding to this decomposition is given by conjugation with the matrix

$$\mathsf{J}_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}\,.$$

4. Consider type I in line 1 of Table 2.1. Recall from Sect. 7.6 of Part I that $\mathrm{U}(n)/\mathrm{O}(n)$ is the space of Lagrangian subspaces of $\mathbb{R}^{2n}$ endowed with its canonical symplectic structure. Correspondingly, $\mathrm{SU}(n)/\mathrm{SO}(n)$ is called the space of special Lagrangian subspaces. Here, we decompose

$$\mathfrak{su}(n) = \mathfrak{o}(n) \oplus \mathfrak{m}\,,$$

with

$$\mathfrak{o}(n) = \left\{ \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \in \mathfrak{su}(n) : X \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ \mathrm{tr}\, X = 0 \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix} \in \mathfrak{su}(n) : Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ Y = Y^{\mathrm{T}} \right\}\,.$$

Here, we have used the embedding $\mathfrak{u}(n) \subset \mathfrak{o}(2n)$ from the previous point. Under this embedding, the involutive automorphism $\lambda : \mathfrak{su}(n) \to \mathfrak{su}(n)$ is given by

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\,.$$

[40] Cf. Example I/7.5.5.

[41] Cf. Example 2.2.19.

5. Consider type III in line 4 of Table 2.1 with $p = 1$, that is, $M = \mathrm{SO}_0(1, n)/\mathrm{SO}(n)$. On the level of Lie algebras, we have to consider the pseudo-Euclidean space $(\mathbb{R}^{1,n}, \eta)$ with $\eta = \mathbb{1}_{1,n}$ given by (2.5.22). Then,

$$\mathfrak{o}(n, 1) = \left\{X \in \mathfrak{gl}(n + 1, \mathbb{R}) : X^{\mathrm{T}}\mathbb{1}_{1,n} + \mathbb{1}_{1,n}X = 0\right\} .$$

Embedding $\mathfrak{o}(n) \subset \mathfrak{o}(1, n)$ via $Y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & Y \end{bmatrix}$, we obtain the canonical decomposition

$$\mathfrak{o}(1, n) = \mathfrak{o}(n) \oplus \mathfrak{m} , \quad \mathfrak{m} = \left\{ \begin{bmatrix} 0 & \mathbf{u}^{\mathrm{T}} \\ \mathbf{u} & 0 \end{bmatrix} \in \mathfrak{o}(1, n) : \ \mathbf{u} \in \mathbb{R}^n \right\} .$$

It is obvious that M may be identified with the hypersurface $H_+(1, n) \subset \mathbb{R}^{1,n}$ defined by

$$\eta(\mathbf{u}, \mathbf{u}) = -1 , \quad u^0 \geq 1 .$$

Therefore, M is referred to as the hyperbolic space form of $(\mathbb{R}^{1,n}, \eta)$. ♦

Remark 2.5.28 Consider the example of the n-sphere above. By Example 1.1.18, under the identification $\mathfrak{m} \cong \mathbb{R}^n$, the bundle of orthonormal frames $O(\mathrm{S}^n)$ coincides with the principal $\mathrm{SO}(n)$-bundle $\mathrm{SO}(n + 1) \to \mathrm{SO}(n + 1)/\mathrm{SO}(n)$ and, by Proposition 2.5.10, the Levi-Civita connection on S^n with respect to the natural metric coincides with the $\mathrm{SO}(n + 1)$-invariant canonical connection on this bundle. The curvature (2.5.10) reads $F(\mathbf{x}, \mathbf{y}) = \mathbf{x} \wedge \mathbf{y}$. Comparing with (2.4.2), this shows that S^n has a constant sectional curvature equal to 1. ♦

For applications of the theory of symmetric spaces in this book, see Sects. 6.8 and 7.9.

Exercises

2.5.1 Prove that λ defined by (2.5.1) is an involutive Lie algebra homomorphism.

2.5.2 Prove that the decomposition (2.5.3) is orthogonal with respect to the Killing form.

2.5.3 Prove Proposition 2.5.3.

2.5.4 Prove Lemma 2.5.12.

2.5.5 Prove the following. For an involutive isometry s with isolated fixed point m, one has $s'_m = -\,\mathrm{id}$. *Hint.* Use the eigenspace decomposition of s'_m.

2.5.6 Check the statements in Example 2.5.27.

2.6 Compatible Connections on Vector Bundles

Here, we take up the discussion of Sect. 2.2. We consider real or complex vector bundles endowed with a fibre metric h and an h-compatible connection ∇. Such a structure will be denoted by (E, h, ∇). In the first part, we will collect what we know already for the case of real (pseudo-)Riemannian base manifolds (M, g), and in the second part we will pass to complex base manifolds and Hermitean vector bundles endowed additionally with a holomorphic structure.

First, recall Examples 2.2.19 and 2.2.27.

(a) $\mathrm{O}(k, l)$-structures are in one-to-one correspondence with pseudo-Riemannian manifolds (M, g) of dimension $(k + l)$, where the $\mathrm{O}(k, l)$-structure coincides with the bundle $O(M)$ of frames which are orthonormal with respect to g. A linear connection ω on M is compatible with the $\mathrm{O}(k, l)$-structure iff g is parallel with respect to ω. Such a connection is called metric.

(b) $\mathrm{U}(n)$-structures are in one-to-one correspondence with $2n$-dimensional almost Hermitean manifolds $(M, \mathsf{g}, \mathsf{J})$ or, equivalently, with Hermitean fibre metrics on $\mathrm{T}M$ relative to a given J. A linear connection ω on M is compatible with the $\mathrm{U}(n)$-structure iff both g and J are parallel with respect to ω. Such a connection is called unitary. Equivalently, ω is unitary iff the Hermitean fibre metric h in $\mathrm{T}M$ defined by g and J is parallel with respect to ω.

More generally, as we know from Examples 1.6.6 and 1.6.12, a connection ∇ on a real or complex vector bundle (E, h) is compatible with h iff

$$\nabla \mathsf{h} = 0 , \tag{2.6.1}$$

which is equivalent to

$$X(\mathsf{h}(s_1, s_2)) = \mathsf{h}(\nabla_X s_1, s_2) + \mathsf{h}(s_1, \nabla_X s_2) , \tag{2.6.2}$$

for any $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in \Gamma^\infty(E)$. Since h may be viewed as a section of the associated bundle $L(E) \times_{\mathrm{GL}(n,\mathbb{K})} \mathscr{F}$, where $\mathscr{F}$ denotes the space of fibre metrics, (2.6.1) is equivalent to

$$D_\omega \tilde{\mathsf{h}} = 0 , \tag{2.6.3}$$

where ω is the connection form on $L(E)$ and $\tilde{\mathsf{h}} : L(E) \to \mathscr{F}$ is the G-homomorphism corresponding to ∇ and h, respectively. The metric h defines a reduction to the subbundle of orthonormal frames

$$O(E) = \left\{ u \in L(E) : \tilde{\mathsf{h}}(u) = \mathsf{h}_0 \right\} ,$$

where $\mathsf{h}_0 = \mathbb{1}_{p,q}$ in the real and $\mathsf{h}_0 = \mathbb{1}$ in the complex case. By compatibility, ω is reducible to $O(E)$. In the (pseudo-)Riemannian case, the restriction of equation (2.6.3) to $O(E)$ reads

$$\left(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \omega^{\mathrm{T}}\right)(\mathsf{h}_0) = 0$$

and in the Hermitean case, we obtain

$$\big(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \overline{\omega^{\mathrm{T}}}\big)(\mathsf{h}_0) = 0\,.$$

Thus, ∇ is h-compatible iff ω is metric or unitary for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively.

Remark 2.6.1

1. By Proposition 1.3.7, $O(E)$ admits a connection. Thus, every (pseudo-) Riemannian or Hermitean vector bundle admits a compatible connection.
2. Using the isomorphisms given by (1.2.4) and by Proposition 1.6.7, we have

$$E \cong L(E) \times_{\mathrm{GL}(k,\mathbb{K})} \mathbb{K}^k \cong O(E) \times_G \mathbb{K}^k\,,$$

   where $G = \mathrm{O}(p, q)$ in the (pseudo-)Riemannian and $G = \mathrm{U}(k)$ in the Hermitean case. Since $\tilde{\mathsf{h}}$ is constant on $O(E)$, without loss of generality, we can limit our attention to the following setting. Let $P(M, G)$ be a principal G-bundle over an oriented (pseudo-)Riemannian manifold (M, g) and let $E = P \times_G F$ be an associated vector bundle such that (F, G, σ) is a finite-dimensional representation space carrying a σ-invariant inner product $\langle\cdot,\cdot\rangle_F$. Then, $\langle\cdot,\cdot\rangle_F$ induces a fibre metric on E via

$$\mathsf{h}(e_1, e_2) := \langle f_1, f_2\rangle_F\,, \tag{2.6.4}$$

   with $e_1 = [(p, f_1)]$ and $e_2 = [(p, f_2)]$. By G-invariance of $\langle\cdot,\cdot\rangle_F$, this definition does not depend on the choice of representatives. ♦

For the remainder, let us assume that M is a complex manifold. Recall that a complex manifold of dimension n is a real manifold of dimension $2n$ endowed with an equivalence class of holomorphic atlases.

Definition 2.6.2 A complex vector bundle E over a complex manifold M is called holomorphic if E admits a system of local trivializations whose transition functions are holomorphic.

Note that such a system of trivializations turns E into a complex manifold such that the projection $\pi : E \to M$ is holomorphic. Also note that, since the composition of anti-holomorphic mappings need not be anti-holomorphic, there is no notion of an anti-holomorphic vector bundle.

Remark 2.6.3

1. For a complex manifold of complex dimension n, one can define the principal $\mathrm{GL}(n, \mathbb{C})$-bundle $C(M)$ of complex linear frames in the same way as in the real case, cf. Example 2.2.10. Correspondingly, any holomorphic vector bundle E of rank k over M may be viewed as associated with its complex linear frame bundle $C(E)$, that is, $E \cong C(E) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}^k$.

2. As in the C^∞-case, any functorial construction in linear algebra gives rise to holomorphic vector bundles. In particular, one can build the dual bundle, direct sums and tensor products, see [336] for details. ♦

The basic example of a holomorphic vector bundle is provided by the holomorphic tangent bundle of a complex manifold M. Let $(U_i, \varphi_i)_{i\in I}$ be a holomorphic atlas of M with transition mappings φ_{ij} and let z^i be the complex coordinates corresponding to φ_i. Consider the Jacobian

$$\mathscr{J}(\varphi_{ij})(\varphi_j(z)) := \frac{\partial \varphi_{ij}^k}{\partial z^l}(\varphi_j(z))$$

of the transition mappings.

Definition 2.6.4 (*Holomorphic tangent bundle*) The holomorphic tangent bundle of a complex manifold M of dimension n is the holomorphic vector bundle $\mathscr{T}M$ over M of rank n given by the transition functions $\psi_{ij}(z) = \mathscr{J}(\varphi_{ij})(\varphi_j(z))$.

The dual $\mathscr{T}^*M$ of $\mathscr{T}M$ is called the holomorphic cotangent bundle. Clearly, $\{\frac{\partial}{\partial z^k}\}$ and $\{\mathrm{d}z^k\}$ provide local frames in $\mathscr{T}M$ and $\mathscr{T}^*M$, respectively.

Let J be the natural almost complex structure of the complex manifold M, cf. Proposition 2.2.11. Consider the decomposition (2.2.17) defined by J. It is easy to see that $\mathrm{T}^{1,0}M$ has the same transition functions as $\mathscr{T}M$ (Exercise 2.6.1). This implies the following.

Proposition 2.6.5 *If M is a complex manifold, then $\mathrm{T}^{1,0}M$ is naturally isomorphic to the holomorphic tangent bundle $\mathscr{T}M$.* ■

Note that the induced tensor bundles $\bigotimes^p \mathrm{T}^{1,0}M$ and $\bigwedge^k \mathrm{T}^{1,0}M$ are holomorphic, whereas $\bigwedge^k \mathrm{T}^{0,1}M$ is not holomorphic.

Next, recall the decomposition (2.2.18). For a complex vector bundle E over a complex manifold M, let $\Omega^{p,q}(M, E)$ be the space of E-valued (p, q)-forms on M.

Proposition 2.6.6 *Let $\pi : E \to M$ be a holomorphic vector bundle. Then, there exists a $\mathbb{C}$-linear differential operator $\bar\partial_E : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ fulfilling $\bar\partial_E^2 = 0$ and the Leibniz rule*

$$\bar\partial_E(f\alpha) = \bar\partial(f) \wedge \alpha + f\bar\partial_E(\alpha)\,,$$

for any function f on M and any $\alpha \in \Omega^{p,q}(M, E)$.

Proof Let $(e_1, \ldots, e_k)$ be a local holomorphic frame[42] in E over $U \subset M$. Then, locally, any $\alpha \in \Omega^{p,q}(M, E)$ may be written as $\alpha = \sum_i \alpha_i \otimes e_i$, with $\alpha_i \in \Omega^{p,q}(M)$. We define

[42] That is, every $e_i : U \to E$ is a holomorphic mapping.

$$\bar{\partial}_E \alpha := \sum_i \bar{\partial}(\alpha_i) \otimes e_i \,.$$

This definition is independent of the choice of frame. Indeed, let $e'_i = g^j{}_i e_j$ be another holomorphic frame. Then, the $g^j{}_i$ are holomorphic functions on M and

$$\bar{\partial}'_E \alpha = \bar{\partial}'_E \big(\sum_i \alpha'_i \otimes g^j{}_i e_j \big) = \sum_i \bar{\partial}\alpha'_i \otimes g^j{}_i e_j = \sum_i \bar{\partial}(g^j{}_i \alpha'_i) \otimes e_j = \sum_i \bar{\partial}(\alpha_i) \otimes e_j \,.$$

Thus, $\bar{\partial}'_E \alpha = \bar{\partial}_E \alpha$. The remaining statements are now obvious. ■

The mapping $\bar{\partial}_E$ is called the Dolbeault operator. It gives rise to a cohomology theory, see Example 5.7.25 and [336] for much more material.[43] Now, let

$$\nabla : \Gamma^\infty(E) \to \Omega^1(M, E)$$

be a connection on E. Taking the complexification of $\mathrm{T}^* M$, we extend it to an operator

$$\nabla : \Gamma^\infty(E) \to \Omega^1_{\mathbb{C}}(M, E) \,.$$

According to (2.2.18), the latter decomposes as follows:

$$\nabla = \nabla^{1,0} + \nabla^{0,1} \,. \tag{2.6.5}$$

Definition 2.6.7 A connection ∇ on a holomorphic vector bundle E is called compatible with the holomorphic structure if $\nabla^{0,1} = \bar{\partial}_E$ on $\Gamma^\infty(E)$.

Note that for a compatible connection, the following are equivalent: for any local section φ of E, $\nabla^{0,1}\varphi = 0$ iff φ is holomorphic.

Proposition 2.6.8 *Let (E, h) be a holomorphic Hermitean vector bundle over the complex manifold M. Then, there exists a unique connection ∇ on E which is compatible both with the holomorphic and with the Hermitean structure.*

Proof Let ∇ be a connection fulfilling the compatibility assumptions and let ω be its connection form. Let $\mathfrak{e} = (e_1, \dots, e_k)$ be a local holomorphic frame, let $\mathscr{A} = \mathfrak{e}^*\omega$ be the local representative of ω and let H be the matrix of h with respect to $\mathfrak{e}$, that is, $H_{ij} = \mathsf{h}(e_i, e_j)$. Taking the pullback of the compatibility condition (2.6.2) under $\mathfrak{e}$, we obtain

$$\mathrm{d}H = \mathscr{A}^{\mathrm{T}} H + H \overline{\mathscr{A}} \,. \tag{2.6.6}$$

To analyze the compatibility of ∇ with the holomorphic structure, we act with ∇ on a local holomorphic section φ. Then,

$$0 = \nabla^{0,1}\varphi = \bar{\partial}\varphi + \mathscr{A}^{0,1}\varphi \,.$$

[43] Note that there is no analogue of the ∂-operator.

Thus, $\mathscr{A}^{0,1} = 0$, that is, $\mathscr{A}$ is of type $(1, 0)$. Now, decomposing both sides of (2.6.6) into their $(1, 0)$ and $(0, 1)$-parts, we read off $\partial H = \mathscr{A}^{\mathrm{T}} H$ and $\bar{\partial} H = H\overline{\mathscr{A}}$ and, thus,

$$\mathscr{A} = \overline{H}^{-1} \partial \overline{H} .$$

This formula defines unique compatible connections on each open subset belonging to a system of local trivializations. It is easy to check that, by passing to another local holomorphic frame, these local 1-forms transform properly. Thus, using a partition of unity, they may be glued together to a compatible connection on $C(M)$. ■

Definition 2.6.9 The unique connection given by Proposition 2.6.8 is called the Chern connection, or the canonical connection, of the holomorphic Hermitean vector bundle (E, h).

Corollary 2.6.10 *Let (E, h) be a holomorphic Hermitean vector bundle, let ∇ be its Chern connection and let ω and Ω be the connection and curvature form of ∇, respectively. Let $\mathscr{A} = \mathfrak{e}^*\omega$ and $\mathscr{F} = \mathfrak{e}^*\Omega$ be the local representatives with respect to a local holomorphic frame $\mathfrak{e}$ and let H be the matrix of h with respect to $\mathfrak{e}$. Then,*

$$\mathscr{A} = \overline{H}^{-1} \partial \overline{H} , \quad \mathscr{F} = \bar{\partial} \mathscr{A} , \tag{2.6.7}$$

that is, $\mathscr{A}$ is of type $(1, 0)$ and $\mathscr{F}$ is of type $(1, 1)$.

Proof The first assertion follows from the proof of Proposition 2.6.8. We show the second one: using the explicit expression for $\mathscr{A}$, together with $\partial^2 = 0$ and $\partial H^{-1} = -H^{-1} \cdot \partial H \cdot H^{-1}$, we obtain $\partial \mathscr{A} = -\mathscr{A} \wedge \mathscr{A}$. Then,

$$\mathscr{F} = \mathrm{d}\mathscr{A} + \mathscr{A} \wedge \mathscr{A} = \bar{\partial} \mathscr{A} .$$

Since $\mathscr{A}$ is of type $(1, 0)$, $\mathscr{F}$ is of type $(1, 1)$. ■

Example 2.6.11 In particular, we may consider the holomorphic tangent bundle $\mathscr{T} M$ of a complex manifold M endowed with its Chern connection. According to (2.2.13), $\mathrm{T}M$ viewed as a complex vector bundle is $\mathbb{C}$-linearly isomorphic to $\mathrm{T}^{1,0}M$. On the other hand, by Proposition 2.6.5, $\mathrm{T}^{1,0}M$ is naturally isomorphic to $\mathscr{T} M$. Thus, we have a vector bundle isomorphism $\Phi : \mathrm{T}M \to \mathscr{T} M$ which can be used to transport the Chern connection to $\mathrm{T}M$. The image can be compared with the Levi-Civita connection, see the Appendix to Chap. 4 in [336] for details. In particular, if (M, g) is Kähler, then under Φ, the Chern connection and the Levi-Civita connection coincide. ♦

The following theorem states a converse of Proposition 2.6.8. Our proof is along the lines of [384], cf. Proposition 1.3.7 there.

Theorem 2.6.12 *Let (E, h) be a Hermitean vector bundle over a complex manifold M and let ∇ be a Hermitean connection on E such that its curvature Ω is of type $(1, 1)$, that is, $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$. Then, there exists a holomorphic structure on E such that ∇ is the canonical connection with respect to this structure.*

Proof Let $C(E)$ be the principal $\mathrm{GL}(k,\mathbb{C})$-bundle of complex linear frames associated with E, that is, $E \cong C(E) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}^k$. Clearly, we may view $\mathrm{GL}(k,\mathbb{C})$ as a complex manifold. Let J_M and J_G be the almost complex structures on M and $\mathrm{GL}(k,\mathbb{C})$, respectively, defined by the complex manifold structures. Let ω be the connection form on $C(E)$ corresponding to ∇ and let $\Gamma \subset \mathrm{T}(C(E))$ be its horizontal distribution. Then, we have a unique almost complex structure on $C(E)$ defined by ω, J_M and J_G as follows: Take the splitting $\mathrm{T}(C(E)) = V \oplus \Gamma$, lift J_M from $\mathrm{T}M$ to Γ and define J on $\mathrm{T}(C(E))$ as the direct sum of this lift and of J_G. By construction, J is invariant under the right $\mathrm{GL}(k,\mathbb{C})$-action. Thus, J and the natural almost complex structure of $\mathbb{C}^k$ combine to an almost complex structure on E denoted by the same symbol.

We prove that $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$ implies that J is integrable. It is enough to give the proof in a local trivialization of E. For a chosen local trivialization $\pi^{-1}(U) \cong U \times \mathbb{C}^k$, let $(z^1, \ldots, z^n)$ be complex local coordinates on $U \subset M$ and let $(w^1, \ldots, w^k)$ be the complex coordinates on $\mathbb{C}^k$ with respect to the standard basis. Let $\mathscr{A}$ be the local representative of ω on U and let $\mathscr{A}^\alpha{}_\beta$ be its components with respect to the standard basis $\{E^\alpha{}_\beta\}$ of the Lie algebra $\mathfrak{gl}(k,\mathbb{C})$. We decompose $\mathscr{A}$ with respect to J_M,

$$\mathscr{A} = \mathscr{A}^{1,0} + \mathscr{A}^{0,1}\,.$$

Then, $\left\{\frac{\partial}{\partial \overline{z}^k}\right\}$ locally span $\Gamma^\infty\big(\mathrm{T}^{0,1}M\big)$ and, thus, $\Gamma^\infty\big(\mathrm{T}^{0,1}\Gamma\big)$ is locally spanned by the following vector fields[44]:

$$\left\{\frac{\partial}{\partial \overline{z}^k} - (\mathscr{A}^{0,1})^\alpha{}_\beta\left(\frac{\partial}{\partial \overline{z}^k}\right)(E^\beta{}_\alpha)_*\right\}\,,\quad k = 1, \ldots, n\,,\ \ \alpha, \beta = 1, \ldots k\,,$$

where $(E^\beta{}_\alpha)_*$ is the Killing vector field generated by $E^\beta{}_\alpha$. Now, the horizontal distribution on E corresponding to Γ is given by (1.3.4). Here, since $\mathbb{C}^k$ is the basic $\mathrm{GL}(k,\mathbb{C})$-module,

$$\iota'_{\mathbf{z}}(A_{*u}) = u(A\mathbf{z})\,,\quad \mathbf{z} \in \mathbb{C}^k\,,\ u \in C(E)\,,\ A \in \mathfrak{gl}(k,\mathbb{C})\,.$$

Thus, $\Gamma^\infty\big(\mathrm{T}^{0,1}E\big)$ is locally spanned by

$$\left\{\frac{\partial}{\partial \overline{z}^k} - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta\left(\frac{\partial}{\partial \overline{z}^k}\right)w^\beta\frac{\partial}{\partial w^\alpha}, \frac{\partial}{\partial \overline{w}^\alpha}\right\}\,.$$

Consequently, its annihilator $\Omega^{1,0}(E)$ is locally spanned by $\{\mathrm{d}z^l, \vartheta^\alpha\}$, where

$$\vartheta^\alpha = \mathrm{d}w^\alpha + \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta w^\beta\,.$$

[44]Recall that the horizontal component of a vector field X on a principal G-bundle is given by $X - \Psi'_p(\omega(X))$, cf. formula (1.3.7).

Now, using $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$, we calculate

$$\begin{aligned} \mathrm{d}\vartheta^\alpha &= w^\beta \mathrm{d}\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \mathrm{d}w^\beta \\ &= w^\beta \mathrm{d}\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \big(\vartheta^\beta - \big(\mathscr{A}^{0,1}\big)^\beta{}_\gamma w^\gamma\big) \\ &= w^\beta \big(\partial\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta + \big(\Omega^{0,2}\big)^\alpha{}_\beta\big) - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \vartheta^\beta \\ &= w^\beta \partial\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \vartheta^\beta , \end{aligned}$$

that is, $\mathrm{d}\vartheta^\alpha \in \Omega^{1,1}(E)$. By Proposition 2.2.14, this is equivalent to the vanishing of the Nijenhuis tensor and, thus, the Newlander–Nirenberg Theorem 2.2.13 implies that J is integrable.

It remains to prove that, with respect to the holomorphic structure defined by J, ∇ coincides with the Chern connection. That is, we have to prove that a local section $\varphi : U \to E$ fulfilling $\nabla^{0,1}\varphi = 0$ is holomorphic. For that purpose, it is enough to show that any φ fulfilling this condition pulls back every $(1,0)$-form on E to a $(1,0)$-form on M.[45] In the above notation, $\nabla^{0,1}\varphi = 0$ reads

$$\bar{\partial}\varphi^\alpha + \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \varphi^\beta = 0 .$$

Using this, we calculate $\varphi^*(\mathrm{d}z^k) = \mathrm{d}z^k$ and

$$\varphi^*(\vartheta^\alpha) = \mathrm{d}\varphi^\alpha + \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \varphi^\beta = \partial\varphi^\alpha .$$

■

For a more general integrability theorem containing Theorem 2.6.12 as a special case, we refer to [35].

Exercises

2.6.1 Prove Proposition 2.6.5.

2.7 Hodge Theory. The Weitzenboeck Formula

Let us recall some basic notions from Sects. 4.4 and 4.5 of Part I. Consider an n-dimensional oriented pseudo-Riemannian manifold (M, g) with signature (r, s). The metric g yields a distinguished volume form v_g, cf. Definition I/4.4.4., and a mapping

$$* : \Omega^k(M) \to \Omega^{n-k}(M) , \quad *\alpha := (-1)^s \mathsf{g}^{-1}(\alpha) \lrcorner \mathsf{v}_\mathsf{g} , \tag{2.7.1}$$

called the Hodge star operator, cf. Definition I/4.5.1. We immediately read off

[45] Recall Exercise 2.2.3.

$$*\,1 = (-1)^s \mathsf{v_g}\,, \quad *\,\mathsf{v_g} = 1\,. \tag{2.7.2}$$

We have the following further basic properties: for any $\alpha, \beta \in \Omega^k(M)$,

$$* * \alpha = (-1)^{k(n-k)+s}\,\alpha\,, \tag{2.7.3}$$

$$\mathsf{g}^{-1}(*\alpha, *\beta) = (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\,, \tag{2.7.4}$$

$$\alpha \wedge *\beta = (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\mathsf{v_g}\,, \tag{2.7.5}$$

cf. Proposition I/4.5.3. Let $\{e_i\}$ be an orthonormal local frame on M and let $\{\vartheta^i\}$ be the dual coframe. Then, locally, we have

$$\mathsf{v_g} = (-1)^s \vartheta^{I_n}\,, \tag{2.7.6}$$

$$*\,\vartheta^I = \eta^{IJ} e_J \lrcorner \vartheta^{I_n} = \operatorname{sign}\begin{pmatrix} I_n \\ J\ J^c \end{pmatrix} \eta^{IJ}\,\vartheta^{J^c}\,. \tag{2.7.7}$$

Using (2.7.7), for any $\alpha \in \Omega^k(M)$, one easily shows the following:

$$(*\alpha)(X_{k+1}, \ldots, X_n)\mathsf{v_g} = \alpha \wedge \mathsf{g}(X_{k+1}) \wedge \ldots \wedge \mathsf{g}(X_n)\,. \tag{2.7.8}$$

This implies

$$X \lrcorner * \alpha = *(\alpha \wedge \mathsf{g}(X))\,, \tag{2.7.9}$$

$$\mathsf{g}^{-1}(\beta) \lrcorner * \alpha = *(\alpha \wedge \beta)\,, \tag{2.7.10}$$

for any $\alpha \in \Omega^*(M)$, $\beta \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ (Exercise 2.7.1). The metric induces a natural fibre metric on $E = \bigwedge^k \mathrm{T}^* M$ via

$$\langle \alpha, \beta \rangle := (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\,,$$

which gives rise to an L^2-inner product on the space of square-integrable k-forms:

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha, \beta \rangle \mathsf{v_g} = \int_M \alpha \wedge * \beta\,. \tag{2.7.11}$$

Using this inner product, one defines the Hodge dual $\mathrm{d}^* : \Omega^k(M) \to \Omega^{k-1}(M)$ of the exterior derivative by

$$\langle \mathrm{d}^*\alpha, \beta \rangle_{L^2} := \langle \alpha, \mathrm{d}\beta \rangle_{L^2}\,, \tag{2.7.12}$$

for all $\beta \in \Omega^{k-1}(M)$. For $\alpha \in \Omega^k(M)$, one has

$$\mathrm{d}^*\alpha = (-1)^{n(k-1)+s+1}\ * \mathrm{d} * \alpha\,. \tag{2.7.13}$$

Given the exterior derivative and its Hodge dual, we build the Hodge–Laplace operator of (M, g):

$$\Box : \Omega^k(M) \to \Omega^k(M)\,, \quad \Box := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\,. \tag{2.7.14}$$

Clearly,

$$\langle \Box\alpha, \alpha\rangle_{L^2} = \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle_{L^2} + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha\rangle_{L^2}\,. \tag{2.7.15}$$

Moreover,

$$\mathrm{d}\,\Box = \Box\,\mathrm{d}\,, \quad \mathrm{d}^*\,\Box = \Box\,\mathrm{d}^*\,, \quad *\Box = \Box\,*\,. \tag{2.7.16}$$

Finally, we note that $\Box$ is symmetric:

$$\langle \Box\alpha, \beta\rangle_{L^2} = \langle \alpha, \Box\beta\rangle_{L^2}\,. \tag{2.7.17}$$

The proof of these elementary properties is left to the reader (Exercise 2.7.2).

Remark 2.7.1 (*Hodge decomposition*) In this Remark, we assume that (M, g) is a compact oriented n-dimensional Riemannian manifold.

Since g is Riemannian, the inner product (2.7.11) is positive definite. Then, (2.7.15) implies that $\Box$ is positive definite and that

$$\Box\alpha = 0 \quad \text{iff} \quad \mathrm{d}\alpha = 0 \;\text{ and }\; \mathrm{d}^*\alpha = 0\,. \tag{2.7.18}$$

Since $\Box = (\mathrm{d} + \mathrm{d}^*)^2$, we also have

$$\ker(\Box) = \ker(\mathrm{d} + \mathrm{d}^*)\,. \tag{2.7.19}$$

A k-form α fulfilling $\Box\alpha = 0$ is called harmonic. We conclude that the only harmonic functions on a compact connected oriented Riemannian manifold are the constant functions. This in turn implies that if, additionally, the first de Rham cohomology of M is trivial, then there does not exist any nontrivial harmonic 1-form on M (Exercise 2.7.3). The space of harmonic k-forms is denoted by

$$\mathscr{H}^k(M) := \left\{\alpha \in \Omega^k(M) : \Box\alpha = 0\right\}\,.$$

In Sect. 5.7 we will see that the Hodge–Laplace operator on a compact oriented Riemannian manifold is elliptic. The theory of elliptic operators implies that, for any k, $\mathscr{H}^k(M)$ is finite-dimensional. Moreover, the following orthogonal direct sum decomposition, called Hodge decomposition, holds.[46]

Theorem 2.7.2 (Hodge Decomposition Theorem)

$$\Omega^k(M) = \mathscr{H}^k(M) \oplus \Box(\Omega^k(M))\,. \tag{2.7.20}$$

[46] Clearly, by the elementary properties of $\Box$ proved above, the second summand can be decomposed further, $\Box(\Omega^k(M)) = \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathrm{d}^*(\Omega^{k+1}(M))$.

The proof will be given in a more general context in Chap. 5, see Theorem 5.7.18. The Hodge decomposition has the following immediate consequences:

1. The natural mapping

$$F : \mathscr{H}^k(M) \to H^k_{\mathrm{dR}}(M)\,, \quad \alpha \mapsto [\alpha]\,,$$

is an isomorphism, that is, every de Rham cohomology class contains a unique harmonic form. To prove injectivity of F, take two harmonic k-forms α and β belonging to the same cohomology class. Then, there exists a $(k-1)$-form τ such that $\alpha - \beta = \mathrm{d}\tau$. Then,

$$\| \alpha - \beta \|^2_{L^2} = \langle \alpha - \beta, \mathrm{d}\tau \rangle_{L^2} = \langle \mathrm{d}^*\alpha - \mathrm{d}^*\beta, \tau \rangle_{L^2} = 0\,,$$

and thus $\alpha = \beta$. To prove surjectivity, take an arbitrary class $[\alpha] \in H^k_{\mathrm{dR}}(M)$ and represent it by some closed form $\alpha \in Z^k(M)$. Then, by the Hodge decomposition (2.7.20), there exists an element $\omega \in \mathscr{H}^k(M)$ and a k-form β such that

$$\alpha = \omega + \Box\beta.$$

Since $\mathrm{d}\omega = 0$, we have $0 = \mathrm{d}\alpha = \mathrm{d}\mathrm{d}^*\mathrm{d}\beta$ and thus

$$\langle \mathrm{d}^*\mathrm{d}\beta, \mathrm{d}^*\mathrm{d}\beta \rangle_{L^2} = \langle \mathrm{d}\beta, \mathrm{d}\mathrm{d}^*\mathrm{d}\beta \rangle_{L^2} = 0\,.$$

This implies $\mathrm{d}^*\mathrm{d}\beta = 0$ and thus $\alpha = \omega + \mathrm{d}\mathrm{d}^*\beta$, showing that $[\omega] = [\alpha]$.
2. The natural pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}\,, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta\,,$$

defines an isomorphism (Poincaré duality) of $H^{n-k}_{\mathrm{dR}}(M)$ with the dual space of $H^k_{\mathrm{dR}}(M)$,

$$H^{n-k}_{\mathrm{dR}}(M) \cong \left(H^k_{\mathrm{dR}}(M)\right)^*\,. \tag{2.7.21}$$

To prove this, given a nonzero cohomology class $[\alpha] \in H^k_{\mathrm{dR}}(M)$, we must find a cohomology class $[\beta] \in H^{n-k}_{\mathrm{dR}}(M)$ such that $\int_M \alpha \wedge \beta \neq 0$. For that purpose, we choose a Riemannian metric g on M. By point 1, we may choose a harmonic representative α of $[\alpha]$ which, of course, cannot vanish identically. Then, by the third identity in (2.7.16), $*\alpha$ is also harmonic and thus, by (2.7.18), it is closed. This means that $*\alpha$ represents a cohomology class in $H^{n-k}_{\mathrm{dR}}(M)$. Pairing this element with $[\alpha]$ yields

$$([\alpha], [*\alpha]) \mapsto \int_M \alpha \wedge *\alpha = \| \alpha \|^2 \neq 0\,.$$

Thus, the above pairing defines an isomorphism of $H^{n-k}_{\mathrm{dR}}(M)$ and $\big(H^k_{\mathrm{dR}}(M)\big)^*$, indeed. ♦

Below, we wish to prove the Weitzenboeck Formula which, combined with the theory of harmonic forms, yields deep insight into the relation between curvature and topology. It compares the Hodge–Laplace operator of (M, g) to the Bochner–Laplace operator built from the Levi-Civita connection ∇ of g. The basic object relating these two quantities is the Weitzenboeck curvature operator built from the curvature endomorphism of ∇. In order to accomplish this goal, we need a unified treatment of these objects in terms of the Koszul calculus. Thus, we consider the vector bundle $E = \bigwedge^k \mathrm{T}^* M$ endowed with its natural fibre metric $\langle \cdot, \cdot \rangle$ defined above and with the natural connection induced from the Levi-Civita connection,[47] which we also denote by ∇. Clearly, ∇ is compatible with $\langle \cdot, \cdot \rangle$. Then, we proceed as follows:

(a) We express the Hodge dual operator d^* in terms of ∇. Recall that d has been already calculated in terms of ∇, cf. formula (2.2.49).
(b) We define the Bochner–Laplace operator and calculate it in terms of ∇. Since this can be done without any modifications for an arbitrary Riemannian (or Hermitean) vector bundle endowed with a compatible connection, we present it for this general case. This will also be useful later on.
(c) We define the Weitzenboeck curvature operator and derive the Weitzenboeck Formula.

(a) Let ω be the connection form of ∇. Let $\mathfrak{e} = \{e_i\}$ be a local frame and let $\{\vartheta^i\}$ be its dual coframe. By (2.1.39), the local representative of ω with respect to $\mathfrak{e}$ is given by $\mathfrak{e}^* \omega^i{}_k = \Gamma^i{}_{jk} \vartheta^j$, where $\Gamma^i{}_{jk}$ are the Christoffel symbols with respect to $\mathfrak{e}$.

Lemma 2.7.3 *For any $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^*(M)$,*

$$\nabla_X \mathsf{v}_{\mathsf{g}} = 0 \,, \quad \nabla_X * \alpha = * \nabla_X \alpha \,. \tag{2.7.22}$$

Proof As an immediate consequence of (2.7.6), (2.1.47) and (2.2.44), for any orthonormal frame $\{e_i\}$, we have

$$\nabla_{e_i} \mathsf{v}_{\mathsf{g}} = (-1)^{s+1} \sum_j \Gamma^j{}_{ij} \vartheta^1 \wedge \ldots \wedge \vartheta^n = 0 \,.$$

This proves the first assertion. To prove the second one, we act with ∇_X on equation (2.7.5). Using $\nabla_X \mathsf{v}_{\mathsf{g}} = 0$, $\nabla_X \mathsf{g} = 0$ and once again (2.7.5), we obtain

$$\begin{aligned} \nabla_X \alpha \wedge *\beta + \alpha \wedge \nabla_X * \beta &= (-1)^s \big(\mathsf{g}^{-1}(\nabla_X \alpha, \beta) + \mathsf{g}^{-1}(\alpha, \nabla_X \beta)\big) \mathsf{v}_{\mathsf{g}} \\ &= \nabla_X \alpha \wedge *\beta + \alpha \wedge *\nabla_X \beta \,, \end{aligned}$$

for arbitrary forms α and β. From this we read off the second assertion. ■

[47] Cf. Exercise 2.1.7.

Lemma 2.7.4 *Let (M, g) be a pseudo-Riemannian manifold and let $\alpha \in \Omega^k(M)$. Let $\{e_i\}$ be a local frame and let $\{\vartheta^i\}$ be its dual coframe. Then,*

$$\mathrm{d}^*\alpha = -\mathsf{g}^{-1}(\vartheta^j)\lrcorner\nabla_{e_j}\alpha \,. \tag{2.7.23}$$

Proof Let $\alpha \in \Omega^k(M)$. Using (2.2.47), Lemma 2.7.3 and (2.7.10), we calculate

$$\begin{aligned} *\mathrm{d} * \alpha &= * \left(\vartheta^j \wedge \nabla_{e_j} * \alpha\right) \\ &= (-1)^{n-k} * \left(*(\nabla_{e_j}\alpha) \wedge \vartheta^j\right) \\ &= (-1)^{n-k} \left(\mathsf{g}^{-1}(\vartheta^j)\lrcorner(*^2\nabla_{e_j}\alpha)\right) \\ &= (-1)^{(n-k)(k+1)+s} \left(\mathsf{g}^{-1}(\vartheta^j)\lrcorner\nabla_{e_j}\alpha\right) \,. \end{aligned}$$

Comparison with (2.7.13) yields the assertion. ■

Remark 2.7.5 Since the operator d^* is intrinsically defined, formula (2.7.23) does not depend on the choice of the frame. Using $\mathsf{g}^{-1}(\vartheta^j) = \mathsf{g}^{jk}e_k$, it reads

$$(\mathrm{d}^*\alpha)(X_2, \ldots, X_k) = -\mathsf{g}^{jl} \left(\nabla_{e_j}\alpha\right)(e_l, X_2, \ldots, X_k) \,. \tag{2.7.24}$$

For some purposes, it is useful to rewrite this as

$$(\mathrm{d}^*\alpha)(X_2, \ldots, X_k) = -\left(\mathrm{tr}^{\mathsf{g}}_{12}(\nabla\alpha)\right)(X_2, \ldots, X_k) \,. \tag{2.7.25}$$

Here, $\nabla\alpha \in \Gamma^\infty(\mathrm{T}^*M \otimes \bigwedge^k\mathrm{T}^*M)$ and $\mathrm{tr}^{\mathsf{g}}_{12}$ means contracting the first two tensor indices of $\nabla\alpha$ with g. The quantity $\mathrm{tr}^{\mathsf{g}}_{12}(\nabla\alpha)$ is called the divergence of α and is denoted by $\mathrm{div}^{\mathsf{g}}\alpha$. In this terminology, we have

$$\mathrm{d}^*\alpha = -\mathrm{div}^{\mathsf{g}}\alpha \,. \tag{2.7.26}$$

In particular, for a 1-form $\alpha \in \Omega^1(M)$, we obtain (Exercise 2.7.4)

$$\mathrm{div}^{\mathsf{g}}(\alpha)\,\mathsf{v}_{\mathsf{g}} = \mathrm{d}(\mathsf{g}^{-1}(\alpha)\lrcorner\,\mathsf{v}_{\mathsf{g}}) \,. \tag{2.7.27}$$

♦

(b) Next, instead of $(\bigwedge^k\mathrm{T}^*M, \langle\cdot,\cdot\rangle, \nabla)$, consider any Riemannian or Hermitean vector bundle E with a fibre metric $\langle\cdot,\cdot\rangle$ and a compatible connection ∇ over a pseudo-Riemannian manifold (M, g). As in the above special case, $\langle\cdot,\cdot\rangle$ and g induce a natural L^2-inner product on $\Gamma^\infty(E)$ via

$$\langle s_1, s_2\rangle_{L^2} := \int_M \langle s_1, s_2\rangle \mathsf{v}_{\mathsf{g}} \,. \tag{2.7.28}$$

If we endow T^*M with the natural fibre metric given by g^{-1}, then we may extend $\langle\cdot,\cdot\rangle_{L^2}$ to an inner product on $\Gamma^\infty(\mathrm{T}^*M \otimes E)$ which we denote by the same symbol.

We define the formal adjoint $\nabla^* : \Gamma^\infty(\mathrm{T}^*M \otimes E) \to \Gamma^\infty(E)$ of ∇ by

$$\langle s, \nabla^* \varphi \rangle_{L^2} = \langle \nabla s, \varphi \rangle_{L^2} ,$$

for any $s \in \Gamma^\infty(E)$ and $\varphi \in \Gamma^\infty(\mathrm{T}^*M \otimes E)$.

Proposition 2.7.6 *For any $\varphi \in \Gamma^\infty(\mathrm{T}^*M \otimes E)$,*

$$\nabla^* \varphi = - \operatorname{tr}^{\mathsf{g}}_{12}(\nabla \varphi) .$$

Proof Let $s \in \Gamma^\infty(E)$. For a given local frame $\{e_i\}$ and its dual coframe $\{\vartheta^i\}$, decompose

$$\nabla s = \vartheta^i \otimes \nabla_{e_i} s , \quad \varphi = \vartheta^j \otimes \varphi(e_j) ,$$

and calculate

$$\langle \nabla s, \varphi \rangle = \langle \vartheta^i \otimes \nabla_{e_i} s, \vartheta^j \otimes \varphi(e_j) \rangle = \mathsf{g}^{ij} \langle \nabla_{e_i} s, \varphi(e_j) \rangle .$$

Since ∇ is compatible with the fibre metric, (2.6.2) implies

$$e_i(\langle s, \varphi(e_j) \rangle) = \langle \nabla_{e_i} s, \varphi(e_j) \rangle + \langle s, \nabla_{e_i}(\varphi(e_j)) \rangle ,$$

and, thus,

$$\begin{aligned} \langle \nabla s, \varphi \rangle &= \mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \nabla_{e_i}(\varphi(e_j)) \rangle \right) \\ &= \mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \varphi(\nabla_{e_i} e_j) \rangle - \langle s, (\nabla_{e_i} \varphi)(e_j) \rangle \right) . \end{aligned}$$

Defining a 1-form $\beta \in \Omega^1(M)$ by $\beta(X) := \langle s, \varphi(X) \rangle$, where $X \in \mathfrak{X}(M)$, we obtain

$$\mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \varphi(\nabla_{e_i} e_j) \rangle \right) = \mathsf{g}^{ij} (\nabla_{e_i} \beta)(e_j) = \operatorname{div}^{\mathsf{g}} \beta .$$

Then, (2.7.27) implies

$$\langle \nabla s, \varphi \rangle = \mathrm{d}(\mathsf{g}^{-1}(\beta) \lrcorner \mathsf{v}_{\mathsf{g}}) - \mathsf{g}^{ij} \langle s, (\nabla_{e_i} \varphi)(e_j) \rangle .$$

Integrating this identity with v_{g} and using Stokes' Theorem, we find

$$\langle \nabla s, \varphi \rangle_{L^2} = - \langle s, \mathsf{g}^{ij} (\nabla_{e_i} \varphi)(e_j) \rangle_{L^2} = - \langle s, \operatorname{tr}^{\mathsf{g}}_{12}(\nabla \varphi) \rangle_{L^2} .$$

■

Remark 2.7.7 By Proposition 2.7.6, $\nabla^* \varphi = -\mathsf{g}^{ij} (\nabla_{e_i} \varphi)(e_j)$ for any local frame $\{e_i\}$ and, thus,

$$\nabla^* \varphi = \mathsf{g}^{ij} \left(\varphi(\nabla_{e_i} e_j) - \nabla_{e_i}(\varphi(e_j)) \right) . \tag{2.7.29}$$

♦

Definition 2.7.8 (*Bochner–Laplace operator*) The mapping

$$\nabla^*\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E)$$

is called the Bochner–Laplace operator.[48]
By Proposition 2.7.6, we have

$$\nabla^*\nabla s = -\operatorname{tr}^{\mathrm{g}}_{12}(\nabla\nabla s)\,, \quad s \in \Gamma^\infty(E)\,, \tag{2.7.30}$$

and, by (2.7.29),

$$\nabla^*\nabla s = -\mathrm{g}^{ij}\left(\nabla_{e_i}\nabla_{e_j}s - \nabla_{\nabla_{e_i}e_j}s\right)\,. \tag{2.7.31}$$

Moreover, since $\langle\nabla^*\nabla s_1, s_2\rangle_{L^2} = \langle\nabla s_1, \nabla s_2\rangle_{L^2} = \langle s_1, \nabla^*\nabla s_2\rangle_{L^2}$, the Bochner–Laplace operator is formally self-adjoint.

(c) It is convenient to consider $\bigwedge^k \mathrm{T}^*M$ as associated with the reduced bundle of orthonormal frames $O(M)$. Then, σ is induced from the basic representation of the orthogonal group $\mathrm{O}(r,s)$ of the pseudo-Euclidean metric η on $\mathbb{R}^n$. It acts on $\bigwedge^k(\mathbb{R}^n)^*$ via

$$\sigma(a)(\xi_1\wedge\ldots\wedge\xi_k) = \left(\left(a^{-1}\right)^{\mathrm{T}}\xi_1\right)\wedge\xi_2\wedge\ldots\wedge\xi_k + \ldots + \xi_1\wedge\ldots\wedge\xi_{k-1}\wedge\left(\left(a^{-1}\right)^{\mathrm{T}}\xi_k\right)\,.$$

Identifying $\bigwedge^k(\mathbb{R}^n)^* \cong \bigwedge^k\mathbb{R}^n$ via the metric, we obtain the representation σ' of the Lie algebra $\mathfrak{o}(r,s)$ on $\bigwedge^k(\mathbb{R}^n)^*$:

$$\sigma'(A)(\xi_1\wedge\ldots\wedge\xi_k) = (A\xi_1)\wedge\xi_2\wedge\ldots\wedge\xi_k + \ldots + \xi_1\wedge\ldots\wedge\xi_{k-1}\wedge(A\xi_k)\,, \tag{2.7.32}$$

that is, $A\in\mathfrak{o}(r,s)$ acts as a derivation on $\bigwedge^k(\mathbb{R}^n)^*$. Accordingly, the curvature endomorphism form

$$\mathsf{R}^{\Lambda}_m(X,Y) = \iota_p\circ\sigma'(\Omega_p(X^h,Y^h))\circ\iota_p^{-1}$$

of ∇ is a 2-form on M with values in $\mathrm{End}(\bigwedge^k\mathrm{T}^*M)$ acting as a derivation. For the convenience of the reader, we recall the following.

Remark 2.7.9 (*Contraction and exterior multiplication*) Let V be a real vector space endowed with a metric $\eta = \langle\cdot,\cdot\rangle$. The contraction mapping $\iota : V^* \to \mathrm{End}(\bigwedge V)$ is defined by $\iota(\xi)1 = 0$ and

$$\iota(\xi)(v_1\wedge\ldots\wedge v_k) = \sum_{i=1}^k(-1)^{i-1}\langle\xi, v_i\rangle v_1\wedge\ldots\hat{v}_i\ldots\wedge v_k\,,$$

where $\xi\in V^*$ and $v_1,\ldots,v_k\in V$. We will also write $\iota(\xi)\equiv\xi\lrcorner$. Since

$$\iota(\xi)\iota(\zeta) + \iota(\zeta)\iota(\xi) = 0$$

[48] Some authors call it the rough Laplacian.

for all $\xi, \zeta \in V^*$, by the universal property of the exterior algebra, ι extends to an algebra morphism $\iota : \bigwedge V^* \to \mathrm{End}(\bigwedge V)$. We denote the operation of exterior multiplication with an element $v \in V$ by

$$\varepsilon(v)(\alpha) := v \wedge \alpha$$

and note the following basic identity (Exercise 2.7.6):

$$\varepsilon(v)\iota(\xi) + \iota(\xi)\varepsilon(v) = \langle \xi, v \rangle \cdot 1 . \tag{2.7.33}$$

Let $\{\mathbf{e}_j\}$ be an orthonormal basis of V, let $\{\vartheta^j\}$ be the dual basis and denote $\varepsilon_j := \varepsilon(\mathbf{e}_j)$ and $\iota^k := \iota(\vartheta^k)$. In this notation, the natural action $\mathrm{End}(V) \to \mathrm{Der}(\bigwedge V)$ of $\mathrm{End}(V)$ by derivations on the exterior algebra,

$$A^\Lambda(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge v_2 \wedge \cdots \wedge v_k + \cdots + v_1 \wedge \cdots \wedge v_{k-1} \wedge Av_k ,$$

is given by

$$A^\Lambda = \eta^{jl}\eta(\mathbf{e}_l, A\mathbf{e}_k)\varepsilon_j\iota^k . \tag{2.7.34}$$

In terms of the matrix elements $A_{ij} = \eta(\mathbf{e}_i, A\mathbf{e}_j)$, we have

$$A^\Lambda = A^j{}_k\varepsilon_j\iota^k . \tag{2.7.35}$$

♦

By (2.7.34), the curvature endomorphism $\mathsf{R}^\Lambda_m(X, Y)$ acts as a derivation on $\bigwedge^k \mathrm{T}^*M$ as follows:

$$\mathsf{R}^\Lambda(e_i, e_j) = \eta^{km}\mathsf{g}(\mathsf{R}(e_i, e_j)e_m, e_l)e^l{}_k , \tag{2.7.36}$$

where $e^l{}_k := \varepsilon^l\iota_k$ and where $\{e_j\}$ is any local orthonormal frame.

Definition 2.7.10 The Weitzenboeck curvature operator $\mathfrak{R}^\Lambda : \Omega^k(M) \to \Omega^k(M)$ of ∇ is defined by

$$\mathfrak{R}^\Lambda(\alpha)(X_1, \ldots, X_k) := \sum_i \eta^{jl}\left(\mathsf{R}^\Lambda(e_j, X_i)\alpha\right)(X_1, \ldots, \overset{\overset{i}{\curlyvee}}{e_l}, \ldots, X_k) , \tag{2.7.37}$$

where $X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $\{e_j\}$ is an arbitrary orthonormal local frame.[49]

Let us calculate $\mathfrak{R}^\Lambda$ in the frame $\{e^k{}_l\}$. Using (2.7.36), together with the symmetry properties of R, we obtain

[49]We have only made the summation over i explicit. The remaining summations are in accordance with the Einstein summation convention.

$$
\begin{aligned}
\mathfrak{R}^{\Lambda}(\alpha)(X_1,\ldots,X_k) &= \sum_i \eta^{jl}\big(\mathsf{R}^{\Lambda}(e_j,X_i)\alpha\big)(X_1,\ldots,e_l,\ldots,X_k)\\
&= \sum_i \eta^{jl}\eta^{kp}\mathsf{g}\big(\mathsf{R}(e_p,e_m)e_j,X_i\big)(e^m{}_k\alpha)(X_1,\ldots,e_l,\ldots,X_k)\\
&= \sum_i (e^m{}_k\alpha)(X_1,\ldots,\eta^{jl}\eta^{kp}\mathsf{g}\big(\mathsf{R}(e_p,e_m)e_j,X_i\big)e_l,\ldots,X_k)\\
&= -\sum_i (e^m{}_k\alpha)(X_1,\ldots,\eta^{kl}\mathsf{R}(e_l,e_m)X_i,\ldots,X_k)\\
&= \big(\eta^{km}\mathsf{R}^{\Lambda}(e_m,e_l)\circ e^l{}_k\big)(\alpha)(X_1,\ldots,X_k)\,.
\end{aligned}
$$

In the last step, we have used that R^{Λ} is a derivation which acts trivially on zero-forms. Using (2.7.36) once again, we obtain

$$
\mathfrak{R}^{\Lambda} = \mathsf{R}_{ijkl}\varepsilon^i\iota^j\varepsilon^k\iota^l\,. \tag{2.7.38}
$$

Now we are able to formulate the main result of the second part of this section.

Theorem 2.7.11 (Weitzenboeck Formula) *Let* $\alpha\in\Omega^k(M)$. *Then,*

$$
\Box\alpha \;=\; \nabla^*\nabla\alpha+\mathfrak{R}^{\Lambda}(\alpha)\,. \tag{2.7.39}
$$

Proof We choose an orthonormal local frame $\{e_i\}$ and the dual coframe $\{\vartheta^i\}$. Using Lemma 2.7.4, (2.2.47), (2.1.46) and and the first equation in (2.2.44), we calculate

$$
\begin{aligned}
\mathrm{dd}^*\alpha &= -\mathrm{d}\left(\eta^{jl}e_l\lrcorner\nabla_{e_j}\alpha\right)\\
&= -\vartheta^i\wedge\nabla_{e_i}\left(\eta^{jl}e_l\lrcorner\nabla_{e_j}\alpha\right)\\
&= -\vartheta^i\wedge\left(\nabla_{e_i}e_l\lrcorner\nabla_{e_j}\alpha+e_l\lrcorner\nabla_{e_i}\nabla_{e_j}\alpha\right)\eta^{jl}\\
&= e^i{}_l\left(\nabla_{\nabla_{e_i}e_j}\alpha-\nabla_{e_i}\nabla_{e_j}\alpha\right)\eta^{jl}\,.
\end{aligned}
$$

On the other hand, again by Lemma 2.7.4, together with (2.1.47), we obtain

$$
\begin{aligned}
\mathrm{d}^*\mathrm{d}\alpha &= \mathrm{d}^*\left(\vartheta^i\wedge\nabla_{e_i}\alpha\right)\\
&= -\eta^{jl}e_l\lrcorner\left(\nabla_{e_j}\left(\vartheta^i\wedge\nabla_{e_i}\alpha\right)\right)\\
&= -\eta^{jl}e_l\lrcorner\left(\nabla_{e_j}\vartheta^i\wedge\nabla_{e_i}\alpha+\vartheta^i\wedge\nabla_{e_j}\nabla_{e_i}\alpha\right)\\
&= \eta^{jl}e_l\lrcorner\left(\vartheta^i\wedge\nabla_{\nabla_{e_j}e_i}\alpha-\vartheta^i\wedge\nabla_{e_j}\nabla_{e_i}\alpha\right)\\
&= \eta^{ji}\left(\nabla_{\nabla_{e_j}e_i}\alpha-\nabla_{e_j}\nabla_{e_i}\alpha\right)-\eta^{jl}e^i{}_l\left(\nabla_{\nabla_{e_j}e_i}\alpha-\nabla_{e_j}\nabla_{e_i}\alpha\right)\,.
\end{aligned}
$$

Adding up these two equations and using (2.7.31) yields

$$
\Box\alpha = \nabla^*\nabla\alpha-\eta^{jl}e^i{}_l\left(\nabla_{e_i}\nabla_{e_j}\alpha-\nabla_{e_j}\nabla_{e_i}\alpha-\nabla_{(\nabla_{e_i}e_j-\nabla_{e_j}e_i)}\alpha\right)\,.
$$

Since the Levi-Civita connection is torsionless, we have $\nabla_{e_i} e_j - \nabla_{e_j} e_i = [e_i, e_j]$ and, thus, by point 2 of Remark 1.5.12 and Eqs. (2.1.32) and (2.7.36),

$$\Box\alpha = \nabla^*\nabla\alpha - \eta^{jl} e^i{}_l \big(\mathsf{R}^\Lambda(e_i, e_j)\alpha\big) = \nabla^*\nabla\alpha + \mathsf{R}_{ijkl}\varepsilon^i \iota^j \varepsilon^k \iota^l \alpha \,.$$

Comparing with (2.7.38), we obtain the assertion. ■

Clearly, the second term in the Weitzenboeck Formula may be analyzed in more detail for every form degree k. To do so, recall the presentation of the Ricci tensor in a local frame given by (2.3.27),

$$\mathsf{Ric}(X, Y) = -\eta^{ij}\mathsf{g}\left(\mathsf{R}(X, e_i)Y, e_j\right) , \quad X, Y \in \mathfrak{X}(M) \,. \tag{2.7.40}$$

Associated with the Ricci tensor, one has the Ricci mapping

$$\mathsf{Ric} : \mathrm{T}M \to \mathrm{T}M \,, \quad \mathsf{Ric}(X) := \eta^{ij}\mathsf{R}(X, e_i)e_j \,. \tag{2.7.41}$$

Being an endomorphism of $\mathrm{T}M$, the Ricci mapping naturally extends to a derivation Ric^Λ of $\bigwedge \mathrm{T}M$. In degree 2, it is common to denote this derivation by $\mathsf{Ric} \wedge \mathrm{id}$. We have

$$(\mathsf{Ric} \wedge \mathrm{id})(X, Y) := \mathsf{Ric}(X) \wedge Y + X \wedge \mathsf{Ric}(Y) \,.$$

Analogously, associated with the curvature endomorphism form, one has the mapping

$$\mathsf{R} : \bigwedge^2 \mathrm{T}M \to \bigwedge^2 \mathrm{T}M \,, \quad X \wedge Y \mapsto \eta^{ij} e_i \wedge \mathsf{R}(X, Y)e_j \,. \tag{2.7.42}$$

In applications, the cases $k = 1$ and $k = 2$ are of special importance.

Corollary 2.7.12

1 *For $k = 1$, the Weitzenboeck Formula* (2.7.39) *reads*

$$\Box\alpha \;=\; \nabla^*\nabla\alpha + \alpha \circ \mathsf{Ric} \,. \tag{2.7.43}$$

2 *For $k = 2$, the Weitzenboeck Formula may be rewritten as follows:*

$$\Box\alpha \;=\; \nabla^*\nabla\alpha + \alpha \circ (\mathsf{R} + \mathsf{Ric} \wedge \mathrm{id}) \,, \tag{2.7.44}$$

where R *is the mapping defined by* (2.7.42).

Proof 1. For $k = 1$, by (2.7.37), (2.7.36) and (2.7.41),

$$\begin{aligned}\mathfrak{R}^{\Lambda}(\alpha)(X) &= \eta^{jl}\big(\mathsf{R}^{\Lambda}(e_j, X)\alpha\big)(e_l)\\ &= \eta^{il}\eta^{km}\mathsf{g}\big(\mathsf{R}(X, e_i)e_l, e_m\big)\alpha(e_k)\\ &= \eta^{km}\mathsf{g}\big(\mathsf{Ric}(X), e_m\big)\alpha(e_k)\\ &= \alpha(\mathsf{Ric}(X))\,.\end{aligned}$$

2. By a similar calculation as under point 1, using additionally the algebraic Bianchi identity (2.3.16), together with (2.1.52) and (2.3.25), one gets:

$$\begin{aligned}\mathfrak{R}^{\Lambda}(\alpha)(e_i, e_j) &= -\mathsf{R}_{kj}\alpha^k{}_i + \mathsf{R}_{ki}\alpha^k{}_j + \mathsf{R}_{ijkl}\alpha^{kl}\\ &= \alpha(\mathsf{Ric}(e_i), e_j) - \alpha(\mathsf{Ric}(e_j), e_i) + \eta^{kl}\alpha(e_k, \mathsf{R}(e_i, e_j)e_l)\\ &= \big(\alpha\circ(\mathsf{Ric}\wedge \mathrm{id}) + \alpha\circ\mathsf{R}\big)(e_i, e_j)\,.\end{aligned}$$

■

The proof of the following example is left to the reader (Exercise 2.7.5).

Example 2.7.13 For S^n, endowed with the canonical Riemannian metric, the mapping (2.7.42) is given by $\mathsf{R} = -\,\mathrm{id}$ and the Ricci mapping reads $\mathsf{Ric}(X) = (n-1)X$. Using (2.7.38), one finds

$$\mathfrak{R}^{\Lambda} = k(n-k)\,\mathrm{id} \tag{2.7.45}$$

on k-forms. ♦

Combining the Weitzenboeck Formula with the theory of harmonic forms, one gets insight into the relation between curvature and topology. Let us discuss a simple application of this type. We will write $\mathsf{Ric} \geq 0$ if $\mathsf{Ric}_m(X, X) \geq 0$ for all $m \in M$ and all $X \in \mathrm{T}_m M$, and $\mathsf{Ric}_m > 0$ if $\mathsf{Ric}_m(X, X) > 0$ for all $0 \neq X \in \mathrm{T}_m M$.

Proposition 2.7.14 (Bochner) *Let (M, g) be an n-dimensional compact connected and oriented Riemannian manifold with $\mathsf{Ric} \geq 0$. Then, the following statements hold.*

1. *Every harmonic 1-form α is parallel and fulfils $\mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha)) = 0$.*
2. *If, additionally, $\mathsf{Ric}_m > 0$ for some point $m \in M$, then all harmonic 1-forms are trivial.*

Proof 1. By formula (2.7.43), for any $\alpha \in \Omega^1(M)$, we have

$$\langle \Box\alpha, \alpha\rangle_{L^2} = \parallel \nabla\alpha \parallel_{L^2}^2 + \int_M \mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha))\mathsf{v}_{\mathsf{g}}\,.$$

If α is harmonic, then the left hand side vanishes. Since both terms on the right hand side are non-negative, they must vanish, too.

2. Let $\alpha \in \Omega^1(M)$ be harmonic. Then, it is parallel. Since, for any $X \in \mathfrak{X}(M)$,

$$\nabla_X(\| \alpha \|) = X(\| \alpha \|) = 2\langle \nabla_X \alpha, \alpha \rangle ,$$

α has locally constant length. Thus, since M is connected, $\alpha_m = 0$ implies $\alpha = 0$ everywhere and, therefore, the evaluation mapping $\alpha \mapsto \alpha_m$ is injective. Also by point 1, $\mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha)) = 0$. Since $\mathsf{Ric}_m > 0$ for some point $m \in M$, we conclude $\alpha_m = 0$ and, by the injectivity of the evaluation mapping, $\alpha = 0$. ■

From the above proof it is clear that the vector space of harmonic 1-forms has at most dimension n. Combining this with point 1 of Remark 2.7.1 we get the following.

Corollary 2.7.15 *Under the assumptions of Proposition 2.7.14 on* (M, g)*, we have*

1. *If* $\mathsf{Ric} \geq 0$*, then* $b_1(M) = \dim H^1_{\mathrm{dR}}(M) \leq n$*.*
2. *If, additionally,* $\mathsf{Ric}_m > 0$ *for some point* $m \in M$*, then* $b_1(M) = 0$*.* ■

Example 2.7.16

1. Since for the torus $b_1(\mathrm{T}^n) = n \neq 0$, we conclude that this manifold does not admit a Riemanian metric with positive Ricci curvature.
2. Using (2.7.45), for S^n endowed with the canonical Riemannian metric, we get $\mathfrak{R}^{\Lambda}(\alpha) = k(n-k)\alpha$, and thus the Weitzenboeck Formula implies $\square > 0$ for $0 < k < n$. Consequently, there are no nontrivial harmonic forms for $0 < k < n$ and the Betti numbers of M vanish for all $k \neq 0, n$. ♦

In the remainder of this section, we show that the Weitzenboeck Formula generalizes in a straightforward way to the case of differential forms on M with values in a Riemannian (or Hermitean) vector bundle E endowed with a fibre metric $\langle \cdot, \cdot \rangle$ and a compatible connection ∇. In this form, it will play a crucial role both for the study of the instanton moduli space and for the investigation of stability of solutions to the Yang-Mills equations.

Recall from point 2 of Remark 2.6.1 that, without loss of generality, we may limit our attention to associated bundles $E = P \times_G F$ with fibre metrics $\langle \cdot, \cdot \rangle$ induced from G-invariant inner products $\langle \cdot, \cdot \rangle_F$ on F. First, note that the fibre metric $\langle \cdot, \cdot \rangle$ induces a pairing $\Omega^k(M, E) \times \Omega^l(M, E) \to \Omega^{k+l}(M)$ as follows. Let $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^l(M, E)$. For any $m \in M$, we choose a local frame $s_i : U \to E, i = 1, \ldots, \dim F$, on an open neighbourhood $U \subset M$ of m, decompose $\alpha = \alpha^i \otimes s_i$ and $\beta = \beta^j \otimes s_j$, and define

$$(\alpha \dot{\wedge} \beta)_m := \alpha^i_m \wedge \beta^j_m \langle s_i(m), s_j(m) \rangle . \tag{2.7.46}$$

Clearly, this definition does not depend on the choice of the local frame.

In particular, using the metric g on M and extending the Hodge-star on M to $\Omega^k(M, E)$ by putting $*\alpha := (*\alpha^i) \otimes s_i$, we obtain a pairing

$$\Omega^k(M, E) \times \Omega^k(M, E) \to \Omega^n(M) , \quad (\alpha, \beta) \mapsto \alpha \dot{\wedge} * \beta . \tag{2.7.47}$$

The latter can be used to define an L^2-inner product[50] on $\Omega^k(M, E)$,

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \dot{\wedge} * \beta \,. \tag{2.7.48}$$

Decomposing $\alpha = \alpha_I \vartheta^I$ and $\beta = \beta_J \vartheta^J$ with respect to a local orthonormal coframe $\{\vartheta^I\}$ in the bundle of k-forms on M, we have

$$\alpha \dot{\wedge} * \beta = \langle \alpha_I, \beta_J \rangle \vartheta^I \wedge * \vartheta^J = \eta^{IJ} \langle \alpha_I, \beta_J \rangle \, \mathsf{v_g} \,. \tag{2.7.49}$$

This shows that to the above pairing, there corresponds a natural inner product on $\Omega^k(M, E)$ given by the tensor product of the fibre metric $\langle \cdot, \cdot \rangle$ with the fibre metric η^{IJ} in $\Omega^k(M)$. If $\langle \cdot, \cdot \rangle$ is positive definite and g is Riemannian, then this inner product is positive definite.

Remark 2.7.17 Let $\tilde{\alpha} \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ and $\tilde{\beta} \in \Omega^l_{\sigma,\mathrm{hor}}(P, F)$ be the horizontal forms corresponding to $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^l(M, E)$ according to Proposition 1.2.12. Then, one easily shows (Exercise 2.7.7)

$$\tilde{\alpha} \dot{\wedge} \tilde{\beta} = \pi^*(\alpha \dot{\wedge} \beta) \,. \tag{2.7.50}$$

♦

Next, recall the covariant exterior derivative $\mathrm{d}_\omega \alpha : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ associated with the connection form ω of ∇, cf. Definition 1.5.1. We define its dual $\mathrm{d}^*_\omega \alpha : \Omega^{k+1}(M, E) \to \Omega^k(M, E)$ in the sense of Hodge by

$$\langle \alpha, \mathrm{d}^*_\omega \beta \rangle_{L^2} = \langle \mathrm{d}_\omega \alpha, \beta \rangle_{L^2} \,, \tag{2.7.51}$$

for $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^{k+1}(M, E)$. The operator d^*_ω will be called the covariant exterior coderivative. Note that, given this operator, we have a natural generalization of the Hodge-Laplacian, cf. (2.7.14),

$$\Box_\omega := \mathrm{d}_\omega \circ \mathrm{d}^*_\omega + \mathrm{d}^*_\omega \circ \mathrm{d}_\omega : \quad \Omega^k(M, E) \to \Omega^k(M, E) \,. \tag{2.7.52}$$

Proposition 2.7.18 *For $\alpha \in \Omega^k(M, E)$,*

$$\mathrm{d}^*_\omega \alpha = (-1)^{n(k-1)+s+1} * \mathrm{d}_\omega * \alpha \,. \tag{2.7.53}$$

Proof Using (2.7.50), (1.5.1) and the G-invariance of $\langle \cdot, \cdot \rangle_F$, for $\beta \in \Omega^{k+1}(M, E)$, we calculate

[50] Again, we must restrict ourselves to square-integrable forms. In particular, we may consider forms with compact support.

$$
\begin{aligned}
\pi^*(\mathrm{d}_\omega \alpha \dot{\wedge} * \beta) &= D_\omega \tilde{\alpha} \dot{\wedge} \widetilde{*\beta} \\
&= (\mathrm{d}\tilde{\alpha} + \sigma'(\omega) \wedge \tilde{\alpha}) \dot{\wedge} \widetilde{*\beta} \\
&= \mathrm{d}(\tilde{\alpha} \dot{\wedge} \widetilde{*\beta}) - (-1)^k \tilde{\alpha} \dot{\wedge} (\mathrm{d}\widetilde{*\beta} + \sigma'(\omega) \wedge \widetilde{*\beta}) \\
&= \mathrm{d}(\tilde{\alpha} \dot{\wedge} \widetilde{*\beta}) - (-1)^k \tilde{\alpha} \dot{\wedge} D_\omega(\widetilde{*\beta}) \\
&= \pi^* \big(\mathrm{d}(\alpha \dot{\wedge} * \beta)\big) - (-1)^k \pi^*(\alpha \dot{\wedge} \mathrm{d}_\omega * \beta)\,.
\end{aligned}
$$

Thus,

$$
\mathrm{d}_\omega \alpha \dot{\wedge} * \beta = \mathrm{d}(\alpha \dot{\wedge} * \beta) - (-1)^k \alpha \dot{\wedge} \mathrm{d}_\omega * \beta\,.
$$

Integrating this identity over M, using Stokes' Theorem, we obtain

$$
\langle \mathrm{d}_\omega \alpha, \beta \rangle_{L^2} = \langle \alpha, (-1)^{nk+s+1} * \mathrm{d}_\omega * \beta \rangle_{L^2}\,.
$$

Comparing with (2.7.51), we read off the assertion. ∎

As above, we need a unified description in terms of the Koszul calculus. For that purpose, it will be convenient to view the space $\Omega^k(M, E)$ as follows. Denote

$$
T^r_s = \mathbb{R}^n \otimes \overset{r}{\cdots} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \overset{s}{\cdots} \otimes \mathbb{R}^{n*}\,.
$$

Consider the fibre product[51] $O(M) \times_M P$ over M with structure group $\mathrm{O}(k,l) \times G$ and the associated bundle with typical fibre $T^r_s \otimes F$,

$$
E_{r,s} = (O(M) \times_M P) \times_{\mathrm{O}(k,l)\times G} (T^r_s \otimes F)\,,
$$

which is clearly isomorphic to the tensor product $\mathrm{T}^r_s(M) \otimes E$ of vector bundles. The left actions of $\mathrm{O}(k,l)$ and G on T^r_s and F are denoted by μ and σ, respectively. By Remark 1.3.17, the Levi-Civita connection form ω^o on $O(M)$ and the gauge connection form ω on P induce a connection form $\omega^o + \omega$ on $O(M) \times_M P$, cf. (1.3.16).[52] As usual, we denote the induced covariant exterior derivative acting on $\Omega^k_{(\mu,\sigma),\mathrm{hor}}(O(M) \times_M P, T^r_s \otimes F)$ by $D_{(\omega^o+\omega)}$, its counterpart acting on $\Omega^k(M, E_{r,s})$ by $\mathrm{d}_{(\omega^o+\omega)}$ and the corresponding covariant derivative acting on sections of $E_{r,s}$ by $\nabla^{(\omega^o+\omega)}$. By the general theory,

$$
D_{\omega^o+\omega} \tilde{\Phi} = \mathrm{d}\tilde{\Phi} + \big(\mu'(\omega^o) \otimes \mathrm{id}_F + \mathrm{id}_{T^r_s} \otimes \sigma'(\omega)\big) \circ \tilde{\Phi}\,, \tag{2.7.54}
$$

cf. (1.4.2). Clearly, $\mu'(\omega^o) \otimes \mathrm{id}_F + \mathrm{id}_{T^r_s} \otimes \sigma'(\omega)$ must be viewed as a 1-form on $Q \times_M P$ with values in $\mathrm{End}(T^r_s \otimes F)$. It is obtained by differentiating the tensor product representation $\mu \otimes \sigma$. Moreover, $\Omega^k_{(\mu,\sigma),\mathrm{hor}}(O(M) \times_M P, T^r_s \otimes F)$ may be viewed as a subspace of

---

[51] Cf. Remark 1.1.9/2.

[52] For simplicity, we omit the canonical projections onto $O(M)$ and P, respectively.

$$\mathrm{Hom}_{\mathrm{O}(k,l)\times G}(O(M)\times_M P, T^r_{s+k}\otimes F)$$

consisting of those elements whose last k covariant tensor indices are anti-symmetric. By Proposition 1.2.12, the latter space in turn may be identified with $\Gamma^\infty(E_{r,s+k})$. Elements of this space may be viewed as tensor fields of type $(r, s+k)$ on M with values in the associated bundle E. In particular, we get the following identification:

$$\Omega^k(M, E) \cong \Omega^k(M, E_{0,0})\,. \tag{2.7.55}$$

Now, the generalization of the Weitzenboeck Formula is straightforward. First, for $(r, s) = (0, 0)$, the action μ is trivial and hence (2.7.54) implies

$$\mathrm{d}_{(\omega^o+\omega)}\alpha = \mathrm{d}_\omega\alpha\,, \quad \mathrm{d}^*_{(\omega^o+\omega)}\alpha = \mathrm{d}^*_\omega\alpha\,,$$

for any $\alpha \in \Omega^k(M, E)$. This implies

$$\Box_\omega = \Box_{(\omega^o+\omega)}\,. \tag{2.7.56}$$

Lemma 2.7.19 *Let $\alpha \in \Omega^k(M, E)$. Then, under the identification (2.7.55),*

$$\mathrm{d}_\omega\alpha(X_0, \ldots, X_k) = \sum_j (-1)^j \left(\nabla^{(\omega^o+\omega)}_{X_j}\alpha\right)\left(X_0, \overset{j}{\check{\ldots}}, X_k\right), \tag{2.7.57}$$

$$(\mathrm{d}^*_\omega\alpha)(X_2, \ldots, X_k) = -\sum_{j,l} \eta^{jl}\left(\nabla^{(\omega^o+\omega)}_{e_j}\alpha\right)(e_l, X_2, \ldots, X_k)\,, \tag{2.7.58}$$

for $X_0, \ldots, X_k \in \mathfrak{X}(M)$ and $\{e_l\}$ being an orthonormal frame on (M, g).

We note the following immediate consequence of (2.7.57):

$$\mathrm{d}_\omega\alpha = \sum_j \vartheta^j \wedge \nabla^{(\omega^o+\omega)}_{e_j}\alpha\,, \tag{2.7.59}$$

where $\{\vartheta^j\}$ is the coframe dual to $\{e_j\}$.

Proof To prove (2.7.57), it is enough to consider elements $\alpha = \phi \otimes \beta$, where $\phi \in \Gamma^\infty(E)$ and $\beta \in \Omega^k(M)$. Then, again using that the action μ is trivial, for the left hand side of (2.7.57) we get

$$\mathrm{d}_\omega\alpha = \mathrm{d}_\omega\phi \wedge \beta + \phi \otimes \mathrm{d}\beta\,.$$

To analyze the right hand side, we use the derivation property of the covariant derivative,

$$\nabla_X^{(\omega^o+\omega)}\alpha = \nabla_X^{\omega}\phi \otimes \beta + \phi \otimes \nabla_X^{\omega^o}\beta\,.$$

This, together with formula (2.2.49), implies the assertion.

The proof of (2.7.58) is analogous to the proof of (2.7.23). We replace d by d_ω and use (2.7.59). ∎

Now, by the same calculation as in the proof of Theorem 2.7.11, we obtain the following Generalized Weitzenboeck Formula

$$\Box_\omega\alpha = \big(\nabla^{(\omega^0+\omega)}\big)^*\nabla^{(\omega^0+\omega)}\alpha + \eta^{jl}e^i{}_l\big(\mathsf{R}^{\nabla^{(\omega^0+\omega)}}(e_j, e_i)\alpha\big)\,, \tag{2.7.60}$$

where $\mathsf{R}^{\nabla^{(\omega^0+\omega)}}$ is the curvature endomorphism form of the connection $\omega^0+\omega$ given by (1.5.13). Here, it reads

$$\mathsf{R}_m^{\nabla^{(\omega^0+\omega)}}(X,Y) := \iota_z \circ \big\{\mu'(\Omega_z^o(X^h,Y^h)) \otimes \mathrm{id} + \mathrm{id} \otimes \sigma'(\Omega_z(X^h,Y^h))\big\} \circ \iota_z^{-1}\,,$$

where $m \in M$, $z \in \pi^{-1}(m) \subset \mathrm{O}(M) \times_M P$, $X, Y \in \mathrm{T}_m M$ and X^h and Y^h are the horizontal lifts of X and Y to z, respectively. Clearly, by the additivity of $\mathsf{R}^{\nabla^{(\omega^0+\omega)}}$, the second term on the right hand side of (2.7.60) is the sum of the Weitzenboeck curvature operators for the representations μ and σ, respectively, cf. Definition 2.7.10. This yields the following.

Theorem 2.7.20 (Generalized Weitzenboeck Formula) *For $\alpha \in \Omega^k(M,E)$,*

$$\Box_\omega\alpha = \big(\nabla^{(\omega^0+\omega)}\big)^*\nabla^{(\omega^0+\omega)}\alpha + \mathfrak{R}^{\nabla^{\omega^o}}(\alpha) + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.61}$$

∎

As above, formula (2.7.61) may be analyzed degreewise. Clearly, the terms coming from the Levi-Civita connection are identical with those in Corollary 2.7.12. Thus, we obtain the following.

Corollary 2.7.21

1. *For $\alpha \in \Omega^1(M,E)$, the Weitzenboeck Formula (2.7.61) reads*

$$\Box_\omega\alpha \;=\; \big(\nabla^{(\omega^0+\omega)}\big)^*\nabla^{(\omega^0+\omega)}\alpha + \alpha \circ \mathsf{Ric} + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.62}$$

2. *For $\alpha \in \Omega^2(M,E)$, formula (2.7.61) may be rewritten as follows:*

$$\Box_\omega\alpha \;=\; \big(\nabla^{(\omega^0+\omega)}\big)^*\nabla^{(\omega^0+\omega)}\alpha + \alpha \circ (\mathsf{R} + \mathsf{Ric} \wedge \mathrm{id}) + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.63}$$

∎

The Generalized Weitenboeck Formula will be taken up again in Example 5.6.7. There, it will be discussed from the point of view of Dirac operator theory. It will play a basic role in the analysis of the stability of Yang-Mills connections.

Exercises

2.7.1 Prove the formulae (2.7.8)–(2.7.10).

2.7.2 Prove the identities contained in (2.7.15)–(2.7.17).

2.7.3 Prove that on a compact connected oriented Riemannian manifold fulfilling $H^1_{\mathrm{dR}}(M) = 0$ there does not exist any nontrivial harmonic 1-form. Construct a nontrivial harmonic 1-form on the 2-torus $\mathrm{T}^2 \subset \mathbb{R}^4$.

2.7.4 Prove formula (2.7.27).

2.7.5 Prove the statements of Example 2.7.13.

2.7.6 Prove formula (2.7.33).

2.7.7 Prove formula (2.7.50).

2.8 Four-Dimensional Riemannian Geometry. Self-duality

In this section, we deal with 4-dimensional (oriented) Riemannian manifolds. We will show that, in contrast to other dimensions, they admit a rich additional structure. Let us explain the reason for that. Given an oriented Riemannian manifold (M, g), we know from Sect. 2.4 that g yields a reduction of the frame bundle $L(M)$ to the principal SO(4)-bundle $O_+(M)$ of oriented orthonormal frames. Correspondingly, all tensor bundles over M become associated with $O_+(M)$ with their typical fibres carrying representations of SO(4). Now, among all rotation groups, SO(4) is the unique group which is not simple. This has striking consequences, as we will see below. Recall from Example I/5.1.10 the isomorphism

$$\mathrm{Sp}(1) \to \mathrm{SU}(2)\,, \quad a = z + \mathbf{j}\, w \mapsto \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}, \tag{2.8.1}$$

where we have identified $\mathbb{C}$ with $\mathrm{span}\{\mathbf{1}, \mathbf{i}\} \subset \mathbb{H}$ and $\mathbb{H}$ with $\mathbb{C}^2$ by writing quaternions in the form $z + \mathbf{j}w$ with $z\,, w \in \mathbb{C}$. Also recall from Example I/5.1.11 that Sp(1) and $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ are the universal (two-fold) covering groups[53] of SO(3) and SO(4), respectively. Denoting by $\iota : \mathrm{Sp}(1) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ the diagonal embedding, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(1) \arrow[r, "\iota"] \arrow[d] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[d, "f"] \\ \mathrm{SO}(3) \arrow[r] & \mathrm{SO}(4) \end{tikzcd} \tag{2.8.2}$$

[53] In Chap. 5, we will see that these are the spin groups in 3 and 4 dimensions, respectively.

This fact reduces the representation theory of SO(4) to that of Sp(1). By the isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, we are led to consider complex representations built from the basic representation of SU(2) on $V \cong \mathbb{C}^2$. By a standard theorem in representation theory [689], up to isomorphisms, the set of irreducible complex SU(2)-modules is

$$\left\{S^r V : r \geq 0\right\} ,$$

where $S^r V$ denotes the subspace of $\otimes^r V$ of totally symmetric tensors. Equivalently, this subspace may be identified with the space of homogeneous polynomials of degree r in two variables. Thus, $\dim_{\mathbb{C}}(S^r V) = r + 1$. Moreover,

$$S^p V \otimes S^q V \cong \bigoplus_{r=0}^{min(p,q)} S^{p+q-2r} V . \tag{2.8.3}$$

Note that $S^2 V$ is the (complexified) adjoint representation space.

Now, any complex SO(4)-module (W, σ) may be viewed as an $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-module via the mapping

$$\sigma \circ f : \quad \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Aut}(W) .$$

Let us denote the basic representation spaces corresponding to the first and the second factor in $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, respectively, by V_+ and V_-. Then, again, by standard representation theory, the irreducible complex $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-modules are given by

$$S^{p,q} = S^p V_+ \otimes S^q V_- , \quad p, q \geq 0 . \tag{2.8.4}$$

Clearly, an irreducible representation $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Aut}(S^{p,q})$ factors through the covering homomorphism f, giving a representation of SO(4), iff $p + q$ is even. Moreover, in that case, $S^{p,q}$ is the complexification of a real representation which we denote by $S_r^{p,q}$. It is common to call $S_r^{p,q}$ the real representation underlying $S^{p,q}$. Also note that

$$\dim_{\mathbb{C}}(S^{p,q}) = \dim_{\mathbb{R}}(S_r^{p,q}) = (p + 1)(q + 1) .$$

In particular, the basic complex SO(4)-module is $S^{1,1} = V_+ \otimes V_-$. We denote $T := S_r^{1,1}$ and write T^* for the dual (contragredient) representation space. Clearly, we may use the Euclidean metric on T to identify $T \cong T^*$. Now, calculating (Exercise 2.8.1)

$$\bigwedge^2 T^*_{\mathbb{C}} \cong \bigwedge^2 (V_+ \otimes V_-) \cong \left(S^2 V_+ \otimes \bigwedge^2 V_-\right) \oplus \left(\bigwedge^2 V_+ \otimes S^2 V_-\right)$$

and using that $\bigwedge^2 V$ is the trivial Sp(1)-module, we obtain

$$\bigwedge^2 T^*_{\mathbb{C}} \cong S^2 V_+ \oplus S^2 V_- . \tag{2.8.5}$$

Since S^2V is the adjoint representation of $\mathrm{Sp}(1)$, $\bigwedge^2 T^*_{\mathbb{C}}$ is the (complexified) adjoint representation space of $\mathrm{SO}(4)$ with (2.8.5) corresponding to the Lie algebra decomposition $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$. Thus, we have the underlying isomorphism of real representations

$$\bigwedge^2 T^* \cong S_r^{2,0} \oplus S_r^{0,2} \tag{2.8.6}$$

corresponding to the decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Next, we will relate the above decompositions to the Hodge star operator. Thus, let $* : \bigwedge^r T^* \to \bigwedge^{4-r} T^*$ be the Hodge star operator with respect to the Euclidean metric on T. By Proposition I/4.5.3,

$$* \circ * = \mathrm{id}_{\bigwedge^2 T^*}\,, \tag{2.8.7}$$

that is, on two-forms, the Hodge star operator is an involution. Thus, we may decompose $\bigwedge^2 T^*$ into an orthogonal direct sum of eigenspaces of $*$ corresponding to the eigenvalues ± 1,

$$\bigwedge^2 T^* = \bigwedge^2{}_+ T^* \oplus \bigwedge^2{}_- T^*\,. \tag{2.8.8}$$

Elements of $\bigwedge^2_+ T^*$ are called self-dual and elements of $\bigwedge^2_- T^*$ are called anti-self-dual. Since the Hodge star operator is invariant under the action of $\mathrm{SO}(4)$, the subspaces $\bigwedge^2_\pm T^*$ are $\mathrm{SO}(4)$-invariant and, thus, they coincide with the direct summands in (2.8.6),

$$\bigwedge^2{}_+ T^* \cong S_r^{2,0}\,, \quad \bigwedge^2{}_- T^* \cong S_r^{0,2}\,. \tag{2.8.9}$$

For the corresponding complexifications, we get

$$\bigwedge^2{}_+ T^*_{\mathbb{C}} \cong S^2 V_+\,, \quad \bigwedge^2{}_- T^*_{\mathbb{C}} \cong S^2 V_-\,. \tag{2.8.10}$$

Remark 2.8.1

1. Let $\vartheta^1, \ldots, \vartheta^4$ be an oriented orthonormal basis in T^*. Then, the irreducible subspaces $\bigwedge^2_\pm T^*$ are spanned by

$$\begin{aligned}
\varphi^1_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^2 \pm \vartheta^3 \wedge \vartheta^4\right),\\
\varphi^2_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^3 \mp \vartheta^2 \wedge \vartheta^4\right),\\
\varphi^3_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^4 \pm \vartheta^2 \wedge \vartheta^3\right).
\end{aligned}$$

2. In the same way as above, we can calculate

$$\begin{aligned} S^2T^*_{\mathbb{C}} &\cong S^2(V_+ \otimes V_-) \\ &\cong (S^2V_+ \otimes S^2V_-) \oplus (\textstyle\bigwedge^2 V_+ \otimes \bigwedge^2 V_-) \\ &\cong (S^2V_+ \otimes S^2V_-) \oplus \mathbb{C}\,. \end{aligned}$$

Thus, using (2.8.10),

$$S^2_0T^* \cong \textstyle\bigwedge^2{}_+T^* \otimes \bigwedge^2{}_-T^*\,, \tag{2.8.11}$$

where the subindex zero refers to tracelessness. ♦

Comparison of the decompositions (2.8.8) with (2.2.16) yields the following deep insight. Let T^* be endowed with the complex structure[54]

$$\mathsf{J} = \begin{bmatrix} \mathsf{J}_1 & 0 \\ 0 & \mathsf{J}_1 \end{bmatrix},$$

where J_1 is the standard complex structure on $\mathbb{R}^2$. With respect to this structure, the decomposition (2.2.16) reads

$$\textstyle\bigwedge^2 T^*_{\mathbb{C}} = \left(\bigwedge^{2,0}T^*_{\mathbb{C}} \oplus \bigwedge^{0,2}T^*_{\mathbb{C}}\right) \oplus \bigwedge^{1,1}T^*_{\mathbb{C}}\,. \tag{2.8.12}$$

As already noted, the left hand side may be identified with the Lie algebra $\mathfrak{o}(4,\mathbb{C})$. In analogy to (2.2.22), J induces an embedding $\mathrm{U}(2) \subset \mathrm{SO}(4)$ and the summands on the right hand side of (2.8.12) carry representations of $\mathrm{U}(2)$. Observe that the almost symplectic form β defined by (2.2.26) belongs to $\bigwedge^{1,1}T^*_{\mathbb{C}}$ and is $\mathrm{U}(2)$-invariant. Thus, we have an orthogonal decomposition

$$\textstyle\bigwedge^{1,1}T^*_{\mathbb{C}} = \mathbb{C} \oplus \bigwedge_0^{1,1}T^*_{\mathbb{C}}$$

into $\mathrm{U}(2)$-irreducible components. By dimension counting, $\bigwedge_0^{1,1}T^*_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C})$ (the complexification of $\mathfrak{su}(2)$) and, thus, (2.8.12) corresponds to the complexification of the Lie algebra decomposition $\mathfrak{o}(4) = \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{m}$, cf. point 3 of Example 2.5.27.

Lemma 2.8.2 *We have*

$$\textstyle\bigwedge^2{}_+T^*_{\mathbb{C}} = \mathbb{C} \oplus \left(\bigwedge^{2,0}T^*_{\mathbb{C}} \oplus \bigwedge^{0,2}T^*_{\mathbb{C}}\right), \quad \bigwedge^2{}_-T^*_{\mathbb{C}} = \bigwedge_0^{1,1}T^*_{\mathbb{C}}\,. \tag{2.8.13}$$

Proof Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_4\}$ be the standard basis in the basic $\mathrm{SO}(4)$-module $T = \mathbb{R}^4$ and let $\{\vartheta^1, \ldots, \vartheta^4\}$ be the dual basis in T^*. Clearly, $\bigwedge^{1,0}T^*_{\mathbb{C}}$ is spanned by

[54] This choice is made in order to be compatible with standard conventions in gauge theory. It is obtained by combining the standard complex structure J_0 on $\mathbb{R}^4$ with the transformation defined by permuting the standard basis vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. Beware that J and J_0 induce different orientations.

$$\psi^1 = \vartheta^1 + i\vartheta^2 \,, \quad \psi^2 = \vartheta^3 + i\vartheta^4 \,.$$

Now, using point 1 of Remark 2.8.1, we express the generators of the $\mathrm{U}(n)$-modules on the right hand side of (2.8.12) in terms of the bases $\{\varphi^i_\pm\}$ of $\bigwedge^2_\pm T^*$:

$$\begin{aligned}
\frac{1}{2} i(\psi^1 \wedge \bar{\psi}^1 + \psi^2 \wedge \bar{\psi}^2) &= \varphi^1_+ \,, \\
\psi^1 \wedge \psi^2 &= \varphi^2_+ + i\varphi^3_+ \,, \\
\bar{\psi}^1 \wedge \bar{\psi}^2 &= \varphi^2_+ - i\varphi^3_+ \,, \\
\frac{1}{2} i(\psi^1 \wedge \bar{\psi}^1 - \psi^2 \wedge \bar{\psi}^2) &= \varphi^1_- \,, \\
\psi^1 \wedge \bar{\psi}^2 &= \varphi^2_- - i\varphi^3_- \,, \\
\bar{\psi}^2 \wedge \bar{\psi}^1 &= -\varphi^2_- + i\varphi^3_- \,.
\end{aligned}$$

■

Corollary 2.8.3 *A 2-form on $\mathbb{R}^4$ is anti-self-dual iff it is of type* $(1, 1)$ *for all compatible complex structures.* ■

As we will see, the following lemma is of basic importance in 4-dimensional Riemannian geometry [592].

Lemma 2.8.4 *We have*

$$S^2\left(\bigwedge^2 T^*\right) \cong S^{0,0}_r \oplus S^{0,0}_r \oplus S^{2,2}_r \oplus S^{4,0}_r \oplus S^{0,4}_r \,. \tag{2.8.14}$$

Proof Using (2.8.8), we calculate

$$S^2\left(\bigwedge^2_+ T^* \oplus \bigwedge^2_- T^*\right) \cong S^2\left(\bigwedge^2_+ T^*\right) \oplus \left(\bigwedge^2_+ T^* \otimes \bigwedge^2_- T^*\right) \oplus S^2\left(\bigwedge^2_- T^*\right) .$$

By (2.8.9), the second term on the right hand side corresponds to $S^{2,2}_r$. The complexification of the first term corresponds via (2.8.10) to the symmetric component of $S^2V_+ \otimes S^2V_+$ and thus has complex dimension 6. By (2.8.3),

$$S^2V_+ \otimes S^2V_+ = S^4V_+ \oplus S^2V_+ \oplus S^0V_+ \,.$$

By counting dimensions, we find that the symmetric component corresponds to $S^4V_+ \oplus S^0V_+$. It follows that

$$S^2\left(\bigwedge^2_+ T^*\right) = S^{4,0}_r \oplus S^{0,0}_r \tag{2.8.15}$$

and, analogously, $S^2\left(\bigwedge^2_+ T^*\right) = S^{0,4}_r \oplus S^{0,0}_r$. ■

Now, we can apply the above results to the 4-dimensional Riemannian manifold (M, g). By Proposition I/4.5.3, the Hodge star operator is an isometric involution on the bundle of two forms, that is, $* : \bigwedge^2 \mathrm{T}^* M \to \bigwedge^2 \mathrm{T}^* M$ fulfils

$$* \circ * = \mathrm{id}_{\bigwedge^2 \mathrm{T}^* M}\,, \quad \langle *\alpha, *\beta\rangle_{L^2} = \langle \alpha, \beta\rangle_{L^2}\,, \tag{2.8.16}$$

and, corresponding to (2.8.8), we have the splitting

$$\bigwedge{}^2 \mathrm{T}^* M = \bigwedge{}^2{}_+ \mathrm{T}^* M \oplus \bigwedge{}^2{}_- \mathrm{T}^* M\,. \tag{2.8.17}$$

Clearly, the decomposition (2.8.17) implies a decomposition of 2-forms on M,

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)\,. \tag{2.8.18}$$

Thus, any $\alpha \in \Omega^2(M)$ may be decomposed as follows:

$$\alpha = \alpha^+ + \alpha^-\,, \quad *\alpha^+ = \alpha^+\,, \quad *\alpha^- = -\alpha^-\,, \tag{2.8.19}$$

where $\alpha^\pm = \frac{1}{2}(\alpha \pm *\alpha)$. Elements of $\Omega^2_+(M)$ are called self-dual and elements of $\Omega^2_-(M)$ are called anti-self-dual 2-forms. Finally, for a local oriented orthonormal frame $\vartheta^1, \ldots, \vartheta^4$ in $\bigwedge^1 \mathrm{T}^* M$, the subbundles $\bigwedge^2_\pm \mathrm{T}^* M$ are locally spanned by $\{\varphi^i_\pm\}$ given by the same formulae as in Remark 2.8.1/2.

Next, let us consider the Riemann curvature endomorphism form

$$\mathsf{R} \in \Omega^2(M, \mathrm{End}(\mathrm{T}M))$$

of (M, g). By Remark 2.3.7, pointwise, it may be viewed as a symmetric endomorphism of $\bigwedge^2 \mathrm{T}^*_m M$,

$$\mathsf{R}(m) \in S^2\left(\bigwedge{}^2 \mathrm{T}^*_m M\right)\,. \tag{2.8.20}$$

Correspondingly, for every $u \in O(M)$, it may be viewed as an element

$$\mathscr{R}(u) \in \bigwedge{}^2 \left(\mathbb{R}^4\right)^* \overset{s}{\otimes} \bigwedge{}^2 \left(\mathbb{R}^4\right)^* \equiv S^2\left(\bigwedge{}^2 \left(\mathbb{R}^4\right)^*\right)\,. \tag{2.8.21}$$

We wish to derive the counterpart of the general decomposition formula (2.3.21) for $n = 4$. Here, according to the additional structures at a our disposal, this can be done in two different ways. First, using (2.8.17), we can write

$$\mathsf{R}(m) = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}\,. \tag{2.8.22}$$

Here, $B \in \mathrm{Hom}(\bigwedge^2_- \mathrm{T}^*_m M, \bigwedge^2_+ \mathrm{T}^*_m M)$, $A \in \mathrm{End}(\bigwedge^2_+ \mathrm{T}^*_m M)$ and $C \in \mathrm{End}(\bigwedge^2_- \mathrm{T}^*_m M)$. Since $\mathsf{R}(m) \in S^2(\bigwedge^2 \mathrm{T}^*_m M)$, both A and C are symmetric endomorphisms. Note that B^{T} is the adjoint of B.

Lemma 2.8.5 *We have*

$$\operatorname{tr} A = \operatorname{tr} C = -\frac{1}{4}\mathsf{Sc}\,,$$

where Sc *denotes the scalar curvature of* ∇.

Proof This is a simple exercise which we leave to the reader (Exercise 2.8.2). ■

Remark 2.8.6 We show that the decomposition (2.8.22) corresponds to the decomposition of $S^2(\bigwedge^2 T^*)$ into irreducible components of SO(4) given by Lemma 2.8.4, with one of the two $S^{0,0} \cong \mathbb{R}$-summands removed. For that purpose, we choose an orthonormal basis in $\mathrm{T}_m M$ and use it to identify $\mathrm{T}_m M$ with T. Using (2.8.15), we obtain

$$A \in S^2\left(\textstyle\bigwedge^2_+ T^*\right) = S^{4,0}_r \oplus S^{0,0}_r\,, \quad C \in S^{0,4}_r \oplus S^{0,0}_r\,.$$

Moreover,

$$B \in \mathrm{Hom}\left(\textstyle\bigwedge^2_- T^*, \bigwedge^2_+ T^*\right) \cong S^{2,0}_r \otimes S^{0,2}_r \cong S^{2,2}_r\,.$$

Finally, by Lemma 2.8.5, one of the summands $S^{0,0}_r$ is removed and we obtain the following 4-dimensional counterpart of the decomposition (2.3.21) of the space of Riemann curvatures

$$\mathfrak{R}(m) = S^{0,0}_r \oplus S^{2,2}_r \oplus S^{4,0}_r \oplus S^{0,4}_r\,,$$

with

$$\mathsf{R}(m) = (\operatorname{tr} A, B, A - \tfrac{1}{3}\operatorname{tr} A, C - \tfrac{1}{3}\operatorname{tr} C)\,. \tag{2.8.23}$$

This result belongs to Singer and Thorpe [592]. ♦

We denote

$$\mathsf{W}_+ := A - \tfrac{1}{3}\operatorname{tr} A\,, \quad \mathsf{W}_- := C - \tfrac{1}{3}\operatorname{tr} C\,, \tag{2.8.24}$$

and call

$$\mathsf{W} := \begin{bmatrix} \mathsf{W}_+ & 0 \\ 0 & \mathsf{W}_- \end{bmatrix}$$

the Weyl tensor. Note that $\mathsf{W}_\pm : \bigwedge^2_\pm \to \bigwedge^2_\pm$ are symmetric endomorphisms with vanishing trace. Summarizing the above discussion, we obtain the following.

Theorem 2.8.7 (Singer-Thorpe) *The Riemann curvature* R *of an oriented* 4-*dimensional Riemannian manifold defines a symmetric endomorphism of* $\bigwedge^2 \mathrm{T}^* M$ *which decomposes as*

$$\mathsf{R} = -\frac{\mathsf{Sc}}{12}\mathbb{1} + \begin{bmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{bmatrix} + \mathsf{W}\,. \tag{2.8.25}$$

The statements of the following remark are left as an exercise to the reader (Exercise 2.8.3).

Remark 2.8.8 In a local orthonormal frame on M, the decomposition (2.8.25) reads as follows:

$$\mathsf{R}_{ijkl} = \frac{\mathsf{Sc}}{6}\big(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}\big) + \frac{1}{2}\big(\mathsf{R}_{il}\delta_{jk} + \mathsf{R}_{jk}\delta_{il} - \mathsf{R}_{ik}\delta_{jl} - \mathsf{R}_{jl}\delta_{ik}\big) + \mathsf{W}_{ijkl}\,, \tag{2.8.26}$$

where R_{ij} are the components of the Ricci tensor. Clearly, the Weyl tensor $\mathsf{W}_{ijkl} = \mathsf{g}(\mathsf{W}(e_i, e_j)e_k, e_l)$ inherits the properties (2.3.15) from the curvature tensor. By construction, we have $\sum_i \mathsf{W}_{ijki} = 0$. ♦

Definition 2.8.9 An oriented Riemannian 4-manifold is called self-dual or anti-self-dual if, respectively, $\mathsf{W}_- = 0$ or $\mathsf{W}_+ = 0$.

By direct inspection of (2.8.26), one can check that M is Einstein if $B = 0$.

Example 2.8.10

1. The manifolds S^4, $\mathrm{S}^1 \times \mathrm{S}^3$ and T^4, endowed with their natural metrics, have a vanishing Weyl tensor and are, thus, both self-dual and anti-self-dual (Exercise 2.8.4).
2. $\mathbb{C}\mathrm{P}^2$ with its standard metric and orientation is self-dual. For a detailed proof we refer to [689]. ♦

Exercises

2.8.1 Prove formula (2.8.5). *Hint.* Construct explicit bases for the occuring representation spaces.

2.8.2 Prove Lemma 2.8.5.

2.8.3 Prove the statements of Remark 2.8.8.

2.8.4 Prove the statements of Example 2.8.10/1.

Chapter 3
Homotopy Theory of Principal Fibre Bundles. Classification

We start with a discussion of fibrations and with the derivation of their exact homotopy sequence from the exact homotopy sequence of a pair. This yields, in particular, an exact sequence for fibre bundles containing the homotopy groups of the typical fibre, the total space and the base space.

Then, we solve the classification problem of principal bundles with a given structure group and a given base manifold up to vertical isomorphisms. This is accomplished in three steps. First, in Sect. 3.3, we prove the Covering Homotopy Theorem, which implies that the pullbacks of a given topological principal G-bundle under homotopic mappings are vertically isomorphic. This leads to the idea of classifying topological principal G-bundles in terms of homotopy classes of mappings to the base space of a universal principal G-bundle. Following this idea, in Sects. 3.4 and 3.5 we prove that there exists a universal topological principal G-bundle for every Lie group G. Finally, in Sect. 3.6, we prove that the smooth vertical isomorphism classes of smooth principal G-bundles over a manifold M are in bijective correspondence with the continuous isomorphism classes of topological principal G-bundles over M. A posteriori, this gives the justification for classifying topological principal bundles first.

In the final section, we discuss connections which are n-universal in the sense that, via pullback, they can produce every connection on a principal G-bundle over a manifold of dimension $\leq n$. We give both the explicit description in terms of the natural connections on the Stiefel bundles and the more abstract description in terms of the tautological connection on the section jet bundle of an n-universal principal G-bundle.

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3.1 Basics

To make the topological concept of homotopy fruitful for the theory of principal bundles, we have to work with topological principal bundles. The definition of topological principal G-bundle is obtained from Definition 1.1.1 in the obvious way, that is, by requiring P and M to be topological spaces, G to be a topological group, the action Ψ to be free and continuous, the projection $\pi : P \to M$ to be continuous and the local trivializations $\chi : \pi^{-1}(U) \to U \times G$ to be equivariant homeomorphisms projecting to the identical mapping. Analogously, the definition of smooth fibre bundle translates into the definition of topological fibre bundle. Sections in these bundles are assumed to be continuous if not otherwise stated. The basic results about smooth principal bundles discussed in Chap. 1 translate in an obvious way to topological principal bundles. In particular, we will need the following.

1. Associated bundles constructed by means of topological group actions are topological fibre bundles.
2. Every vertical G-morphism is an isomorphism (Remark 1.1.8/2).
3. The pullback of a topological principal G-bundle by a continuous mapping is a topological principal G-bundle (Remark 1.1.9/1). Moreover, $f^*(g^*P)$ is vertically isomorphic to $(g \circ f)^*P$.
4. If $\vartheta : Q \to P$ is a G-morphism of principal bundles with projection $\tilde{\vartheta}$, then the mapping
$$Q \to \tilde{\vartheta}^*P, \quad q \mapsto \big(\pi_Q(q), \vartheta(q)\big),$$
is a vertical isomorphism and ϑ decomposes into the composition of this isomorphism with the natural principal G-bundle morphism $\tilde{\vartheta}^*P \to P$ (Remark 1.1.9/1).
5. G-bundle morphisms $P \to Q$ are in bijective correspondence with sections in $P \times_G Q$ (Proposition 1.2.6). If P and Q have the same base space, then vertical G-bundle morphisms $P \to Q$ are in bijective correspondence with sections in $P \times_{G,M} Q$ (Corollary 1.2.7).
6. If $H \subset G$ is a closed subgroup, the action of G on P restricts to an action of H and the latter makes P into a principal H-bundle over the topological quotient P/H. The induced projection $P/H \to M$ is a topological fibre bundle with typical fibre G/H (Example 1.2.4/1).

In this and in the next chapter, topological spaces will usually be denoted by X, Y, Z etc. Continuous mappings $X \to Y$ will be denoted by f, g, h etc. and their totality will be denoted by $C(X, Y) \equiv C^0(X, Y)$. The set of homotopy classes of continuous mappings $X \to Y$ will be denoted by $[X, Y]$. That is, $[X, Y] = C(X, Y)/\sim$, where $\sim$ refers to the equivalence relation of being homotopic. Every continuous mapping $g : Y \to Z$ induces a mapping $g_* : [X, Y] \to [X, Z]$ by

$$g_*([f]) := [g \circ f]. \tag{3.1.1}$$

Recall that a pointed topological space is a topological space X together with a point $*_X$. A mapping $f : X \to Y$ of pointed spaces is pointed if $f(*_X) = *_Y$. The subset of continuous pointed mappings will be denoted by $C_*(X, Y) \subset C(X, Y)$. A pointed homotopy is a homotopy through pointed mappings. The set of pointed homotopy classes of pointed mappings $X \to Y$ will be denoted by $[X, Y]_*$. Every pointed continuous mapping $g : Y \to Z$ induces a mapping $g_* : [X, Y]_* \to [X, Z]_*$ given by (3.1.1).

Recall further that a topological pair (X, A) is a topological space X together with a subset A endowed with the relative topology. A pair mapping $(X, A) \to (Y, B)$ is a mapping $f : X \to Y$ satisfying $f(A) \subset B$. The subset of continuous pair mappings will be denoted by $C\big((X, A), (Y, B)\big) \subset C(X, Y)$. A pair homotopy is a homotopy through pair mappings. The set of pair homotopy classes of pair mappings $(X, A) \to (Y, B)$ will be denoted by $[(X, A), (Y, B)]$. A pointed pair is a pair (X, A) with base point in A. The subset of continuous pointed pair mappings will be denoted by $C_*\big((X, A), (Y, B)\big) \subset C\big((X, A), (Y, B)\big)$ and the set of pointed pair homotopy classes of pointed pair mappings $(X, A) \to (Y, B)$ will be denoted by $[(X, A), (Y, B)]_*$. Every pair mapping (pointed pair mapping) $g : (Y, B) \to (Z, C)$ induces a mapping $g_* : [(X, A), (Y, B)] \to [(X, A), (Z, C)]$ ($g_* : [(X, A), (Y, B)]_* \to [(X, A), (Z, C)]_*$) given by (3.1.1).

Let $I = [0, 1]$. Recall that homotopies $f, g : X \times I \to Y$ which satisfy $f(x, 1) = g(x, 0)$ for all $x \in X$ can be concatenated and that their concatenation is usually defined to be the homotopy

$$f \cdot g : X \times I \to Y, \quad (f \cdot g)(x, t) := \begin{cases} f(x, 2t) & \mid \ t \leq \frac{1}{2}, \\ g(x, 2t - 1) & \mid \ t > \frac{1}{2}. \end{cases} \tag{3.1.2}$$

The concatenation of pointed homotopies, pair homotopies or pointed pair homotopies is, respectively, a pointed homotopy, a pair homotopy or a pointed pair homotopy. In the special case where X is the one-point space, (3.1.2) boils down to the concatenation of curves $\gamma, \delta : I \to Y$ satisfying $\gamma(1) = \delta(0)$, cf. formula (1.7.1).

Finally, recall the homotopy groups of a pointed topological space X,

$$\pi_n(X) := [(I^n, \partial I^n), (X, \{*_X\})]_*, \quad n = 0, 1, 2, \dots,$$

with the origin 0 as the base point of $(I^n, \partial I^n)$. In case $n = 0$, we put $I^0 = \{0, 1\}$ and $\partial I^0 = \varnothing$. Thus, $\pi_0(X)$ is the set of pathwise connected components of X. In case $n \geq 1$, the set $\pi_n(X)$ carries a group structure with multiplication defined by concatenation (3.1.2), where the mappings $I^k \to X$ are viewed as homotopies $I^{k-1} \times I \to X$. In case $n \geq 2$, the multiplication is Abelian. Alternatively, since $I^n/\partial I^n$ is homotopic to S^n, the elements of $\pi_n(X)$ can be viewed as (homotopy classes of) pointed mappings $\mathrm{S}^n \to X$.

Accordingly, the relative homotopy groups of a pointed topological pair (X, A) are defined by

$$\pi_n(X, A) := [(I^n, \partial I^n), (X, A)]_*, \quad n = 1, 2, 3, \ldots .$$

Here, the multiplication is Abelian for $n \geq 3$.

First, we discuss the compact-open topology on mapping spaces. Let X and Y be Hausdorff spaces and assume X to be locally compact. The compact-open topology on the space $C(X, Y)$ of continuous mappings $X \to Y$ is generated by the subsets

$$M(K, O) = \{f \in C(X, Y) : f(K) \subset O\}$$

with $K \subset X$ compact and $O \subset Y$ open. These subsets form a subbasis, meaning that the topology is generated by taking finite intersections and arbitrary unions. In case X and Y are pointed, the compact-open topology on $C_*(X, Y)$ is defined likewise. It coincides with the relative topology induced from $C(X, Y)$. We will need the following properties.

Proposition 3.1.1 *Let X, Y and Z be Hausdorff spaces and assume X to be locally compact.*

1. *$C(X, Y)$ is Hausdorff.*
2. *The evaluation mapping $C(X, Y) \times X \to Y$, $(f, x) \mapsto f(x)$, is continuous.*
3. *A mapping $f : X \times Z \to Y$ is continuous iff so are all the mappings $f_z : X \to Y$, $x \mapsto f(x, z)$, with $z \in Z$ and the mapping $Z \to C(X, Y)$, $z \mapsto f_z$.*
4. *Let $\mathrm{pr}_Y : Y \times Z \to Y$ and $\mathrm{pr}_Z : Y \times Z \to Z$ denote the natural projections. The mapping $C(X, Y \times Z) \to C(X, Y) \times C(X, Z)$ defined by $f \mapsto (\mathrm{pr}_Y \circ f, \mathrm{pr}_Z \circ f)$ is a homeomorphism.*

Similar statements hold in the pointed case.

Proof 1. Let f and g be two distinct elements of $C(X, Y)$. There exists $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open neighbourhoods U_1 of $f(x)$ and U_2 of $g(x)$. Then, $f^{-1}(U_1) \cap g^{-1}(U_2)$ is an open neighbourhood of x, because f and g are continuous. Since X is locally compact, this neighbourhood contains a compact neighbourhood K. Then, $M(K, U_1)$ and $M(K, U_2)$ are neighbourhoods of f and g, respectively. They are disjoint, because U_1 and U_2 are disjoint.

2. We have to show that for every open subset $O \subset Y$, the subset

$$\tilde{O} = \{(f, x) \in C(X, Y) \times X : f(x) \in O\}$$

of $C(X, Y) \times X$ is open in the product topology. Let $(f_0, x_0) \in \tilde{O}$. Then, $x_0 \in f_0^{-1}(O)$. Since f_0 is continuous, $f_0^{-1}(O)$ is an open neighbourhood of x_0. Since X is locally compact, $f_0^{-1}(O)$ contains a compact neighbourhood K of x_0. Then, $M(K, O) \times K$ is a neighbourhood of (f_0, x_0) which is contained in $\tilde{O}$. Therefore, $\tilde{O}$ is open.

3. First, assume that f is continuous. For given $z \in Z$, the mapping f_z is continuous, because it arises by composing f with the mapping $X \to X \times Z$, $x \mapsto (x, z)$, whose continuity is immediate from the definition of the product topology. To prove that the mapping $z \mapsto f_z$ is continuous, denote this mapping by φ. Since taking preimages commutes with taking intersections or unions, it suffices to show that $\varphi^{-1}\big(M(K, O)\big)$ is open for all compact $K \subset X$ and all open $O \subset Y$. Thus, let K and O be given and let $z \in \varphi^{-1}\big(M(K, O)\big)$. Then, $f(K \times \{z\}) \subset O$. By continuity of f, for every $x \in K$, there exist open neighbourhoods U_x of x in X and V_x of z in Z such that $f(U_x \times V_x) \subset O$. Since K is compact, we can find $x_1, \dots, x_r$ such that $K \subset \bigcup_{i=1}^r U_{x_i}$. Then, $V := \bigcap_{i=1}^r V_{x_i}$ is an open neighbourhood of z satisfying $f(K \times V) \subset O$, that is, $V \subset \varphi^{-1}\big(M(K, O)\big)$. Therefore, $\varphi^{-1}\big(M(K, O)\big)$ is open, as asserted.

The converse implication follows by observing that f can be written as the composition of the mapping $X \times Z \to C(X, Y) \times X$, $(x, z) \mapsto (f_z, x)$ with the evaluation mapping $C(X, Y) \times X \to Y$, $(f, x) \mapsto f(x)$ and by applying point 2.

4. Denote the mapping under consideration by φ. Obviously, φ is bijective with inverse $(f, g) \mapsto (f \times g) \circ \Delta_X$, where $\Delta_X : X \to X \times X$ denotes the diagonal mapping.

To prove that φ is continuous and open, since application of φ^{-1} and φ commutes with taking intersections or unions, it suffices to show that the subsets $\varphi^{-1}\big(M(K_1, O_Y) \times M(K_2, O_Z)\big)$ of $C(X, Y \times Z)$ and $\varphi\big(M(K, O)\big)$ of $C(X, Y) \times C(X, Z)$ are open for all compact $K_1, K_2, K \subset X$ and all open $O_Y \subset Y$, $O_Z \subset Z$ and $O \subset Y \times Z$. Since the first subset coincides with $M(K_1, O_Y \times Z) \cap M(K_2, Y \times O_Z)$, this part is immediate. To see that $\varphi\big(M(K, O)\big)$ is open, write $O = \bigcup_\alpha O_{A,\alpha} \times O_{Z,\alpha}$ with appropriate open subsets $O_{Y,\alpha} \subset Y$ and $O_{Z,\alpha} \subset Z$. Then, $M(K, O) = \bigcup_\alpha M(K, O_{Y,\alpha} \times O_{Z,\alpha})$ and hence

$$\varphi\big(M(K, O)\big) = \bigcup_\alpha \varphi\big(M(K, O_{Y,\alpha} \times O_{Z,\alpha})\big) = \bigcup_\alpha M(K, O_{Y,\alpha}) \times M(K, O_{Z,\alpha}).$$

■

Point 3 of Proposition 3.1.1 implies the following.

Corollary 3.1.2 *Let X and Y be Hausdorff spaces and assume X to be locally compact. A mapping $f : X \times I \to Y$ is a homotopy iff the mapping $t \mapsto f_t$ is a continuous curve in $C(X, Y)$. In particular, $[X, Y]$ coincides with the set of pathwise connected components of $C(X, Y)$. In case X and Y are pointed, $[X, Y]_*$ coincides with the set of pathwise connected components of $C_*(X, Y)$.* ■

Under the correspondence of homotopies on X with values in Y with continuous curves in $C(X, Y)$, the concatenation of homotopies corresponds to the concatenation of curves. Together with point 4 of Proposition 3.1.1, this implies the following.

Corollary 3.1.3 *Let X, Y and Z be Hausdorff spaces and assume X to be locally compact. Let $\mathrm{pr}_Y : Y \times Z \to Y$ and $\mathrm{pr}_Z : Y \times Z \to Z$ denote the natural projections. The mapping $[X, Y \times Z] \to [X, Y] \times [X, Z]$ defined by*

$$[f] \mapsto \big(\mathrm{pr}_{Y*}[f], \mathrm{pr}_{Z*}[f]\big)$$

is a bijection. A similar statement holds in the pointed case. ■

Next, we discuss loop spaces and their homotopy groups. The loop space of a pointed Hausdorff space X is the mapping space

$$\Omega X := C\big((I, \partial I), (X, \{*\})\big)$$

endowed with the compact-open topology induced from $C(I, X)$. By Proposition 3.1.1/1, ΩX is Hausdorff. It is pointed with base point given by the constant loop at $* \in X$. Thus, for $n = 0, 1, 2, \dots$, we can consider the space of pointed pair mappings $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$. For $n \geq 1$, we may identify $I^n = I^{n-1} \times I$ and thus view the elements of this space as homotopies. As such, any two of them can be concatenated. Hence, through this identification, concatenation of homotopies defines an operation on $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$. This operation descends to the ordinary multiplication in $\pi_n(\Omega X) = [(I^n, \partial I^n), (\Omega X, \{*\})]_*$. We will therefore refer to this operation as ordinary concatenation.

On the other hand, one can check that the operation of concatenation in ΩX, given by (1.7.1), is continuous (Exercise 3.1.2). Hence, by pointwise application, it induces an operation $\odot$ in $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$,

$$f \odot g : I^n \to \Omega X, \quad (f \odot g)(\mathbf{t}) := f(\mathbf{t}) \cdot g(\mathbf{t}). \tag{3.1.3}$$

We will refer to this operation as pointwise concatenation. One can further check that loop inversion $\gamma \mapsto \gamma^{-1}$ defines a continuous mapping $\Omega X \to \Omega X$ (Exercise 3.1.2). Hence, for every $f \in C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$, the mapping $f^{-\odot} : I^n \to \Omega X$ defined by $f^{-\odot}(\mathbf{t}) = f(\mathbf{t})^{-1}$ (inverse loop) is continuous and hence an element of $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$.

The following lemma collects the homotopy properties of the operation of pointwise concatenation. The proof is analogous to that for ordinary concatenation and is therefore left to the reader.

Lemma 3.1.4 *Let $n = 0, 1, 2, \dots,$ let X be a pointed Hausdorff space and let $f, g, h, k \in C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$. Let $\sim$ denote the equivalence relation of being pointed homotopic and let $e \in C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$ be defined by assigning to every $\mathbf{t} \in I^n$ the constant loop at $*$.*

1. *If $f \sim h$ and $g \sim k$, then $f \odot g \sim h \odot k$ and $f^{-\odot} \sim h^{-\odot}$.*
2. *$(f \odot g) \odot h \sim f \odot (g \odot h)$.*
3. *$f \odot e \sim e \odot f \sim f$.*
4. *$f^{-\odot} \odot f \sim f \odot f^{-\odot} \sim e$.* ■

By an elementary calculation, one finds that for $n \geq 1$, pointwise concatenation and ordinary concatenation are related by

$$(f \odot g) \cdot (h \odot k) = (f \cdot h) \odot (g \cdot k). \tag{3.1.4}$$

Theorem 3.1.5 *Let X be a pointed Hausdorff space and let $n = 0, 1, 2, \ldots$.*

1. *The operation of pointwise concatenation in $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$ induces a group operation in $\pi_n(\Omega X) \equiv [(I^n, \partial I^n), (\Omega X, \{*\})]_*$. For $n \geq 1$, the latter operation coincides with that induced by ordinary concatenation.*
2. *The group $\pi_n(\Omega X)$ is isomorphic to $\pi_{n+1}(X)$. An isomorphism is induced by the mapping $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big) \to C_*\big((I^{n+1}, \partial I^{n+1}), (X, \{*\})\big)$, $f \mapsto \tilde{f}$, where $\tilde{f}(\mathbf{t}, t) := f(\mathbf{t})(t)$ for all $\mathbf{t} \in I^n$ and $t \in I$.*

According to point 1, the operation of pointwise concatenation provides an alternative view on the homotopy groups of the loop space ΩX and, in addition, a natural group operation on $\pi_0(\Omega X)$.

Proof 1. That $\odot$ induces a group operation on $\pi_n(\Omega X)$ for all $n \geq 0$ follows from Lemma 3.1.4. In case $n \geq 1$, using this lemma, (3.1.4) and the homotopy properties of ordinary concatenation, we find

$$f \odot g \sim (f \cdot e) \odot (e \cdot g) = (f \odot e) \cdot (e \odot g) \sim f \cdot g.$$

Hence, the operations induced on $[(I^n, \partial I^n), (\Omega X, \{*\})]_*$ coincide.

2. By Proposition 3.1.1/3, $\tilde{f}$ is continuous for every f. Hence, the mapping $f \mapsto \tilde{f}$ is well defined. It is easy to see that this mapping is bijective.

To check that $f \sim g$ iff $\tilde{f} \sim \tilde{g}$, according to Corollary 3.1.2, it suffices to show that a curve γ in $C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$ is continuous iff so is the corresponding curve $\tilde{\gamma}$ in $C_*\big((I^{n+1}, \partial I^{n+1}), (X, \{*\})\big)$. Applying Proposition 3.1.1/3 twice, we find that γ is continuous iff so is the mapping

$$I^n \times I \times I \to X, \quad (\mathbf{t}, t, s) \mapsto \big(\gamma(s)(\mathbf{t})\big)(t).$$

Since $\big(\gamma(s)(\mathbf{t})\big)(t) = \big(\tilde{\gamma}(s)\big)(\mathbf{t}, t)$, this mapping is continuous iff so is the mapping

$$I^{n+1} \times I \to X, \quad (\mathbf{t}', s) \mapsto \big(\tilde{\gamma}(s)\big)(\mathbf{t}').$$

Applying Proposition 3.1.1/3 once again, we find that the latter holds iff $\tilde{\gamma}$ is continuous. As a result, the mapping $f \mapsto \tilde{f}$ descends to a bijection $\pi_n(\Omega X) \to \pi_{n+1}(X)$.

It remains to check that the latter is a group homomorphism. For that purpose, it suffices to check that $(f \odot g)^{\sim} = \tilde{f} \cdot \tilde{g}$ for all f, g. We leave this to the reader. ■

Remark 3.1.6 By Lemma 3.1.4 and formula (3.1.4), one finds

$$f \odot g \sim (e \cdot f) \odot (g \cdot e) = (e \odot g) \cdot (f \odot e) \sim g \cdot f \sim g \odot f$$

for all $f, g \in C_*\big((I^n, \partial I^n), (\Omega X, \{*\})\big)$. Hence, the group operation on $\pi_n(\Omega X)$ inherited from pointwise concatenation is Abelian. As a consequence, Theorem 3.1.5 implies that the homotopy groups $\pi_n(X)$ of a pointed Hausdorff space X are Abelian for $n \geq 2$. This is in fact the standard argument used in textbooks, cf. [104]. ♦

Next, we discuss *CW*-complexes. Recall that the direct sum of a family of topological spaces $\{X_\alpha : \alpha \in A\}$ is given by the disjoint union $\bigsqcup_{\alpha\in A} X_\alpha$ endowed with the final topology[1] defined by the natural inclusion mappings $X_\alpha \to \bigsqcup_{\alpha\in A} X_\alpha$.

Definition 3.1.7 Let X be a set and let $r_0, r_1, r_2, \dots$ be a sequence of non-negative integers. A *CW*-structure on X with r_n cells in dimension n is a family $\mathscr{F}$ of mappings $f_i^n : \mathrm{D}^n \to X$, where $n = 0, 1, 2, \dots$ and $i = 1, \dots, r_n$ whenever $r_n > 0$, such that the following conditions hold. Let $X^{(n)}$ denote the union of the images of the mappings f_i^k with $k \leq n$.

1. For every n with $r_n \neq 0$, $\bigsqcup_{i=1}^{r_n} f_i^n$ maps $\bigsqcup_{i=1}^{r_n}(\mathrm{Int}\,\mathrm{D}^n)$ injectively to $X \setminus X^{(n-1)}$.[2]
2. Every f_i^n maps $\partial\mathrm{D}^n$ to $X^{(n-1)}$.
3. $X = \bigcup_n X^{(n)}$.

A *CW*-complex is a Hausdorff topological space X together with a *CW*-structure $\mathscr{F}$ on the underlying set such that the topology of X coincides with the final topology defined by $\mathscr{F}$.

The mappings f_i^n are referred to as the characteristic mappings and their restrictions to $\partial\mathrm{D}^n \subset \mathrm{D}^n$ as the attaching mappings of the *CW*-structure $\mathscr{F}$. The images $f_i^n(\mathrm{D}^n)$ are referred to as the closed cells and the subsets $f_i^n(\mathrm{Int}\,\mathrm{D}^n)$ as the open cells of $\mathscr{F}$. The subsets $X^{(n)} \subset X$ are called the n-skeleta of $\mathscr{F}$. A *CW*-complex $(X, \mathscr{F})$ is said to be finite if only finitely many of the numbers r_n are nonzero. In this case, the largest n such that $r_n \neq 0$ is called the dimension. The *CW*-complex $(X, \mathscr{F})$ is said to be pointed if X is pointed and the base point is a 0-cell. A subcomplex of $(X, \mathscr{F})$ is a subspace $\tilde{X} \subset X$ endowed with the relative topology, together with a subfamily $\tilde{\mathscr{F}} \subset \mathscr{F}$ such that $(\tilde{X}, \tilde{\mathscr{F}})$ is a *CW*-complex.

Remark 3.1.8 The acronym *CW* refers to the following properties.

1. Closure-finiteness: every closed cell meets only finitely many open cells (because by the defining property 1, it can meet only open cells of lower dimension and these are finite in number).
2. Weak topology: X carries the final topology defined by the family $\mathscr{F}$. Thus, a subset of X is open iff all of its preimages under the mappings f_i^n are open. An analogous statement holds for closed subsets. Since D^n is compact and X is Hausdorff, the latter is equivalent to the statement that a subset of X is closed iff its intersection with every closed cell is closed. ♦

Proposition 3.1.9 *Let X be a Hausdorff topological space and let $\mathscr{F}$ be a finite CW-structure on X. For that $\mathscr{F}$ makes X into a CW-complex it suffices that every $f_i^n \in \mathscr{F}$ is continuous.*

[1]The final topology defined on X by a family of mappings $f_\alpha : X_\alpha \to X$ is the finest topology in which all f_α are continuous. That is, a subset $A \subset X$ is open iff $f_\alpha^{-1}(A) \subset X_\alpha$ is open for all α.

[2]For $n = 0$, we put $\mathrm{Int}\,\mathrm{D}^0 = \mathrm{D}^0$ and $X^{(-1)} = \varnothing$.

Proof We show that a subset $A \subset X$ is closed iff $(f_i^n)^{-1}(A) \subset \mathrm{D}^n$ is closed for all n and i. The 'only if' direction is obvious. To prove the 'if' direction, assume that $(f_i^n)^{-1}(A)$ is closed for all n and i. Since a continuous mapping from a compact space to a Hausdorff space is closed (Exercise 3.1.1), it follows that $f_i^n\big((f_i^n)^{-1}(A)\big) \subset X$ is closed for all n and i. Since A is the union over all these subsets, and since their number is finite, we conclude that A is closed. ∎

Example 3.1.10 Proofs are left to the reader (Exercise 3.1.3).

1. The n-sphere S^n admits a CW-structure with one cell in dimension 0 and one cell in dimension n. The characteristic mappings can be chosen as

$$f^0(*) = \mathbf{e}_1, \quad f^n(\mathbf{x}) = (2\mathbf{x}^2 - 1, 2\sqrt{1-\mathbf{x}^2}\mathbf{x}).$$

   There is another CW-structure, with two cells in each dimension up to n. Its characteristic mappings can be chosen as

$$f_\pm^0(*) = \pm\mathbf{e}_1, \quad f_\pm^k(\mathbf{x}) = (\mathbf{x}, \pm\sqrt{1-\mathbf{x}^2}, 0, \ldots, 0). \tag{3.1.5}$$

   Correspondingly, the two closed cells in dimension k are given by

$$\{(x_1, \ldots, x_{k+1}, 0, \ldots, 0) \in \mathrm{S}^n : \pm x_{k+1} \geq 0\}.$$

   This CW-structure has the advantage that the lower dimensional spheres

$$\mathrm{S}^k = \{(x_1, \ldots, x_{k+1}, 0, \ldots, 0) \in \mathrm{S}^n\}, \quad k = 0, 1, 2, \ldots, n-1,$$

   are subcomplexes.
2. The closed n-disk D^n has a tautological CW-structure with one cell in dimension n and the identical mapping as the characteristic mapping. It is however sometimes convenient to have the boundary S^{n-1} as a subcomplex. This can be achieved by just adding either one of the two CW-structures of S^{n-1} of point 1, with S^{n-1} being viewed as a subset of D^n and the characteristic mappings as mappings to D^n.
3. The one-point union[3] of two pointed CW-complexes $(X_1, \mathscr{F}_1)$ and $(X_2, \mathscr{F}_2)$ is a CW-complex with underlying space $X_1 \vee X_2$ and CW-structure $\mathscr{F}_1 \cup \mathscr{F}_2$, where the elements of $\mathscr{F}_i$ are viewed as mappings to $X_1 \vee X_2$ via the natural inclusion mappings $X_i \to X_1 \vee X_2$. This way, the characteristic mappings of the base points get identified and thus yield one element of $\mathscr{F}_1 \cup \mathscr{F}_2$. As an application, from the CW-structures on S^1 we obtain CW-structures on the figure eight and, more generally, on the one-point union of a finite number of 1-spheres. However, one cannot obtain a CW-structure on the one-point union of a countably

[3]The one-point union $X \vee Y$ of pointed topological spaces X and Y with base points $*_X$ and $*_Y$ is the quotient of $X \sqcup Y$ by the subset $\{*_X, *_Y\}$. It is pointed with base point $[*_X] = [*_Y]$.

infinite number of copies, a space which is known as the Hawaiian earring, in this way. In fact, this space does not admit any CW-structure.

4. The direct product of two CW-complexes $(X_1, \mathscr{F}_1)$ and $(X_2, \mathscr{F}_2)$ is a CW-complex with underlying space $X_1 \times X_2$ and CW-structure $\mathscr{F}_1 \times \mathscr{F}_2$ with elements $(f^n_{1i} \times f^m_{2j}) \circ p_{n+m}$, where $p_{n+m} : \mathrm{D}^{n+m} \to \mathrm{D}^n \times \mathrm{D}^m$ is some chosen homeomorphism. This makes sense, because, as a homeomorphism, p_{n+m} maps the boundary S^{n+m-1} of D^{n+m} onto the boundary $(\mathrm{S}^{n-1} \times \mathrm{D}^m) \cup (\mathrm{D}^n \times \mathrm{S}^{m-1})$ of $\mathrm{D}^n \times \mathrm{D}^m$. The number of cells of $\mathscr{F}_1 \times \mathscr{F}_2$ in dimension n is
$$\sum_{k=0}^{n} r_{1,k}\, r_{2,n-k} .$$
For example, the direct product of two copies of the CW-complex S^1 with one cell in dimensions 0 and 1 yields a CW-structure on the 2-torus $\mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ with one cell in dimensions 0 and 2 and two cells in dimension 1. This CW-structure coincides with the one obtained by means of Morse theory in Example 8.9.9 of Part I.
5. Let $(X, \mathscr{F})$ be a CW-complex and let G be a finite group acting freely on X by homeomorphisms. If one can define a free action of G on $\mathscr{F}$ by permutations of characteristic mappings of the same dimension such that
$$(a \cdot f^n_i)(\mathbf{x}) = a \cdot \big(f^n_i(\mathbf{x})\big)$$
for all $a \in G$ and $\mathbf{x} \in \mathrm{D}$, then by choosing one representative in each G-orbit in $\mathscr{F}$ and composing it with the natural projection onto the quotient, one obtains a CW-structure on that quotient. For example, consider the action of the cyclic group $G = \mathbb{Z}_2$ of order two on S^n generated by the antipodal mapping. The quotient of this action is the real projective space $\mathbb{R}\mathrm{P}^n$. One can define a free action of $\mathbb{Z}_2$ on the CW-structure with two cells in each dimension up to n, cf. point 5, by exchanging cells of the same dimension. By choosing one cell in each dimension, f^k_+ say, and composing it with the natural projection $\mathrm{S}^n \to \mathbb{R}\mathrm{P}^n$ we obtain a CW-structure on $\mathbb{R}\mathrm{P}^n$ with one cell in each dimension. ♦

Proposition 3.1.11 *Let $(X, \mathscr{F})$ be a CW-complex, let Y be a topological space and let $f : X \to Y$ be a mapping. The following statements are equivalent.*

1. *The mapping f is continuous.*
2. *The mappings $f \circ f^n_i$ are continuous for all n and i.*
3. *The restrictions of f to the closed cells of $\mathscr{F}$ are continuous.*

Proof The implication $1 \Rightarrow 3$ is obvious.
$3 \Rightarrow 2$. Let n and i be given. Since f^n_i is continuous, so is its restriction in range to the closed cell $f^n_i(\mathrm{D}^n)$. Composition of the latter with the restriction of f in domain to that closed cell yields $f \circ f^n_i$.
$2 \Rightarrow 1$. Let $A \subset Y$ be open. By assumption, then $(f \circ f^n_i)^{-1}(A)$ is open in D^n for all n and i. Since $(f \circ f^n_i)^{-1}(A) = (f^n_i)^{-1}\big(f^{-1}(A)\big)$, then $f^{-1}(A) \subset X$ is open. ■

Proposition 3.1.12 *Let $(X, \mathscr{F})$ be a CW-complex, let Y be a topological space and let $f_n : X^{(n)} \to Y$, $n = 0, 1, 2, \ldots$, be a family of continuous mappings satisfying $f_{n+1 \restriction X^{(n)}} = f_n$ for all n. Then, there exists a unique mapping $f : X \to Y$ such that $f_{\restriction X^{(n)}} = f_n$ for all n and this mapping is continuous.*

Proof Since the assumption implies that $f_{m \restriction X^{(n)}} = f_n$ for all $m > n$, and since X is the union over the n-skeleta, we can define f by $f_{\restriction X^{(n)}} = f_n$. Uniqueness is then obvious. To check continuity, we observe that for all n, i and $\mathbf{x} \in \mathrm{D}^n$, we have $f\big(f_i^n(\mathbf{x})\big) = f_n\big(f_i^n(\mathbf{x})\big)$. It follows that $f \circ f_i^n$ is continuous for all n and i and hence, by Proposition 3.1.11/2, that f is continuous. ■

Using Morse theory, one can show the following, cf. the discussion for compact manifolds on page 420 in Part I.

Proposition 3.1.13 *Every smooth manifold M is homotopy equivalent to a CW-complex of the same dimension.*

Proof See [449, p. 36]. ■

Finally, we discuss direct limits. A directed system of topological spaces consists of a directed set[4] $(A, \leq)$, a topological space X_α for every $\alpha \in A$ and a continuous mapping $f_{\alpha\beta} : X_\alpha \to X_\beta$ for every pair $(\alpha, \beta) \in A \times A$ with $\alpha \leq \beta$ such that $f_{\alpha\alpha} = \mathrm{id}_{X_\alpha}$ for all α and $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$. The direct limit

$$X = \lim_{\rightarrow} X_\alpha$$

of a directed system $\{X_\alpha, f_{\alpha\beta}\}$ is the topological quotient of the direct sum $\bigsqcup_{\alpha \in A} X_\alpha$ with respect to the equivalence relation that $x \in X_\alpha$ is equivalent to $y \in X_\beta$ iff $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$ for some γ. Composition of the natural inclusion mappings $X_\alpha \to \bigsqcup_{\alpha \in A} X_\alpha$ with the natural projection to equivalence classes yields continuous mappings

$$\varphi_\alpha : X_\alpha \to X$$

and the topology of X coincides with the final topology defined by these mappings. That is, a subset of X is open iff its preimage under φ_α is open in X_α for every α. The proofs of the following two propositions are left to the reader (Exercises 3.1.4 and 3.1.5).

Proposition 3.1.14 *Let $\{X_\alpha, f_{\alpha\beta}\}$ and $\{Y_\alpha, g_{\alpha\beta}\}$ be directed systems of topological spaces over the same index set A and let X and Y, respectively, be the direct limits. Every family of continuous mappings $h_\alpha : X_\alpha \to Y_\alpha$ satisfying $h_\beta \circ f_{\alpha\beta} = g_{\alpha\beta} \circ h_\alpha$ whenever $\alpha \leq \beta$ descends to a continuous mapping $h : X \to Y$.* ■

---

[4] A set with a partial ordering $\leq$ which has the property that for any two elements $\alpha_1, \alpha_2 \in I$ there exists $\alpha_3 \in I$ such that $\alpha_1 \leq \alpha_3$ and $\alpha_2 \leq \alpha_3$.

Proposition 3.1.15 *Let $\{X_\alpha, f_{\alpha\beta}\}$ be directed systems of topological spaces and let X be the direct limit. If for some i one has $\pi_i(X_\alpha) = 0$ for all but finitely many α, then $\pi_i(X) = 0$.* ■

Example 3.1.16 The family of skeleta $\{X^{(k)} : k = 0, 1, 2, \dots\}$ of a CW-complex $(X, \mathscr{F})$, together with the natural inclusion mappings $f_{kl} : X^{(k)} \to X^{(l)}$ for $k \leq l$, forms a directed system of topological spaces. The direct limit of this system is homeomorphic to X (Exercise 3.1.6). As a consequence, Proposition 3.1.14 reproduces Proposition 3.1.12. ♦

Exercises

3.1.1 Prove that a continuous mapping from a compact space to a Hausdorff space is closed. *Hint.* First, prove the following. A closed subset of a compact space is compact. The image of a compact set under a continuous mapping is compact. A compact subset of a Hausdorff space is closed.

3.1.2 Show that the mappings $m : \Omega X \times \Omega X \to \Omega X$ and $i : \Omega X \to \Omega X$ defined by concatenation of loops and loop inversion, respectively, are continuous.

3.1.3 Prove the statements of Example 3.1.10.

3.1.4 Prove Proposition 3.1.14.

3.1.5 Prove the statement about the homotopy groups of the direct limit of a directed system of topological spaces given in Proposition 3.1.15.

3.1.6 Show that the direct limit of the directed system made up by the skeleta of a CW-structure on a topological space is homeomorphic to that space, cf. Example 3.1.16.

3.2 Fibrations

In this section, let X, Y be topological spaces and let $\pi : Y \to X$ be a continuous mapping. Given a topological space Z and a continuous mapping $f : Z \to X$, every continuous mapping $\tilde{f} : Z \to Y$ satisfying

$$\pi \circ \tilde{f} = f$$

is called a lift of f through π. Let there be given a topological pair (Z, A), a continuous mapping $f : Z \to X$ and a lift $\tilde{f}_0 : A \to Y$ of $f_{\restriction A}$ through π. The quest for an extension of $\tilde{f}_0$ to a lift $\tilde{f}$ of f through π is called the lifting problem for π defined by the mapping f and the initial condition $\tilde{f}_0$. The situation can be summarized in the diagram

$$
\begin{tikzcd}
A \arrow[r, "\tilde{f}_0"] \arrow[d, "\cap"'] & Y \arrow[d, "\pi"] \\
Z \arrow[r, "f"'] \arrow[ur, dotted, "\tilde{f}"] & X
\end{tikzcd}
\tag{3.2.1}
$$

If for a certain class of topological pairs (Z, A) every lifting problem for π has a solution, one says that π has the lifting property with respect to that class of pairs.

Of particular interest is the special situation where the pair under consideration is of the form $(Z \times I, Z \times \{0\})$. In this case, the lifting problem is referred to as the homotopy lifting problem. The corresponding diagram (3.2.1) reads

$$
\begin{tikzcd}
Z \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, "\cap"'] & Y \arrow[d, "\pi"] \\
Z \times I \arrow[r, "f"'] \arrow[ur, dotted, "\tilde{f}"] & X
\end{tikzcd}
\tag{3.2.2}
$$

If for a certain class of pointed topological spaces Z every homotopy lifting problem for π has a solution, one says that π has the homotopy lifting property with respect to that class of spaces.

Definition 3.2.1 A continuous mapping $\pi : Y \to X$ is called a Hurewicz fibration if it has the homotopy lifting property with respect to all topological spaces. It is called a Serre fibration if it has the homotopy lifting property with respect to D^n for all n.

Example 3.2.2

1. The natural projections in a direct product are Hurewicz fibrations. Indeed, for $Y = X \times F$ and the natural projection $\pi : Y \to X$, the homotopy lifting problem defined by some mapping $f : Z \times I \to X$ and an appropriate initial condition $\tilde{f}_0 : Z \times \{0\} \to Y$ is solved by the mapping
$$\tilde{f} : Z \times I \to Y, \quad \tilde{f}(z, t) := \big(f(z, t), \tilde{f}_0(z, 0)\big).$$
2. Topological fibre bundles are Serre fibrations, see Corollary 3.2.5 below. ♦

In what follows, we first collect the basic properties of Serre fibrations. Then, we prove that topological fibre bundles are Serre fibrations. Thereafter, we show that the homotopy sequence for pairs induces a homotopy sequence for Serre fibrations. Finally, we discuss the path-loop fibration of a topological space and pullbacks of fibrations.

Proposition 3.2.3 *Serre fibrations have the lifting property with respect to all pairs of the form*

1. $\big(K \times I, (K \times \{0\}) \cup (L \times I)\big)$*, where K is a CW-complex and L is a subcomplex,*
2. (K, L)*, where K is a CW-complex and L is a subcomplex which is a strong deformation retract of K.*

Proof Let $\pi : Y \to X$ be a Serre fibration, let K be a CW-complex and let L be a subcomplex of K.

1. Consider the lifting problem defined by some $f : K \times I \to X$ and an appropriate initial condition $\tilde{f}_0 : (K \times \{0\}) \cup (L \times I) \to Y$. We prove the assertion by induction on the dimension k of the cells attached to L to build K. Let $K^{(k)}$ denote the k-skeleton of K. The case $k = 0$ is trivial. Thus, assume that we have constructed a lift $\tilde{f}$ of f over the subspace $(K^{(k)} \cup L) \times I \subset K \times I$ for some $k \geq 0$ and consider a $(k+1)$-cell C not contained in L, with characteristic mapping $\chi : \mathrm{D}^{k+1} \to K$. Since C is not contained in L, we have $C \cap (K^{(k)} \cup L) = C \cap K^{(k)}$. Hence, we wish to extend $\tilde{f}$ from

$$(C \times \{0\}) \cup \big((C \cap K^{(k)}) \times I\big) \subset C \times I$$

to a lift of f on $C \times I$. Assume that we can extend

$$\tilde{f} \circ \big(\chi \times \mathrm{id}_I\big)_{\restriction (\mathrm{D}^{k+1} \times \{0\}) \cup (\partial \mathrm{D}^{k+1} \times I)}$$

to a lift of $f \circ (\chi \times \mathrm{id}_I\big)$ on $\mathrm{D}^{k+1} \times I$. Since χ is injective on $\operatorname{Int} \mathrm{D}^{k+1}$, this lift uniquely determines a lift of f on $C \times I$. By Proposition 3.1.11, applied to the CW-complex $C \times I$, the latter is continuous.

This argument shows that in order to prove that $\tilde{f}$ extends to a lift of f over $(K^{(k+1)} \cup L) \times I$, it suffices to show that π has the lifting property with respect to the pair $\big(\mathrm{D}^{k+1} \times I, (\mathrm{D}^{k+1} \times \{0\}) \cup (\partial \mathrm{D}^{k+1} \times I)\big)$. It is not hard to see that this pair is homeomorphic to the pair $(\mathrm{D}^{k+1} \times I, \mathrm{D}^{k+1} \times \{0\})$ (Exercise 3.2.1). Since π is a Serre fibration, this yields the assertion.

2. Let $F : K \times I \to K$ be a strong deformation retraction from K to L and consider the lifting problem defined by some $f : K \to X$ and an appropriate initial condition $\tilde{f}_0 : L \to Y$. Define $g : K \times I \to X$ by $g := f \circ F$. Since F maps the subsets $K \times \{1\}$ and $L \times I$ to L, we can also define

$$\tilde{g}_0 : (K \times \{1\}) \cup (L \times I) \to Y, \quad \tilde{g}_0(x, t) := \tilde{f}_0\big(F(x, t)\big).$$

A brief calculation shows that $\tilde{g}_0$ is a lift of g over the subset $(K \times \{1\}) \cup (L \times I)$. Hence, according to point 1, it can be extended to a lift $\tilde{g}$ of g. Then, another brief calculation shows that the mapping $\tilde{f} : K \to Y$ defined by $\tilde{f}(x) := \tilde{g}(x, 0)$ is a lift of f through π extending $\tilde{f}_0$. ■

To be a Serre fibration is a local property in the following sense.

Proposition 3.2.4 *For a continuous mapping $\pi : Y \to X$ to be a Serre fibration it suffices that every $x \in X$ admits a neighbourhood U such that the mapping $\pi^{-1}(U) \to U$ induced by restriction of π is a Serre fibration.*

Proof Consider the homotopy lifting problem defined by some $f : \mathrm{D}^n \times I \to X$ and some appropriate initial condition $\tilde{f}_0 : \mathrm{D}^n \times \{0\} \to Y$. Since $\mathrm{D}^n \times I$ is compact, we can find open subsets $U_1, \ldots, U_r$ of X such that the mappings $\pi^{-1}(U_i) \to U_i$ induced by restriction of π are Serre fibrations and such that the preimages $f^{-1}(U_i)$ cover $\mathrm{D}^n \times I$. We find a cell complex structure of D^n and numbers $t_1, \ldots, t_s \in I$ such that for every cell C and every $j = 0, \ldots, s$, there exists i such that $f(C \times [t_j, t_{j+1}]) \subset U_i$. Here, $t_0 = 0$ and $t_{s+1} = 1$. We prove the assertion by induction on the dimension k of the cells and, for each fixed cell, by induction on j. The case $k = 0$ is trivial for all j. For a given cell C of dimension $k \geq 1$ and given j, via the characteristic mapping of C, the induction argument boils down to solving a lifting problem for the Serre fibration $\pi^{-1}(U_i) \to U_i$, defined on the pair $\big(C \times [t_j, t_{j+1}], (C \times \{0\}) \cup (\partial C \times I)\big)$. Thus, the assertion follows from Proposition 3.2.3/1. ■

In view of Example 3.2.2/1, Proposition 3.2.4 implies

Corollary 3.2.5 *Topological fibre bundles are Serre fibrations.* ■

If the base space is assumed to be paracompact, one has the following stronger result, originally proved independently in [322] and [332].

Proposition 3.2.6 (Huebsch and Hurewicz) *Topological fibre bundles over paracompact base spaces are Hurewicz fibrations.*

Proof See [598, Theorem 2.7.13]. ■

Now, we show that the homotopy sequence for pointed pairs induces a homotopy sequence for Serre fibrations. Let $\pi : Y \to X$ be a Serre fibration. Let $*_X$ be a base point in X, let $F := \pi^{-1}(*_X)$ and let $*_F$ be a base point in F. The latter will be taken as a base point in Y, too. This way, π is turned into a pointed mapping. The subset F is referred to as the fibre of π. Recall that for the pointed pair (Y, F), one has the following natural homomorphisms of homotopy groups:

1. the boundary homomorphism defined by

$$\partial : \pi_n(Y, F) \to \pi_{n-1}(F), \quad \partial[f] := [f_{\restriction \partial I^n}], \tag{3.2.3}$$

2. the homomorphism $i_* : \pi_n(Y) \to \pi_n(Y, F)$ induced from the natural inclusion mapping $(Y, \{*_F\}) \to (Y, F)$,
3. the homomorphism $j_* : \pi_n(F) \to \pi_n(Y)$ induced from the natural inclusion mapping $j : F \to Y$.

Recall further that these homomorphisms fit into an exact sequence

$$\begin{array}{l}\cdots \xrightarrow{\partial} \pi_n(F) \xrightarrow{j_*} \pi_n(Y) \xrightarrow{i_*} \pi_n(Y, F) \\ \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{j_*} \pi_{n-1}(Y) \xrightarrow{i_*} \pi_{n-1}(Y, F) \xrightarrow{\partial} \cdots \\ \cdots \xrightarrow{j_*} \pi_1(Y) \xrightarrow{i_*} \pi_1(Y, F) \xrightarrow{\partial} \pi_0(F) \xrightarrow{j_*} \pi_0(Y),\end{array} \tag{3.2.4}$$

referred to as the homotopy sequence of the pair (Y, F). Except for the last two, all mappings are group homomorphisms.

Lemma 3.2.7 *For every $n \geq 1$, composition with π defines a mapping*

$$C_*\big((I^n, \partial I^n), (Y, F)\big) \to C_*\big((I^n, \partial I^n), (X, \{*_X\})\big), \quad f \mapsto \pi \circ f,$$

and this mapping descends to a group isomorphism $\pi_n(Y, F) \to \pi_n(X)$.

Proof Since $\pi(F) = \{*_X\}$, the mapping is well defined. Moreover, for every pointed pair homotopy $H : I^n \times I \to Y$, the mapping $\pi \circ H : I^n \times I \to X$ is a pointed homotopy. Hence, the assignment $f \mapsto \pi \circ f$ descends to a mapping $\iota : \pi_n(Y, F) \to \pi_n(X)$. Clearly, ι is a group homomorphism.

The mapping ι is injective: let $f, g \in C_*\big((I^n, \partial I^n), (Y, F)\big)$ and $H : I^n \times I \to X$ be a pointed homotopy from $\pi \circ f$ to $\pi \circ g$. Every solution $\tilde{H}$ of the homotopy lifting problem for π defined by H and the initial condition

$$\tilde{H}_0 : (I^n \times \{0, 1\}) \cup (\{\mathbf{e}_1\} \times I) \to Y$$

given by

$$\tilde{H}_{0\restriction I^n \times \{0\}} = f, \quad \tilde{H}_{0\restriction I^n \times \{1\}} = g, \quad \tilde{H}_0(\mathbf{e}_1, t) = *_F$$

defines a homotopy $\tilde{H} : \mathrm{D}^n \times I \to Y$ from f to g. Since the subset $(I^n \times \{0, 1\}) \cup (\{\mathbf{e}_1\} \times I)$ is a strong deformation retract of $I^n \times I$ (Exercise 3.2.2), Proposition 3.2.3/2 yields that $\tilde{H}$ exists. Since $\pi \circ \tilde{H}_t = H_t$ sends ∂I^n to $\{*_X\}$, $\tilde{H}$ is a pair homotopy. Since $\tilde{H}(\mathbf{e}_1, t) = \tilde{H}_0(\mathbf{e}_1, t) = *_F$, it is pointed.

The mapping ι is surjective: let $f \in C_*\big((I^n, \partial I^n), (X, \{*_X\})\big)$. By Proposition 3.2.3/2, the lifting problem for π defined by the mapping $f : I^n \to X$ and the initial condition $\tilde{f}_0 : \{\mathbf{e}_1\} \to Y$, $\tilde{f}_0(\mathbf{e}_1) := *_F$, has a solution $\tilde{f} : I^n \to Y$. By construction, $\tilde{f} \in C_*\big((I^n, \partial I^n), (Y, F)\big)$ and $\iota[\tilde{f}] = [f]$. ■

As a consequence, in the homotopy sequence of the pair (Y, F), we can replace the relative homotopy groups $\pi_n(Y, F)$ by the ordinary homotopy groups $\pi_n(X)$, the homomorphism i_* by $\iota \circ i_*$ and the homomorphism ∂ by $\partial \circ \iota^{-1}$. The homomorphism $\partial \circ \iota^{-1}$ will be referred to as the boundary homomorphism of the fibration π and will be denoted by ∂. We determine these homomorphisms explicitly. On the one hand, for $f \in C_*\big((I^n, \partial I^n), (Y, \{*_F\})\big)$, we have $\iota \circ i_*([f]) = [\pi \circ f]$. On the other hand, for $f \in C_*\big((I^n, \partial I^n), (X, \{*_X\})\big)$, we have

$$\partial([f]) = [\tilde{f}_{\restriction \partial I^n}], \tag{3.2.5}$$

where $\tilde{f} \in C_*\big((I^n, \partial I^n), (Y, \{*_F\})\big)$ is a lift of f through π. Thus, (3.2.4) translates into the sequence

$$
\begin{aligned}
\cdots \xrightarrow{\partial}\ & \pi_n(F) \xrightarrow{j_*} \pi_n(Y) \xrightarrow{\pi_*} \pi_n(X) \\
& \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{j_*} \pi_{n-1}(Y) \xrightarrow{\pi_*} \pi_{n-1}(X) \xrightarrow{\partial} \cdots \\
& \cdots \xrightarrow{j_*} \pi_1(Y) \xrightarrow{\pi_*} \pi_1(X) \xrightarrow{\partial} \pi_0(F) \xrightarrow{j_*} \pi_0(Y),
\end{aligned}
\tag{3.2.6}
$$

where up to the last two, all mappings are group homomorphisms. This sequence is referred to as the homotopy sequence of the fibration π. Exactness of (3.2.4) implies the following.

Theorem 3.2.8 *The homotopy sequence of a Serre fibration is exact.* ■

According to Corollary 3.2.5, this sequence applies in particular to a principal G-bundle $\pi : P \to M$. In this case, one can identify $\pi_n(F)$ with $\pi_n(G)$ by means of an equivariant diffeomorphism $\kappa : F \to G$ sending the base point of F to the unit element $\mathbb{1}$. Under this identification, the boundary homomorphism reads

$$
\partial : \pi_n(M) \to \pi_{n-1}(G), \quad \partial([f]) = [\kappa \circ \tilde{f}_{\restriction \partial I^n}], \tag{3.2.7}
$$

where $f \in C_*\big((I^n, \partial I^n), (M, \{*_M\})\big)$ and $\tilde{f} \in C_*\big((I^n, \partial I^n), (P, F)\big)$ is a lift of f through π.

Recall that $\pi_0(G)$ can be identified with the set of connected components of G and thus carries a natural group structure. This group acts on $\pi_n(G)$ by those automorphisms which are induced by the inner automorphisms of G:

$$
(aG_0) \cdot ([f]) = (\mathrm{C}_a)_*([f]), \tag{3.2.8}
$$

where $a \in G$ and $f \in C_*\big((I^n, \partial I^n), (G, \mathbb{1})\big)$. Here, G_0 denotes the identity component of G and C_a denotes conjugation by a.

Recall further that $\pi_1(M)$ acts on $\pi_n(M)$ from the right by automorphisms as follows. Given $[\gamma] \in \pi_1(M)$ and $[f] \in \pi_n(M)$, choose an extension $\tilde{h} : I^n \times I \to M$ of the mapping

$$
h : (I^n \times \{0\}) \cup (\{0\} \times I) \to M
$$

defined by $h(\mathbf{t}, 0) = f(\mathbf{t})$ and $h(0, t) = \gamma(t)$ and put $\varphi_{[\gamma]}([f]) := [g]$, where $g(\mathbf{t}) = \tilde{h}(\mathbf{t}, 1)$. Then, $\varphi_{[\gamma]}$ is a group automorphism of $\pi_n(M)$ for every $[\gamma] \in \pi_1(M)$ and the assignment of $\varphi_{[\gamma]}$ to $[\gamma]$ is a group anti-homomorphism $\pi_1(M) \to \mathrm{Aut}\,\big(\pi_n(M)\big)$. For simplicity, we will write $[\gamma] \cdot [f] := \varphi_{[\gamma]}([f])$.

Proposition 3.2.9 *Let P be a topological principal G-bundle over M. For every $[\gamma] \in \pi_1(M)$ and $[f] \in \pi_n(M)$,*

$$
\partial\big([\gamma] \cdot [f]\big) = \partial\big([\gamma]\big) \cdot \partial\big([f]\big).
$$

That is, via the boundary homomorphism in dimension 1*, the boundary homomorphism in dimension n intertwines the action of $\pi_1(M)$ on $\pi_n(M)$ with the action of $\pi_0(G)$ on $\pi_{n-1}(G)$.*

Proof Let γ and f be given and choose a representative $g \in C_*\big((I^n, \partial I^n), (M, *_M)\big)$ of $[\gamma] \cdot [f] \in \pi_n(M)$. There exists a homotopy $H : I^n \times I \to M$ satisfying $H(0, t) = \gamma(t)$ for all t.[5] Choose a lift $\tilde{f}$ of f through $\pi : P \to M$ and let $\tilde{H} : I^n \times I \to P$ be a lift of H through π with initial condition $\tilde{f}$. Then, the curve $\tilde{\gamma} : I \to P$ defined by

$$\tilde{\gamma}(t) := \tilde{H}(0, t)$$

is a lift of γ and belongs to $C_*\big((I, \partial I), (P, F)\big)$. Hence, according to (3.2.7), under the identification of $\pi_0(G)$ with the group of connected components of G,

$$\partial([\gamma]) = [\kappa \circ \tilde{\gamma}_{\restriction \partial I}] \equiv a G_0, \quad a := \kappa\big(\tilde{\gamma}(1)\big).$$

Thus, on the one hand, according to (3.2.7) and (3.2.8), we have

$$\partial([\gamma]) \cdot \partial([f]) = [\mathrm{C}_a \circ \kappa \circ \tilde{f}_{\restriction \partial I^n}].$$

On the other hand, $\Psi_{a^{-1}} \circ \tilde{H}_1$ is a lift of g and belongs to $C_*\big((I^n, \partial I^n), (P, F)\big)$. Hence,

$$\partial\big([\gamma] \cdot [f]\big) = \partial([g]) = [\kappa \circ (\Psi_{a^{-1}} \circ \tilde{H}_1)_{\restriction \partial I^n}] = [\mathrm{R}_{a^{-1}} \circ \kappa \circ (\tilde{H}_1)_{\restriction \partial I^n}],$$

where $\mathrm{R}_{a^{-1}}$ denotes right translation by a^{-1}. Thus, to prove the assertion, we have to show that $\mathrm{C}_a \circ \kappa \circ \tilde{f}_{\restriction \partial I^n}$ is pointed homotopic to $\mathrm{R}_{a^{-1}} \circ \kappa \circ (\tilde{H}_1)_{\restriction \partial I^n}$.

To see this, consider the (topological) principal G-bundle $\gamma^* P$ over I. Since $\tilde{\gamma}$ is a global section of $\gamma^* P$, it defines a global trivialization and hence a continuous equivariant mapping $\tilde{\kappa} : \gamma^* P \to G$ which sends $\tilde{\gamma}(t)$ to $\mathbb{1}$ for all $t \in I$. Since, by construction, $\pi \circ \tilde{H}(\mathbf{t}, t) = H(\mathbf{t}, t) = \gamma(t)$ for all $\mathbf{t} \in \partial I^n$, we can define a continuous mapping

$$h : \partial I^n \times I \to G, \quad h(\mathbf{t}, t) := \tilde{\kappa}\left(\gamma(t), \tilde{H}(\mathbf{t}, t)\right).$$

Since $\tilde{\kappa}\big(\gamma(0), \tilde{\gamma}(0)\big) = \mathbb{1} = \kappa\big(\tilde{\gamma}(0)\big)$, the equivariant mappings $\tilde{\kappa}$ and κ coincide on the fibre over $t = 0$. Hence,

$$h_0 = \kappa \circ (\tilde{H}_0)_{\restriction \partial I^n} = \kappa \circ \tilde{f}_{\restriction \partial I^n}.$$

Since $\tilde{\kappa}\big(\tilde{\gamma}(1)\big) = \mathbb{1} = a^{-1}\kappa\big(\tilde{\gamma}(1)\big)$, the equivariant mappings $\tilde{\kappa}$ and $\mathrm{L}_{a^{-1}} \circ \kappa$ coincide on the fibre over $t = 1$.[6] Therefore,

$$h_1 = \mathrm{L}_{a^{-1}} \circ \kappa \circ (\tilde{H}_1)_{\restriction \partial I^n}.$$

---

[5]A homotopy with this property is referred to as a homotopy along γ.

[6]Note that we could write $\tilde{\kappa}\big(\tilde{\gamma}(1)\big) = \mathbb{1} = \kappa\big(\tilde{\gamma}(1)\big)a^{-1}$ as well. This does however not mean that $\tilde{\kappa}$ coincides with $\mathrm{R}_{a^{-1}} \circ \kappa$, because the latter mapping is not equivariant.

Since, in addition, we have $h(0,t) = \tilde{\kappa}\big(\tilde{\gamma}(t)\big) = \mathbb{1}$ for every $t \in I$, it follows that $\mathrm{C}_a \circ h$ yields the desired homotopy. ∎

Example 3.2.10 The exact homotopy sequence (3.2.6) can be used to compute the homotopy groups of the quotients of free group actions. Here, we give three examples.

1. Consider the complex Hopf bundle $\mathrm{S}^3 \xrightarrow{\mathrm{S}^1} \mathrm{S}^2$, cf. Example 1.1.20. Here, (3.2.6) reads
$$\ldots \to \pi_i(\mathrm{S}^1) \to \pi_i(\mathrm{S}^3) \to \pi_i(\mathrm{S}^2) \to \pi_{i-1}(\mathrm{S}^1) \to \ldots .$$
Since $\pi_i(\mathrm{S}^1) = \pi_{i-1}(\mathrm{S}^1) = 0$ for $i > 2$, we find $\pi_i(\mathrm{S}^3) \cong \pi_i(\mathrm{S}^2)$ for all $i > 2$, where the isomorphism is induced by the projection (the Hopf mapping). This implies, in particular,
$$\pi_3(\mathrm{S}^2) = \pi_3(\mathrm{S}^3) = \mathbb{Z},$$
where the generator is given by the Hopf mapping itself, because the generator of $\pi_3(\mathrm{S}^3)$ is the identical mapping.
2. Consider the action of the cyclic group of order two on S^n generated by the antipodal mapping. The quotient manifold is the real projective space $\mathbb{R}\mathrm{P}^n$. Since $\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$ and $\pi_k(\mathbb{Z}_2) = 0$ for $k > 0$, we find $\pi_k(\mathbb{R}\mathrm{P}^n) \cong \pi_k(\mathrm{S}^n)$ for all $k > 1$ and $\pi_1(\mathbb{R}\mathrm{P}^n) = 0$ for $n > 1$. For $k = 1$ and $n = 1$, we obtain the piece
$$0 \to \mathbb{Z} \to \pi_1(\mathbb{R}\mathrm{P}^1) \to \mathbb{Z}_2 \to 0, \tag{3.2.9}$$
so that the sequence does not give sufficient information about $\pi_1(\mathbb{R}\mathrm{P}^1)$. However, we know that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1 and hence $\pi_1(\mathbb{R}\mathrm{P}^1) = \mathbb{Z}$. In fact, under this identification, the second arrow in (3.2.9) is induced from the mapping $\mathrm{S}^1 \to \mathrm{S}^1$ defined by taking the square.
3. By a similar analysis, using $\pi_i\big(\mathrm{U}(1)\big) = \pi_i(\mathrm{S}^1) = \mathbb{Z}$ for $i = 1$ and $\pi_i\big(\mathrm{U}(1)\big) = 0$ otherwise, one finds
$$\pi_i(\mathbb{C}\mathrm{P}^n) = \begin{cases} 0 & k = 0, 1, \\ \mathbb{Z} & k = 2, \\ \pi_k(\mathrm{S}^{2n+1}) & k > 2. \end{cases} \tag{3.2.10}$$
The argument for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ breaks down for $\mathbb{H}\mathrm{P}^n$, because the group acting is $\mathrm{Sp}(1)$ which is homeomorphic to S^3 and thus has nontrivial higher homotopy groups (which are not even known in full).
4. The exact homotopy sequence is also used to prove the vanishing of the lower homotopy groups of the Stiefel manifolds, see the proof of Theorem 3.4.10. ◆

Next, we discuss the path-loop fibration associated with a pointed Hausdorff space X. As before, let $t = 0$ be the base point of I. By definition, the path space of X is
$$\mathrm{P}X := C_*(I, X)$$

endowed with the compact-open topology. This space consists of the continuous curves in X starting at the base point $*$. As a base point of $\mathrm{P}X$, we take the constant curve at $*$. By assigning to every curve its endpoint, we obtain a pointed mapping

$$\pi : \mathrm{P}X \to X, \quad \pi(\gamma) := \gamma(1). \tag{3.2.11}$$

This mapping is continuous, because the preimage of an open subset $O \subset X$ is given by the open subset $M(\{1\}, O)$ of $\mathrm{P}X$.

Theorem 3.2.11 *The mapping* (3.2.11) *is a Hurewicz fibration with fibre*

$$\pi^{-1}(*) = \Omega X.$$

Therefore, the mapping (3.2.11) is referred to as the path-loop fibration of X.

Proof Let Z be a topological space and consider the lifting problem for π defined by some $f : Z \times I \to X$ and an appropriate initial condition $\tilde{f}_0 : Z \times \{0\} \to \mathrm{P}X$. By Proposition 3.1.1/3, via the relation

$$\hat{f}(z, t, s) = \big(\tilde{f}(z, t)\big)(s),$$

solutions $\tilde{f} : Z \times I \to \mathrm{P}X$ of the lifting problem correspond to continuous mappings $\hat{f} : Z \times I \times I \to X$. In terms of $\hat{f}$, the condition that $\tilde{f}$ maps $Z \times I$ to $\mathrm{P}X$ reads

$$\hat{f}(z, t, 0) = *, \tag{3.2.12}$$

the lifting condition $\pi \circ \tilde{f} = f$ reads

$$\hat{f}(z, t, 1) = f(z, t) \tag{3.2.13}$$

and the initial condition $\tilde{f}_{\restriction Z \times \{0\}} = \tilde{f}_0$ reads

$$\hat{f}(z, 0, s) = \big(\tilde{f}_0(z, 0)\big)(s). \tag{3.2.14}$$

Hence, by passing from $\tilde{f}$ to $\hat{f}$, we have turned the lifting problem into an extension problem: the desired solution $\hat{f}$ is an extension to $Z \times I \times I$ of the mapping

$$(Z \times I \times \{0, 1\}) \cup (Z \times \{0\} \times I) \to X$$

defined by (3.2.12)–(3.2.14). Since the subset $(I \times \{0, 1\}) \cup (\{0\} \times I)$ is a retract of $I \times I$ (Exercise 3.2.2), the subset $(Z \times I \times \{0, 1\}) \cup (Z \times \{0\} \times I)$ is a retract of $Z \times I \times I$. Therefore, the existence of $\hat{f}$, and hence of $\tilde{f}$, follows from the fact that if a topological space X is a retract of $A \subset X$, then every continuous mapping $f : A \to Y$ to a topological space Y has a continuous prolongation to X. ∎

Proposition 3.2.12 *The path space* $\mathrm{P}X$ *is contractible.*

Proof Consider the mapping

$$F : \mathrm{P}X \times I \to \mathrm{P}X, \quad \big(F(\gamma, t)\big)(s) := \gamma\big((1-t)s\big).$$

Using Proposition 3.1.1/3, one can check that F is continuous. Since it is a strong deformation retraction of PX to the constant curve at $*$, the assertion follows. ■

As a consequence, the homotopy groups of PX are trivial. Thus, in view of Theorems 3.2.8 and 3.2.11, Proposition 3.2.12 implies the following.

Corollary 3.2.13 *For $n \geq 1$, the boundary homomorphism $\partial : \pi_n(X) \to \pi_{n-1}(\Omega X)$ is an isomorphism.* ■

In fact, the boundary homomorphism coincides with the isomorphism provided by Theorem 3.1.5 (Exercise 3.2.3).

Now, we turn to the discussion of pullbacks of fibrations.[7] To begin with, let $\pi : Y \to X$ be a continuous mapping (not necessarily a fibration). Let Z be a topological space and let $f : Z \to X$ be a continuous mapping. Define

$$f^*Y := \{(z, y) \in Z \times Y : f(z) = \pi(y)\}$$

with the induced topology. By restriction, the natural projections to the factors of $Z \times Y$ induce continuous mappings

$$\pi_f : f^*Y \to Z, \quad F_f : f^*Y \to Y,$$

fitting into the commutative diagram

$$\begin{tikzcd} f^*Y \arrow[r, "F_f"] \arrow[d, "\pi_f"'] & Y \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & X \end{tikzcd} \tag{3.2.15}$$

The mapping π_f is referred to as the pullback of π by f. Pullbacks have the following universal property.

Proposition 3.2.14 *Let W be a topological space. For every pair of mappings $\rho : W \to Z$ and $F : W \to Y$ such that $\pi \circ F = f \circ \rho$, there exists a unique mapping $\tilde{f} : W \to f^*Y$ such that $F = F_f \circ \tilde{F}$ and $\rho = \pi_f \circ \tilde{F}$ and this mapping is continuous.*

[7]This generalizes the pullback construction for principal bundles, cf. Remark 1.1.9.

The situation can be summarized in the diagram

$$\begin{tikzcd}
W \arrow[drr, "F"] \arrow[dr, dotted] \arrow[ddr, "\rho"'] & & \\
& f^*Y \arrow[r, "F_f"'] \arrow[d, "\pi_f"] & Y \arrow[d, "\pi"] \\
& Z \arrow[r, "f"] & X
\end{tikzcd}$$

with $\tilde{F}$ being represented by the dotted arrow.

Proof Since $\pi \circ F = f \circ \rho$, the mapping

$$W \xrightarrow{\Delta} W \times W \xrightarrow{\rho \times F} Z \times Y$$

takes values in the subset $f^*Y \subset Z \times Y$. Hence, it induces a continuous mapping $\tilde{F} : W \to f^*Y$. It is immediate that $\tilde{F}$ fulfils $F_f \circ \tilde{F} = F$ and $\pi_f \circ \tilde{F} = \rho$ and that any mapping fulfilling these two relations must coincide with $\tilde{F}$. ∎

Proposition 3.2.15 *The pullback of a Serre fibration is a Serre fibration. An analogous statement holds for Hurewicz fibrations.*

Proof Since the argument does not depend on the type of fibration, we give it for Serre fibrations. Thus, assume that π is a Serre fibration and consider the homotopy lifting problem for π_f defined by a mapping $g : \mathrm{D}^n \times I \to Z$ and an appropriate initial condition $\tilde{g}_0 : \mathrm{D}^n \times \{0\} \to f^*Y$. Using (3.2.15), we check that the induced mapping $f \circ g : \mathrm{D}^n \times I \to X$ and the induced initial condition $F_f \circ \tilde{g}_0 : \mathrm{D}^n \times \{0\} \to Y$ define a homotopy lifting problem for π. Let $\tilde{h} : \mathrm{D}^n \times I \to Y$ be a solution. Then, $\pi \circ \tilde{h} = f \circ g$. Hence, application of Proposition 3.2.14 to $W = \mathrm{D}^n \times I$, $F = \tilde{h}$ and $\rho = g$ yields a unique continuous mapping $\tilde{g} : \mathrm{D}^n \times I \to f^*Y$ such that $F_f \circ \tilde{g} = \tilde{h}$ and $\pi_f \circ \tilde{g} = g$. Then,

$$F_f \circ \tilde{g}_{\restriction \mathrm{D}^n \times \{0\}} = \tilde{h}_{\restriction \mathrm{D}^n \times \{0\}} = F_f \circ \tilde{g}_0, \quad \pi_f \circ \tilde{g}_{\restriction \mathrm{D}^n \times \{0\}} = g_{\restriction \mathrm{D}^n \times \{0\}} = \pi_f \circ \tilde{g}_0$$

and hence, by uniqueness, $\tilde{g}_{\restriction \mathrm{D}^n \times \{0\}} = \tilde{g}_0$. Since, furthermore, the second equation means that $\tilde{g}$ is a lift of g through π_f, it follows that $\tilde{g}$ is a solution of the homotopy lifting problem under consideration. ∎

To conclude this section, we show how to turn an arbitrary continuous mapping into a Hurewicz fibration. Given $f : Y \to X$, define

$$E_f := \{(y, \gamma) \in Y \times C(I, X) : f(y) = \gamma(0)\} \tag{3.2.16}$$

and the mappings

$$p_f : E_f \to X,\ p_f(y,\gamma) := \gamma(1), \qquad j_f : Y \to E_f,\ j_f(y) := (y, \gamma_{f(y)}), \tag{3.2.17}$$

where $\gamma_{f(y)}$ denotes the constant path at $f(y)$. By construction,

$$f = p_f \circ j_f .$$

We endow E_f with the relative topology induced from the product topology of $Y \times C(I, X)$, where $C(I, X)$ carries the compact-open topology.

Proposition 3.2.16 *The mapping p_f is a Hurewicz fibration. The mapping j_f is a homeomorphism onto its image and the image is a strong deformation retract of E_f.*

Proof First, consider the mapping p_f. Let pr_1 and pr_2 denote the natural projections to the first and the second factor of $Y \times C(I, X)$, respectively.

By Proposition 3.1.1/2, p_f is continuous. To see that it is a fibration, consider the lifting problem given by some $g : Z \times I \to X$ and an appropriate initial condition $\tilde{g}_0 : Z \times \{0\} \to E_f$. Then, $p_f \circ \tilde{g}_0(z) = g(z, 0)$ for all $z \in Z$, meaning that the curve $\mathrm{pr}_2 \circ \tilde{g}_0(z)$ in X runs from $f \circ \mathrm{pr}_1 \circ \tilde{g}_0(z)$ to $g(z, 0)$. Hence, for every $t \in I$, we may take the concatenation with the curve

$$\gamma_{z,t} : I \to X, \quad \gamma_{z,t}(s) := g(z, st)$$

running from $g(z, 0)$ to $g(z, t)$. That the curves $\gamma_{z,t}$ are indeed continuous follows from point 3 of Proposition 3.1.1, because the mapping $I \times (Z \times I) \to X$ sending $(s, (z, t))$ to $g(z, st)$ is certainly continuous. In addition, this point yields that the mapping

$$Z \times I \to C(I, X), \quad (z, t) \mapsto \gamma_{z,t},$$

is continuous. Define

$$\tilde{g} : Z \times I \to E_f, \quad \tilde{g}(z, t) := \left(\mathrm{pr}_1 \circ \tilde{g}_0(z), \mathrm{pr}_2 \circ \tilde{g}_0(z) \cdot \gamma_{z,t}\right).$$

Clearly, $p_f \circ \tilde{g}(z, t) = \gamma_{z,t}(1) = g(z, t)$, hence $\tilde{g}$ is a lift of g. To see that $\tilde{g}$ is continuous, it remains to show that the mapping from the subset

$$\varDelta := \{(\gamma_1, \gamma_2) \in C(I, X) \times C(I, X) : \gamma_1(1) = \gamma_2(0)\} \subset C(I, X) \times C(I, X)$$

to $C(I, X)$ defined by concatenation is continuous. In view of point 3 of Proposition 3.1.1, it suffices to check that the mapping

$$\varDelta \times I \to X, \quad ((\gamma_1, \gamma_2), t) \mapsto \gamma_1 \cdot \gamma_2(t),$$

is continuous. Continuity in t for all fixed (γ_1, γ_2) is obvious. Continuity in (γ_1, γ_2) for each fixed t follows from point 2 of Proposition 3.1.1, because the image is either $\gamma_1(2t)$ or $\gamma_2(2t - 1)$.

Now, consider the mapping j_f. For every subset $A \subset Y$, one has $j_f(A) = \mathrm{pr}_1^{-1}(A) \cap j_f(Y)$. Hence, if A is open, so is $j_f(A)$ in $j_f(Y)$. This shows that j_f is a homeomorphism onto its image. Given $\gamma \in C(I, X)$ and $s \in I$, define $\gamma_s \in C(I, X)$ by $\gamma_s(t) := \gamma((1-s)t)$. Thus, $\gamma_0 = \gamma$ and γ_1 is the constant curve at $\gamma(0)$. Since the mapping $I \times (C(I, X) \times I) \to X$ sending $\big(t, (\gamma, s)\big)$ to $\gamma_s(t)$ is continuous, Proposition 3.1.1/3 implies that the mapping $C(I, X) \times I \to C(I, X)$ sending (γ, s) to γ_s is continuous. Hence, so is the mapping

$$E_f \times I \to E_f, \quad \big((y, \gamma), s\big) \mapsto (y, \gamma_s).$$

It provides a strong deformation retraction of E_f to the subset $j_f(Y)$. ■

Remark 3.2.17 One can show that the fibres $p^{-1}(x)$ over a pathwise connected component of a fibration $f : Y \to X$ are all homotopy equivalent [288, Prop. 4.61]. The homotopy type of the fibres is usually referred to as the homotopy fibre of the fibration. Proposition 3.2.16 allows to extend this notion to arbitrary mappings $f : Y \to X$ by defining the homotopy fibre of f to be the homotopy fibre of the associated fibration p_f. ♦

Exercises

3.2.1 Complete the proof of Proposition 3.2.3/1 by showing that for all $k \geq 1$, the pair $\big(\mathrm{D}^k \times I, (\mathrm{D}^k \times \{0\}) \cup (\partial \mathrm{D}^k \times I)\big)$ is homeomorphic to $(\mathrm{D}^k \times I, \mathrm{D}^k \times \{0\})$. *Hint.* Solve the case $k = 0$ first.

3.2.2 Complete the proof of Lemma 3.2.7 by showing that the subset $(I^n \times \{0, 1\}) \cup (\{0\} \times I)$ is a strong deformation retract of $I^n \times I$.

3.2.3 Prove that for every pointed Hausdorff space X, the boundary homomorphism $\partial : \pi_n(X) \to \pi_{n-1}(\Omega X)$ associated with the path-loop fibration of X coincides with the isomorphism provided by Theorem 3.1.5.

3.3 The Covering Homotopy Theorem

We are now addressing the classification problem of principal bundles. The final result will be that, for a given Lie group G and a given smooth base manifold M, the vertical isomorphism classes of smooth principal G-bundles over M are in bijective correspondence with the homotopy classes of continuous mappings from M to some topological space BG to be constructed. We will solve the classification problem for topological principal bundles under the additional assumptions that G is a Lie group with finitely many connected components and that the base space is paracompact Hausdorff and of *CW*-homotopy type, meaning that it is homotopy equivalent to a *CW*-complex. This situation is particularly simple, and it is all we need.

We will proceed in three steps. First, in the present section, we prove the Covering Homotopy Theorem. Then, in Sect. 3.4, we classify topological principal bundles.

Finally, in Sect. 3.6, we show that the vertical isomorphism classes of smooth principal G-bundles over M are in bijective correspondence with the vertical isomorphism classes of topological principal bundles over M.

Let X be a paracompact Hausdorff space and let $I = [0, 1]$. Since we want to relate bundle isomorphisms with homotopies of mappings defined on X, we need to know how topological principal bundles over $X \times I$ look like. One particular type is given by bundles of the form $Q \times I$, where Q is a topological principal G-bundle over X and where G acts trivially on I.

Theorem 3.3.1 (Topological principal bundles over $X \times I$) *Let G be a Lie group and let X be a paracompact Hausdorff space. Every topological principal G-bundle P over $X \times I$ is vertically isomorphic to $P_0 \times I$, where $P_0 = P_{\restriction X \times \{0\}}$ is viewed as a bundle over X. The isomorphism can be chosen so that its restriction to $P_0 \subset P$ coincides with the inclusion $P_0 \to P_0 \times I$ given by $p_0 \mapsto (p_0, 0)$.*

Under the assumption that X is a CW-complex, the assertion follows from Proposition 3.2.3/1 and the fact that topological fibre bundles are Serre fibrations (Exercise 3.3.1). For the proof, we need the following fact.

Lemma 3.3.2 *Under the assumptions of Theorem 3.3.1, there exists a locally finite open covering $\{U_i : i = 1, 2, \dots\}$ of X such that P is trivial over $U_i \times I$ for all i.*

Proof of the Lemma. We proceed in two steps. First, we show that every $x \in X$ possesses an open neighbourhood U such that P is trivial over $U \times I$. Second, from the open covering so obtained, we construct a locally finite and countable one.

Let $x \in X$ be given. By local triviality, for every $t \in I$, there exists an open neighbourhood V_t of x and an open interval I_t containing t such that P is trivial over $V_t \times I_t$. By compactness of I, we can find $0 < t_1 < \cdots < t_k < 1$ such that $I_{t_1}, \dots, I_{t_k}$ cover I. Denote $V_i := V_{t_i}$ and $I_i := I_{t_i}$ and define

$$U_i := \bigcap_{j=1}^{i} V_j, \quad J_i := \bigcup_{j=1}^{i} I_j.$$

Clearly, P is trivial over $U_1 \times J_1$. We will show that a trivialization χ_1 of P over $U_1 \times J_1$ and a trivialization $\tilde{\chi}_2$ of P over $V_2 \times I_2$ induce a trivialization χ_2 of P over $U_2 \times J_2$. We have

$$(U_1 \times J_1) \cap (V_2 \times I_2) = U_2 \times (J_1 \cap I_2).$$

If $J_1 \cap I_2$ is empty, we can choose $\chi_2 = \chi_1$ on $P_{\restriction U_1 \times J_1}$ and $\chi_2 = \tilde{\chi}_2$ on $P_{\restriction V_2 \times I_2}$. If $J_1 \subset I_2$ or $I_2 \subset J_1$, we can choose $\chi_2 = \tilde{\chi}_2$ or $\chi_2 = \chi_1$, respectively. Otherwise, consider the transition function $\rho : U_2 \times (J_1 \cap I_2) \to G$ defined by $\chi_1(p) = \tilde{\chi}_2(p) \cdot \rho\big(\pi(p)\big)$, where on the right hand side, $\rho\big(\pi(p)\big)$ acts by right translation on the second factor. Choose $c \in J_1 \cap I_2$ and a continuous function $f : I_2 \to J_1 \cap I_2$ such that $f(t) = t$ for all $t \leq c$ to define

$$\tilde{\rho} : U_2 \times I_2 \to G, \quad \tilde{\rho}(x,t) := \rho\big(x, f(t)\big).$$

By construction, ρ and $\tilde{\rho}$ coincide on $U_2 \times \big([0,c] \cap I_2\big)$. Hence, the mapping

$$\chi_2 : P_{\restriction U_2 \times J_2} \to (U_2 \times J_2) \times G$$

defined by

$$\chi_2(p) = \begin{cases} \chi_1(p) & \mid \pi(p) \in U_2 \times [0,c] \\ \tilde{\chi}_2(p) \cdot \tilde{\rho}\big(\pi(p)\big) & \mid \pi(p) \in U_2 \times I_2 \end{cases}$$

yields a trivialization of P over $U_2 \times J_2$. By iterating this argument, we finally obtain that P is trivial over $U \times I$, where $U = U_k$. As a result, we find an open covering $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ of X such that P is trivial over $U_\alpha \times I$ for all α.

Next, from $\mathscr{U}$, we construct an open covering which is locally finite and countable. Since X is paracompact, we may assume that $\mathscr{U}$ is locally finite. Since X is in addition Hausdorff, there exists a subordinate partition of unity $\{f_\alpha : \alpha \in A\}$, that is, $\mathrm{supp}(f_\alpha) \subset U_\alpha$ for all α. For a given finite subset $S \subset A$, define a subset U_S of X by

$$U_S := \{x \in X : \big(f_\alpha - f_{\alpha'}\big)(x) > 0 \quad \text{for all} \quad \alpha \in S, \alpha' \notin S\}.$$

The subsets U_S are open: for every $x \in U_S$, there exists an open neighbourhood V_x of x in X such that $f_{\alpha'}(x) \neq 0$ for only finitely many α'. Hence, $U_S \cap V_x$ is the subset of V_x where a given finite number of continuous functions take nonzero values. It follows that $U_S \cap V_x$ is open in V_x and hence in X. This shows that U_S is open in X.

Now, for $i = 1, 2, \ldots$, let U_i be the union of all U_S with $S \subset A$ having i elements. The family $\{U_i : i = 1, 2, \ldots\}$ covers X, because $x \in U_{i_x}$, where i_x is the number of elements α of A such that $f_\alpha \neq 0$ in some neighbourhood of x. It is locally finite, because $x \notin U_i$ for all $i > i_x$.

It remains to show that P is trivial over $U_i \times I$ for each i. On the one hand, P is trivial over $U_S \times I$ for all S, because $U_S \subset \mathrm{supp}(f_\alpha)$ and hence $U_S \subset U_\alpha$ for every $\alpha \in S$. On the other hand, the U_S with $S \subset A$ having i elements form a disjoint decomposition of U_i, because if $S \neq S'$, then $S \setminus S'$ contains an element α and $S' \setminus S$ contains an element α'. Elements x of $U_S \cap U_{S'}$ would fulfil $f_\alpha(x) > f_{\alpha'}(x)$ and $f_\alpha(x) < f_{\alpha'}(x)$, which is a contradiction. ■

Proof of Theorem 3.3.1. By Lemma 3.3.2, there exists a locally finite open covering $\mathscr{U} = \{U_i : i = 1, 2, \ldots\}$ of X such that P is trivial over $U_i \times I$ for all i. For each i, let χ_i be a trivialization of $P_{\restriction U_i \times I}$ and let $\hat{\chi}_i$ denote the induced trivialization of $(P_{0 \restriction U_i}) \times I$.

Since X is paracompact Hausdorff, there exists a closed covering $\{W_i : i = 1, 2, \ldots\}$ subordinate to $\mathscr{U}$, that is, $W_i \subset U_i$,[8] and this covering is locally finite, too. Consider the nested sequence of closed subsets covering X which is formed by the unions $\tilde{W}_i := \bigcup_{j=1}^{i} W_j$. We will construct the desired isomorphism by induction

[8]For example, one may choose $W_i = \mathrm{supp}(f_i)$ for a partition of unity subordinate to $\mathscr{U}$.

on i, that is, we will successively construct open neighbourhoods V_i of $\tilde{W}_i$ and vertical isomorphisms Φ_i over $V_i \times I$. Since $P_{\restriction U_1 \times I}$ and hence $(P_{0\restriction U_1}) \times I$ are trivial, we may put $V_1 = U_1$ and choose a vertical isomorphism

$$\Phi_1 : P_{\restriction V_1 \times I} \to (P_{0\restriction V_1}) \times I$$

so that $(\Phi_1)_{\restriction (P_{0\restriction V_1})} = \mathrm{id}_{(P_{0\restriction V_1})}$. Now, assume that we have found an open neighbourhood V_i of $\tilde{W}_i$ and a vertical isomorphism

$$\Phi_i : P_{\restriction V_i \times I} \to (P_{0\restriction V_i}) \times I$$

satisfying $(\Phi_i)_{\restriction (P_{0\restriction V_i})} = \mathrm{id}_{(P_{0\restriction V_i})}$. Via the trivializations χ_{i+1} and $\hat{\chi}_{i+1}$, Φ_i is represented over $(V_i \times I) \cap (U_{i+1} \times I) = (V_i \cap U_{i+1}) \times I$ by a continuous mapping

$$g : (V_i \cap U_{i+1}) \times I \to G$$

satisfying $g(x, 0) = \mathbb{1}_G$ for all $x \in V_i \cap U_{i+1}$. Since paracompact Hausdorff spaces are normal, there exist open subsets O_1, O_2 such that

$$\tilde{W}_i \subset O_1, \quad \overline{O_1} \subset O_2, \quad \overline{O_2} \subset V_i$$

and, by Urysohn's Lemma, a continuous function $h : V_i \cup U_{i+1} \to I$ which takes the constant value 1 on O_1 and has support in O_2. Using h, we define a mapping

$$\tilde{g} : U_{i+1} \times I \to G, \quad \tilde{g}(x,t) := \begin{cases} g\big(x, h(x)t\big) & \mid x \in O_2, \\ \mathbb{1}_G & \mid x \notin O_2. \end{cases}$$

Via the trivializations χ_{i+1} and $\hat{\chi}_{i+1}$, the mapping $\tilde{g}$ represents a vertical isomorphism

$$\tilde{\Phi} : P_{\restriction U_{i+1} \times I} \to (P_{0\restriction U_{i+1}}) \times I.$$

Let $V_{i+1} := O_1 \cup U_{i+1}$. Since on $(U_{i+1} \times I) \cap (O_1 \times I) = (U_{i+1} \cap O_1) \times I$, the mapping $\tilde{g}$ coincides with g, the isomorphisms $\tilde{\Phi}$ and $(\Phi_i)_{\restriction (P_{\restriction O_1 \times I})}$ coincide on their common domain. Hence, they combine to a vertical isomorphism

$$\Phi_{i+1} : P_{\restriction V_{i+1} \times I} \to (P_{0\restriction V_{i+1}}) \times I.$$

Finally, since $\tilde{g}(x, 0) = \mathbb{1}_G$ for all $x \in U_{i+1}$, we have $\tilde{\Phi}_{\restriction (P_{0\restriction U_{i+1}})} = \mathrm{id}_{(P_{0\restriction U_{i+1}})}$ and hence $(\Phi_{i+1})_{\restriction (P_{0\restriction V_{i+1}})} = \mathrm{id}_{(P_{0\restriction V_{i+1}})}$. This proves the theorem. ■

Remark 3.3.3 By analogy, the proof of Lemma 3.3.2 carries over to smooth principal G-bundles. ♦

There are two consequences of Theorem 3.3.1 which are important for what follows.

Corollary 3.3.4 (Covering Homotopy Theorem) *Let X and Y be paracompact Hausdorff spaces and let P and Q be topological principal G-bundles over X and Y, respectively. Let $H : X \times I \to Y$ be a continuous mapping. Every principal G-bundle morphism $P \to Q$ covering H_0 has a prolongation to a principal G-bundle morphism $P \times I \to Q$ covering H.*

Since the projection $Q \to Y$ is a Serre fibration, we already know from Proposition 3.2.3/1 that the lifting problem defined by the mapping $H \circ (\pi_P \times \mathrm{id}_I) : P \times I \to M$ and the initial condition $\tilde{H}_0$ has a solution. What the Covering Homotopy Theorem states in addition is that the solution can be chosen to consist of principal G-bundle morphisms.

Proof Let $\tilde{H}_0 : P \to Q$ be a principal G-bundle morphism covering H_0. Then, according to Remark 1.1.9/1, the mapping

$$\lambda : P \to H_0^* Q, \quad \lambda(p) := \big(\pi(p), \tilde{H}_0(p)\big),$$

is a vertical isomorphism over X. Moreover, by Theorem 3.3.1, there exists a vertical isomorphism

$$\Phi : H^* Q \to H_0^* Q \times I$$

over $X \times I$ satisfying $\Phi\big((x,0), q\big) = \big((x,q), 0\big)$. Together with the natural morphism $\mathrm{pr}_2 : H^* Q \to Q$, the isomorphisms λ and Φ combine to a morphism

$$\tilde{H} : P \times I \xrightarrow{\lambda \times \mathrm{id}_I} H_0^* Q \times I \xrightarrow{\Phi^{-1}} H^* Q \xrightarrow{\mathrm{pr}_2} Q$$

covering H. Since

$$\tilde{H}(p,0) = \mathrm{pr}_2 \circ \Phi^{-1}\Big(\big(\pi(p), \tilde{H}_0(p)\big), 0\Big) = \mathrm{pr}_2\Big(\big(\pi(p), 0\big), \tilde{H}_0(p)\Big) = \tilde{H}_0(p),$$

$\tilde{H}$ is a prolongation of $\tilde{H}_0$. ■

The other consequence of Theorem 3.3.1 leads, in effect, to the idea of classifying principal bundles in terms of homotopy classes of mappings.

Corollary 3.3.5 (Homotopy implies isomorphism) *Let G be a Lie group and let Q be a topological principal G-bundle over a topological space B. Let X be a paracompact Hausdorff space and let $f, g : X \to B$ be continuous mappings. If f and g are homotopic, then the topological principal G-bundles $f^* Q$ and $g^* Q$ over K are vertically isomorphic.*

Proof Let $H : X \times I \to B$ be a homotopy from f to g, that is, $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. Consider the topological principal G-bundle $P := H^* Q$ over $X \times I$. Let $P_t := P_{\restriction X \times \{t\}}$, viewed as a bundle over X. Clearly, $P_0 = f^* Q$ and $P_1 = g^* Q$. By Theorem 3.3.1, P is vertically isomorphic to $P_0 \times I$. By restricting an isomorphism to the subbundle $P_1 \subset P$, we obtain a vertical isomorphism from P_1, viewed as a bundle over X, to P_0. ■

Exercises

3.3.1 Use Proposition 3.2.3/1 and the fact that topological fibre bundles are Serre fibrations to prove Theorem 3.3.1 under the assumption that the base space is a *CW*-complex.

3.4 Universal Principal Bundles

In this section, we classify topological principal bundles over paracompact Hausdorff spaces of *CW*-homotopy type up to vertical isomorphisms.

For a Lie group G and a topological space X, let $\mathrm{PFB}(G, X)$ denote the totality of vertical isomorphism classes of topological principal G-bundles over X. As a consequence of Corollary 3.3.5, given a topological principal G-bundle Q over B, for every paracompact Hausdorff space X, the assignment of the pullback bundle f^*Q to a continuous mapping $f : X \to B$ induces a mapping

$$[X, B] \to \mathrm{PFB}(G, X). \tag{3.4.1}$$

Definition 3.4.1 (*Topological universal bundle*) Let G be a Lie group and let E be a pathwise connected topological principal G-bundle over a paracompact Hausdorff space B of *CW*-homotopy type.

1. E is called a universal bundle for G and B is called a classifying space for G if the mapping (3.4.1) is a bijection for all paracompact Hausdorff spaces X of *CW*-homotopy type.
2. For $n = 1, 2, \ldots,$ E is called an n-universal bundle for G and B is called an n-classifying space for G if the mapping (3.4.1) is a bijection for all paracompact Hausdorff spaces X of the homotopy type of a *CW*-complex of dimension n or less.

In either case, given a principal G-bundle P over a space X, any mapping $f : X \to B$ such that $P \cong f^*E$ is said to be a classifying mapping for P.

Clearly, a topological principal G-bundle is universal iff it is n-universal for all n.

In what follows, we first discuss uniqueness and then existence of universal bundles. Uniqueness is a direct consequence of the fact that, by our definition, the base space of a universal bundle is paracompact Hausdorff of *CW*-homotopy type.

Definition 3.4.2 Two topological principal G-bundles P_1 over X_1 and P_2 over X_2 are said to be G-homotopy equivalent if there exist G-morphisms $F_1 : P_1 \to P_2$ and $F_2 : P_2 \to P_1$ such that $F_2 \circ F_1$ and $F_1 \circ F_2$ are homotopic through G-morphisms to vertical automorphisms of P_1 and P_2, respectively.

Clearly, G-homotopy equivalent principal G-bundles have homotopy equivalent base spaces.

Proposition 3.4.3 *Any two classifying spaces for G are homotopy equivalent. Any two universal bundles for G are G-homotopy equivalent.*

It is, therefore, common to speak of *the* universal bundle and *the* classifying space for G, and to write $\mathrm{E}G$ and $\mathrm{B}G$ for (representatives[9] of) the corresponding equivalence classes.

Proof Let E_i be universal G-bundles over B_i, $i = 1, 2$. Since E_2 is universal, and since B_1 is paracompact Hausdorff of CW-homotopy type, E_1 is vertically isomorphic to $f_1^* E_2$ for an appropriate classifying mapping $f_1 : B_1 \to B_2$. By analogy, E_2 is vertically isomorphic to $f_2^* E_1$ for an appropriate classifying mapping $f_2 : B_2 \to B_1$. Then, E_1 is vertically isomorphic to $f_1^*(f_2^* E_1)$ and hence to $(f_2 \circ f_1)^* E_1$. Since E_1 is universal, and since $E_1 = \mathrm{id}_{B_1}^* E_1$, it follows that $f_2 \circ f_1$ is homotopic to id_{B_1}. An analogous argument shows that $f_1 \circ f_2$ is homotopic to id_{B_2}. Hence, f_1 and f_2 provide a homotopy equivalence between B_1 and B_2.

Now, consider the total spaces E_1 and E_2. The natural G-morphism $f_1^* E_2 \to E_2$ combines with a vertical isomorphism $E_1 \to f_1^* E_2$ to a G-morphism $F_1 : E_1 \to E_2$ covering f_1. Analogously, we obtain a G-morphism $F_2 : E_2 \to E_1$ covering f_2. Then, $F_2 \circ F_1$ is a G-automorphism of E_1 covering $f_2 \circ f_1$. Since $f_2 \circ f_1$ is homotopic to id_{B_1}, by the Covering Homotopy Theorem 3.3.4, there exists a homotopy through G-morphisms from $F_2 \circ F_1$ to some G-morphism of E_1 covering id_{B_1}, that is, to some vertical automorphism of E_1. An analogous argument shows that $F_1 \circ F_2$ is homotopic through G-morphisms to a vertical automorphism of E_2. Thus, F_1 and F_2 provide a G-homotopy equivalence between E_1 and E_2. ■

Now, we are going to discuss the existence of universal bundles. Before entering the actual construction, we derive a criterion for universality in terms of the homotopy groups of the total space E. We start with a criterion for the extendability of sections[10] in topological fibre bundles over CW-complexes.

Lemma 3.4.4 (Prolongation of sections) *Let K be a CW-complex and let $L \subset K$ be a subcomplex. Let E be a topological fibre bundle over K with typical fibre V. If $\pi_i(V) = 0$ for all $i < \dim K$, then every section of $E_{\restriction L}$ can be extended to a section of E.*

Proof We give the argument for an infinite dimensional CW-complex. The finite dimensional case is then obvious.

By possibly refining the CW-complex structure, we may assume that E is trivial over every cell of K. For $i = 0, 1, 2, \ldots$, let $K^{(i)}$ denote the i-skeleton of K. We will prove the assertion by induction on i. Since $K^{(0)}$ is discrete, the given section of E over L extends to a section s_0 over $L \cup K^{(0)}$. Thus, for $i > 0$, assume that

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[9] A particular representative is called a model of the classifying space for G.

[10] Recall that sections in topological fibre bundles are assumed to be continuous.

s_i is a section of E over $L \cup K^{(i)}$ and let $\alpha : \mathrm{D}^{i+1} \to K$ be an $(i+1)$-cell. Since $\alpha(\partial\mathrm{D}^{i+1}) \subset K^{(i)}$, s_i induces a section $\tilde{s}_i$ of $(\alpha^*E)_{\restriction \partial\mathrm{D}^{i+1}}$. We have to show that $\tilde{s}_i$ extends to a section of α^*E. By the assumption that E is trivial over every cell of K, α^*E is a trivial fibre bundle over D^{i+1} with typical fibre V. Hence, sections correspond to mappings $\mathrm{D}^{i+1} \to V$ and we have to show that every continuous mapping $f_i : \partial\mathrm{D}^{i+1} \to V$ extends to a continuous mapping $f_{i+1} : \mathrm{D}^{i+1} \to V$. Since $\pi_i(V) = 0$, f_i is homotopic to a constant mapping via $H : \partial\mathrm{D}^{i+1} \times I \to V$. Since H_1 maps $\partial\mathrm{D}^{i+1}$ to the base point of V, H induces a continuous mapping $\hat{H}$ from the cone over $\partial\mathrm{D}^{i+1}$ to V. Since the cone over $\partial\mathrm{D}^{i+1}$ is homeomorphic to D^{i+1}, we obtain an extension f_{i+1}.

As a result, we obtain a family of sections $\{s_i : i = 0, 1, 2, \dots\}$ over the skeleta. By Proposition 3.1.12, this family defines a continuous mapping $s : K \to E$. By construction, s is a section. ■

By applying Lemma 3.4.4 to the case where L consists of a single point, we obtain the following.

Corollary 3.4.5 (Existence of sections) *Let E be a topological fibre bundle over a CW-complex K with typical fibre V. If $\pi_i(V) = 0$ for all $i < \dim K$, then E admits a section.* ■

Now, we can formulate the criterion for universality announced above.

Theorem 3.4.6 (Universality criterion) *Let G be a Lie group and let E be a topological principal G-bundle over a paracompact Hausdorff space B of CW-homotopy type. If $\pi_i(E) = 0$ for all $i \leq n$, then E is n-universal for G. If $\pi_i(E) = 0$ for all i,*[11] *then E is universal.*

Proof Clearly, it suffices to prove n-universality. Let K be a CW-complex of dimension $\dim(K) \leq n$. First, we show that the mapping (3.4.1) is bijective for $X = K$.

To check surjectivity, let P be a topological principal G-bundle over K. It suffices to find a G-morphism $P \to E$, because the pullback of E by the projection of such a morphism is vertically isomorphic to P. For that purpose, recall that the G-morphisms $P \to E$ correspond to the sections in the associated fibre bundle $P \times_G E$. Since this bundle has typical fibre E and since $\pi_i(E) = 0$ for all $i < n$, the assertion follows from Corollary 3.4.5.

To check injectivity, let there be given continuous mappings $f_0, f_1 : K \to B$ and assume that there exists a vertical isomorphism $\Phi : f_0^*E \to f_1^*E$. Let $F_i : f_i^*E \to E$ be the natural G-morphisms. Then, $F_1 \circ \Phi : f_0^*E \to E$ is a G-morphism covering f_1. It suffices to find a G-morphism $H : f_0^*E \times I \to E$ such that

$$H(\cdot, 0) = F_0, \quad H(\cdot, 1) = F_1 \circ \Phi,$$

because the projection $h : K \times I \to B$ of H then satisfies

[11] That is, if E is weakly contractible.

$$h(\cdot, 0) = f_0, \quad h(\cdot, 1) = f_1$$

and thus yields a homotopy from f_0 to f_1. To find H, it suffices to find a section in the associated fibre bundle $(f_0^* E \times I) \times_G E$ whose restrictions to $K \times \{0\}$ and $K \times \{1\}$ correspond to the morphisms F_0 and $F_1 \circ \Phi$, respectively. Since this bundle has typical fibre E, and since $\pi_i(E) = 0$ for all $i < \dim(K \times I) \leq n+1$, the existence of such a section follows from the Prolongation Lemma 3.4.4. This proves injectivity.

Now, let X be a paracompact Hausdorff space which is homotopy equivalent to K. Let $h : X \to K$ and $k : K \to X$ be a homotopy equivalence.

To see that the mapping (3.4.1) is surjective, let P be a topological principal G-bundle over X. Then, $k^* P$ is a topological principal G-bundle over K and hence vertically isomorphic to $f^* E$ for some continuous mapping $f : K \to B$. It follows that we have the vertical isomorphisms

$$(f \circ h)^* E \to h^*(f^* E) \to h^*(k^* P) \to (k \circ h)^* P \to P,$$

where the last one is due to Corollary 3.3.5.

To see that the mapping (3.4.1) is injective, let there be given continuous mappings $f_1, f_2 : X \to B$ and assume that $f_1^* E$ be vertically isomorphic to $f_2^* E$. Then, the topological principal G-bundles $k^*(f_1^* E)$ and $k^*(f_2^* E)$ over K are vertically isomorphic. Hence, $f_1 \circ k$ is homotopic to $f_2 \circ k$ and thus $f_1 \circ k \circ h$ is homotopic to $f_2 \circ k \circ h$. Since $k \circ h$ is homotopic to id_X, then f_1 is homotopic to f_2. ■

In addition to Theorem 3.4.6, we will also need a criterion which applies to universal bundles for closed subgroups of G. Let E be a topological principal G-bundle over B and let $H \subset G$ be a closed subgroup. Recall that the action of G on E reduces to an action of H and that the latter makes E into a principal H-bundle over the topological quotient E/H.

Lemma 3.4.7 *Let G be a compact Lie group and let E be a topological principal G-bundle over a paracompact Hausdorff space B of CW-homotopy type. For every closed subgroup $H \subset G$, the quotient space E/H is paracompact Hausdorff of CW-homotopy type.*

Proof We use that the induced projection $E/H \to B$ is a topological fibre bundle with typical fibre being the homogeneous space G/H. Since B and G/H are Hausdorff, so is E/H (Exercise 3.4.1). Since B is paracompact, by Proposition 3.2.6, the induced projection is a Hurewicz fibration. Since, in addition, B is pathwise connected and both G/H and B are of CW-homotopy type, Theorem 5.4.2 in [221] yields that E/H is of CW-homotopy type.

To see that E/H is paracompact, let $\mathscr{U} = \{U_i : i \in I\}$ be an open covering of E/H. We have to find a locally finite open refinement of $\mathscr{U}$. Since B is paracompact, π admits a system of local trivializations $\{(V_\alpha, \chi_\alpha) : \alpha \in A\}$ such that the open covering $\mathscr{V} = \{V_\alpha : \alpha \in A\}$ is locally finite. For each α, we find an open subset $W_\alpha \subset V_\alpha$ such that the closure $\overline{W_\alpha} \subset V_\alpha$ and $\{W_\alpha : \alpha \in A\}$ is an open covering of B; for example one may put $W_\alpha = \{x \in B : f_\alpha(x) \neq 0\}$ for a partition of unity $\{f_\alpha : \alpha \in A\}$ subordinate to $\mathscr{V}$. By intersecting the members of $\mathscr{U}$ with $\pi^{-1}(\overline{W_\alpha})$, we obtain an open covering $\mathscr{U}_\alpha$ of $\pi^{-1}(\overline{W_\alpha})$. Since this space is homeomorphic to the direct product of the paracompact space $\overline{W_\alpha}$ with the compact space G/H, it is paracompact.[12] Hence, $\mathscr{U}_\alpha$ admits a locally finite refinement. By intersecting the members of $\mathscr{U}_\alpha$ with $\pi^{-1}(W_\alpha)$, we obtain a locally finite family of open subsets of E/H covering $\pi^{-1}(W_\alpha)$. By taking the union of these families over all α, we then obtain an open refinement of $\mathscr{U}$. It is locally finite, because so is $\mathscr{V}$ and hence the family $\{\pi^{-1}(W_\alpha) : \alpha \in A\}$. ■

In view of Lemma 3.4.7, Theorem 3.4.6 implies the following.

Corollary 3.4.8 *Let G be a compact Lie group and let $H \subset G$ be a closed subgroup. Let E be a topological principal G-bundle over a paracompact Hausdorff space B of CW-homotopy type. If $\pi_i(E) = 0$ for all $i \leq n$, the induced principal H-bundle $E \to E/H$ is n-universal for H. If $\pi_i(E) = 0$ for all i, this bundle is universal for H.* ■

Remark 3.4.9 If $\pi_i(E) = 0$ for all $i \leq n$, the exact homotopy sequence (3.2.6) implies

$$\pi_1(\mathrm{B}G) = G/G_0, \quad \pi_i(\mathrm{B}G) = \pi_{i-1}(G) \text{ for } 2 \leq i \leq n, \tag{3.4.2}$$

where G_0 denotes the identity component of G. ♦

Now, we are going to prove that universal bundles exist for all Lie groups with a finite number of connected components. We start with discussing the classical compact Lie groups $\mathrm{O}(k)$, $\mathrm{U}(k)$ and $\mathrm{Sp}(k)$.

Let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $k < l$ be positive integers. Consider the Stiefel bundle

$$\mathrm{S}_{\mathbb{K}}(k, l) \to \mathrm{G}_{\mathbb{K}}(k, l),$$

where $\mathrm{S}_{\mathbb{K}}(k, l)$ denotes the Stiefel manifold of k-frames in $\mathbb{K}^l$ and $\mathrm{G}_{\mathbb{K}}(k, l)$ denotes the Graßmann manifold of k-dimensional subspaces of $\mathbb{K}^l$ and the projection assigns to a frame the subspace spanned by that frame. According to Example 1.1.24, the Stiefel bundle is a smooth principal bundle with structure group $\mathrm{O}(k)$ in case $\mathbb{K} = \mathbb{R}$, $\mathrm{U}(k)$ in case $\mathbb{K} = \mathbb{C}$ and $\mathrm{Sp}(k)$ in case $\mathbb{K} = \mathbb{H}$.

Theorem 3.4.10 *The Stiefel bundle $\mathrm{S}_{\mathbb{K}}(k, l) \to \mathrm{G}_{\mathbb{K}}(k, l)$ fulfils $\pi_i\big(\mathrm{S}_{\mathbb{K}}(k, l)\big) = 0$ for all $i \leq n$ and is thus n-universal*

1. *for $\mathrm{O}(k)$ in case $\mathbb{K} = \mathbb{R}$ and $l \geq n + 1 + k$,*
2. *for $\mathrm{U}(k)$ in case $\mathbb{K} = \mathbb{C}$ and $l \geq n/2 + k$,*
3. *for $\mathrm{Sp}(k)$ in case $\mathbb{K} = \mathbb{H}$ and $l \geq n/4 - 1/2 + k$.*

Proof Since the base spaces are manifolds, they are paracompact Hausdorff of *CW*-homotopy type.[13] Hence, according to Theorem 3.4.6, it suffices to check the homotopy groups of the Stiefel manifolds. We claim that

[12] This argument is the reason why G is assumed to be compact.

[13] In fact, $\mathrm{G}_{\mathbb{K}}(k, l)$ admits a canonical *CW*-complex structure, see Sect. 6 in [451].

$$\pi_i(S_{\mathbb{K}}(k,l)) = 0 \quad \text{for all} \quad i \leq d(l-k) + d - 2, \tag{3.4.3}$$

where d denotes the dimension of $\mathbb{K}$ over $\mathbb{R}$. From this, one obtains points 1–3 by plugging in n for i.

Consider the case $\mathbb{K} = \mathbb{R}$. The exact sequence of homotopy groups (3.2.6) for the principal $\mathrm{O}(l-k)$-bundle

$$\mathrm{O}(l) \to \mathrm{O}(l)/\mathrm{O}(l-k) \cong \mathrm{S}_{\mathbb{R}}(k,l)$$

contains the pieces

$$\pi_i(\mathrm{O}(l-k)) \xrightarrow{\iota_*} \pi_i(\mathrm{O}(l)) \to \pi_i(\mathrm{S}_{\mathbb{R}}(k,l)) \xrightarrow{\partial} \pi_{i-1}(\mathrm{O}(l-k)) \xrightarrow{\iota_*} \pi_{i-1}(\mathrm{O}(l)), \tag{3.4.4}$$

where

$$\iota : \mathrm{O}(l-k) \to \mathrm{O}(l), \quad \iota(a) = \begin{bmatrix} \mathbb{1}_k & 0 \\ 0 & a \end{bmatrix}.$$

We decompose ι into the sequence of embeddings

$$\mathrm{O}(l-k) \xrightarrow{\iota_1} \mathrm{O}(l-k+1) \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_k} \mathrm{O}(l).$$

Since ι_1 makes $\mathrm{O}(l-k+1)$ into a principal bundle over S^{l-k} with structure group $\mathrm{O}(l-k)$, from (3.2.6) we obtain exact sequences

$$\pi_{i+1}(\mathrm{S}^{l-k}) \longrightarrow \pi_i(\mathrm{O}(l-k)) \xrightarrow{\iota_{1*}} \pi_i(\mathrm{O}(l-k+1)) \longrightarrow \pi_i(\mathrm{S}^{l-k}),$$

showing that ι_{1*} is an isomorphism for $i < l-k-1$ and surjective for $i = l-k-1$. By replacing k by $k-1, \ldots, 1$ in this argument, we obtain that the homomorphisms of the i-th homotopy groups induced by, respectively, $\iota_2, \ldots, \iota_k$ are isomorphisms for all $i \leq l-k-1$. Consequently,

$$\iota_* = \iota_{1*} \circ \cdots \circ \iota_{k*} : \pi_i\big(\mathrm{O}(l-k)\big) \to \pi_i\big(\mathrm{O}(l)\big)$$

is an isomorphism for all $i < l-k-1$ and surjective for $i = l-k-1$. Now, exactness of (3.4.4) implies that for $i \leq l-k-1$, we have $\pi_i\big(\mathrm{S}_{\mathbb{R}}(k,l)\big) = 0$. This proves (3.4.3) for $\mathbb{K} = \mathbb{R}$. The arguments for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$ are similar (Exercise 3.4.2). ∎

For the Lie groups $\mathrm{O}(1) \cong \mathbb{Z}_2$, $\mathrm{U}(1)$ and $\mathrm{Sp}(1)$, Theorem 3.4.10 yields, respectively, the n-universal bundles

$$\begin{aligned} \mathrm{S}^l &\to \mathbb{R}\mathrm{P}^l, \quad l \geq n+2, \\ \mathrm{S}^{2l+1} &\to \mathbb{C}\mathrm{P}^l, \quad l \geq n/2, \\ \mathrm{S}^{4l+3} &\to \mathbb{H}\mathrm{P}^l \quad l \geq n/4 - 1/2. \end{aligned}$$

As a consequence of Corollary 3.4.8, by embedding the cyclic group $\mathbb{Z}_r$ as

$$\mathbb{Z}_r \to \mathrm{U}(1), \quad s \mapsto \mathrm{e}^{2\pi \mathrm{i} s/r}, \tag{3.4.5}$$

from the Stiefel bundle $\mathrm{S}^{2l+1} \to \mathbb{C}\mathrm{P}^l$, we obtain the n-universal bundle

$$\mathrm{S}^{2l+1} \to \mathrm{L}_r^{2l+1} \equiv \mathrm{S}^{2l+1}/\mathbb{Z}_r, \quad l \geq n/2,$$

for $\mathbb{Z}_r$. The quotient manifold L_r^{2l+1} is referred to as a lens space. It has the structure of a smooth principal bundle over $\mathbb{C}\mathrm{P}^l$ with structure group $\mathrm{U}(1)/\mathbb{Z}_r \cong \mathrm{U}(1)$.

As another consequence of Corollary 3.4.8, for the subgroups $\mathrm{SO}(k) \subset \mathrm{O}(k)$ and $\mathrm{SU}(k) \subset \mathrm{U}(k)$, Theorem 3.4.10 yields, respectively, the n-universal bundles

$$\begin{aligned} \mathrm{S}_{\mathbb{R}}(k,l) &\to \tilde{\mathrm{G}}_{\mathbb{R}}(k,l) \equiv \mathrm{S}_{\mathbb{R}}(k,l)/\mathrm{SO}(k), \quad l \geq n+k+1, \\ \mathrm{S}_{\mathbb{C}}(k,l) &\to \tilde{\mathrm{G}}_{\mathbb{C}}(k,l) \equiv \mathrm{S}_{\mathbb{C}}(k,l)/\mathrm{SU}(k), \quad l \geq n/2+k. \end{aligned}$$

The quotient manifold $\tilde{\mathrm{G}}_{\mathbb{R}}(k,l)$ has the structure of a smooth principal bundle over $\mathrm{G}_{\mathbb{R}}(k,l)$ with structure group $\mathrm{O}(k)/\mathrm{SO}(k) \cong \mathrm{O}(1)$. Accordingly, the quotient manifold $\tilde{\mathrm{G}}_{\mathbb{C}}(k,l)$ has the structure of a smooth principal bundle over $\mathrm{G}_{\mathbb{C}}(k,l)$ with structure group $\mathrm{U}(k)/\mathrm{SU}(k) \cong \mathrm{U}(1)$.

Remark 3.4.11

1. The principal bundle structure in the classifying spaces of $\mathbb{Z}_r$, $\mathrm{SO}(k)$ and $\mathrm{SU}(k)$ observed here generalizes to arbitrary closed normal subgroups, see Proposition 3.7.5 below.
2. In the situation of a closed subgroup H of $\mathrm{O}(k)$, it is actually not necessary to use Lemma 3.4.7 to prove Corollary 3.4.8, because the action of $\mathrm{O}(k)$ on $\mathrm{S}_{\mathbb{R}}(k,l)$ restricts to a smooth free proper action of H on $\mathrm{S}_{\mathbb{R}}(k,l)$ and Corollary 6.5.1 in Part I implies that the topological quotient $\mathrm{S}_{\mathbb{R}}(k,l)/H$ is a smooth manifold. Hence, it is automatically paracompact Hausdorff of CW-homotopy type. A similar statement holds true for closed subgroups of $\mathrm{U}(k)$ and $\mathrm{Sp}(k)$. ♦

The existence of n-universal bundles for the orthogonal groups entails the existence of n-universal bundles for all Lie groups with a finite number of connected components by the following argument.

First, by a theorem due to Iwasawa [341] and Malcev [420], there exists a maximal compact subgroup $K \subset G$ and a submanifold $N \subset G$, diffeomorphic to a real vector space, such that the mapping

$$\mu : K \times N \to G, \quad \mu(k,n) := kn, \tag{3.4.6}$$

is a diffeomorphism.[14] This generalizes the polar decomposition of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, cf. Exercise 5.1.9 in Part I. In the case where G is simply connected, the theorem is due to É. Cartan [122].

Second, being compact, K admits a finite-dimensional faithful real representation, see Remark 3.4.14 below. By Proposition 5.5.6 in Part I, this representation admits an invariant scalar product. Thus, by choosing an orthonormal basis, we obtain a Lie subgroup embedding $K \to \mathrm{O}(k)$ for some k. Then, the action of $\mathrm{O}(k)$ on $\mathrm{S}_{\mathbb{R}}(k, l)$ restricts to a smooth free proper action of K, thus turning $\mathrm{S}_{\mathbb{R}}(k, l)$ into a smooth principal K-bundle over $\mathrm{S}_{\mathbb{R}}(k, l)/K$. By extending the structure group from K to G, we finally obtain the smooth principal G-bundle

$$\mathrm{S}_{\mathbb{R}}(k, l) \times_K G \to \mathrm{S}_{\mathbb{R}}(k, l)/K. \tag{3.4.7}$$

Corollary 3.4.12 *Let G be a Lie group with finitely many connected components, let K be a maximal compact subgroup of G and let $K \subset \mathrm{O}(k)$ via a faithful orthogonal representation. For $l \geq n + k + 1$, the topological principal G-bundle underlying (3.4.7) fulfils $\pi_i(\mathrm{S}_{\mathbb{R}}(k, l) \times_K G) = 0$ for all $i \leq n$ and is thus n-universal for G.*

In particular, n-universal bundles exist for all Lie groups with a finite number of connected components and all n.

Proof Denote $E := \mathrm{S}_{\mathbb{R}}(k, l)$. Since by Theorem 3.4.10, the assertion holds true for E, it suffices to show that $E \times_K G$ is a deformation retract of E. In the following argument, details are left to the reader (Exercise 3.4.3).

Let $I = [0, 1]$ and let $\mathrm{pr}_K : G \to K$ and $\mathrm{pr}_N : G \to N$ denote the mappings obtained by composing the inverse of the diffeomorphism (3.4.6) with the natural projections in the direct product $K \times N$. Since N is diffeomorphic to a vector space, there exists a strong deformation retraction $\varphi_N : N \times I \to N$ of N onto the one-point subset $\{\mathbb{1}\}$. Then,

$$\varphi_G : G \times I \to G, \quad \varphi_G(a, t) := \mu\left(\mathrm{pr}_K(a), \varphi_N\left(\mathrm{pr}_N(a), t\right)\right),$$

is a strong deformation retraction of G onto K. For all $a \in G$, $k \in K$ and $t \in I$, we have $\mathrm{pr}_K(ka) = k\,\mathrm{pr}_K(a)$ and $\mathrm{pr}_N(ka) = \mathrm{pr}_N(a)$, and hence $\varphi_G(ka, t) = k\varphi_G(a, t)$. Therefore, φ_G induces a mapping

$$\varphi : (E \times_K G) \times I \to E \times_K G, \quad \varphi\left([(e, a)], t\right) := [(e, \varphi_G(a, t))].$$

It is not hard to see that φ is a strong deformation retraction of $E \times_K G$ onto the subset $E \times_K K = E$. This proves the corollary. ∎

Example 3.4.13 (*Principal bundles over spheres*) Consider the case where $M = \mathrm{S}^n$ and G is a connected Lie group. Let $E \to B$ be an n-universal bundle for G. Since E is pathwise connected, so is B. Since G is connected, (3.4.2) yields $\pi_1(B) = 0$.

[14] For a detailed proof, see Sect. 3.7 of [302] or Theorem XV.3.1 of [310].

Since for a pathwise connected topological space X, the mapping $\pi_n(X) \to [\mathrm{S}^n, X]$ induced by the natural inclusion mapping $C_*(\mathrm{S}^n, X) \to C(\mathrm{S}^n, X)$ descends to a bijection from $\pi_n(X)/\pi_1(X)$ onto $[\mathrm{S}^n, X]$, this implies $[\mathrm{S}^n, B] = \pi_n(B)$.[15] On the other hand, according to (3.4.2), we have $\pi_n(B) \cong \pi_{n-1}(G)$. It follows that the vertical isomorphism classes of principal G-bundles over S^n are in bijective correspondence with the elements of $\pi_{n-1}(G)$. This is consistent with the Čech cohomological description of these bundles in terms of transition mappings: we can cover S^n by two contractible open subsets whose intersection can be retracted to the equator S^{n-1}. Hence, according to Proposition 1.1.10 and Theorem 1.1.11, since G is connected, vertical isomorphism classes of topological principal G-bundles over S^n are in bijective correspondence with homotopy classes of continuous mappings from $\mathrm{S}^{n-1} \to G$.

For example, in case $G = \mathrm{U}(1)$, we obtain that nontrivial $\mathrm{U}(1)$-bundles over S^n exist for $n = 2$ only and that, in this case, they are classified by an integer. For a detailed discussion of principal bundles over spheres, we refer to [599]. ♦

Remark 3.4.14 The fact that every compact Lie group G admits a finite-dimensional faithful representation is a consequence of a central result in the theory of compact Lie groups, the Peter-Weyl Theorem. This theorem states that the representative functions form a dense subset of the Hilbert space $L^2(G, \mathsf{v})$ of real or complex valued functions on G which are square integrable with respect to a bi-invariant volume form[16] v, see for example [105, Theorem III.3.1]. Recall that a representative function is a linear combination of functions of the form

$$G \to \mathbb{K}, \quad a \mapsto \langle \eta, \rho(a)v \rangle \tag{3.4.8}$$

where $v \in V$ and $\eta \in V^*$ for some $\mathbb{C}$-vector space V carrying a finite-dimensional irreducible representation $\rho : G \to \mathrm{Aut}(V)$. To conclude from this that G admits a finite-dimensional faithful representation, let $a_1 \in G$ such that $a_1 \neq \mathbb{1}$. Since the elements of $L^2(G, \mathsf{v})$ separate the points of G and since functions of the form (3.4.8) are dense in $L^2(G, \mathsf{v})$, there exists a function of this form satisfying $f(a_1) \neq f(\mathbb{1})$. This means that there exists a finite-dimensional irreducible $\mathbb{K}$-representation $\rho_1 : G \to \mathrm{Aut}(V_1)$ such that $a_1 \notin \ker(\rho_1)$ and hence $K_1 := \ker(\rho_1)$ is properly contained in G. If ρ_1 is faithful, we are done. Otherwise, there exists $a_2 \in K_1$ such that $a_2 \neq \mathbb{1}$. By the same argument as above, we can find a function f of the form (3.4.8) such that $f(a_2) \neq f(\mathbb{1})$, and hence a finite-dimensional irreducible $\mathbb{K}$-representation $\rho_2 : G \to \mathrm{Aut}(V_2)$ such that $a_2 \notin \ker(\rho_2)$. Then, $K_2 := K_1 \cap \ker(\rho_2)$ is properly contained in K_1. Iterating this argument, we obtain a sequence $K_1, K_2, \ldots$ of closed subgroups of G with K_{i+1} being properly contained in K_i. Since G is compact, so are the K_i. By the Theorem on Invariance of Domain,[17] if K_i and K_{i+1} have the same dimension, then K_{i+1} is open in K_i and hence is a union of connected components of

[15] The quotient $\pi_n(X)/\pi_1(X)$ is the set of orbits of the natural action of $\pi_1(X)$ on $\pi_n(X)$. The latter was explained prior to Proposition 3.2.9.

[16] A Haar measure, cf. Sect. 5.5 in Part I.

[17] See the footnote on page 159 in Part I.

K_i. Thus, each K_{i+1} must have smaller dimension or fewer connected components than K_i. Since, by compactness, the number of connected components is finite, the sequence must be finite, and hence must end with the subgroup $K_r = \{\mathbb{1}\}$. Then, the representation

$$\rho_1 \times \cdots \times \rho_r : G \to \operatorname{Aut}(V_1 \oplus \cdots \oplus V_r)$$

has kernel $\ker(\rho_1) \cap \cdots \cap \ker(\rho_r) = K_1 \cap \cdots \cap K_r = \{\mathbb{1}\}$ and is thus faithful. ♦

From the Stiefel bundles $\mathrm{S}_{\mathbb{K}}(k, l) \to \mathrm{G}_{\mathbb{K}}(k, l)$ we can construct universal bundles by taking the direct limits $l \to \infty$. To be definite, let us explain the construction for the case $\mathbb{K} = \mathbb{R}$.

Let $\mathbb{R}^\infty$ be the direct sum of countably many copies of $\mathbb{R}$. Recall that $\mathbb{R}^\infty$ is a real vector space whose elements are infinite sequences $(x_1, x_2, \dots)$ with $x_i \neq 0$ for only finitely many i. It carries an obvious scalar product. Let $\mathrm{S}_{\mathbb{R}}(k, \infty)$ denote the set of orthonormal k-frames in $\mathbb{R}^\infty$ and let $\mathrm{G}_{\mathbb{R}}(k, \infty)$ denote the set of k-dimensional subspaces of $\mathbb{R}^\infty$. $\mathrm{G}_{\mathbb{R}}(k, \infty)$ is known as the infinite Graßmannian. Every element of $\mathbb{R}^l$ can be made into an element of $\mathbb{R}^\infty$ by appending zero entries. This way, we may identify $\mathbb{R}^l$ with a subset of $\mathbb{R}^\infty$, $\mathrm{S}_{\mathbb{R}}(k, l)$ with a subset of $\mathrm{S}_{\mathbb{R}}(k, \infty)$ and $\mathrm{G}_{\mathbb{R}}(k, l)$ with a subset of $\mathrm{G}_{\mathbb{R}}(k, \infty)$. By construction, then $\mathbb{R}^l$ is a subset of $\mathbb{R}^{l+1}$, $\mathrm{S}_{\mathbb{R}}(k, l)$ is a subset of $\mathrm{S}_{\mathbb{R}}(k, l+1)$ and $\mathrm{G}_{\mathbb{R}}(k, l)$ is a subset of $\mathrm{G}_{\mathbb{R}}(k, l+1)$ for every l. We topologize $\mathrm{S}_{\mathbb{R}}(k, \infty)$ and $\mathrm{G}_{\mathbb{R}}(k, \infty)$ by the final topologies defined by the natural inclusion mappings $\mathrm{S}_{\mathbb{R}}(k, l) \to \mathrm{S}_{\mathbb{R}}(k, \infty)$ and $\mathrm{G}_{\mathbb{R}}(k, l) \to \mathrm{G}_{\mathbb{R}}(k, \infty)$, respectively. That is, a subset of $\mathrm{S}_{\mathbb{R}}(k, \infty)$ is open iff its intersection with the subset $\mathrm{S}_{\mathbb{R}}(k, l)$ is open for all l. An analogous statement holds for $\mathrm{G}_{\mathbb{R}}(k, \infty)$. Note that $\mathrm{S}_{\mathbb{R}}(k, \infty)$ may be identified with the direct limit of the directed system given by the topological spaces $\mathrm{S}_{\mathbb{R}}(k, l)$ and the natural inclusion mappings $\mathrm{S}_{\mathbb{R}}(k, l) \to \mathrm{S}_{\mathbb{R}}(k, l+1)$, $l = 1, 2, \dots$. Again, a similar statement holds for $\mathrm{G}_{\mathbb{R}}(k, \infty)$. To make $\mathrm{S}_{\mathbb{R}}(k, \infty)$ into a principal $\mathrm{O}(k)$-bundle over $\mathrm{G}_{\mathbb{R}}(k, \infty)$, we define a mapping

$$\pi : \mathrm{S}_{\mathbb{R}}(k, \infty) \to \mathrm{G}_{\mathbb{R}}(k, \infty) \tag{3.4.9}$$

by assigning to a k-frame in $\mathbb{R}^\infty$ the subspace spanned by this k-frame and a mapping

$$\Psi : \mathrm{S}_{\mathbb{R}}(k, \infty) \times \mathrm{O}(k) \to \mathrm{S}_{\mathbb{R}}(k, \infty) \tag{3.4.10}$$

by letting $\mathrm{O}(k)$ act on the first k entries of the elements of $\mathbb{R}^\infty$. This is a free action. Then, denoting the natural projection $\mathrm{S}_{\mathbb{R}}(k, l) \to \mathrm{G}_{\mathbb{R}}(k, l)$ by $\pi^{(l)}$ and the action of $\mathrm{O}(k)$ on $\mathrm{S}_{\mathbb{R}}(k, l)$ by $\Psi^{(l)}$, we have

$$\pi_{\restriction \mathrm{S}_{\mathbb{R}}(k,l)} = \pi^{(l)}, \quad \Psi_{\restriction \mathrm{S}_{\mathbb{R}}(k,l) \times \mathrm{O}(k)} = \Psi^{(l)}, \tag{3.4.11}$$

where we have omitted the natural inclusion mappings. This implies that π is continuous and that Ψ is a topological right action (Exercise 3.4.4).

Lemma 3.4.15 *The tuple* $\big(\mathrm{S}_{\mathbb{R}}(k,\infty), \mathrm{G}_{\mathbb{R}}(k,\infty), \mathrm{O}(k), \pi, \Psi\big)$ *is a principal fibre bundle.*

This bundle is referred to as the infinite real Stiefel bundle.

Proof It remains to construct local trivializations. Thus, let $W_0 \in \mathrm{G}_{\mathbb{R}}(k,\infty)$ be given. We construct a local section in (3.4.9) at W_0 as follows. There exists l_0 such that $W_0 \in \mathrm{G}_{\mathbb{R}}(k,l_0)$. Define

$$U_{l_0} := \{W \in \mathrm{G}_{\mathbb{R}}(k,l_0) : \dim P_W(W_0) = k\},$$

where $P_W : \mathbb{R}^{l_0} \to \mathbb{R}^{l_0}$ denotes orthogonal projection to W. Choose an orthonormal basis $\{\mathbf{e}_i\}$ in $\mathbb{R}^{l_0}$ whose first k elements span W_0. For every $W \in U_{l_0}$,

$$\big\{P_W(\mathbf{e}_1), \ldots, P_W(\mathbf{e}_k), P_{W^\perp}(\mathbf{e}_{k+1}), \ldots, P_{W^\perp}(\mathbf{e}_{l_0})\big\}$$

is a basis in $\mathbb{R}^{l_0}$. By applying the standard orthonormalization procedure to this basis, we obtain an orthonormal basis $\{\mathbf{e}_i(W)\}$ whose first k elements span W and thus define an element $s_{l_0}(W)$ belonging to the fibre over W of the Stiefel bundle $\mathrm{S}_{\mathbb{R}}(k,l_0) \to \mathrm{G}_{\mathbb{R}}(k,l_0)$. To see that the mapping $W \mapsto s_{l_0}(W)$ is continuous and hence a local section in that bundle, we view $\mathrm{S}_{\mathbb{R}}(k,l_0)$ as the homogeneous space $\mathrm{O}(l_0)/\mathrm{O}(l_0-k)$. Then, $s_{l_0}(W)$ is given by the coset of the matrix built from the columns $\mathbf{e}_1(W), \ldots, \mathbf{e}_{l_0}(W)$. Using that $P_W(\mathbf{v})$ depends continuously on W for all $\mathbf{v} \in \mathbb{R}^{l_0}$, it is not hard to see that each of the vectors $\mathbf{e}_i(W)$ depends continuously on W. Hence, so does the corresponding matrix and, therefore, its coset. Now, we view W_0 as an element of $\mathrm{G}_{\mathbb{R}}(k,l_0+1)$, define U_{l_0+1} in the same way as before and construct a local section $s_{l_0+1} : U_{l_0+1} \to \mathrm{S}_{\mathbb{R}}(k,l_0+1)$ using an orthonormal basis in $\mathbb{R}^{l_0+1}$ whose first l_0 elements coincide with the elements of the basis used before. Then, $U_{l_0} \subset U_{l_0+1}$ and s_{l_0+1} coincides with s_{l_0} on U_{l_0}. Continuing in this way, we obtain a family of continuous mappings $s_l : U_l \to \mathrm{S}_{\mathbb{R}}(k,l)$, $l \geq l_0$, where U_l is an open neighbourhood of W_0 in $\mathrm{G}_{\mathbb{R}}(k,l)$ and

$$U_l \subset U_{l+1}, \quad s_{l+1\restriction U_l} = s_l$$

for all l. By Proposition 3.1.14, this family defines a continuous mapping s from $\bigcup_{l \geq l_0} U_l$ to $\mathrm{S}_{\mathbb{R}}(k,\infty)$ and this mapping is a local section of the projection (3.4.9). ■

In a similar way, one constructs the infinite complex and quaternionic Stiefel bundles.

Theorem 3.4.16 *The infinite Stiefel bundle fulfils* $\pi_i\big(\mathrm{S}_{\mathbb{K}}(k,\infty)\big) = 0$ *for all* i. *It is universal for* $\mathrm{O}(k)$ *in case* $\mathbb{K} = \mathbb{R}$, *for* $\mathrm{U}(k)$ *in case* $\mathbb{K} = \mathbb{C}$, *and for* $\mathrm{Sp}(k)$ *in case* $\mathbb{K} = \mathbb{H}$.

Proof Again, we apply Theorem 3.4.6. The Graßmannian $\mathrm{G}_{\mathbb{R}}(k,l)$ admits a canonical cell decomposition consisting of a total of $\binom{l}{k}$ cells [177, 451]. The cell complex structure so obtained has the property that $\mathrm{G}_{\mathbb{R}}(k,l)$ is a subcomplex of $\mathrm{G}_{\mathbb{R}}(k,l+1)$

for every $l > k$. It follows that the infinite Graßmannian $\mathrm{G}_{\mathbb{R}}(k, \infty)$ inherits a natural *CW*-complex structure.[18] In particular, it is paracompact Hausdorff. To check that $\pi_i\big(\mathrm{S}_{\mathbb{K}}(k, \infty)\big) = 0$ for all i, let $f : \mathrm{S}^i \to \mathrm{S}_{\mathbb{K}}(k, \infty)$ be a continuous mapping. Since S^i and hence $f(\mathrm{S}^i)$ is compact, there exists l_0 such that $f(\mathrm{S}^i)$ is contained in $\mathrm{S}_{\mathbb{K}}(k, l_0)$ and hence in $\mathrm{S}_{\mathbb{K}}(k, l)$ for any $l \geq l_0$. By (3.4.3), for large enough l, f is homotopic in $\mathrm{S}_{\mathbb{K}}(k, l)$ to a constant mapping. Hence, it is so in $\mathrm{S}_{\mathbb{K}}(k, \infty)$. ■

Example 3.4.17

1. For $k = 1$, Theorem 3.4.16 states that the bundle $\mathrm{S}^\infty \to \mathbb{K}\mathrm{P}^\infty$ is universal for $\mathrm{O}(1)$ in case $\mathbb{K} = \mathbb{R}$, $\mathrm{U}(1)$ in case $\mathbb{K} = \mathbb{C}$ and $\mathrm{Sp}(1)$ in case $\mathbb{K} = \mathbb{H}$.
2. In view of the fact that $\pi_i(\mathrm{S}_{\mathbb{K}}(k, \infty) = 0$ for all i, Corollary 3.4.8 yields that via the embedding (3.4.5), from the case $\mathbb{K} = \mathbb{C}$ we obtain the universal bundle

$$\mathrm{S}^\infty \to \mathrm{L}_r^\infty \equiv \mathrm{S}^\infty / \mathbb{Z}_r$$

for the cyclic group $\mathbb{Z}_r$. Here, L_r^∞ is the infinite lense space. Like in finite dimension, L_r^∞ is a topological principal $\mathrm{U}(1)$-bundle over the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$.
3. Corollary 3.4.8 yields the universal bundles

$$\mathrm{S}_{\mathbb{R}}(k, \infty) \to \tilde{\mathrm{G}}_{\mathbb{R}}(k, \infty) \equiv \mathrm{S}_{\mathbb{R}}(k, \infty)/\mathrm{SO}(k),$$
$$\mathrm{S}_{\mathbb{C}}(k, \infty) \to \tilde{\mathrm{G}}_{\mathbb{C}}(k, \infty) \equiv \mathrm{S}_{\mathbb{C}}(k, \infty)/\mathrm{SU}(k)$$

for $\mathrm{SO}(k)$ and $\mathrm{SU}(k)$, respectively. Here, $\tilde{\mathrm{G}}_{\mathbb{R}}(k, \infty)$ is a topological principal $\mathrm{O}(1)$-bundle over $\mathrm{G}_{\mathbb{R}}(k, \infty)$ and $\tilde{\mathrm{G}}_{\mathbb{C}}(k, \infty)$ is a topological principal $\mathrm{U}(1)$-bundle over $\mathrm{G}_{\mathbb{C}}(k, \infty)$. ♦

As in the n-universal case, the existence of universal bundles for the orthogonal groups entails the existence of universal bundles for all Lie groups with a finite number of connected components.

Corollary 3.4.18 *Let G be a Lie group with a finite number of connected components, let K be a maximal compact subgroup of G and let $K \subset \mathrm{O}(k)$ via a faithful orthogonal representation. Then, the principal G-bundle*

$$\mathrm{S}_{\mathbb{R}}(k, \infty) \times_K G \to \mathrm{S}_{\mathbb{R}}(k, \infty)/K$$

is universal for G. In particular, universal bundles exist for all Lie groups with a finite number of connected components.

Proof By Lemma 3.4.7, $\mathrm{S}_{\mathbb{R}}(k, \infty)/K$ is paracompact Hausdorff of *CW*-homotopy type. By the argument used in the proof of Corollary 3.4.12, $\mathrm{S}_{\mathbb{R}}(k, \infty) \times_K G$ is a deformation retract of $\mathrm{S}_{\mathbb{R}}(k, \infty)$. ■

[18] The number of r-cells of this structure coincides with the number of ways to write r as a sum of at most k positive integers. For a detailed description, see Sect. 6 in [451].

Remark 3.4.19 According to Proposition 3.4.3, the classifying space $\mathrm{B}G$ may be chosen to be a *CW*-complex. ♦

Having constructed universal bundles, we can now show that the universality criterion given in Theorem 3.4.6 is sharp. By Proposition 3.4.3, every universal G-bundle E is G-homotopy equivalent to $\mathrm{S}_{\mathbb{R}}(k, \infty) \times_K G$ for some faithful k-dimensional orthogonal representation of a maximal compact subgroup K. As was shown in the proof of Theorem 3.4.16, then $\pi_i(E) = 0$ for all i. Hence, Theorem 3.4.6 implies the following.

Proposition 3.4.20 *Let G be a Lie group with finitely many connected components. A topological principal G-bundle E is universal iff $\pi_i(E) = 0$ for all i.* ■

In view of this, Corollary 3.4.8 translates into the following statement.

Corollary 3.4.21 *Let G be compact and let $H \subset G$ be a closed subgroup. Then, the induced bundle $\mathrm{E}G \to \mathrm{E}G/H$ is universal for H and the quotient space $\mathrm{E}G/H$ is a classifying space for H.* ■

Remark 3.4.22 Let G_1 and G_2 be Lie groups with a finite number of connected components. By Proposition 3.4.20, one has

$$\pi_i(\mathrm{E}G_1 \times \mathrm{E}G_2) = \pi_i(\mathrm{E}G_1) \times \pi_i(\mathrm{E}G_2) = 0$$

for all i, and this implies that the topological principal $(G_1 \times G_2)$-bundle $\mathrm{E}G_1 \times \mathrm{E}G_2$ over $\mathrm{B}G_1 \times \mathrm{B}G_2$ is universal. It follows that the classifying space $\mathrm{B}(G_1 \times G_2)$ of the direct product of Lie groups may be realized by the direct product of the classifying spaces $\mathrm{B}G_1 \times \mathrm{B}G_2$. For an alternative proof, see Exercise 3.4.5.

Under this assumption, if P_1 and P_2 are principal G_i-bundles over the same base space B and if $f_i \to \mathrm{B}G_i$ are classifying mappings for P_i, then $(f_1 \times f_2) \circ \Delta$ is a classifying mapping for the principal $(G_1 \times G_2)$-bundle $P_1 \times_B P_2$ (fibre product). Indeed, one can check that the assignment $\big(b, (y_1, y_2)\big) \mapsto \big((b, y_1), (b, y_2)\big)$ induces a vertical isomorphism from $\big((f_1 \times f_2) \circ \Delta\big)^*(\mathrm{E}G_1 \times \mathrm{E}G_2)$ onto the fibre product $(f_1^*\mathrm{E}G_1) \times_B (f_2^*\mathrm{E}G_2)$. ♦

Finally, let us summarize the discussion of this section.

Theorem 3.4.23 (Classification Theorem) *For every Lie group G with a finite number of connected components, there exists a topological principal G-bundle $\mathrm{E}G \to \mathrm{B}G$ with the following property. For every paracompact Hausdorff topological space X of CW-homotopy type, the vertical isomorphism classes of topological principal G-bundles over X are in bijective correspondence with homotopy classes of continuous mappings $f : X \to \mathrm{B}G$. The correspondence is given by assigning to f the bundle $f^*\mathrm{E}G$.* ■

Exercises

3.4.1 Let E be a topological fibre bundle over B with typical fibre F. Show that if B and F are Hausdorff, then E is Hausdorff.

3.4.2 Work out the proof of (3.4.3) for the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$.

3.4.3 Check that the mappings φ_N, φ_G and φ defined in the proof of Corollary 3.4.12 are strong deformation retractions.

3.4.4 Use the relations (3.4.11) to show that the mapping π defined by (3.4.9) is continuous and that the mapping Ψ defined by (3.4.10) is a topological right action.

3.4.5 Let G_1 and G_2 be Lie groups and let P be a principal $(G_1 \times G_2)$-bundle over a smooth manifold M. By embedding G_1 and G_2 in the obvious way into $G_1 \times G_2$, the action of $G_1 \times G_2$ on P induces actions of G_1 and G_2. Convince yourself that P/G_1 can be made into a principal G_2-bundle over M, and vice versa. Show that P is vertically isomorphic to the principal $(G_1 \times G_2)$-bundle $\Delta^*(P/G_2 \times P/G_1)$, where $\Delta : M \to M \times M$ denotes the diagonal mapping. Use this to prove that the classifying space of $G_1 \times G_2$ may be realized as the direct product $\mathrm{B}G_1 \times \mathrm{B}G_2$, cf. Remark 3.4.22.

3.5 The Milnor Construction

In this section, we discuss the Milnor construction, which provides a topological principal G-bundle whose total space is contractible rather than weakly contractible. While being less intuitive than the construction of the infinite Stiefel bundles, the Milnor construction has two advantages. First, it applies to any Hausdorff topological group. In particular, in the case of a Lie group there is no need to assume a finite number of connected components. Second, it classifies topological principal bundles over all paracompact Hausdorff spaces, and not just those of CW-homotopy type. In fact, it classifies all principal bundles admitting a system of trivializations with a subordinate partition of unity. Such bundles are called numerable.

In a first step, we construct a topological principal G-bundle $G(l) \to B(l)$ for every positive integer l. Let $I = [0, 1]$. In what follows, elements of the direct products G^l and I^l will be denoted by $\mathbf{a} = (a_1, \ldots, a_l)$ and $\mathbf{t} = (t_1, \ldots, t_l)$, respectively. The l-join $G(l)$ is the topological quotient of the subset

$$\left\{(\mathbf{a}, \mathbf{t}) \in G^l \times I^l : t_1 + \cdots + t_l = 1\right\} \tag{3.5.1}$$

of $G^l \times I^l$ with respect to the equivalence relation

$$(\mathbf{a}, \mathbf{t}) \sim (\mathbf{b}, \mathbf{u}) \;\text{ iff }\; \mathbf{t} = \mathbf{u} \;\text{ and }\; a_i = b_i \;\text{ for all }\; i \;\text{ such that }\; t_i > 0. \tag{3.5.2}$$

Elements of $G(l)$ will be denoted by $[(\mathbf{a},\mathbf{t})]$. The free right action of G on $G^l \times I^l$ given by

$$\big(g, (\mathbf{a},\mathbf{t})\big) \mapsto \big((a_1 g, \ldots, a_l g), (t_1, \ldots, t_l)\big)$$

leaves the subset (3.5.1) invariant and hence descends to a topological free right action $\Psi^{(l)}$ of G on $G(l)$. Let $B(l)$ denote the topological quotient of this action and let $\pi^{(l)} : G(l) \to B(l)$ denote the natural projection to orbits. To see that $\Psi^{(l)}$ makes $G(l)$ into a topological principal G-bundle over $B(l)$, it suffices to cover $B(l)$ by local sections of $\pi^{(l)}$, that is, by continuous mappings $s : U \to G(l)$, where $U \subset B(l)$ is open, satisfying $\pi^{(l)} \circ s = \mathrm{id}_U$. For $i = 1, \ldots, l$, define subsets

$$S_i^{(l)} := \{[(\mathbf{a},\mathbf{t})] : a_i = \mathbb{1}, t_i > 0\}, \quad U_i^{(l)} := \pi^{(l)}\big(\{[(\mathbf{a},\mathbf{t})] : t_i > 0\}\big)$$

of $G(l)$ and $B(l)$, respectively. The subsets $U_i^{(l)}$ cover $B(l)$. They are open, because $\pi^{(l)}$ is an open mapping.[19] Since $U_i^{(l)} = \pi^{(l)}\big(S_i^{(l)}\big)$, by restriction, $\pi^{(l)}$ induces a continuous surjective mapping

$$\pi_i^{(l)} : S_i^{(l)} \to U_i^{(l)}.$$

It is easy to see that $\pi_i^{(l)}$ is injective. We show that it is open. Let $[(\mathbf{a},\mathbf{t})] \in S_i^{(l)}$. We have to show that $\pi_i^{(l)}$ maps neighbourhoods of $[(\mathbf{a},\mathbf{t})]$ in $S_i^{(l)}$ to neighbourhoods of $\pi_i^{(l)}\big([(\mathbf{a},\mathbf{t})]\big)$ in $B(l)$. For an open neighbourhood W of $\mathbb{1}$ in G and $\varepsilon > 0$, let $V(W,\varepsilon)$ denote the open subset of $G(l)$ obtained by intersecting

$$\big\{(\mathbf{b},\mathbf{s}) \in G^l \times I^l \ : \ b_i \in a_i W, \ s_i \in (t_i - \varepsilon, t_i + \varepsilon) \cap I\big\}$$

with the subset (3.5.1) and passing to classes with respect to the equivalence relation (3.5.2). Every neighbourhood of $[(\mathbf{a},\mathbf{t})]$ in $S_i^{(l)}$ contains a neighbourhood of the form $V(W,\varepsilon) \cap S_i^{(l)}$ with appropriately chosen W and ε. By continuity of the multiplication and inversion mappings of G, we can find an open neighbourhood $\tilde{W} \subset W$ of $\mathbb{1}$ in G such that $\tilde{W}\tilde{W}^{-1} \subset W$. Then, $V(\tilde{W},\varepsilon)$ is a neighbourhood of $[(\mathbf{a},\mathbf{t})]$ in $G(l)$ and hence, since $\pi_i^{(l)}$ is an open mapping, $\pi_i^{(l)}\big(V(\tilde{W},\varepsilon)\big)$ is a neighbourhood of $\pi_i^{(l)}\big([(\mathbf{a},\mathbf{t})]\big)$ in $B(l)$. Then, so is $\pi_i^{(l)}\big(V(W,\varepsilon) \cap S_i^{(l)}\big)$, because

$$\pi_i^{(l)}\big(V(\tilde{W},\varepsilon)\big) \subset \pi_i^{(l)}\big(V(\tilde{W}\tilde{W}^{-1},\varepsilon) \cap S_i^{(l)}\big) \subset \pi_i^{(l)}\big(V(W,\varepsilon) \cap S_i^{(l)}\big).$$

This shows that the mappings $\pi_i^{(l)}$ are open and hence homeomorphisms. By inverting them, we obtain the desired local sections

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[19] See Proposition 6.1.5/2 in Part I. The argument given there for Lie group actions applies to topological group actions as well.

$$s_i^{(l)} : U_i^{(l)} \to G(l).$$

As a result, for every l, $G(l)$ is a topological principal G-bundle over $B(l)$.

In a second step, we use the bundles $G(l) \to B(l)$ to construct a topological principal bundle G-bundle $G(\infty) \to B(\infty)$ in much the same way as the infinite Stiefel bundles. Let G^∞ denote the set of infinite sequences $\mathbf{a} = (a_1, a_2, \dots)$ with $a_i \in G$. Let I^∞ denote the set of infinite sequences $\mathbf{t} = (t_1, t_2, \dots)$ with $t_i \in I$ and only finitely many $t_i > 0$. Define the infinite join $G(\infty)$ to be the set of equivalence classes of the elements of the subset

$$\{(\mathbf{a}, \mathbf{t}) \in G^\infty \times I^\infty : t_1 + t_2 + \cdots = 1\} \tag{3.5.3}$$

of $G^\infty \times I^\infty$ with respect to the equivalence relation (3.5.2). The free right action of G on $G^l \times I^l$ given by

$$\big(g, (\mathbf{a}, \mathbf{t})\big) \mapsto \big((a_1 g, a_2 g, \dots), (t_1, t_2, \dots)\big)$$

leaves invariant the subset (3.5.3) and hence descends to a free right action $\Psi^{(\infty)}$ of G on the set $G(\infty)$. Let $B(\infty)$ denote the set of orbits and let $\pi^{(\infty)} : G(\infty) \to B(\infty)$ denote the natural projection. To equip $G(\infty)$ and $B(\infty)$ with a topology, we observe that every element of G^l can be made into an element of G^∞ by appending an infinite sequence with entries $\mathbb{1}$ and every element of I^l can be made into an element of I^∞ by appending an infinite sequence with zero entries. It is easy to check that, in this way, $G(l)$ and $B(l)$ are made into subsets of $G(\infty)$ and $B(\infty)$, respectively, for every n. Thus, we can topologize $G(\infty)$ and $B(\infty)$ by the final topologies defined by the corresponding natural inclusion mappings. These topologies coincide with those inherited from the final topology on $G^\infty \times I^\infty$ induced by the family of natural inclusion mappings $G^l \times I^l \to G^\infty \times I^\infty$ by taking subsets and quotients. Then, the obvious relations

$$\pi^{(\infty)}_{\restriction G(l)} = \pi^{(l)}, \quad \Psi^{(\infty)}_{\restriction G \times G(l)} = \Psi^{(l)},$$

holding for all $l \geq i$, imply that $\pi^{(\infty)}$ and $\Psi^{(\infty)}$ are continuous. To construct local sections of the projection $\pi^{(\infty)} : G(\infty) \to B(\infty)$, for every positive integer i we define a subset

$$U_i := \pi^{(\infty)}\big(\{[(\mathbf{a}, \mathbf{t})] \in G(\infty) : t_i > 0\}\big)$$

of $B(\infty)$ and a mapping

$$s_i : U_i \to G(\infty)$$

by assigning to $\pi^{(\infty)}\big([(\mathbf{a}, \mathbf{t})]\big)$ the unique representative with $a_i = \mathbb{1}$. This mapping is continuous, because its restriction to $U_i^{(l)}$ coincides with $s_i^{(l)}$ for all $l \geq i$, and it satisfies $\pi^{(\infty)} \circ s_i = \mathrm{id}_{U_i}$. Since the subsets U_i cover $B(\infty)$, this shows that $G(\infty)$ is a topological principal G-bundle over $B(\infty)$.

Theorem 3.5.1 *The assignment* (3.4.1) *induced by the topological principal G-bundle $G(\infty)$ over $B(\infty)$ is a bijection for all paracompact Hausdorff spaces X.*

Proof First, assume that we are given a topological principal G-bundle $\pi : P \to X$. We aim at constructing a classifying mapping $f : X \to B(\infty)$.

By applying Lemma 3.3.2 to the bundle $P \times I$, we find a locally finite open covering $\{U_i : i = 1, 2, \dots\}$ of X such that P is trivial over U_i for each i. Since X is paracompact, there exists a subordinate partition of unity $\{\varphi_i : i = 1, 2, \dots\}$. That is, $\mathrm{supp}(\varphi_i) \subset U_i$ for all i.

Using a system of local trivializations $\{\chi_i\}$, we can define the associated mappings $\kappa_i := \mathrm{pr}_G \circ \chi_i : \pi^{-1}(U_i) \to G$. Extending them to all of P by assigning to $p \notin \pi^{-1}(U_i)$ the value $\mathbb{1}$, we can define a mapping

$$P \to G(\infty), \quad p \mapsto \Big[\big(\big(\kappa_1(p), \kappa_2(p), \dots\big), \big(\varphi_1 \circ \pi(p), \varphi_2 \circ \pi(p), \dots\big)\big)\Big].$$

It is easy to see that this mapping is continuous and a principal G-bundle morphism. According to Remark 1.1.9/1, the projection $f : X \to B(\infty)$ yields the desired classifying mapping.

Conversely, let $f_0, f_1 : X \to B(\infty)$ be continuous mappings such that there exists an isomorphism $\lambda : f_0^* G(\infty) \to f_1^* G(\infty)$ and let $F_i : f_i^* G(\infty) \to G(\infty), i = 0, 1,$ denote the corresponding natural morphisms. To prove that f_0 and f_1 are homotopic, we define mappings $F^{\pm} : G(\infty) \to G(\infty)$ by

$$F^-\big([(\mathbf{a}, \mathbf{t})]\big) := \big[\big((a_1, \mathbb{1}, a_2, \mathbb{1}, \dots), (t_1, 0, t_2, 0, \dots)\big)\big],$$
$$F^+\big([(\mathbf{a}, \mathbf{t})]\big) := \big[\big((\mathbb{1}, a_1, \mathbb{1}, a_2, \dots), (0, t_1, 0, t_2, \dots)\big)\big].$$

Since, for every l, the restriction of $F^{\pm}$ to $G(l)$ is a composition of the natural inclusion mappings $G(l) \to G(2l)$ and $G(2l) \to G(\infty)$ with an intermediate mapping $G(2l) \to G(2l)$ induced by a simultaneous permutation of the entries of the elements of $G^{2l} \times I^{2l}$, the mappings $F^{\pm}$ are continuous. In fact, they are principal G-bundle morphisms. We show that they are homotopic through principal G-bundle morphisms to $\mathrm{id}_{G(\infty)}$, and hence that their projections $f^{\pm} : B(\infty) \to B(\infty)$ are homotopic to $\mathrm{id}_{B(\infty)}$. Consider the mappings

$$H^{\pm} : G(\infty) \times I \to G(\infty), \quad H^{\pm}\big([(\mathbf{a}, \mathbf{t})], s\big) := [(\mathbf{a}', \mathbf{t}')],$$

where for any positive integer n such that $s \in [1 - 2^{-n}, 1 - 2^{-n-1}]$,

$$\big(a_i', t_i'\big) = \begin{cases} (a_i, t_i) & \mid\ i - n \leq 1 \\ \Big(a_{n+\frac{i-n+1}{2}}, (2^{n+1} - 1 - 2^{n+1}s) t_{n+\frac{i-n+1}{2}}\Big) & \mid\ i - n > 1 \text{ and odd}, \\ \Big(a_{n+\frac{i-n+2}{2}}, (2^{n+1}s - 2^{n+1} + 2) t_{n+\frac{i-n+2}{2}}\Big) & \mid\ i - n > 1 \text{ and even}, \end{cases}$$

in case of H^- and

$$\left(a_i', t_i'\right) = \begin{cases} (a_i, t_i) & \mid \; i - n \leq 0 \\ \left(a_{n+\frac{i-n+1}{2}}, (2^{n+1}s - 2^{n+1} + 2)t_{n+\frac{i-n+1}{2}}\right) & \mid \; i - n > 0 \text{ and odd,} \\ \left(a_{n+\frac{i-n}{2}}, (2^{n+1} - 1 - 2^{n+1}s)t_{n+\frac{i-n}{2}}\right) & \mid \; i - n > 0 \text{ and even,} \end{cases}$$

in case of H^+. By an explicit calculation one can check that the definition is consistent for $s = 2^k$, where n can be chosen as k or as $k-1$. Continuity follows by observing that, for every l, the restriction of $H^\pm$ to $G(l) \times I$ is a composition of a certain mapping $G(l) \times I \to G(2l)$, which can be read off from the definition of $H^\pm$ and which is obviously continuous, with the natural inclusion mapping $G(2l) \to G(\infty)$. Since $H^\pm$ is equivariant with respect to the action of G, it yields the desired homotopy through principal G-bundle morphisms between $F^\pm$ and $\mathrm{id}_{G(\infty)}$.

As a result of these considerations, it suffices to show that $f^- \circ f_0$ is homotopic to $f^+ \circ f_1$. For that purpose, we define a mapping $H : f_0^* G(\infty) \times I \to G(\infty)$ by

$$H(x, s) := \big[\big((a_1, b_1, a_2, b_2, \dots), ((1-s)t_1, su_1, (1-s)t_2, su_2, \dots)\big)\big],$$

where $[(\mathbf{a}, \mathbf{t})] = F_0(x)$ and $[(\mathbf{b}, \mathbf{u})] = F_1 \circ \lambda(x)$. The definition makes sense, because if $t_i = 0$ or $u_i = 0$ (so that a_i or b_i are indeterminate), then, respectively, $(1-s)t_i = 0$ or $su_i = 0$. To see that H is continuous, we write it as a composition of the mapping

$$(F^- \circ F_0) \times (F^+ \circ F_1 \circ \lambda) : G(\infty) \to G(\infty) \times G(\infty)$$

with the mapping $G(\infty) \times G(\infty) \to G(\infty)$ which assigns to a pair $\big([(\mathbf{a}, \mathbf{t})], [(\mathbf{b}, \mathbf{u})]\big)$ the single element

$$\Big[\Big(\big(a_1, b_2, a_3, b_4, \dots\big), \; \big((1-s)(t_1 + t_2), s(u_1 + u_2), \\ (1-s)(t_3 + t_4), s(u_3 + u_4), \dots\big)\Big)\Big]$$

and check that the restriction to $G(l) \times G(l)$ of the latter mapping is continuous for all l. Since H is equivariant, it yields a homotopy through principal G-bundle morphisms between $F^- \circ F_0$ and $F^+ \circ F_1 \circ \lambda$ and hence a homotopy between the respective projections, that is between $f^- \circ f_0$ and $f^+ \circ f_1$. This completes the proof of the theorem. ∎

Remark 3.5.2

1. From the construction of classifying mappings used in the proof of Theorem 3.5.1, it is clear that none of the topological principal G-bundles $G(l) \to B(l)$ can be n-universal for some $n > 1$, in contrast to the Stiefel bundles.
2. Let G be a Hausdorff topological group. On the one hand, the principal G-bundle $G(\infty) \to B(\infty)$ is numerable, because for $i = 1, 2, \dots$, the assignment of t_i to $[(\mathbf{a}, \mathbf{t})]$ descends to a continuous function f_i on $B(\infty)$. Clearly, the family of these functions is a partition of unity. Moreover, $G(\infty)$ is trivial over $U_i := f_i^{-1}(0, 1]$, with trivialization given by

$$\pi^{-1}(U_i) \to U_i \times G, \quad [(\mathbf{a}, \mathbf{t})] \mapsto \big(\pi\big([(\mathbf{a}, \mathbf{t})]\big), a_i\big) .$$

On the other hand, it is not hard to see that the proofs of the Covering Homotopy Theorem 3.3.1 and of Theorem 3.5.1 work for numerable principal G-bundles as well. As a consequence, the assignment (3.4.1) induced by $G(\infty) \to B(\infty)$ is well defined for all topological spaces X and it maps $[X, B(\infty)]$ bijectively onto the isomorphism classes of numerable topological principal G-bundles over X. Thus, for an arbitrary Hausdorff topological group G, and in particular for a Lie group, the principal G-bundle $G(\infty) \to B(\infty)$ is universal in the realm of numerable principal G-bundles.

3. The total space $G(\infty)$ is contractible [628, Theorem 14.4.6]. ♦

3.6 Classification of Smooth Principal Bundles

In this section, we use the classification result for topological principal bundles to complete the classification of smooth principal bundles. In addition, from the classification of principal bundles, we derive the classification of vector bundles.

We start with showing that for a given Lie group G, every topological principal G-bundle over a smooth manifold admits a compatible smooth structure. The argument is based on the following fact.

Theorem 3.6.1 *For smooth manifolds M and N, every continuous mapping $M \to N$ is homotopic to a smooth mapping.*

Proof This is an immediate consequence of the fact that $C^\infty(M, N)$ is dense in $C^0(M, N)$ in the strong topology[20] and hence in the weaker compact-open topology [303, Theorem 2.2.6]. ■

Combining this with the observation that the n-universal G-bundle provided by Corollary 3.4.12 happens to be smooth, we obtain the following result.

Proposition 3.6.2 *Let G be a Lie group and let M be a smooth manifold. Every topological principal G-bundle over M is continuously vertically isomorphic to a smooth principal G-bundle over M.*

Thus, every topological principal G-bundle over a smooth manifold admits a compatible smooth structure.

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[20] For given $f \in C^0(M, N)$, a basis for the neighbourhoods of f in the strong topology is given by the following subsets. Let $\{(U_i, \kappa_i) : i \in I\}$ be a locally finite atlas on M, let $\{K_i : i \in I\}$ be a family of compact subsets of M satisfying $K_i \subset U_i$ for all i, let $\{(V_i, \kappa_i) : i \in I\}$ be an atlas on N satisfying $f(K_i) \subset V_i$ for all i, and let $\{\varepsilon_i : i \in I\}$ be a sequence of positive numbers. The neighbourhood of f defined by these data consists of all mappings $g : M \to N$ such that $g(K_i) \subset V_i$ and $\sup_{K_i} |\rho_i \circ g \circ \kappa_i - \rho_i \circ f \circ \kappa_i| < \varepsilon_i$ for all $i \in I$. See [303, Sect. 2.1] for details.

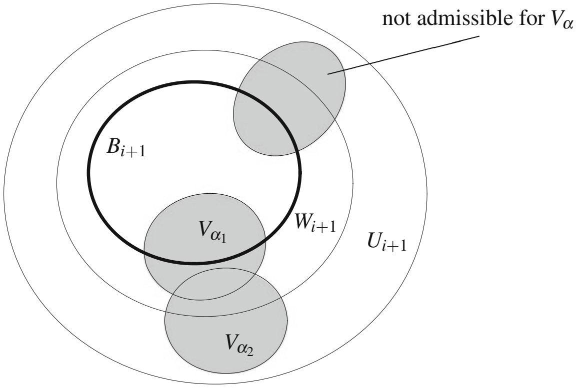

Fig. 3.1 Smoothening procedure for the mapping f_i in the proof of Lemma 3.6.3: admissible choices of the elements of the open covering $\{V_\alpha : \alpha = 1, 2, \ldots\}$ of U_{i+1}. The mapping is smoothened in step α_1, but is left unchanged in step α_2

Proof According to Corollary 3.4.12, there exists a smooth principal G-bundle $E \to B$ such that the mapping $[M, B] \to \mathrm{PFB}(G, M)$ defined by $f \mapsto f^*E$ is a bijection for M. Hence, for every topological principal G-bundle P over M, there exists a continuous mapping $f : M \to B$ such that P is vertically isomorphic to the topological principal G-bundle f^*E. By Theorem 3.6.1, f is homotopic to a smooth mapping $g : M \to B$. Hence, P is continuously vertically isomorphic to the smooth principal G-bundle g^*E. ■

Next, we show that smooth principal G-bundles over M are vertically isomorphic if so are their underlying topological principal bundles. The crucial step is the following smoothening result.

Lemma 3.6.3 *Let E be a smooth fibre bundle over a smooth manifold M. If E admits a continuous section, then it admits a smooth section.*

Proof Let F be the typical fibre of E. Let φ_0 be a continuous section in E. By Lemma 3.3.2 and Remark 3.3.3, we can choose a locally finite open covering $\{\mathrm{U}_i : i = 1, 2, \ldots\}$ of M such that E is trivial over each U_i. Then, there exists a closed covering $\{B_i : i = 1, 2, \ldots\}$ such that $B_i \subset U_i$; for example given by the supports of a partition of unity subordinate to the U_i. Since manifolds are normal spaces, for every i, there exists an open subset W_i such that $B_i \subset W_i$ and $\overline{W_i} \subset U_i$. Starting with φ_0, by induction on i, we will construct continuous sections φ_i of E which are smooth on a neighbourhood $\tilde{U}_i$ of $\tilde{B}_i := \bigcup_{j=1}^{i} B_j$ and coincide with φ_i outside W_i. Clearly, φ_0 may be chosen to be smooth. Thus, assume that we have constructed φ_i. Since E is trivial over U_{i+1}, the restriction $\varphi_{i\restriction U_{i+1}}$ is represented by a continuous mapping $f_i : U_{i+1} \to F$ which is smooth on $\tilde{U}_i \cap U_{i+1}$. We choose a countable atlas for F and cover U_{i+1} by open subsets $V_\alpha, \alpha = 1, 2, \ldots$, such that $f_i(V_\alpha)$ is contained in the domain of a single chart of that atlas on F and such that either $V_\alpha \subset W_{i+1}$ or $V_\alpha \cap B_{i+1} = \varnothing$; see Fig. 3.1. Now, we apply to f_i the usual smoothening procedure by induction on α from the proof that $C^\infty(U_{i+1}, F)$ is dense in $C^0(U_{i+1}, F)$ in the strong topology,[21] with the following modification. If $V_\alpha \cap B_{i+1} = \emptyset$, the mapping is left unchanged in step α. This way, we obtain a continuous mapping $f_{i+1} : U_{i+1} \to F$ which is smooth in a neighbourhood $\tilde{V}_{i+1}$ of B_{i+1} and coincides with f_{i+1} outside W_{i+1}. This mapping corresponds to a continuous section $\tilde{\varphi}_{i+1}$ in $E_{\restriction U_{i+1}}$ which is smooth on $\tilde{V}_{i+1}$ and coincides with φ_i on $U_{i+1} \setminus \overline{W_{i+1}}$. It follows that

$$\varphi_{i+1}(m) := \begin{cases} \varphi_i(m) & | \; m \in M \setminus \overline{W_{i+1}}, \\ \tilde{\varphi}_{i+1}(m) & | \; m \in U_{i+1} \end{cases}$$

defines a continuous section in E which is smooth on the neighbourhood $\tilde{U}_{i+1} := \tilde{U}_i \cup \tilde{V}_{i+1}$ of $\tilde{B}_{i+1}$ and which coincides with φ_i outside W_{i+1}. This completes the proof of the existence of the sections φ_i.

Now, let $m \in M$. Since $\overline{W_i} \subset U_i$ for all i and since the covering $\{U_i : i = 1, 2, \dots\}$ is locally finite, there exists a neighbourhood V of m in M such that $V \cap \overline{W_i}$ is nonempty for only finitely many i. Out of these, let $i_1, \dots, i_r$ be the numbers for which $m \notin \overline{W_i}$. Then, $\tilde{V} := V \setminus (\overline{W_{i_1}} \cup \dots \cup \overline{W_{i_r}})$ is an open neighbourhood of m. This shows that the function

$$m \mapsto i(m) := \max\{i \in \mathbb{N} : m \in \overline{W_i}\}$$

is well defined and locally constant. We define a section φ of E by

$$\varphi(m) := \varphi_{i(m)}(m), \quad m \in M.$$

Since the function $m \mapsto i(m)$ is locally constant, φ coincides with $\varphi_{i(m)}$ on some neighbourhood of any m. On the other hand, since $B_i \subset W_i$ and $m \notin W_i$ for all $i > i(m)$, every m belongs to some B_i with $i \leq i(m)$ and hence to $\tilde{B}_{i(m)}$. Since $\varphi_{i(m)}$ is smooth in a neighbourhood of $\tilde{B}_{i(m)}$, it follows that φ is smooth in a neighbourhood of m for every $m \in M$. This proves the lemma. ■

Let P and Q be smooth principal G-bundles over M. By Corollary 1.2.7, smooth vertical isomorphisms $P \to Q$ correspond bijectively to smooth sections of the smooth fibre bundle $P \times_{G,M} Q$ over M. An analogous statement holds for continuous vertical isomorphisms and continuous sections of this bundle. Hence, Lemma 3.6.3 implies

Proposition 3.6.4 *Let G be a Lie group, let M be a smooth manifold and let P and Q be smooth principal G-bundles over M. If P and Q are vertically isomorphic as topological principal G-bundles, they are vertically isomorphic as smooth principal G-bundles.* ■

Remark 3.6.5 Using Proposition 1.2.6, one can prove a similar result for G-morphisms $P \to Q$, where P and Q are smooth principal G-bundles over different manifolds. Since we do not need this, we leave it to the interested reader to

[21] See, for example, [303, Theorem 2.2.6].

work out a proof. The problem is that such morphisms need not be isomorphisms, so that one has to make sure that the smoothened morphism can be chosen to be an isomorphism. This requires the following arguments.

1. The smoothened section of Lemma 3.6.3 can be chosen arbitrarily close to the original section in the strong topology induced from $C^0(M, P \times_G Q)$.
2. The assignment of morphisms to sections is continuous in the strong topologies induced from $C^0(M, P \times_G Q)$ and $C^0(P, Q)$, respectively.

The assertion then follows from the fact that the subset of homeomorphisms is open in $C^0(P, Q)$ in the strong topology [303, Theorem 1.1.7]. ♦

Now, we can prove that vertical isomorphism classes of smooth principal G-bundles over a smooth manifold M correspond bijectively to vertical isomorphism classes of topological principal G-bundles over M.

Theorem 3.6.6 *Let G be a Lie group and let M be a smooth manifold. Forgetting about the smooth structure defines a bijection from the set of vertical isomorphism classes of smooth principal G-bundles over M onto the set of vertical isomorphism classes of topological principal G-bundles over M.*

Proof Forgetting about the smooth structure clearly defines an assignment on the level of vertical isomorphism classes. By Proposition 3.6.2, this assignment is surjective. By Proposition 3.6.4, it is injective. ■

Combining this with Corollary 3.4.12, we obtain that, given a smooth manifold of dimension $\dim(M) \leq n$, every n-universal bundle $E \to B$ for G establishes a bijection between vertical isomorphism classes of smooth principal G-bundles over M and homotopy classes of continuous mappings $M \to B$. However, since smooth n-universal bundles exist, it makes sense to use them for directly classifying smooth principal G-bundles in terms of smooth classifying mappings, without taking the detour through topological bundles.

Theorem 3.6.7 (Classification Theorem) *Let G be a Lie group, let $E \to B$ be an n-universal bundle for G which is smooth, and let M be a smooth manifold with* $\dim(M) \leq n$. *Then, the assignment of f^*E to $f : M \to B$ induces a bijection from the set of (continuous) homotopy classes of smooth mappings to vertical isomorphism classes of smooth principal G-bundles over M.*

Proof The assignment induces a mapping of the classes: if $f, g : M \to B$ are smooth mappings which are homotopic through a continuous homotopy, then f^*E and g^*E are vertically isomorphic as topological principal bundles. By Proposition 3.6.4, they are isomorphic as smooth principal bundles then.

The induced mapping is surjective: let P be a smooth principal G-bundle over M. Since E is n-universal, P is vertically isomorphic, as a topological principal G-bundle, to f^*E for some continuous mapping $f : M \to B$. By Theorem 3.6.1, f is homotopic to a smooth mapping $g : M \to B$. By Corollary 3.3.5, then P and g^*E

are vertically isomorphic as topological principal G-bundles. By Proposition 3.6.4, they are vertically isomophic as smooth principal G-bundles then.

The induced mapping is injective: let $f, g : M \to B$ be smooth mappings. If f^*E and g^*E are vertically isomorphic as smooth principal G-bundles, they are vertically isomorphic as topological principal G-bundles. Since E is n-universal, then f and g are homotopic. ∎

To conclude this section, we use the classification results for principal bundles obtained above to classify vector bundles.

Let M be a smooth manifold and let k be a positive integer. As in Example 1.2.9/2, for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, let $\mathrm{U}_{\mathbb{K}}(k)$ denote, respectively, the group $\mathrm{O}(k)$, $\mathrm{U}(k)$ or $\mathrm{Sp}(k)$. Given a principal $\mathrm{U}_{\mathbb{K}}(k)$-bundle P over M, one has the associated $\mathbb{K}$-vector bundle of rank k given by $P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$, where $\mathrm{U}_{\mathbb{K}}(k)$ acts on $\mathbb{K}^k$ via the basic representation. This bundle carries a natural fibre metric, induced from the natural scalar product on $\mathbb{K}^k$.

Theorem 3.6.8 *For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, the assignment $P \mapsto P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ induces a bijection between the isomorphism classes of principal $\mathrm{U}_{\mathbb{K}}(k)$-bundles over M and the isomorphism classes of $\mathbb{K}$-vector bundles of rank k over M.*

Proof By Proposition 1.2.8/3, the assignment $P \mapsto P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ induces a mapping of isomorphism classes. By Example 1.2.9/2, the induced mapping is surjective.

The induced mapping is injective: it suffices to show that for every principal $\mathrm{U}_{\mathbb{K}}(k)$-bundle P over M, P is isomorphic to the orthonormal frame bundle $O(E)$ of the associated $\mathbb{K}$-vector bundle $E = P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$, equipped with its natural fibre metric. Consider the mapping

$$P \to O(E), \quad p \mapsto \big([(p, \mathbf{e}_1)], \ldots, [(p, \mathbf{e}_k)]\big), \tag{3.6.1}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are the elements of the standard basis of $\mathbb{K}^k$. It is injective, because $[(p, \mathbf{e}_1)] = [(q, \mathbf{e}_1)]$ implies $p = q$. It is surjective, because every ordered orthonormal basis in a fibre of $P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ is of the form $[(p, a^j{}_i \mathbf{e}_j)] = [(\Psi_a(p), \mathbf{e}_i)]$ for some $p \in P$ and some $a \in \mathrm{U}_{\mathbb{K}}(k)$ and hence is the image of $\Psi_a(p)$. Finally, the local representative of (3.6.1) with respect to the local trivialization of P induced by a local section s and a local trivialization of $P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ induced by the local frame $m \mapsto \big[\big(s(m), \mathbf{e}_i\big)\big]$ is given by the identical mapping. Thus, (3.6.1) is an isomorphism of principal $\mathrm{U}_{\mathbb{K}}(k)$-bundles over M. This proves the theorem. ∎

Combining Theorem 3.6.8 with Theorems 3.6.6 and 3.6.7, as well as Theorems 3.4.10 and 3.4.16, we obtain the following.

Corollary 3.6.9 *Let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, let M be a smooth manifold of dimension n, let k be a positive integer and let $d = \dim_{\mathbb{R}}(\mathbb{K})$.*

1. *For every $l \geq \frac{n+2}{d} + k - 1$, the assignment $f \mapsto f^*\big(\mathrm{S}_{\mathbb{K}}(k, l) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k\big)$ induces a bijection between homotopy classes of smooth mappings $M \to \mathrm{G}_{\mathbb{K}}(k, l)$ and isomorphism classes of smooth $\mathbb{K}$-vector bundles over M of rank k.*

2. *Forgetting about the smooth structure induces a bijection from the set of isomorphism classes of smooth $\mathbb{K}$-vector bundles over M of rank k onto the set of isomorphism classes of topological $\mathbb{K}$-vector bundles over M of rank k.*
3. *The assignment $f \mapsto f^*\big(\mathrm{S}_{\mathbb{K}}(k, \infty) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k\big)$ induces a bijection between homotopy classes of continuous mappings $M \to \mathrm{G}_{\mathbb{K}}(k, \infty)$ and isomorphism classes of topological $\mathbb{K}$-vector bundles over M of rank k.* ■

Remark 3.6.10

1. According to point 3 of Corollary 3.6.9, for every topological $\mathbb{K}$-vector bundle E of rank k over B, there exists a classifying mapping, that is, a continuous mapping $f : B \to \mathrm{G}_{\mathbb{K}}(k, \infty)$ such that E is vertically isomorphic to $f^*\big(\mathrm{S}_{\mathbb{K}}(k, \infty) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k\big)$, and this mapping is unique up to homotopy. According to Proposition 1.2.8/4, up to homotopy, a principal $\mathrm{U}_{\mathbb{K}}(k)$-bundle P has the same classifying mapping as the associated vector bundle $P \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$, and a $\mathbb{K}$-vector bundle of rank k has the same classifying mapping as its orthonormal frame bundle $O(E)$ with respect to some chosen positive definite fibre metric.
2. The notions of n-universal principal bundle and universal principal bundle carry over in an obvious way to vector bundles of a prescribed rank. Using this, points 1 and 3 of Corollary 3.6.9 may be restated as follows.
   1. For every $l \geq \frac{n+2}{d} + k - 1$, the associated vector bundle $\mathrm{S}_{\mathbb{K}}(k, l) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ is n-universal for $\mathbb{K}$-vector bundles of rank k.
   3. The associated vector bundle $\mathrm{S}_{\mathbb{K}}(k, \infty) \times_{\mathrm{U}_{\mathbb{K}}(k)} \mathbb{K}^k$ is universal for $\mathbb{K}$-vector bundles of rank k.

   If, on the other hand, one just wants to classify vector bundles, one may skip the detour through principal bundles and construct smooth n-universal vector bundles $E_{\mathbb{K}}(k, l) \to \mathrm{G}_{\mathbb{K}}(k, l)$ of rank k directly by defining

$$E_{\mathbb{K}}(k, l) := \{(W, \mathbf{v}) \in \mathrm{G}_{\mathbb{K}}(k, l) \times \mathbb{K}^k : \mathbf{v} \in W\},$$

   where $l \geq \frac{n+2}{d} + k - 1$. In complete analogy with the construction of the infinite Stiefel bundles, by taking the direct limit $l \to \infty$, one obtains a universal topological vector bundle $E_{\mathbb{K}}(k, \infty) \to \mathrm{G}_{\mathbb{K}}(k, \infty)$ of rank k, see Sect. 1.2 in [287]. ♦

3.7 Classifying Mappings Associated with Lie Group Homomorphisms

Throughout this section, let G, H be Lie groups with finitely many connected components and let X be a paracompact Hausdorff space of CW-homotopy type. We choose universal bundles $\mathrm{E}G \to \mathrm{B}G$ and $\mathrm{E}H \to \mathrm{B}H$.

Given a Lie group homomorphism $\lambda : G \to H$, we can form the associated bundle $\mathrm{E}G^{[\lambda]} = \mathrm{E}G \times_G H$, cf. (1.2.8). Right translation on H defines an action of H on $\mathrm{E}G^{[\lambda]}$ and this action makes $\mathrm{E}G^{[\lambda]}$ into a topological principal H-bundle over $\mathrm{B}G$.

Definition 3.7.1 The mapping $\mathrm{B}\lambda : \mathrm{B}G \to \mathrm{B}H$ associated with λ is defined to be the classifying mapping of the principal H-bundle $\mathrm{E}G^{[\lambda]}$, that is, the mapping fulfilling

$$\mathrm{E}G^{[\lambda]} \cong (\mathrm{B}\lambda)^*(\mathrm{E}H). \tag{3.7.1}$$

Clearly, the mapping $\mathrm{B}\lambda$ is determined up to homotopy. In what follows, we discuss its properties.

Proposition 3.7.2 *Let $\lambda : G \to H$ be a Lie group homomorphism.*

1. *Let P be a topological principal G-bundle over X and let f be a classifying mapping for P. Then, $\mathrm{B}\lambda \circ f$ is a classifying mapping for $P^{[\lambda]}$.*
2. *Let Q be a topological principal H-bundle over X and let g be a classifying mapping for Q. The vertical isomorphism classes of topological principal G-bundles P over X having the property that $P^{[\lambda]}$ is vertically isomorphic to Q correspond bijectively to the homotopy classes of mappings $f : X \to \mathrm{B}G$ such that $\mathrm{B}\lambda \circ f$ is homotopic to g.*

Proof 1. By definition of $\mathrm{B}\lambda$, the principal H-bundle $(\mathrm{B}\lambda)^*\mathrm{E}H$ over $\mathrm{B}G$ is vertically isomorphic to $\mathrm{E}G^{[\lambda]}$. Combining this with Proposition 1.2.5, we obtain the vertical isomorphisms

$$(\mathrm{B}\lambda \circ f)^*\mathrm{E}H \cong f^*(\mathrm{B}\lambda)^*\mathrm{E}H \cong f^*(\mathrm{E}G^{[\lambda]}) \cong (f^*\mathrm{E}G)^{[\lambda]} \cong P^{[\lambda]}.$$

2. It suffices to show that for a continuous mapping $f : X \to \mathrm{B}G$, the composition $\mathrm{B}\lambda \circ f$ is homotopic to g iff $(f^*\mathrm{E}G)^{[\lambda]}$ is vertically isomorphic to Q. By point 1, $\mathrm{B}\lambda \circ f$ is a classifying mapping for $(f^*\mathrm{E}G)^{[\lambda]}$. Hence, the assertion follows from the universality of $\mathrm{E}H$. ∎

Point 2 of Proposition 3.7.2 applies in particular to Lie subgroup embeddings.

Corollary 3.7.3 *The vertical isomorphism classes of reductions of a topological principal H-bundle Q over X to a Lie subgroup $\lambda : G \to H$ correspond bijectively to the homotopy classes of mappings $f : X \to \mathrm{B}G$ such that $\mathrm{B}\lambda \circ f$ is a classifying mapping for Q.* ∎

Next, we discuss the functorial properties of $\mathrm{B}\lambda$.

Proposition 3.7.4 *Up to homotopy, the following holds true.*

1. *For $\lambda_1 : G \to H$ and $\lambda_2 : H \to K$, one has $\mathrm{B}(\lambda_2 \circ \lambda_1) = \mathrm{B}\lambda_2 \circ \mathrm{B}\lambda_1$.*
2. *One has $\mathrm{B}\,\mathrm{id}_G = \mathrm{id}_{\mathrm{B}G}$. More generally, if λ is an inner automorphism of G, then $\mathrm{B}\lambda = \mathrm{id}_{\mathrm{B}G}$.*
3. *For the constant homomorphism, $\mathrm{B}\lambda$ is homotopic to a constant mapping.*

Proof 1. By definition, $\mathrm{B}(\lambda_2 \circ \lambda_1)$ is a classifying mapping for $\mathrm{E}G^{[\lambda_2 \circ \lambda_1]}$. By Proposition 3.7.2/1, $\mathrm{B}\lambda_2 \circ \mathrm{B}\lambda_1$ is a classifying mapping for $\big(\mathrm{E}G^{[\lambda_1]}\big)^{[\lambda_2]}$. We leave it to the reader to check that the mapping

$$\mathrm{E}G \times K \to (\mathrm{E}G \times H) \times K, \quad (y, k) \mapsto \big((y, \mathbb{1}_H), k\big) \tag{3.7.2}$$

descends to a vertical K-morphism, and hence isomorphism, from $\mathrm{E}G^{[\lambda_2 \circ \lambda_1]}$ onto $\big(\mathrm{E}G^{[\lambda_1]}\big)^{[\lambda_2]}$ (Exercise 3.7.1).

2. The action mapping $\Psi : \mathrm{E}G \times G \to \mathrm{E}G$ descends to a vertical isomorphism $\mathrm{E}G^{[\mathrm{id}_G]} \to \mathrm{E}G$. More generally, let λ be given by conjugation by $b \in G$, that is, $\lambda(a) = bab^{-1}$. We leave it to the reader to prove that the mapping $\mathrm{E}G \times G \to \mathrm{E}G$ defined by $(y, g) \mapsto \Psi_{b^{-1}g}(y)$ descends to a vertical isomorphism from $\mathrm{E}G^{[\lambda]}$ onto $\mathrm{E}G$.

3. The mapping $\pi \times \mathrm{id}_H : \mathrm{E}G \times H \to \mathrm{B}G \times H$ descends to a vertical isomorphism $\mathrm{E}G^{[\lambda]} \to \mathrm{B}G \times H$. ∎

In the special case where $\lambda : G \to H$ is a Lie subgroup embedding, $\mathrm{B}\lambda$ inherits a bundle structure.

Proposition 3.7.5 *Let G be a compact Lie group.*

1. *If $\lambda : H \to G$ is a Lie subgroup embedding, then the classifying mapping* $\mathrm{B}\lambda : \mathrm{B}H \to \mathrm{B}G$ *can be realized as the projection in the topological fibre bundle* $\mathrm{E}G/H \to \mathrm{B}G$ *with typical fibre G/H.*
2. *If $\lambda : H \to G$ is a normal Lie subgroup embedding, then* $\mathrm{B}H$ *can be realized as a topological principal bundle over* $\mathrm{B}G$ *with structure group G/H and projection* $\mathrm{B}\lambda$. *This bundle has classifying mapping* $\mathrm{B}p$, *where $p : G \to G/H$ is the natural projection.*

Proof 1. By Corollary 3.4.21, the induced bundle $\mathrm{E}G \to \mathrm{E}G/H$, with H acting via λ, is universal for H. Hence, $\mathrm{E}H = \mathrm{E}G$ up to H-homotopy equivalence and $\mathrm{B}H = \mathrm{E}G/H$ up to homotopy equivalence. Moreover, the induced projection $f : \mathrm{B}H \equiv \mathrm{E}G/H \to \mathrm{B}G$ is a topological fibre bundle with typical fibre G/H. To see that f realizes $\mathrm{B}\lambda$, it suffices to check that the principal G-bundles $f^*\mathrm{E}G$ and $\mathrm{E}H^{[\lambda]}$ are vertically isomorphic. We leave it to the reader to show that the mapping

$$\mathrm{E}G \times G \to \mathrm{B}H \times \mathrm{E}G, \quad (y, h) \mapsto \big(\pi(y), \Psi_h(y)\big) \tag{3.7.3}$$

where $\pi : \mathrm{E}G \to \mathrm{E}G/H \equiv \mathrm{B}H$ denotes projection to orbits and Ψ denotes the action of G on $\mathrm{E}G$, induces a vertical isomorphism $\mathrm{E}H^{[\lambda]} \to f^*\mathrm{E}G$ (Exercise 3.7.2).

2. The first assertion follows from point 1 by recalling that, in the present case, the induced projection $\mathrm{E}G/H \to \mathrm{B}G$ has the structure of a principal bundle with structure group G/H. For the second assertion, we observe that the mapping $\mathrm{E}G \to \mathrm{E}G \times (G/H)$ descends to a vertical G/H-morphism, and hence isomorphism, from $\mathrm{B}H$ to $\mathrm{E}G^{[p]}$. ∎

Proposition 3.7.6 *Let P, Q be topological principal bundles over topological spaces B_P, B_Q with structure groups G, H, respectively, and let $F : P \to Q$ be a morphism with Lie group homomorphism $\lambda : G \to H$ and projection $f : B_P \to B_Q$.*

1. *If $f_P : B_P \to \mathrm{B}G$ and $f_Q : B_Q \to \mathrm{B}H$ are classifying mappings for P and Q, respectively, then $f_Q \circ f$ is homotopic to $\mathrm{B}\lambda \circ f_P$. That is, the diagram*

$$\begin{tikzcd} B_P \arrow[r, "f_P"] \arrow[d, "f"'] & \mathrm{B}G \arrow[d, "\mathrm{B}\lambda"] \\ B_Q \arrow[r, "f_Q"] & \mathrm{B}H \end{tikzcd}$$

   commutes up to homotopy.
2. *If f is a homeomorphism, then the mapping $P \times H \to Q$, $(p, h) \mapsto \Psi_h^Q\big(F(p)\big)$ descends to a principal H-bundle isomorphism $P^{[\lambda]} \to Q$ projecting to f.*

Proof 1. Consider the associated principal H-bundle $P^{[\lambda]}$. One can check that the mapping

$$P \times H \to B_P \times Q, \quad (p, h) \mapsto \big(\pi_P(p), \Psi_h^Q(F(p))\big) \tag{3.7.4}$$

takes values in $f^*Q \subset B_P \times Q$ and that it descends to a vertical isomorphism of principal H-bundles from $P^{[\lambda]}$ to f^*Q (Exercise 3.7.3). Hence, according to Proposition 3.7.2/1, $\mathrm{B}\lambda \circ f_P : B_P \to \mathrm{B}H$ is a classifying mapping for f^*Q. It is therefore homotopic to $f_Q \circ f$.

2. The mapping under consideration is the composition of (3.7.4), viewed as a mapping to $f^*Q \subset B_P \times Q$, with the natural principal bundle morphism $f^*Q \to Q$ given by projecting to the second entry. Since the latter is an isomorphism if f is a homeomorphism, the assertion follows. ∎

Finally, recall from Remark 3.4.22 that the universal bundle and the classifying space for a direct product $G_1 \times G_2$ of Lie groups with finitely many connected components may be realized by the direct products $\mathrm{E}G_1 \times \mathrm{E}G_2$ and $\mathrm{B}G_1 \times \mathrm{B}G_2$, respectively. Under this assumption, we have the following.

Proposition 3.7.7 *Up to homotopy, the following holds true.*

1. *For $i = 1, 2$, let G_i, H_i be Lie groups with finitely many connected components and let $\lambda_i : G_i \to H_i$ be Lie group homomorphisms. Then,*

$$\mathrm{B}(\lambda_1 \times \lambda_2) = \mathrm{B}\lambda_1 \times \mathrm{B}\lambda_2.$$

2. *For the diagonal mappings $\Delta_G : G \to G \times G$ and $\Delta_{\mathrm{B}G} : \mathrm{B}G \to \mathrm{B}G \times \mathrm{B}G$, one has $\mathrm{B}\Delta_G = \Delta_{\mathrm{B}G}$.*

Proof 1. By Proposition 1.2.5/3, we have the vertical isomorphisms

$$\begin{aligned}(\mathrm{E}G_1 \times \mathrm{E}G_2)^{[\lambda_1\times\lambda_2]} &\cong \mathrm{E}G_1{}^{[\lambda_1]} \times \mathrm{E}G_2{}^{[\lambda_2]}\\ &\cong (\mathrm{B}\lambda_1)^*\mathrm{E}H_1 \times (\mathrm{B}\lambda_2)^*\mathrm{E}H_2\\ &\cong (\mathrm{B}\lambda_1 \times \mathrm{B}\lambda_2)^*(\mathrm{E}H_1 \times \mathrm{E}H_2),\end{aligned}$$

where the last one is induced by the rearrangement

$$(\mathrm{B}G_1 \times \mathrm{E}H_1) \times (\mathrm{B}G_2 \times \mathrm{E}H_2) \to (\mathrm{B}G_1 \times \mathrm{B}G_2) \times (\mathrm{E}H_1 \times \mathrm{E}H_2).$$

2. Let $\Psi : \mathrm{E}G \times G \to \mathrm{E}G$ denote the principal action mapping. We leave it to the reader to check that the mapping

$$\mathrm{E}G \times (G \times G) \to \mathrm{E}G \times \mathrm{E}G, \quad \big(y, (a, b)\big) \mapsto \big(\Psi_a(y), \Psi_b(y)\big),$$

descends to a principal $(G \times G)$-bundle morphism from $\mathrm{E}G^{[\Delta]}$ to $\mathrm{E}G \times \mathrm{E}G$ covering $\Delta_{\mathrm{B}G}$. Then, the assertion follows from Remark 1.1.9/1. ■

Exercises

3.7.1 Show that the mapping (3.7.2) induces a vertical G-morphism $\mathrm{E}G^{[\lambda_2\circ\lambda_1]} \to \big(\mathrm{E}G^{[\lambda_1]}\big)^{[\lambda_2]}$.

3.7.2 Show that the mapping (3.7.3) induces a vertical H-morphism, and hence isomorphism, from $\mathrm{E}G^{[\lambda]}$ onto $f^*\mathrm{E}H$.

3.7.3 Complete the proof of Proposition 3.7.6 by showing that the mapping defined in (3.7.4) takes values in f^*Q and that it descends to a vertical morphism of principal H-bundles from $P^{[\lambda]}$ to f^*Q.

3.8 Universal Connections

In this section, we extend the discussion of n-universal objects from bundles to bundles with connections.

Definition 3.8.1 A connection ω_0 on an n-universal principal G-bundle $E \to B$ is called n-universal if for every connection ω on a principal G-bundle $P \to M$ with $\dim(M) \le n$ there exists a G-morphism $\vartheta : P \to E$ such that $\omega = \vartheta^*\omega_0$.

Necessarily, the projection of ϑ is then a classifying mapping for P. We will proceed in two steps.

(a) We present the classical result of Narasimhan and Ramanan [476] for compact Lie groups, see also [560] and, for an algebraic reformulation, [396]. In particular, the natural connections on the Stiefel bundles, given in Example 1.3.20, provide n-universal connections for the classical compact Lie groups.

(b) For the case of an arbitrary Lie group, there are at least two different approaches. The one of Narasimhan and Ramanan [477] is, similarly to their method used in the compact case, by patching together local solutions. The one presented in [81] is more geometric and uses the tautological connection on the section jet bundle of an n-universal G-bundle.[22] This is the approach we follow here.

Unfortunately, the result in the compact case seemingly cannot be obtained directly as a special case of (b).

To start with, recall the canonical connection ω^c on the Stiefel bundle

$$\mathrm{S}_{\mathbb{K}}(k,n) \cong \mathrm{U}_{\mathbb{K}}(n)/\mathrm{U}_{\mathbb{K}}(n-k) \to \mathrm{G}_{\mathbb{K}}(k,n) \cong \mathrm{U}_{\mathbb{K}}(n)/(\mathrm{U}_{\mathbb{K}}(n-k) \times \mathrm{U}_{\mathbb{K}}(k))\,,$$

cf. Example 1.3.20. According to (1.3.19), in terms of the matrix-valued function u which assigns to the k-frame built from $u_\alpha = a^j{}_\alpha \mathbf{e}_j$ the $(n \times k)$-matrix $a^j{}_\alpha$, it reads

$$\omega^c = u^\dagger \mathrm{d}u. \tag{3.8.1}$$

By Theorem 3.4.10, the Stiefel bundles are n-universal for the classical compact Lie groups. We will show that ω^c provides universal connections for these groups. In our presentation we follow [476]. To be definite, we restrict attention to the unitary group, that is, $\mathbb{K} = \mathbb{C}$. The starting point is the following technical lemma. For the proof we refer to Sect. 3 in [476].

Lemma 3.8.2 *Let $U \subset \mathbb{R}^n$ be an open subset and let $V \subset U$ be a relatively compact subset whose closure is contained in U. Let $l = (2n+1)k^2$. For every 1-form α with values in $\mathfrak{u}(k)$, there exist smooth mappings $f_1, \dots, f_l : V \to \mathrm{M}_k(\mathbb{C})$ such that*

$$\sum_{i=1}^{l} f_i^\dagger f_i = \mathbb{1}_k, \quad \sum_{i=1}^{l} f_i^\dagger \mathrm{d} f_i = \alpha.$$

■

Lemma 3.8.3 *Let P be a principal $\mathrm{U}(k)$-bundle over a manifold M of dimension $\leq n$ and let ω be a connection form on P. Let $V \subset M$ be a relatively compact open subset whose closure is contained in a coordinate neighbourhood U over which P is trivial. Then, there exists a bundle morphism $\vartheta : P_V \to \mathrm{S}_{\mathbb{C}}(k, lk)$ such that*

$$\vartheta^* \omega^c = \omega_{\restriction P_V},$$

where P_V is the restriction of P to V.

Proof Recall that the matrix-valued function u on $\mathrm{S}_{\mathbb{C}}(k, lk)$ mentioned above realizes $\mathrm{S}_{\mathbb{C}}(k, lk)$ as the subset of the vector space of complex $(lk) \times k$-matrices which is

[22] This point of view has already been outlined before in [169].

defined by the relation $A^\dagger A = \mathbb{1}_k$. In this realization, the action of the structure group $\mathrm{U}(k)$ is given by right multiplication.

Let $\pi : P \to M$ denote the canonical projection. Choose a local trivialization of P over U, let $\kappa : P_U \to \mathrm{U}(k)$ be the corresponding equivariant mapping and let $s : U \to P$ be the associated local section. Consider the local representative $\mathscr{A} := s^*\omega$. By Lemma 3.8.2, using a chart on U, we can find smooth mappings $f_1, \ldots, f_l : V \to \mathrm{M}_k(\mathbb{C})$ satisfying

$$\sum\nolimits_{i=1}^{l} f_i^\dagger f_i = \mathbb{1}_k, \quad \sum\nolimits_{i=1}^{l} f_i^\dagger \mathrm{d} f_i = \mathscr{A}.$$

Define a mapping

$$\vartheta : P_V \to \mathrm{S}_{\mathbb{C}}(k, lk), \quad \vartheta(p) := \begin{bmatrix} f_1\big(\pi(p)\big)\kappa(p) \\ \vdots \\ f_l\big(\pi(p)\big)\kappa(p) \end{bmatrix}.$$

This makes sense, because

$$\vartheta(p)^\dagger \vartheta(p) = \kappa(p)^\dagger \left(\sum\nolimits_{i=1}^{l} \big(f_i\big(\pi(p)\big)\big)^\dagger f_i\big(\pi(p)\big)\right) \kappa(p) = \mathbb{1}_k,$$

so that ϑ takes values in $\mathrm{S}_{\mathbb{C}}(k, lk)$, indeed. It is easy to see that ϑ is equivariant and hence a morphism of principal $\mathrm{U}(k)$-bundles. Using (3.8.1) and $\kappa \circ s = \mathbb{1}_k$, we finally compute

$$s^*\vartheta^*\omega^c = s^*(\vartheta^\dagger \mathrm{d}\vartheta) = (\vartheta \circ s)^\dagger \mathrm{d}(\vartheta \circ s) = \sum\nolimits_{i=1}^{l} f_i^\dagger \mathrm{d} f_i = \mathscr{A}.$$

Hence, $\vartheta^*\omega^c = \omega$ over V. ∎

Theorem 3.8.4 *Let P be a principal $\mathrm{U}(k)$-bundle over a manifold M of dimension $\leq n$ and let $m := (n+1)kl$. For every connection form ω on P, there exists a bundle morphism $\vartheta : P \to \mathrm{S}_{\mathbb{C}}(k, m)$ such that $\vartheta^*\omega^c = \omega$.*

Proof According to [480], there exists an open covering $\{W_1, \ldots, W_{n+1}\}$ of M such that each W_i decomposes into a disjoint union of relatively compact open subsets V_{ij} whose closure is contained in a coordinate neighbourhood over which P is trivial. To each V_{ij}, we can apply Lemma 3.8.3. For each fixed i, the resulting $\mathrm{U}(k)$-morphisms $\vartheta_{ij} : P_{V_{ij}} \to \mathrm{S}_{\mathbb{C}}(k, lk)$ combine to $\mathrm{U}(k)$-morphisms $\vartheta_i : P_{V_i} \to \mathrm{S}_{\mathbb{C}}(k, lk)$ satisfying $\omega_{\restriction P_{V_i}} = \vartheta_i^*\omega^c$. Extend the ϑ_i arbitrarily to smooth mappings from P to the space of complex $(lk \times k)$-matrices. Choose a partition of unity $\{\varphi_1, \ldots, \varphi_{n+1}\}$ subordinate to the covering $\{W_1, \ldots, W_{n+1}\}$ and define a mapping

$$\vartheta : P \to \mathrm{S}_{\mathbb{C}}\big(k, (n+1)lk\big), \quad \vartheta(p) := \begin{bmatrix} \sqrt{\varphi_1\big(\pi(p)\big)}\ \vartheta_1(p) \\ \vdots \\ \sqrt{\varphi_{n+1}\big(\pi(p)\big)}\ \vartheta_{n+1}(p) \end{bmatrix}.$$

This makes sense, because

$$\vartheta(p)^\dagger \vartheta(p) = \sum_{i=1}^{n+1} \varphi_i\big(\pi(p)\big)\, \vartheta_i(p)^\dagger \vartheta_i(p) = \mathbb{1}_k,$$

as $\vartheta_i(p)^\dagger \vartheta_i(p) = \mathbb{1}_k$ whenever $\varphi\big(\pi(p)\big) \neq 0$. Finally, we compute

$$\vartheta^* \omega^c = \vartheta^\dagger \, \mathrm{d}\vartheta = \sum_{i=1}^{n+1} (\pi^*\varphi_i)\ \vartheta_i^\dagger \, \mathrm{d}\vartheta_i + \sum_{i=1}^{n+1} (\vartheta_i^\dagger \vartheta_i) \sqrt{\pi^*\varphi_i}\ \mathrm{d}\sqrt{\pi^*\varphi_i}.$$

Since $(\pi^*\varphi_i)\,\vartheta_i^\dagger \mathrm{d}\vartheta_i = \vartheta_i^*\omega^c = \omega$ whenever $\pi^*\varphi_i \neq 0$, the first term yields ω. Since $\vartheta_i^\dagger \vartheta_i = \mathbb{1}_k$ whenever $\pi^*\varphi_i \neq 0$, and since

$$\sum_{i=1}^{n+1} \sqrt{\pi^*\varphi_i}\ \mathrm{d}\sqrt{\pi^*\varphi_i} = \tfrac{1}{2}\, \mathrm{d}\left(\sum_{i=1}^{n+1} \pi^*\varphi_i\right) = 0,$$

the second term vanishes. ■

Corollary 3.8.5 *Let G be a compact Lie group and let P be a principal G-bundle. There exists a principal G-bundle $E \to B$ and a connection form ω_0 on E such that for every connection form ω on P, there exists a bundle morphism $\vartheta : P \to E$ fulfilling $\vartheta^*\omega_0 = \omega$.*

Proof By [105, Theorem 4.1], G admits a faithful unitary representation on $\mathbb{C}^k$ for some k. Let $\lambda : G \to \mathrm{U}(k)$ be the corresponding Lie subgroup embedding. Consider the principal $\mathrm{U}(k)$-bundle $P^{[\lambda]}$ and let $j : P \to P^{[\lambda]}$ be the induced mapping, given by $j(p) = [(p, \mathbb{1}_k)]$. By Corollary 1.3.14, there exists a unique connection ω_1 on $P^{[\lambda]}$ such that $j^*\omega_1 = \mathrm{d}\lambda \circ \omega$. By Theorem 3.8.4, there exists a positive integer m and a $\mathrm{U}(k)$-morphism $\vartheta_1 : P^{[\lambda]} \to \mathrm{S}_{\mathbb{C}}(k, m)$ such that $\vartheta_1^*\omega^c = \omega_1$. Via λ, the structure group G acts freely and properly on $E := \mathrm{S}_{\mathbb{C}}(k, m)$ and thus turns E into a principal G-bundle over the quotient manifold $B := E/G$. On the one hand, there exists a unique G-morphism $\vartheta : P \to E$ satisfying $\vartheta_1 \circ j = \vartheta$. On the other hand, since $\mathrm{U}(k)$ is compact, we have a reductive decomposition $\mathfrak{u}(k) = \mathrm{d}\lambda(\mathfrak{g}) \oplus \mathfrak{m}$. Let $\mathrm{pr}_{\mathfrak{g}} : \mathfrak{u}(k) \to \mathrm{d}\lambda(\mathfrak{g}) \to \mathfrak{g}$ denote the corresponding projection. One can check that $\omega_0 := \mathrm{pr}_{\mathfrak{g}} \circ \omega^c$ is a connection form on the principal G-bundle $E \to B$ (Exercise 3.8.1, cf. also Example 1.3.19). We compute

$$\vartheta^*\omega_0 = \mathrm{pr}_{\mathfrak{g}} \circ (\vartheta^*\omega^c) = \mathrm{pr}_{\mathfrak{g}} \circ (j^*\vartheta_1^*\omega^c) = \mathrm{pr}_{\mathfrak{g}} \circ (j^*\omega_1) = \mathrm{pr}_{\mathfrak{g}} \circ \mathrm{d}\lambda \circ \omega = \omega.$$

■

Now, we turn to the discussion of universal connections for arbitrary Lie groups. Let P be a principal G-bundle over the base manifold M with action Ψ and projection π. By point 2 of Remark 1.3.3, every connection ω on P defines a horizontal lift $\ell_p^\omega : \mathrm{T}_{\pi(p)}M \to \mathrm{T}_p P$ for every $p \in P$. It is evident that the assignment $p \mapsto \ell_p^\omega$ defines a smooth section in the vector bundle $\mathrm{Hom}(\pi^*\mathrm{T}M, \mathrm{T}P)$ over P and that this section takes values in the subset

$$\mathrm{J}^1 P := \left\{ \ell_p \in \mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P) : \pi'_p \circ \ell_p = \mathrm{id}_{\mathrm{T}_{\pi(p)}M} \right\}, \tag{3.8.2}$$

where the lower index p means that ℓ_p belongs to the fibre over p. We show that $\mathrm{J}^1 P$ inherits the structure of a fibre bundle over P from $\mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P)$. There are natural surjective mappings

$$\pi_1 : \mathrm{J}^1 P \to P, \quad \pi_1(\ell_p) := p, \qquad \pi_0 : \mathrm{J}^1 P \to M, \quad \pi_0 := \pi \circ \pi_1,$$

called the target projection and the source projection, respectively. For $p \in P$ and $m \in M$, denote

$$\mathrm{J}^1_p P := \pi_1^{-1}(p), \quad \mathrm{J}^1_m P := \pi_0^{-1}(m).$$

Consider the vertical vector bundle morphism

$$\tau : \mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P) \to \mathrm{End}(\pi^* \mathrm{T}M), \quad \tau(\ell_p) := \pi'_p \circ \ell_p.$$

According to Example 2.7.7 of Part I, $\ker \tau$ is a vertical vector subbundle of $\mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P)$ of rank $r = \dim(M) \cdot \dim(G)$. It is not hard to see that, for every $p \in P$, the subset $\mathrm{J}^1_p P$ is an affine subspace of $\mathrm{Hom}\left((\pi^* \mathrm{T}M)_p, \mathrm{T}_p P\right)$ with translation vector space given by the linear subspace $\ker \tau_p$. Given $p_0 \in P$, we find an open neighbourhood $U \subset P$, a local frame $\{s_1, \ldots, s_r\}$ in $\ker \tau$ over $U \subset P$ and a local section s in $\mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P)$ over U taking values in $\mathrm{J}^1 P$ (for example, the section defined by a connection on P). Then, the mapping

$$\psi : U \times \mathbb{R}^r \to \pi_1^{-1}(U), \quad \psi(p, \mathbf{x}) := s(p) + \sum_{i=1}^{r} x^i s_i(p), \tag{3.8.3}$$

is a bijection. We leave it to the reader to check that the transition mappings between two such bijections are smooth (Exercise 3.8.2). Hence, the collection of mappings (3.8.3) defines on $\mathrm{J}^1 P$ the structure of a smooth manifold. With respect to this structure, $\mathrm{J}^1 P$ endowed with the projection $\pi_1 : \mathrm{J}^1 P \to P$ is a fibre bundle with typical fibre $\mathbb{R}^r$. More precisely, it is an affine bundle with translation vector bundle $\ker \tau$ and an affine subbundle of $\mathrm{Hom}(\pi^* \mathrm{T}M, \mathrm{T}P)$. Note that $\mathrm{J}^1 P$ is a concrete realization of the first jet manifold of sections in P. It is therefore referred to as the first section jet bundle of P.

Next, we are going to endow $\mathrm{J}^1 P$ with the structure of a principal G-bundle. Recall from Example 6.1.2/5 of Part I that the action Ψ of G on P induces an action of G on $\mathrm{T}P$ by the tangent mappings $(\Psi_a)'$, $a \in G$. Moreover, G acts on the vector bundle $\pi^* \mathrm{T}M$ by $\left(a, (p, X)\right) \mapsto \left(\Psi_a(p), X\right)$. Since both these actions cover Ψ, they induce a smooth action of G on $\mathrm{Hom}\left(\pi^* \mathrm{T}M, \mathrm{T}P\right)$ by assigning to an element ℓ_p in the fibre over p the element $(\Psi_a)'_p \circ \ell_p$ in the fibre over $\Psi_a(p)$ (Exercise 3.8.3). Since for $\ell_p \in \mathrm{J}^1 P$ we have

$$\pi'_{\Psi_a(p)} \circ (\Psi_a)'_p \circ \ell_p = \pi'_p \circ \ell_p = \mathrm{id}_{\mathrm{T}_{\pi(p)}M},$$

the submanifold J^1P is invariant under this action. Since J^1P is a vertical subbundle of $\mathrm{Hom}(\pi^*\mathrm{T}M, \mathrm{T}P)$, an argument similar to that for vertical vector subbundles in Example 2.7.2 of Part I shows that J^1P is an embedded submanifold of $\mathrm{Hom}(\pi^*\mathrm{T}M, \mathrm{T}P)$ (Exercise 3.8.4). Hence, by restriction, the action of G on $\mathrm{Hom}(\pi^*\mathrm{T}M, \mathrm{T}P)$ induces the action

$$\Psi^1 : G \times \mathrm{J}^1P \to \mathrm{J}^1P, \quad \Psi^1_a(\ell_p) = (\Psi_a)'_p \circ \ell_p, \tag{3.8.4}$$

of G on J^1P. By construction, the action Ψ^1 covers Ψ. By Remark 6.3.9 of Part I, this implies that it is free and proper. As a consequence, the orbit space

$$\mathrm{C}^1P := (\mathrm{J}^1P)/G$$

carries a unique smooth manifold structure such that the natural projection

$$\rho : \mathrm{J}^1P \to \mathrm{C}^1P$$

to classes is a submersion. With respect to this structure, J^1P is a principal G-bundle over C^1P with action Ψ^1 and projection ρ, cf. Sect. 6.5 of Part I.

Another consequence of the fact that Ψ^1 covers Ψ is that the projection $\pi^1 : \mathrm{J}^1P \to P$ is a morphism of G-bundles. The induced mapping of the base manifolds

$$\delta : \mathrm{C}^1P \to M$$

is a surjective submersion. To summarize, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{J}^1P \arrow[r, "\pi_1"] \arrow[d, "\rho"'] \arrow[dr, "\pi_0"] & P \arrow[d, "\pi"] \\
\mathrm{C}^1P \arrow[r, "\delta"] & M
\end{tikzcd}$$

Using the mappings (3.8.3) and local sections σ of P over $V \subset M$, one can cover C^1P by local diffeomorphisms of the type

$$\rho \circ \chi \circ (\sigma \times \mathrm{id}_{\mathbb{R}^r}) : V \times \mathbb{R}^r \to \delta^{-1}(V).$$

Hence, C^1P inherits from J^1P the structure of a fibre bundle over M with typical fibre $\mathbb{R}^r$ (Exercise 3.8.5). In fact, one can show that it inherits the structure of an affine bundle with translation vector bundle $\mathrm{Hom}(\mathrm{T}M, \mathrm{Ad}(P))$.

Lemma 3.8.6 *If (U, χ) is a local trivialization of P, then*

$$\tilde{\chi} : \pi_0^{-1}(U) \to \delta^{-1}(U) \times G, \quad \tilde{\chi}(\ell) := \big(\rho(\ell), \mathrm{pr}_G \circ \chi \circ \pi_1(\ell)\big)$$

is a local trivialization of ρ.

Proof As in Chap. 1, we denote $\kappa := \mathrm{pr}_G \circ \chi$. It suffices to show that the subset

$$S := \{\ell \in \pi_0^{-1}(U) : \kappa \circ \pi_1(\ell) = \mathbb{1}\}$$

of $\pi_0^{-1}(U)$ is an embedded submanifold transversal to the fibres of ρ, because, then, ρ induces a diffeomorphism from S onto $\delta^{-1}(U)$. Obviously, the inverse of this diffeomorphism is a local section of ρ over $\delta^{-1}(U)$ and $\tilde{\chi}$ is the corresponding local trivialization.

Since κ and π_1 are submersions, $\kappa \circ \pi_1$ is a submersion, too. Hence, by Corollary 1.8.3 of Part I, S is an embedded submanifold and $\mathrm{T}_\ell S = \ker(\kappa \circ \pi_1)'_\ell$ for all $\ell \in S$. To check that S is transversal to the fibres of ρ, let $A \in \mathfrak{g}$ and let A^1_* denote the Killing vector field on $\mathrm{J}^1 P$ generated by A. We have to show that $(\kappa \circ \pi_1)'_\ell (A^1_*)_\ell = 0$ implies $(A^1_*)_\ell = 0$. Since π_1 and κ are equivariant, we have

$$(\kappa \circ \pi_1)'_\ell (A^1_*)_\ell = \left(\mathrm{L}_{\kappa \circ \pi_1(\ell)}\right)'_{\mathbb{1}} A .$$

Since left translation by $\kappa \circ \pi_1(\ell)$ is a diffeomorphism of G and hence $\left(\mathrm{L}_{\kappa \circ \pi_1(\ell)}\right)'_{\mathbb{1}}$ is bijective, it follows that $A = 0$ and hence $(A^1_*)_\ell = 0$, as asserted. ∎

Now, we will relate connections on P to sections of π_1 and δ. As noted above, every connection ω on P defines a section $\check{\omega}$ of π_1 by assigning to $p \in P$ the horizontal lift $\ell^\omega_p : \mathrm{T}_{\pi(p)} M \to \mathrm{T}_p P$. By the equivariance property of connections, we have

$$\ell^\omega_{\Psi_a(p)} = (\Psi_a)'_p \ell^\omega_p .$$

Therefore, $\check{\omega}$ is equivariant and hence a morphism of principal G-bundles. As a consequence, it projects to a smooth mapping $\hat{\omega} : M \to \mathrm{C}^1 P$ and we have the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\check{\omega}"] \arrow[d, "\pi"'] & \mathrm{J}^1 P \arrow[d, "\rho"] \\ M \arrow[r, "\hat{\omega}"] & \mathrm{C}^1 P \end{tikzcd}$$

Since $\check{\omega}$ is a section of π_1, $\hat{\omega}$ is a section of δ.

Proposition 3.8.7 *Let P be a principal G-bundle over M.*

1. *The assignment $\omega \mapsto \check{\omega}$ defines a bijection between connections on P and G-equivariant sections of $\pi_1 : \mathrm{J}^1 P \to P$ or, equivalently, principal G-bundle morphisms $P \to \mathrm{J}^1 P$ satisfying $\pi_1 \circ \check{\omega} = \mathrm{id}_P$.*
2. *The assignment $\omega \mapsto \hat{\omega}$ defines a bijection between connections on P and sections of $\delta : \mathrm{C}^1 P \to M$.*

Proof 1. Every G-equivariant section σ of π_1 defines an equivariant distribution on P by assigning to p the subspace $\mathrm{im}\left(\sigma(p)\right) \subset \mathrm{T}_p P$. Since $\pi'_p \circ \sigma(p) = \mathrm{id}_{\mathrm{T}_{\pi(p)} M}$, this distribution is complementary to $\mathrm{V}P$ and hence defines a connection $\overline{\sigma}$. We have $\overline{\check{\sigma}} = \sigma$ and $\check{\overline{\omega}} = \omega$.

2. Given a section σ of δ, we define a section $\check{\sigma} : P \to \mathrm{J}^1 P$ by assigning to p the unique representative of the class $\sigma\big(\pi(p)\big)$ in the fibre $(\mathrm{J}^1 P)_p$. Since Ψ^1 projects to Ψ, this section is equivariant. Let χ be a local trivialization of P. Composing $\check{\sigma}$ with the induced local trivialization $\tilde{\chi}$ of ρ provided by Lemma 3.8.6, we obtain

$$\tilde{\chi} \circ \check{\sigma}(p) = \big(\sigma \circ \pi(p), \mathrm{pr}_G \circ \chi(p)\big).$$

Hence, $\check{\sigma}$ is smooth. This shows that for every section σ of δ there exists a unique equivariant section of π_1 projecting to σ. In view of point 1, this yields the assertion. ■

As a consequence of point 2 of Proposition 3.8.7, $\mathrm{C}^1 P$ is usually referred to as the bundle of connections. However, more appropriately, it could also be called the manifold of equivariant tangent lifts of P. Accordingly, the jet manifold $\mathrm{J}^1 P$ could also be called the manifold of tangent lifts of P.

Our next aim is to show that the principal G-bundle $\rho : \mathrm{J}^1 P \to \mathrm{C}^1 P$ carries a tautological connection. The key observation is that we have a tautological mapping

$$\mathrm{h} : \mathrm{T}(\mathrm{J}^1 P) \to \mathrm{T}P, \quad \mathrm{h}(X_\ell) := \ell(\pi_0' X_\ell). \tag{3.8.5}$$

Associated with h, we have the mapping

$$\mathrm{v} : \mathrm{T}(\mathrm{J}^1 P) \to \mathrm{T}P, \quad \mathrm{v} := \pi_1' - \mathrm{h}\,. \tag{3.8.6}$$

Lemma 3.8.8 *The mappings* h *and* v *are equivariant*[23] *vector bundle morphisms covering* π_1.

Proof It suffices to prove the assertion for h, because, then, both h and π_1' project to π_1 and the assertion for v follows.

Obviously, h preserves fibres, projects to π_1 and is fibrewise linear. To see that it is smooth, we decompose it into the smooth mapping

$$\mathrm{T}(\mathrm{J}^1 P) \to \mathrm{J}^1 P \times \pi^*\mathrm{T}M, \quad X_\ell \mapsto \big(\ell, \tau(\pi_1' X_\ell)\big),$$

restricted in range to the embedded submanifold $\mathrm{J}^1 P \times_P \pi^*\mathrm{T}M$, and the evaluation mapping

$$\mathrm{Hom}(\pi^*\mathrm{T}M, \mathrm{T}P) \times_P \pi^*\mathrm{T}M \to \mathrm{T}P, \quad (\ell, X) \mapsto \ell(X).$$

For a proof that the latter is smooth, see Exercise 3.8.6. Equivariance is obvious. ■

The vector bundle morphisms h and v satisfy the obvious relations

[23] W.r.t. the actions induced by Ψ^1 and Ψ on the tangent bundles $\mathrm{T}(\mathrm{J}^1 P)$ and $\mathrm{T}P$, respectively.

$$\mathrm{h} + \mathrm{v} = \pi_1', \quad \pi' \circ \mathrm{h} = \pi_0', \quad \pi' \circ \mathrm{v} = 0. \tag{3.8.7}$$

According to the last relation, v maps $\mathrm{T}(\mathrm{J}^1P)$ to the vertical subbundle $\mathrm{V}P \subset \mathrm{T}P$. Hence, we can view it as a mapping $\mathrm{v} : \mathrm{T}(\mathrm{J}^1P) \to \mathrm{V}P$ and thus compose it with the mapping $\mathrm{K} : \mathrm{V}P \to \mathfrak{g}$ defined on the fibre over $p \in P$ as the inverse of Ψ_p' to obtain a smooth mapping

$$\omega_0 := \mathrm{K} \circ \mathrm{v} : \mathrm{T}(\mathrm{J}^1P) \to \mathfrak{g}. \tag{3.8.8}$$

Since v and K are fibrewise linear, so is ω_0. Hence, it defines a 1-form on J^1P with values in $\mathfrak{g}$. This 1-form will be denoted by the same symbol.

Proposition 3.8.9 *The* 1*-form* ω_0 *defined by* (3.8.8) *is a connection form on the principal* G*-bundle* $\rho : \mathrm{J}^1P \to \mathrm{C}^1P$.

Proof According to Proposition 1.3.6, we have to check conditions 2 and 3 of Proposition 1.3.5. First, let $a \in G$. Since v and K are equivariant, one has $\omega_0 \circ (\Psi_a^1)' = \mathrm{Ad}(a^{-1}) \circ \omega_0$. If we interpret ω_0 as a $\mathfrak{g}$-valued 1-form, this equation reads $(\Psi_a^1)^*\omega_0 = \mathrm{Ad}(a^{-1}) \circ \omega_0$.

Now, let $A \in \mathfrak{g}$ and let A_*^1 and A_* denote the Killing vector fields generated by A on J^1P and P, respectively. Since $\pi_0 \circ \Psi_a^1 = \pi_0$, for every $\ell \in \mathrm{J}^1P$, we have $\pi_0'\big(A_*^1(\ell)\big) = 0$ and hence $\mathrm{v}\big(A_*^1(\ell)\big) = \pi_1'\big(A_*^1(\ell)\big)$. Since π_1 is equivariant, the transformation property of Killing vector fields[24] yields

$$\mathrm{v}\big(A_*^1(\ell)\big) = A_*\big(\pi_1(\ell)\big).$$

Applying K to both sides of this equation, we obtain $\omega_0\big(A_*^1(\ell)\big) = A$. ∎

Definition 3.8.10 The connection defined by ω_0 is called the tautological connection of the principal G-bundle $\rho : \mathrm{J}^1P \to \mathrm{C}^1P$.

The most important property of ω_0 is that via pullback it can reproduce every connection on P.

Proposition 3.8.11 *For every connection* ω *on* P *and the corresponding equivariant section (or principal* G*-bundle morphism)* $\check{\omega} : P \to \mathrm{J}^1P$*, one has* $\check{\omega}^*\omega_0 = \omega$.

Proof Let ω be given and let $p \in P$ and $X \in \mathrm{T}_pP$. Then, $\check{\omega}'(X)$ is a tangent vector of J^1P at ℓ_p^ω. Using this and the obvious relations $\pi_1 \circ \check{\omega} = \mathrm{id}_P$ and $\pi_0 \circ \check{\omega} = \pi$, we calculate

$$\begin{aligned}
(\check{\omega}^*\omega_0)(X) &= \mathrm{K} \circ \mathrm{v}\big(\check{\omega}'(X)\big) \\
&= \mathrm{K}\left((\pi_1 \circ \check{\omega})_p'(X) - \ell_p^\omega \circ (\pi_0 \circ \check{\omega})_p'(X)\right) \\
&= \mathrm{K}\big(X - \mathrm{hor}_\omega X\big) \\
&= \omega(X).
\end{aligned}$$

∎

[24] See Proposition I/6.2.4.

Remark 3.8.12 The tautological connection ω_0 is the unique connection on the principal G-bundle $\rho : \mathrm{J}^1 P \to \mathrm{C}^1 P$ with the property stated in Proposition 3.8.11, see Exercise 3.8.7. That is, one may define it by that property. ♦

With the tautological connection of $\rho : \mathrm{J}^1 P \to \mathrm{C}^1 P$ we have a connection at hand which is universal for the connections on P. To obtain the desired n-universal connection, we apply the above construction to the following principal G-bundle. By Corollary 3.4.12, there exists a smooth principal G-bundle $\pi_E : E \to B$ with $\pi_i(E) = 0$ for all $i \leq n$. Define

$$\tilde{B} := B \times \mathbb{R}^{2n}, \quad \tilde{E} := E \times \mathbb{R}^{2n}$$

and let G act on $\tilde{E}$ by acting on the first factor. Obviously, $\tilde{E}$ is a principal G-bundle over $\tilde{B}$, where the projection is given by the direct product of π_E with the identical mapping of $\mathbb{R}^{2n}$. It is not hard to see that the mapping

$$\tilde{E} \to \mathrm{pr}_B^* E, \quad (e, \mathbf{x}) \mapsto \big(\pi_E(e), \mathbf{x}\big)$$

is a principal G-bundle isomorphism over $\tilde{B}$.

Theorem 3.8.13

1. *The principal G-bundle $\tilde{\rho} : \mathrm{J}^1\tilde{E} \to \mathrm{C}^1\tilde{E}$ is n-universal.*
2. *The tautological connection on $\tilde{\rho} : \mathrm{J}^1\tilde{E} \to \mathrm{C}^1\tilde{E}$ is n-universal.*

Proof 1. Since $\mathrm{J}^1\tilde{E} \to \tilde{E}$ is a fibre bundle with contractible fibres, the exact homotopy sequence (3.2.6) yields $\pi_i(\mathrm{J}^1\tilde{E}) = \pi_i(\tilde{E})$ for all i. Since $\mathbb{R}^{2n}$ is contractible, we have $\pi_i(\tilde{E}) = \pi_i(E)$ for all i. Since $\pi_i(E) = 0$ for all $i \leq n$, Theorem 3.4.6 yields the assertion.

2. Denote the tautological connection on $\mathrm{J}^1\tilde{E}$ by ω_0. Let $\pi : P \to M$ be a principal G-bundle over M with $\dim(M) \leq n$ and let ω be a connection on P. We have to construct a morphism of principal G-bundles $\vartheta : P \to \mathrm{J}^1\tilde{E}$ such that $\vartheta^*\omega_0 = \omega$.

By Corollary 3.4.12, there exists a smooth mapping $f_1 : M \to B$ such that $P \cong f_1^* E$. By the strong Whitney Embedding Theorem,[25] there exists a smooth embedding $f_2 : M \to \mathbb{R}^{2n}$. Define

$$f : M \to \tilde{B}, \quad f(m) := \big(f_1(m), f_2(m)\big).$$

Since f_2 is an embedding, so is f. Hence, by the Tubular Neighbourhood Theorem for embedded submanifolds,[26] there exists a diffeomorphism χ from an open neighbourhood U of $f(M)$ in $\tilde{B}$ onto an open neighbourhood of the zero section s_0 in the normal bundle $\mathrm{N}M \subset f^*\mathrm{T}\tilde{B}$ such that $\chi \circ f = s_0$. Define

$$H : U \times [0, 1] \to U, \quad H(x, t) := \chi^{-1}\big((1 - t)\chi(x)\big).$$

[25] Every smooth n-dimensional manifold can be smoothly embedded into $\mathbb{R}^{2n}$ [7, Theorem II.2.2].

[26] See Remark 6.4.7 in Part I.

This mapping is a smooth strong deformation retraction of U to the subset $f(M)$. There exists a unique mapping $\varphi : U \to M$ such that $f \circ \varphi = H_1$. In the terminology of Part I, φ is the restriction in range of H_1 to the embedded submanifold (M, f). By Proposition 1.6.10 of Part I, φ is smooth. Consider the pullback bundle $\varphi^* P$ over U. Using $\varphi^* P$, we will construct three morphisms ϑ_1, ϑ_2 and ϑ_3 whose composition will yield the desired morphism ϑ.

First, since $\varphi \circ f = \mathrm{id}_M$, we have $\varphi \circ f \circ \pi = \pi$. Hence, we can define a mapping

$$\vartheta_1 : P \to \varphi^* P, \quad \vartheta_1(p) := \big(f \circ \pi(p), p\big).$$

This mapping is easily seen to be a principal G-bundle morphism projecting to f.

Second, let $\tilde{E}_U$ denote the restriction of $\tilde{E}$ to U. We claim that there exists an isomorphism

$$\vartheta_2 : \varphi^* P \to \tilde{E}_U.$$

To see this, let $\mathrm{pr}_B : \tilde{B} \to B$ denote the projection to the first factor. It is easy to see that $\mathrm{pr}_B^* E$ is isomorphic over $\tilde{B}$ to $\tilde{E}$. Using this and $\mathrm{pr}_B \circ f = f_1$, we find that

$$P \cong f_1^* E = f^* \mathrm{pr}_B^* E \cong f^* \tilde{E}$$

over M. Viewing f as a mapping to U rather than to $\tilde{B}$, we may replace $\tilde{E}$ by $\tilde{E}_U$ on the right hand side. Taking now the pullback by φ, using that $f \circ \varphi = H_1$ is homotopic to $H_0 = \mathrm{id}_U$ and applying Corollary 3.3.5, we find that

$$\varphi^* P \cong \varphi^* f^* \tilde{E}_U = H_1^* \tilde{E}_U \cong \tilde{E}_U$$

over U, as asserted.

Third, via the natural bundle morphism $\Phi : \varphi^* P \to P$ given by $\Phi(x, p) = p$, the connection ω on P induces the connection $\Phi^* \omega$ on the pullback bundle $\varphi^* P$ and hence the connection $(\vartheta_2^{-1})^* \Phi^* \omega$ on $\tilde{E}_U$. Let

$$\vartheta_3 : \tilde{E}_U \to \mathrm{J}^1 \tilde{E}_U \subset \mathrm{J}^1 \tilde{E}$$

be the corresponding principal G-bundle morphism provided by Proposition 3.8.7.

Finally, we compose ϑ_1, ϑ_2 and ϑ_3 to obtain the principal G-bundle morphism

$$\vartheta : P \xrightarrow{\vartheta_1} \varphi^* P \xrightarrow{\vartheta_2} \tilde{E}_U \xrightarrow{\vartheta_3} \mathrm{J}^1 \tilde{E}.$$

To check that it has the desired property, using Proposition 3.8.11 and the obvious identity $\Phi \circ \vartheta_1 = \mathrm{id}_P$, we compute

$$\vartheta^* \omega_0 = \vartheta_1^* \vartheta_2^* \vartheta_3^* \omega_0 = \vartheta_1^* \vartheta_2^* \big((\vartheta_2^{-1})^* \Phi^* \omega\big) = \omega.$$

This completes the proof of Theorem 3.8.13. ■

Exercises

3.8.1 Let P be a principal G-bundle and let $H \subset G$ be a closed subgroup admitting a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with associated projection $\mathrm{pr}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$. Show that for every connection form ω on P, the $\mathfrak{h}$-valued 1-form $\mathrm{pr}_{\mathfrak{h}} \circ \omega$ is a connection form on the principal H-bundle $P \to P/H$.

3.8.2 Complete the construction of a smooth manifold structure on the jet manifold $\mathrm{J}^1 P$ by showing that the transition mappings between the inverses of the mappings (3.8.3) are smooth.

3.8.3 Prove the following. For $i = 1, 2$, let E_i be vector bundles over a smooth manifold M and let $\Psi^{(i)}$ be actions of G on E_i by vector bundle automorphisms. If both these actions project to the same action Ψ of G on M, they define an action of G on $\mathrm{Hom}(E_1, E_2)$ by

$$(a, \ell_m) \mapsto \left(\Psi_a^{(2)}\right)_m \circ \ell_m \circ \left(\Psi_{a^{-1}}^{(1)}\right)_{\Psi_a(m)}$$

and this action projects to Ψ.
Hint. To prove smoothness, use that $\mathrm{Hom}(E_1, E_2) \cong E_2 \otimes E_1^*$.

3.8.4 Use the argument for vertical vector subbundles in Example 2.7.2 of Part I to show that a vertical subbundle is always embedded.

3.8.5 Show that $\mathrm{C}^1 P$ inherits from $\mathrm{J}^1 P$ the structure of a fibre bundle over M.

3.8.6 Let E_1, E_2 be vector bundles over M. Prove that the evaluation mapping $\mathrm{Hom}(E_1, E_2) \times_M E_1 \to E_2$, $(\mu, x) \mapsto \mu(x)$, is smooth.
Hint. Use the local trivializations of E_1, E_2 and $\mathrm{Hom}(E_1, E_2)$ induced by a local frame in E_1 and a local frame in E_2, both defined over the same open subset of M.

3.8.7 Prove that the tautological connection ω_0 on the principal G-bundle $\rho : \mathrm{J}^1 P \to \mathrm{C}^1 P$ associated with a principal G-bundle P is uniquely determined by the property that $\check{\omega}^* \omega_0 = \omega$ for all connections ω on P.
Hint. Show that for a vector bundle $\pi : E \to M$, the tangent space at a point $e \in E$ is spanned by vertical vectors and by tangent vectors of the form $\sigma' X$, where σ is a (global) section of E with e in its image and $X \in \mathrm{T}_{\pi(e)} M$. Use a section in the affine bundle $\delta : \mathrm{C}^1 P \to M$ to carry over this statement from the translation vector bundle. Use this and Proposition 3.8.7/2 to prove that for every $\ell \in \mathrm{J}^1 P$, the subspace of $\mathrm{T}_\ell(\mathrm{J}^1 P)$ spanned by vectors of the form $\check{\omega}'(X)$, where ω is a connection on P and $X \in \mathrm{T}_{\pi_1(\ell)} P$, contains a complement of the tangent space of the G-orbit through ℓ. Since connections on $\rho : \mathrm{J}^1 P \to \mathrm{C}^1 P$ necessarily coincide on the latter subspace, this proves the assertion.

Chapter 4
Cohomology Theory of Fibre Bundles. Characteristic Classes

In Chap. 3, we have seen that principal bundles with a given structure group and a given base manifold are classified up to vertical isomorphisms by the homotopy classes of continuous mappings from the base manifold to the classifying space of the structure group. While this description is complete in that it provides exact labels for the isomorphism classes, for many problems it is ineffective. Characteristic classes, on the other hand, allow for applying the machinery of algebraic topology. The price one has to pay for this is that they are only able to distinguish between the isomorphism classes to the extent to which cohomology can resolve homotopy. We will view characteristic classes as being defined by generators of the cohomology ring of the classifying space, rather than being defined axiomatically. Accordingly, we proceed as follows. In Sect. 4.2, we study the cohomology rings with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$ of the classifying spaces for the classical compact Lie groups and use their generators to define the Chern, Pontryagin and Stiefel–Whitney classes. The main tool here is the Euler class of an oriented real vector bundle and the corresponding Gysin sequence. Then, in Sects. 4.3 and 4.4, we derive the main properties of the characteristic classes so constructed, including the Whitney Sum Formula, the Splitting Principle and the relations induced by field extension and field restriction. In Sect. 4.6, we discuss the Weil homomorphism, which provides a geometric description of characteristic classes in terms of de Rham cohomology. In Sect. 4.7, we deal with genera and the Chern character as examples of formal power series in the characteristic classes. Finally, in Sect. 4.8, we explain a method to construct an approximation of the classifying space in terms of Eilenberg–MacLane spaces, known as the Postnikov tower. It allows for proving, for example, that the Chern classes classify $\mathrm{U}(n)$-bundles over manifolds of small dimension. This method will also be used in the discussion of gauge orbit types in Chap. 8.

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4.1 Basics

We assume the reader to be familiar with the basics of homology and cohomology theory. To fix the notation, let a topological space X be given. We denote

- the group of singular k-chains in X by $C_k(X)$,
- the k-th singular homology group of X by $H_k(X)$,
- the k-th singular cohomology group with coefficients in the commutative ring R by $H^k_R(X)$,
- $H^*_R(X) = \bigoplus_{k=0}^{\infty} H^k_R(X)$.

Recall that $H^k_R(X)$ is a module over R and that $H^*_R(X)$ is a ring with respect to the cup product. The cup product of $\alpha, \beta \in H^*_R(X)$ will be denoted by $\alpha \cup \beta$ or simply by $\alpha\beta$. We use the convention $H^k_R(X) = 0$ for $k < 0$. Given a subset $A \subset X$, let $C_k(X, A)$, $H_k(X, A)$, $H^k_R(X, A)$ and $H^*_R(X, A)$ denote the corresponding relative objects. We will use a number of basic tools from algebraic topology, notably

- the Hurewicz Theorem, cf. Sect. VII.10 in [104],
- the Universal Coefficient Theorems, cf. Sect. 5.5 in [598],
- the Künneth Theorem for cohomology, cf. [598, Theorem 5.5.11].

In the first part of this section, we introduce the notion of characteristic class and discuss its basic properties. While characteristic classes can be defined for general fibre bundles, we restrict our attention to the case of principal bundles and vector bundles.

Definition 4.1.1 Let G be a Lie group and let R be a commutative ring. An R-valued characteristic class for principal G-bundles assigns to every topological principal G-bundle $P \to B$ a cohomology class $\alpha(P) \in H^*_R(B)$ such that the following holds. For every continuous mapping $f : B' \to B$ one has

$$\alpha(f^*P) = f^*\alpha(P)\,.$$

Characteristic classes for vector bundles are defined by analogy.

First, let us discuss characteristic classes for principal G-bundles. These are closely related to the cohomology of the classifying space $\mathrm{B}G$. Let $\xi \in H^*_R(\mathrm{B}G)$. For a topological principal G-bundle P over B, define

$$\alpha(P) := f_P^*\xi \tag{4.1.1}$$

with some classifying mapping $f_P : B \to \mathrm{B}G$ for P. This makes sense, because f_P is determined up to homotopy, and homotopic mappings induce the same homomorphism in cohomology. For the same reason, by the universality of the principal G-bundle $\mathrm{E}G \to \mathrm{B}G$, one has $\alpha(P_1) = \alpha(P_2)$ whenever P_1 and P_2 are vertically isomorphic.

Proposition 4.1.2 *For every cohomology class $\xi \in H^*_R(\mathrm{B}G)$, the assignment* (4.1.1) *defines an R-valued characteristic class for principal G-bundles. Every R-valued characteristic class for principal G-bundles arises in this way.*

Proof To see that α defined by (4.1.1) is a characteristic class, let $f : B' \to B$ be given. If $f_P : B \to \mathrm{B}G$ is a classifying mapping for P, then $f_P \circ f : B' \to \mathrm{B}G$ is a classifying mapping for f^*P. Hence,

$$\alpha(f^*P) = (f_P \circ f)^*\xi = f^* \circ f_P^*(\xi) = f^*\big(\alpha(P)\big)\,.$$

Conversely, let $\tilde{\alpha}$ be a characteristic class for principal G-bundles. Define

$$\xi := \tilde{\alpha}(\mathrm{E}G) \in H^*_R(\mathrm{B}G)$$

and let α denote the characteristic class defined by ξ via (4.1.1). Since $\tilde{\alpha}$ is a characteristic class, for any principal G-bundle P with classifying mapping f_P, we have

$$\tilde{\alpha}(P) = f_P^*\tilde{\alpha}(\mathrm{E}G) = f_P^*\xi = \alpha(P)\,.$$

■

Since the cohomology elements of the classifying space $\mathrm{B}G$ correspond bijectively to the characteristic classes for principal G-bundles, they are often referred to as the universal characteristic classes for G.

Proposition 4.1.3 *Let $\lambda : G_1 \to G_2$ be a Lie group homomorphism and let $\xi \in H^*_R(\mathrm{B}G_2)$. Let α be the characteristic class for G_2-bundles defined by ξ and let $\tilde{\alpha}$ be the characteristic class for G_1-bundles defined by* $(\mathrm{B}\lambda)^*\xi \in H^*_R(\mathrm{B}G_1)$. *Then, given topological principal G_i-bundles P_i over B_i and a morphism $\vartheta : P_1 \to P_2$ whose group homomorphism coincides with λ, we have*

$$\tilde{\alpha}(P_1) = f^*\alpha(P_2)\,,$$

where $f : B_1 \to B_2$ denotes the projection of ϑ.

Proof Let $f_i : B_i \to \mathrm{B}G_i$ be classifying mappings for P_i, $i = 1, 2$. According to Proposition 3.7.6, then $f_2 \circ f$ is homotopic to $\mathrm{B}\lambda \circ f_1$. Hence,

$$f^*\alpha(P_2) = f^*(f_2^*\xi) = f_1^*\big((\mathrm{B}\lambda)^*\xi\big) = \tilde{\alpha}(P_1)\,.$$

■

In the special case where $P_2 = {P_1}^{[\lambda]}$ and $\vartheta : P_1 \to P_2$ is the natural morphism sending p to $[(p, \mathbb{1}_{G_2})]$, Proposition 4.1.3 yields the following.

Corollary 4.1.4 *Let $\lambda : G_1 \to G_2$ be a Lie group homomorphism and let $\xi \in H^*_R(\mathrm{B}G_2)$. Let α be the characteristic class for G_2-bundles defined by ξ and let $\tilde{\alpha}$ be the characteristic class for G_1-bundles defined by* $(\mathrm{B}\lambda)^*\xi \in H^*_R(\mathrm{B}G_1)$. *Then,*

$$\tilde{\alpha}(P) = \alpha\left(P^{[\lambda]}\right)$$

for every topological principal G_1-bundle P. ■

Now, let us turn to characteristic classes for vector bundles. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let $\mathrm{U}_{\mathbb{K}}(n)$ denote $\mathrm{O}(n)$ for $\mathbb{K} = \mathbb{R}$, $\mathrm{U}(n)$ for $\mathbb{K} = \mathbb{C}$ and $\mathrm{Sp}(n)$ for $\mathbb{K} = \mathbb{H}$. Let $\xi \in H^*_R(\mathrm{BU}_{\mathbb{K}}(n))$. For a $\mathbb{K}$-vector bundle E of rank n over a topological space B, define

$$\alpha(E) := f_E^* \xi \tag{4.1.2}$$

with some classifying mapping $f_E : B \to \mathrm{BU}_{\mathbb{K}}(n) = \mathrm{G}_{\mathbb{K}}(n, \infty)$ for E, cf. Remark 3.6.10/1. By the same argument as in the case of principal bundles, this makes sense and one has $\alpha(E_1) = \alpha(E_2)$ whenever E_1 and E_2 are vertically isomorphic.

Proposition 4.1.5 *For every $\xi \in H^*_R(\mathrm{BU}_{\mathbb{K}}(n))$, the assignment (4.1.2) defines an R-valued characteristic class for $\mathbb{K}$-vector bundles of rank n. Every R-valued characteristic class for $\mathbb{K}$-vector bundles of rank n arises in this way.*

Proof The argument proving that α defined by (4.1.2) is a characteristic class is analogous to that for principal bundles. To see that every characteristic class arises in this way, let $\tilde{\alpha}$ be a characteristic class for $\mathbb{K}$-vector bundles of rank n. Define

$$\xi := \tilde{\alpha}\left(\mathrm{EU}_{\mathbb{K}}(n) \times_{\mathrm{U}_{\mathbb{K}}(n)} \mathbb{K}^n\right) \in H^*_R(\mathrm{BU}_{\mathbb{K}}(n))$$

and let α denote the characteristic class defined by ξ via (4.1.2). Since $\tilde{\alpha}$ is a characteristic class, for any $\mathbb{K}$-vector bundle E of rank n with classifying mapping f_E, we have

$$\tilde{\alpha}(E) = f_E^* \tilde{\alpha}\left(\mathrm{EU}_{\mathbb{K}}(n) \times_{\mathrm{U}_{\mathbb{K}}(n)} \mathbb{K}^n\right) = f_E^* \xi = \alpha(E)\,,$$

because E is vertically isomorphic to $f_E^*\left(\mathrm{EU}_{\mathbb{K}}(n) \times_{\mathrm{U}_{\mathbb{K}}(n)} \mathbb{K}^n\right)$. ■

Remark 4.1.6

1. Let $\xi \in H^*_R(\mathrm{BU}_{\mathbb{K}}(n))$ and let α stand for both the corresponding characteristic class for principal $\mathrm{U}_{\mathbb{K}}(n)$-bundles and the corresponding characteristic class for $\mathbb{K}$-vector bundles of rank n. According to Remark 3.6.10/1, for every principal $\mathrm{U}_{\mathbb{K}}(n)$-bundle P, one has $\alpha(P) = \alpha(P \times_{\mathrm{U}(n)} \mathbb{K}^n)$ and for every $\mathbb{K}$-vector bundle E of rank n one has $\alpha(E) = \alpha(O(E))$, where the orthonormal frame bundle is taken with respect to some chosen positive definite fibre metric.
2. Since a trivial principal G-bundle P over M has constant classifying mapping, for any characteristic class of degree $k > 0$, one has $\alpha(P) = 0$. An analogous statement holds for trivial vector bundles. ♦

In the second part of this section, we discuss the main tools needed for the study of the characteristic classes for the classical compact Lie groups. Our discussion is based, in effect, on the Leray–Hirsch Theorem, which we cite here for completeness.

Given a topological fibre bundle E over B with projection π and a commutative ring R, we can define a mapping

$$H_R^*(B) \times H_R^*(E) \to H_R^*(E)\,, \quad (\alpha, \gamma) \mapsto (\pi^*\alpha) \cup \gamma\,.$$

This yields an action of the ring $H_R^*(B)$ on the Abelian group $H_R^*(E)$ and thus makes $H_R^*(E)$ into a module over $H_R^*(B)$. For $b \in B$, let $j_b : E_b \to E$ denote the natural inclusion mapping of the fibres. Recall that a finitely generated module $\mathfrak{M}$ over a ring R is said to be free if it is isomorphic to the Cartesian product R^r for some r, and that a basis of $\mathfrak{M}$ is said to be free if it corresponds to such an isomorphism.

Theorem 4.1.7 (Leray–Hirsch) *Let E be a topological fibre bundle over B with projection π and typical fibre F and let R be a commutative ring. Assume that $H_R^*(F)$ is a finitely generated free R-module and that there exist elements $\tau_1, \dots, \tau_r \in H_R^{k_i}(E)$ such that $j_b^*\tau_1, \dots, j_b^*\tau_r$ form a free basis of $H_R^*(E_b)$ as an R-module for every $b \in B$. Then, $\tau_1, \dots, \tau_r$ form a free basis of $H_R^*(E)$ as a $H_R^*(B)$-module.* ■

For the proof we refer to [287]. Explicitly, this theorem states that the mapping

$$(H_R^*(B))^r \to H_R^*(E)\,, \quad (\alpha_1, \dots, \alpha_r) \mapsto \sum_{i=1}^{r} (\pi^*\alpha_i) \cup \tau_i\,, \tag{4.1.3}$$

is an isomorphism of $H_R^*(B)$-modules, and thus in particular of Abelian groups. As a consequence of the Leray–Hirsch Theorem, $H_R^*(E)$ is isomorphic, as an R-module, to $H_R^*(B) \otimes_R H_R^*(F)$. Thus, the cohomology of a topological fibre bundle whose typical fibre meets the assumption of the theorem is completely determined by that of the base and the typical fibre and does not depend on the topological type of that bundle.

As we have seen above, to determine the characteristic classes for the classical compact Lie groups, we have to determine the cohomology of the corresponding classifying spaces. This will be done by induction on the rank. Without loss of generality, for given n, we choose the inclusion $j_{n-1,n} : \mathrm{U}_{\mathbb{K}}(n-1) \to \mathrm{U}_{\mathbb{K}}(n)$ to be induced by the inclusion mapping

$$\mathbb{K}^{n-1} \to \mathbb{K}^n\,, \quad (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0)\,.$$

According to Proposition 3.7.5/1, the classifying space $\mathrm{BU}_{\mathbb{K}}(n-1)$ can be realized as a topological fibre bundle over $\mathrm{BU}_{\mathbb{K}}(n)$ with projection $\mathrm{B}j_{n-1,n}$ and typical fibre $\mathrm{U}_{\mathbb{K}}(n)/\mathrm{U}_{\mathbb{K}}(n-1)$. Since

$$\mathrm{U}_{\mathbb{K}}(n)/\mathrm{U}_{\mathbb{K}}(n-1) \cong \mathrm{S}^{dn-1},$$

where d is the dimension of $\mathbb{K}$ over $\mathbb{R}$, the typical fibres are spheres. For sphere bundles, the Gysin sequence, to be discussed below, connects the cohomology groups of the base space with those of the total space, and it does so using the Euler class.

Both the Euler class and the Gysin sequence are provided by the Thom Isomorphism Theorem, which we will give now without proof.

Let B be a topological space and let E be a Riemannian vector bundle of rank n over B with projection π. For $e \in E$, let $\|e\|$ denote the corresponding fibre norm. Define subsets

$$\mathrm{D}E := \{e \in E : \|e\| \leq 1\}\,, \quad \mathrm{S}E := \{e \in E : \|e\| = 1\}\,.$$

In the respective relative topology, $\mathrm{D}E$ is a vertical subbundle of E with typical fibre D^n and $\mathrm{S}E$ is a vertical subbundle of $\mathrm{D}E$ with typical fibre S^{n-1}. Let

$$\pi_{\mathrm{D}} : \mathrm{D}E \to B\,, \quad \pi_{\mathrm{S}} : \mathrm{S}E \to B$$

denote the corresponding projections, induced from π by restriction. Thus, $\mathrm{D}E$ is a disk bundle and $\mathrm{S}E$ is its boundary sphere bundle. Recall that an orientation of E is given by a covering by local trivializations whose transition mappings have positive determinant. Via these local trivializations, an orientation of E defines an orientation of every fibre E_b of E. The latter defines a generator of $H^n_{\mathbb{Z}}(\mathrm{D}E_b, \mathrm{S}E_b)$ as follows. By the Universal Coefficient Theorem, $H^n_{\mathbb{Z}}(\mathrm{D}E_b, \mathrm{S}E_b) = \mathrm{Hom}\big(H_n(\mathrm{D}E_b, \mathrm{S}E_b), \mathbb{Z}\big)$. The desired generator corresponds to the homomorphism which assigns the value 1 to the generator of $H_n(\mathrm{D}E_b, \mathrm{S}E_b)$ represented by the orientation preserving homeomorphisms of the n-simplex to $\mathrm{D}E_b \subset E_b$.

In what follows, let $p : C_k(\mathrm{D}E) \to C_k(\mathrm{D}E, \mathrm{S}E)$ denote the natural projection to classes. Recall that the cup product induces a bi-additive mapping

$$\cup : H^*_R(\mathrm{D}E) \times H^*_R(\mathrm{D}E, \mathrm{S}E) \to H^*_R(\mathrm{D}E, \mathrm{S}E)$$

by the condition that

$$p^*(\alpha \cup \beta) = \alpha \cup p^*\beta \tag{4.1.4}$$

for all $\alpha \in H^*_R(\mathrm{D}E)$ and $\beta \in H^*_R(\mathrm{D}E, \mathrm{S}E)$. Note that j_b induces a pair mapping $(\mathrm{D}E_b, \mathrm{S}E_b) \to (\mathrm{D}E, \mathrm{S}E)$ denoted by the same symbol.

Theorem 4.1.8 (Thom Isomorphism Theorem) *Let E be a Riemannian vector bundle of rank n over a connected topological space B with projection π. Let $R = \mathbb{Z}_2$, or let $R = \mathbb{Z}$ and assume that E is oriented.*

1. *There exists a unique element $\tau \in H^n_R(\mathrm{D}E, \mathrm{S}E)$ such that for every $b \in B$, $j_b^*\tau$ is the generator of $H^n_R(\mathrm{D}E_b, \mathrm{S}E_b)$ (defined by the induced orientation in case $R = \mathbb{Z}$).*
2. *The homomorphism $H^k_R(B) \to H^{k+n}_R(\mathrm{D}E, \mathrm{S}E)$ defined by $\alpha \mapsto (\pi_{\mathrm{D}}^*\alpha) \cup \tau$ is an isomorphism for all $k \geq 0$ and $H^k_R(\mathrm{D}E, \mathrm{S}E) = 0$ for all $k < n$.*

Proof The proof uses a relative version of the Leray–Hirsch Theorem, see Theorem 4D.10 and Corollary 4D.9 in [287]. ∎

Point 2 states that the single element τ forms a free basis of the module $H^*_R(\mathrm{D}E, \mathrm{S}E)$ over $H^*_R(B)$. For further use, we note that the theorem implies, in particular, that $j_b^* : H^k_R(\mathrm{D}E, \mathrm{S}E) \to H^k_R(\mathrm{D}E_b, \mathrm{S}E_b)$ is an isomorphism in dimension $k \leq n$ for all $b \in B$.

Definition 4.1.9 (*Thom class and Euler class*) Let E be a Riemannian vector bundle of rank n over a connected topological space B. Let $R = \mathbb{Z}_2$, or let $R = \mathbb{Z}$ and assume that E is oriented.

1. The class $\tau \in H^n_R(\mathrm{D}E, \mathrm{S}E)$ provided by Theorem 4.1.8 is called the Thom class of E.
2. Let $s_\mathrm{D} : B \to \mathrm{D}E$ denote the zero section. The class $\mathsf{e}(E) := s_\mathrm{D}^* \circ p^*(\tau)$ in $H^n_R(B)$ is called the Euler class of E.

Thus, every Riemannian vector bundle has an Euler class in $\mathbb{Z}_2$-cohomology. If it is oriented, it has in addition an Euler class in $\mathbb{Z}$-cohomology.

For a topological space X and a commutative ring R, let

$$\delta : \mathrm{Hom}\left(C_k(X), R\right) \to \mathrm{Hom}\left(C_{k+1}(X), R\right)$$

denote the coboundary operator and let $Z^k_R(X) \subset \mathrm{Hom}\left(C_k(X), R\right)$ denote its kernel (that is, the subgroup of closed singular cochains in X with coefficients in R).

Theorem 4.1.10 (Gysin sequence) *Let E be a Riemannian vector bundle of rank n over a connected topological space B. Let $R = \mathbb{Z}_2$, or let $R = \mathbb{Z}$ and assume that E is oriented. Then one has a long exact sequence of Abelian groups*

$$\cdots \xrightarrow{\varphi} H^k_R(B) \xrightarrow{\cup \mathsf{e}(E)} H^{k+n}_R(B) \xrightarrow{\pi_\mathrm{S}^*} H^{k+n}_R(\mathrm{S}E) \xrightarrow{\varphi} H^{k+1}_R(B) \xrightarrow{\cup \mathsf{e}(E)} \cdots$$

where $k \in \mathbb{Z}$ and the connecting homomorphism φ is defined by the condition

$$\varphi([\gamma]) \cup \mathsf{e}(E) = s_\mathrm{D}^*[\delta\tilde{\gamma}]\,.$$

Here, $\gamma \in Z^{k+n}_R(\mathrm{S}E)$ and $\tilde{\gamma} \in \mathrm{Hom}(C_{k+n}(\mathrm{D}E), R)$ is some extension of γ from $C_{k+n}(\mathrm{S}E)$ to $C_{k+n}(\mathrm{D}E)$.

Proof Consider the long exact cohomology sequence of the pair $(\mathrm{D}E, \mathrm{S}E)$, cf. [287, Sect. 3.1]

$$\cdots \to H^k_R(\mathrm{D}E, \mathrm{S}E) \xrightarrow{p^*} H^k_R(\mathrm{D}E) \xrightarrow{j^*} H^k_R(\mathrm{S}E) \xrightarrow{\tilde{\varphi}} H^{k+1}_R(\mathrm{D}E, \mathrm{S}E) \to \cdots \tag{4.1.5}$$

where, on the level of representatives $\gamma \in Z^k_R(\mathrm{S}E)$, the connecting homomorphism $\tilde{\varphi}$ is determined by

$$p^*\tilde{\varphi}([\gamma]) = [\delta\tilde{\gamma}]\,, \tag{4.1.6}$$

with $\tilde{\gamma} \in \mathrm{Hom}(C_k(\mathrm{D}E), R)$ being some extension of γ from $C_k(\mathrm{S}E)$ to $C_k(\mathrm{D}E)$. According to the Thom Isomorphism Theorem, we can replace $H^k_R(\mathrm{D}E, \mathrm{S}E)$ by $H^{k-n}_R(B)$. Since DE is homotopy equivalent to B via the mappings π_{D} and s_{D}, the induced homomorphisms $s^*_{\mathrm{D}} : H^k_R(\mathrm{D}E) \to H^k_R(B)$ and $\pi^*_{\mathrm{D}} : H^k_R(B) \to H^k_R(\mathrm{D}E)$ are mutually inverse isomorphisms. Hence, we can furthermore replace $H^k_R(\mathrm{D}E)$ by $H^k_R(B)$. Thus, from (4.1.5), we obtain the long exact sequence

$$\cdots \longrightarrow H^{k-n}_R(B) \xrightarrow{\varphi_1} H^k_R(B) \xrightarrow{\varphi_2} H^k_R(\mathrm{S}E) \xrightarrow{\varphi} H^{k-n+1}_R(B) \longrightarrow \cdots$$

with the homomorphisms φ_1, φ_2 and φ given by, respectively,

$$\varphi_1(\alpha) = s^*_{\mathrm{D}} \circ p^*(\pi^*_{\mathrm{D}}\alpha \cup \tau), \qquad \varphi_2(\alpha) = j^* \circ \pi^*_{\mathrm{D}}, \qquad \pi^*_{\mathrm{D}}\big(\varphi([\gamma])\big) \cup \tau = \tilde{\varphi}([\gamma])$$

for all $\alpha \in H^{k-n}_R(B)$ and $\gamma \in Z^k_R(\mathrm{S}E)$. Clearly, $\varphi_2 = \pi^*_{\mathrm{S}}$. According to (4.1.4),

$$\varphi_1(\alpha) = s^*_{\mathrm{D}}\big((\pi^*_{\mathrm{D}}\alpha) \cup p^*\tau\big) = \alpha \cup \mathsf{e}(E) .$$

To read off φ from the last equation, we apply $s^*_{\mathrm{D}} \circ p^*$ to both sides. By (4.1.4) and (4.1.6), we obtain

$$\varphi([\gamma]) \cup \mathsf{e}(E) = s^*_{\mathrm{D}}[\delta\tilde{\gamma}] .$$

This yields the asserted formula for φ. ■

Remark 4.1.11 The Euler class and the Gysin sequence exist for any sphere bundle (fibre bundle with typical fibre a sphere). In the general case, the role of DE is played by the mapping cone of the projection of that sphere bundle. While the Thom class depends on the fibre metric of E, the Euler class does not. The latter follows by observing that the disk bundles DE defined by different fibre metrics are deformation retracts of each other. ♦

Proposition 4.1.12 (Properties of the Euler class)

1. *If the rank of a connected oriented real vector bundle E is odd, then* $2\mathsf{e}(E) = 0$.
2. *The $\mathbb{Z}_2$-Euler class of a connected oriented real vector bundle coincides with the $\mathbb{Z}_2$-reduction of the integral Euler class of that bundle.*
3. *Let E_1, E_2 be connected real vector bundles over B_1, B_2, respectively. Let $F : E_1 \to E_2$ be a vector bundle morphism and let $f : B_1 \to B_2$ be its projection. Let $R = \mathbb{Z}_2$ or let $R = \mathbb{Z}$ and assume that E_1 and E_2 are oriented and that F preserves the orientations. If the fibre mappings of F are isomorphisms, then* $f^*\mathsf{e}(E_2) = \mathsf{e}(E_1)$.
4. *Let E_1 and E_2 be connected real vector bundles over B. Let $R = \mathbb{Z}_2$ or let $R = \mathbb{Z}$ and assume that E_1 and E_2 are oriented and that $E_1 \oplus E_2$ carries the orientation induced by concatenation of frames. Then,* $\mathsf{e}(E_1 \oplus E_2) = \mathsf{e}(E_1) \cup \mathsf{e}(E_2)$.

Proof 1. This holds trivially true in the case $R = \mathbb{Z}_2$. Thus, assume that E is oriented and that $R = \mathbb{Z}$. Choose an auxiliary Riemannian fibre metric on E and let τ denote the corresponding Thom class. Multiplication by -1 defines an isometric vertical vector bundle automorphism of E and, thus, a vertical bundle automorphism F of $\mathrm{D}E$. Since F is vertical, for all $b \in B$, we have $j_b^* \circ F^*(\tau) = F_b^* \circ j_b^*(\tau)$. Since n is odd, F_b reverses the orientation of the fibre $(\mathrm{D}E)_b$. Hence, $F_b^* \circ j_b^*(\tau) = -j_b^*(\tau)$. By uniqueness of τ, this implies $F^*\tau = -\tau$. Using $F \circ s_{\mathrm{D}} = s_{\mathrm{D}}$ and $p^* \circ F^* = F^* \circ p^*$, we finally obtain $\mathsf{e} = -\mathsf{e}$. This yields the assertion.

2. It suffices to show that the $\mathbb{Z}_2$-Thom class of an oriented connected vector bundle coincides with the $\mathbb{Z}_2$-reduction of the integral Thom class. In view of the Thom Isomorphism Theorem, this follows from the fact that the $\mathbb{Z}_2$-reduction of a generator of $H^n_{\mathbb{Z}}((\mathrm{D}E)_b, (\mathrm{S}E)_b)$ is a generator of $H^n_{\mathbb{Z}_2}((\mathrm{D}E)_b, (\mathrm{S}E)_b)$.

3. Since F is fibrewise a vector space isomorphism, E_1 and E_2 must have the same rank n. We can choose Riemannian fibre metrics on E_1 and E_2 such that F is isometric (Exercise 4.1.1). Then, F induces a bundle morphism $F : \mathrm{D}E_1 \to \mathrm{D}E_2$, denoted by the same symbol, which sends $\mathrm{S}E_1$ to $\mathrm{S}E_2$ and whose fibre mappings are homeomorphisms. Let $\tau_i \in H^n_R(\mathrm{D}E_i, \mathrm{S}E_i)$ denote the Thom class of E_i, $i = 1, 2$. For every $b \in B_1$,

$$j_b^* \circ F^*(\tau_2) = F_b^* \circ j_{f(b)}^*(\tau_2)\,.$$

As a consequence of the Thom Isomorphism Theorem, $j_{f(b)}^*(\tau_2)$ is a generator of $H^n_R((\mathrm{D}E_2)_{f(b)}, (\mathrm{S}E_2)_{f(b)})$. Since F_b is a homeomorphism, F_b^* maps this generator to a generator of $H^n_R((\mathrm{D}E_1)_b, (\mathrm{S}E_1)_b)$. Since F_b preserves the orientations, the latter coincides with $j_b^*\tau_1$. Since j_b^* is an isomorphism in dimension n, we conclude that $F^*(\tau_2) = \tau_1$. The assertion now follows by observing that the zero sections $s_i : B_i \to \mathrm{D}E_i$ fulfil $F \circ s_1 = s_2 \circ f$.

4. We give the proof for $R = \mathbb{Z}$. The proof for $R = \mathbb{Z}_2$ can be obtained by forgetting about the orientation.

Denote $E_\oplus := E_1 \oplus E_2$. We choose auxiliary Riemannian fibre metrics on E_1 and E_2 and equip $E_\oplus$ with their direct sum. For $i = 1, 2, \oplus$, let n_i denote the rank of E_i, write $D_i := \mathrm{D}E_i$ and $S_i := \mathrm{S}E_i$, and let $\tau_i \in H^{n_i}_{\mathbb{Z}}(D_i, S_i)$ denote the Thom class of E_i. Moreover, define $D_\times := D_1 \times_B D_2$ and $S_\times := (S_1 \times_B D_2) \cup (D_1 \times_B S_2)$. For $i = 1, 2, \oplus, \times$, let $j_{i,b} : D_{i,b} \to D_i$ denote the natural inclusion mappings of the fibres.

Define a mapping

$$F : E_\oplus \to E_\oplus\,, \quad F(e_1, e_2) := \begin{cases} \frac{\max\{\|e_1\|, \|e_2\|\}}{\|(e_1, e_2)\|}(e_1, e_2) & (e_1, e_2) \neq 0\,, \\ 0 & (e_1, e_2) = 0\,. \end{cases}$$

Since

$$\max\{\|e_1\|, \|e_2\|\} \leq \|(e_1, e_2)\| \leq \sqrt{2}\,\max\{\|e_1\|, \|e_2\|\}\,,$$

F is a homeomorphism and thus a fibre bundle isomorphism. Since

$$\|F(e_1, e_2)\| = \max\{\|e_1\|, \|e_2\|\}\,,$$

it maps $D_\times$ onto $D_\oplus$ and $S_\times$ onto $S_\oplus$ and hence induces a pair homeomorphism $(D_\times, S_\times) \to (D_\oplus, S_\oplus)$, denoted by the same symbol. For every $b \in B$, we have

$$F \circ j_{\times,b} = j_{\oplus,b} \circ F_b \,, \tag{4.1.7}$$

and hence

$$j^*_{\times,b}(F^*\tau_\oplus) = F^*_b\left(j^*_{\oplus,b}\tau_\oplus\right) \,.$$

Since the fibre mappings F_b are pair homeomorphisms and since they preserve orientations, the class on the right hand side is the generator of $H^{n_\oplus}_{\mathbb{Z}}(D_{\times,b}, S_{\times,b})$ corresponding to the product orientation. On the other hand, consider the relative cohomology cross product[1]

$$\tau_1 \times \tau_2 \in H^{n_1+n_2}_{\mathbb{Z}}(D_1 \times D_2, S_1 \times D_2 \cup D_1 \times S_2)$$

and the natural inclusion mapping $\iota : D_\times \to D_1 \times D_2$. One can check that one has $j_{1,b} \times j_{2,b} = \iota \circ j_{\times,b}$ and that $j_{1,b} \times j_{2,b}$ and ι induce pair mappings from $(D_{\times,b}, S_{\times,b})$ and $(D_\times, S_\times)$, respectively, to $\big(D_1 \times D_2, S_1 \times D_2 \cup D_1 \times S_2\big)$. As a consequence,

$$j^*_{\times,b}\left(\iota^*(\tau_1 \times \tau_2)\right) = (j^*_{1,b}\tau_1) \times (j^*_{2,b}\tau_2) \,.$$

Since $H^k_{\mathbb{Z}}(D_{i,b}, S_{i,b})$ is finitely and freely generated over $\mathbb{Z}$ for all k, the relative Künneth Theorem for cohomology yields that the right hand side provides a generator of $H^{n_\oplus}_{\mathbb{Z}}(D_{\times,b}, S_{\times,b})$. Clearly, this is the generator corresponding to the product orientation. Thus,

$$j^*_{\times,b}\left(\iota^*(\tau_1 \times \tau_2)\right) = j^*_{\times,b}(F^*\tau_\oplus) \,. \tag{4.1.8}$$

According to (4.1.7), since F and F_b are pair homeomorphisms and since $j^*_{\oplus,b}$ is an isomorphism in degree $n_\oplus$, so is $j^*_{\times,b}$. Therefore, (4.1.8) implies

$$\iota^*(\tau_1 \times \tau_2) = F^*\tau_\oplus \,. \tag{4.1.9}$$

Now, for $i = 1, 2, \oplus, \times$, let $p_i : C_{n_i}(D_i) \to C_{n_i}(D_i, S_i)$ denote the natural projections to relative chains (with $n_\times = n_\oplus$) and let $s_i : B \to D_i$, $i = 1, 2, \oplus$ denote the zero sections. We apply $s^*_\times \circ p^*_\times$ to (4.1.9). Using $F \circ s_\times = s_\oplus$, for the right hand side, we obtain

$$s^*_\times \circ p^*_\times\left(F^*\tau_\oplus\right) = s^*_\oplus \circ p^*_\oplus(\tau_\oplus) = \mathsf{e}(E_1 \oplus E_2) \,.$$

Using $\iota \circ s_\times = (s_1 \times s_2) \circ \Delta$, where $\Delta : B \to B \times B$ denotes the diagonal mapping, for the left hand side we obtain

$$s^*_\times \circ p^*_\times\left(\iota^*(\tau_1 \times \tau_2)\right) = \Delta^*\Big((s^*_1 \circ p^*_1(\tau_1)) \times (s^*_2 \circ p^*_2(\tau_2))\Big) = \mathsf{e}(E_1) \cup \mathsf{e}(E_2) \,.$$

This yields the assertion. ∎

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[1] For $\alpha \in H^k_R(X, A)$ and $\beta \in H^l_R(Y, B)$, $\alpha \times \beta = (\mathrm{pr}^*_X\alpha) \cup (\mathrm{pr}^*_Y\beta) \in H^{k+l}_R(X \times Y, A \times Y \cup X \times B)$.

Exercises

4.1.1 Complete the proof of point 3 of Proposition 4.1.12 by showing that one can choose Riemannian fibre metrics on E_1 and E_2 such that the vector bundle isomorphism F under consideration is isometric. *Hint*. Choose an arbitrary Riemannian fibre metric on E_2 and pull it back to f^*E_2. Then, show that the vertical vector bundle morphism $F : E_1 \to f^*E_2$ defined by $F(e) = \big(\pi_1(e), F(e)\big)$, where $\pi_1 : E_1 \to B_1$ is the bundle projection, is an isomorphism.

4.2 Characteristic Classes for the Classical Groups

In this section, we determine the integral cohomology rings for $\mathrm{BU}(n)$, $\mathrm{BSU}(n)$ and $\mathrm{BSp}(n)$ and the $\mathbb{Z}_2$-cohomology rings of $\mathrm{BO}(n)$ and $\mathrm{BSO}(n)$. All of these rings will turn out to be polynomial. The integral cohomology of $\mathrm{BO}(n)$ and $\mathrm{BSO}(n)$ is more involved and will be given without proof in Theorem 4.2.23.

To begin with, let us introduce some terminology and notation. Given a finite set $X = \{x_1, \dots, x_N\}$ and an Abelian group A, the ring of formal polynomials in the commuting variables x_i with coefficients from A will be referred to as the polynomial ring generated over A by X.

For a $\mathbb{K}$-vector space V and a subfield $\mathbb{L} \subset \mathbb{K}$, let $V_{\mathbb{L}}$ denote the $\mathbb{L}$-vector space obtained from V by field restriction, that is, by restricting multiplication by scalars to the subfield $\mathbb{L}$. The same notation will be used for vector bundles. For details about field restriction and field extension we refer to Appendix A. For our construction of characteristic classes for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, we will use the concrete real vector space isomorphisms $\mathbb{R}^{2n} \to \mathbb{C}^n_{\mathbb{R}}$ given by

$$(x_1, \dots, x_{2n}) \mapsto (x_1 + x_2\mathrm{i}, \dots, x_{2n-1} + x_{2n}\mathrm{i}) , \tag{4.2.1}$$

and $\mathbb{R}^{4n} \to \mathbb{H}^n_{\mathbb{R}}$ given by mapping $(x_1, \dots, x_{4n})$ to

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \,, \ \dots \ , \ x_{4n-3} + x_{4n-2}\mathbf{i} + x_{4n-1}\mathbf{j} + x_{4n}\mathbf{k}) , \tag{4.2.2}$$

as well as the complex vector space isomorphism $\mathbb{C}^{2n} \to \mathbb{H}^n_{\mathbb{C}}$ given by

$$(z_1, \dots, z_{2n}) \mapsto \big(z_1 + \mathbf{j}z_2, \dots, z_{2n-1} + \mathbf{j}z_{2n}\big) . \tag{4.2.3}$$

By further field restriction to $\mathbb{R}$, the isomorphism (4.2.3) yields a real vector space isomorphism $\mathbb{C}^{2n}_{\mathbb{R}} \to \mathbb{H}^n_{\mathbb{R}}$. Composition of the latter with the isomorphism $\mathbb{R}^{4n} \to \mathbb{C}^{2n}_{\mathbb{R}}$ given by (4.2.1) yields the isomorphism $\mathbb{R}^{4n} \to \mathbb{H}^n_{\mathbb{R}}$ given by sending $(x_1, \dots, x_{4n})$ to

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} - x_4\mathbf{k} \,, \ \dots \ , \ x_{4n-3} + x_{4n-2}\mathbf{i} + x_{4n-1}\mathbf{j} - x_{4n}\mathbf{k}) . \tag{4.2.4}$$

This isomorphism does not coincide with the one defined by (4.2.2).

Given a complex vector bundle E, the real vector bundle $E_{\mathbb{R}}$ obtained from E by field restriction is called the realification of E. We endow $E_{\mathbb{R}}$ with an orientation[2] as follows. If $(e_1, \ldots, e_n)$ is an ordered local frame in E, then

$$(e_1, \mathrm{i}e_1, \ldots, e_n, \mathrm{i}e_n) \tag{4.2.5}$$

is an ordered local frame in $E_{\mathbb{R}}$. Clearly, the transition mapping between two local frames of the form (4.2.5) is given by the composition of the transition mapping between the original frames in E with the Lie subgroup embedding $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$ defined by the isomorphism (4.2.1). Since the latter takes values in the identity connected component, the transition mapping has positive determinant. Thus, the ordered local frames in $E_{\mathbb{R}}$ of the form (4.2.5) define an orientation in $E_{\mathbb{R}}$. We will refer to this orientation as the orientation induced by E.[3] In terms of local trivializations, the induced orientation is given by the family of local trivializations of $E_{\mathbb{R}}$ which are obtained from local trivializations of E by composition with the isomorphisms (4.2.1).

An analogous argument applies in the case where E is a quaternionic vector bundle. Here, the induced orientation of the realification $E_{\mathbb{R}}$ is defined by the ordered local frames of the form

$$(e_1, e_1\mathbf{i}, e_1\mathbf{j}, e_1\mathbf{k}, \ldots, e_n, e_n\mathbf{i}, e_n\mathbf{j}, e_n\mathbf{k}) \,, \tag{4.2.6}$$

for some ordered local frame $(e_1, \ldots, e_n)$ in E.

Both for complex and quaternionic vector bundles, the induced orientation on the realification has the property that every real vector bundle morphism $(E_1)_{\mathbb{R}} \to (E_2)_{\mathbb{R}}$ which is obtained by field restriction from a complex or quaternionic vector bundle morphism $E_1 \to E_2$ automatically preserves the orientations. Moreover, in case E_1 and E_2 have the same base, the induced orientation in $(E_1 \oplus E_2)_{\mathbb{R}}$ coincides with the orientation in $(E_1)_{\mathbb{R}} \oplus (E_2)_{\mathbb{R}}$ defined by the induced orientations in $(E_1)_{\mathbb{R}}$ and $(E_2)_{\mathbb{R}}$ and by concatenation of ordered local frames.

Now, we are prepared to discuss the characteristic classes of the classical groups. We start with the integral characteristic classes for the unitary groups $\mathrm{U}(n)$. Define

$$E_n^{\mathrm{U}} := \big(\mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}^n\big)_{\mathbb{R}} \,,$$

where $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ in the basic representation, and endow E_n^{U} with the induced orientation. From the characterization of this orientation in terms of local trivializations it is clear that, fibrewise and via the isomorphism (4.2.1), this orientation corresponds to the standard orientation of $\mathbb{R}^{2n}$. Let $\mathsf{c}_n^{\mathrm{U}(n)} \in H^{2n}_{\mathbb{Z}}(\mathrm{BU}(n))$ denote the corresponding Euler class, that is,

[2] Recall the notion of orientation of a $\mathbb{K}$-vector bundle from Example 1.6.6/1.

[3] Note that E itself need not be oriented here.

$$\mathsf{c}_n^{\mathrm{U}(n)} = \mathsf{e}(E_n^{\mathrm{U}})\,.$$

For $0 < k \leq n$, let

$$j_{k,n}^{\mathrm{U}} : \mathrm{U}(k) \to \mathrm{U}(n)$$

denote the Lie subgroup embedding induced by the linear subspace embedding

$$\mathbb{C}^k \to \mathbb{C}^n\,, \quad (z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)\,. \tag{4.2.7}$$

By construction, for $0 < k \leq l \leq n$,

$$j_{l,n}^{\mathrm{U}} \circ j_{k,l}^{\mathrm{U}} = j_{k,n}^{\mathrm{U}}\,. \tag{4.2.8}$$

Theorem 4.2.1 (Integral cohomology of $\mathrm{BU}(n)$) *For $k = 1, \ldots, n-1$, there exists a unique element $\mathsf{c}_k^{\mathrm{U}(n)}$ of $H_{\mathbb{Z}}^{2k}(\mathrm{BU}(n))$ such that*

$$(\mathrm{B}j_{k,n}^{\mathrm{U}})^* \mathsf{c}_k^{\mathrm{U}(n)} = \mathsf{c}_k^{\mathrm{U}(k)}\,.$$

The ring $H_{\mathbb{Z}}^*(\mathrm{BU}(n))$ is the polynomial ring over $\mathbb{Z}$ in the generators $\mathsf{c}_1^{\mathrm{U}(n)}, \ldots, \mathsf{c}_n^{\mathrm{U}(n)}$.

Proof As explained earlier, the strategy of the proof is to relate the cohomology of $\mathrm{BU}(n-1)$ with that of $\mathrm{BU}(n)$ by means of the Gysin sequence. Taking the real part of the standard scalar product on the complex vector space $\mathbb{C}^n$, we obtain a scalar product on the real vector space $\mathbb{C}^n_{\mathbb{R}}$ and thus a Riemannian fibre metric on the real vector bundle E_n^{U}. Then,

$$\mathrm{D}E_n^{\mathrm{U}} = \mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathrm{D}^{2n}\,, \quad \mathrm{S}E_n^{\mathrm{U}} = \mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathrm{S}^{2n-1}\,,$$

where D^{2n} and S^{2n-1} stand for the unit disk and the unit sphere in $\mathbb{C}^n_{\mathbb{R}}$. The sphere bundle $\mathrm{S}E_n^{\mathrm{U}}$ is related to the sphere bundle $\mathrm{B}j_{n-1,n}^{\mathrm{U}} : \mathrm{BU}(n-1) \to \mathrm{BU}(n)$ as follows. By Proposition 3.7.5/1, in the latter, the total space $\mathrm{BU}(n-1)$ is realized as $\mathrm{EU}(n)/\mathrm{U}(n-1)$.[4] By point 1 of Example 1.2.4, there exists a vertical fibre bundle isomorphism

$$F : \mathrm{BU}(n-1) \to \mathrm{S}E_n^{\mathrm{U}}\,,$$

for all $n = 1, 2, \ldots$ Using this isomorphism, in the Gysin sequence of E_n^{U}, we can replace $H_{\mathbb{Z}}^k(\mathrm{S}E_n^{\mathrm{U}})$ by $H_{\mathbb{Z}}^k(\mathrm{BU}(n-1))$, the homomorphism π_{S}^* by $F^* \circ \pi_{\mathrm{S}}^* = (\mathrm{B}j_{n-1,n}^{\mathrm{U}})^*$ and the connecting homomorphism φ by $\varphi \circ (F^*)^{-1}$, which we continue to denote by φ. This way, for $l \in \mathbb{Z}$ and $n = 1, 2, \ldots$, we obtain the exact sequence

$$\cdots \xrightarrow{\varphi} H_{\mathbb{Z}}^l(\mathrm{BU}(n)) \xrightarrow{\cup \mathsf{c}_n^{\mathrm{U}(n)}} H_{\mathbb{Z}}^{l+2n}(\mathrm{BU}(n)) \xrightarrow{(\mathrm{B}j_{n-1,n}^{\mathrm{U}})^*} H_{\mathbb{Z}}^{l+2n}(\mathrm{BU}(n-1)) \xrightarrow{\varphi} H_{\mathbb{Z}}^{l+1}(\mathrm{BU}(n)) \to \cdots \tag{4.2.9}$$

Now, we can prove the assertion of the theorem by induction on n.

[4]This holds also for $n = 1$, provided we define $\mathrm{U}(0)$ as the trivial group consisting of one element.

For $n = 1$, we have to show that $H^*_{\mathbb{Z}}(\mathrm{BU}(1))$ is the free polynomial ring in the generator $\mathsf{c}_1^{\mathrm{U}(1)}$. Since $\mathrm{U}(0)$ is the trivial group, the total space $\mathrm{BU}(0)$ coincides with $\mathrm{EU}(1)$ and $\mathrm{B}j^{\mathrm{U}}_{0,1}$ coincides with the bundle projection $\mathrm{EU}(1) \to \mathrm{BU}(1)$. Since $\mathrm{EU}(1)$ is contractible, $H^k_{\mathbb{Z}}(\mathrm{EU}(1)) = 0$ for $k \neq 0$. Hence, (4.2.9) yields that multiplication by $\mathsf{c}_1^{\mathrm{U}(1)}$ defines an isomorphism from $H^l_{\mathbb{Z}}(\mathrm{BU}(1))$ onto $H^{l+2}_{\mathbb{Z}}(\mathrm{BU}(1))$ for all $l \neq -2$. It follows that $H^l_{\mathbb{Z}}(\mathrm{BU}(1)) = 0$ for odd l. Moreover, since $\mathrm{BU}(1)$ is connected and hence $H^0_{\mathbb{Z}}(\mathrm{BU}(1))$ is the free Abelian group generated by $1_{\mathrm{BU}(1)}$, it follows that for every non-negative integer k, $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(1))$ is the free Abelian group generated by $\left(\mathsf{c}_1^{\mathrm{U}(1)}\right)^k$. Thus, $H^*_{\mathbb{Z}}(\mathrm{BU}(1))$ is the free polynomial ring in the generator $\mathsf{c}_1^{\mathrm{U}(1)}$, indeed.

Now, assume that the assertion holds for $n-1$. We aim at showing that it holds true for n. First, we construct the classes $\mathsf{c}_k^{\mathrm{U}(n)}$ for $k = 1, \dots, n-1$. If, for such k, we plug $l = 2(k-n)$ into (4.2.9), we find that $(\mathrm{B}j^{\mathrm{U}}_{n-1,n})^*$ is an isomorphism in degree $2k$. Hence, there exists a unique element $\mathsf{c}_k^{\mathrm{U}(n)}$ of $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n))$ such that

$$\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^* \mathsf{c}_k^{\mathrm{U}(n)} = \mathsf{c}_k^{\mathrm{U}(n-1)} .$$

By the induction assumption, $\mathsf{c}_k^{\mathrm{U}(n-1)}$ is the unique element of $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n-1))$ fulfilling $\left(\mathrm{B}j^{\mathrm{U}}_{k,n-1}\right)^* \mathsf{c}_k^{\mathrm{U}(n-1)} = \mathsf{c}_k^{\mathrm{U}(k)}$. Now, the relation (4.2.8) yields the first assertion.

It remains to prove that $H^*_{\mathbb{Z}}(\mathrm{BU}(n))$ is the polynomial ring in the generators $\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_n^{\mathrm{U}(n)}$. For that purpose, we first show that $\varphi = 0$ in (4.2.9). By the induction assumption, $H^l_{\mathbb{Z}}(\mathrm{BU}(n-1))$ is the free Abelian group generated by all monomials of degree l in $\mathsf{c}_1^{\mathrm{U}(n-1)}, \dots, \mathsf{c}_{n-1}^{\mathrm{U}(n-1)}$. Since $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ preserves products, each of these monomials is the image under $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ of the corresponding monomial in $\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_{n-1}^{\mathrm{U}(n)}$. Therefore, $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ is surjective. By exactness of (4.2.9), it follows that $\varphi = 0$, indeed. As a result, (4.2.9) splits into the short exact sequences

$$0 \to H^l_{\mathbb{Z}}(\mathrm{BU}(n)) \xrightarrow{\cup\, \mathsf{c}_n^{\mathrm{U}(n)}} H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n)) \xrightarrow{(\mathrm{B}j^{\mathrm{U}}_{n-1,n})^*} H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n-1)) \to 0 . \tag{4.2.10}$$

We use this sequence to show that, for all k, $H^k_{\mathbb{Z}}(\mathrm{BU}(n))$ is the free Abelian group generated by the monomials of degree k in $\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_n^{\mathrm{U}(n)}$.

For $l < 0$, (4.2.10) implies that $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ is an isomorphism in degree $k < 2n$. Hence, here the assertion follows from the induction assumption and the fact that a monomial of degree $k < 2n$ cannot contain $\mathsf{c}_n^{\mathrm{U}(n)}$.

For $l \geq 0$, since we know $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n-1))$, the sequence (4.2.10) allows for reconstructing $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n))$ from $H^l_{\mathbb{Z}}(\mathrm{BU}(n))$. As a consequence, it suffices to show that if the assertion holds in degree l, then it holds in degree $l+2n$. Thus, assume that the assertion holds for l. Then, by exactness, $H^l_{\mathbb{Z}}(\mathrm{BU}(n)) \cup \mathsf{c}_n^{\mathrm{U}(n)}$ is the free Abelian group generated by the monomials of degree $l+2n$ in $\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_n^{\mathrm{U}(n)}$ with at least one factor $\mathsf{c}_n^{\mathrm{U}(n)}$. On the other hand, consider the subgroup A of $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n))$ generated by the monomials of degree $l+2n$ in $\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_{n-1}^{\mathrm{U}(n)}$. These monomials are mapped via $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ to the corresponding monomials in $\mathsf{c}_1^{\mathrm{U}(n-1)}, \dots, \mathsf{c}_{n-1}^{\mathrm{U}(n-1)}$. By the induction assumption, the latter are free generators of $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n-1))$. Hence, the former are free generators of A and $\left(\mathrm{B}j^{\mathrm{U}}_{n-1,n}\right)^*$ maps A isomorphically onto $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n-1))$. This finally implies that $H^{l+2n}_{\mathbb{Z}}(\mathrm{BU}(n))$ is the direct sum of $H^{l}_{\mathbb{Z}}(\mathrm{BU}(n)) \cup \mathsf{c}^{\mathrm{U}(n)}_n$ and A. Thus, the assertion is true for $l + 2n$. It follows that the assertion holds for all $k \geq 2n$. ■

Definition 4.2.2 (*Chern classes*) For $k = 1, \ldots, n$, the element $\mathsf{c}^{\mathrm{U}(n)}_k$ of $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n))$ provided by Theorem 4.2.1 is called the k-th universal Chern class of $\mathrm{U}(n)$. The element

$$\mathsf{c}^{\mathrm{U}(n)} := 1 + \mathsf{c}^{\mathrm{U}(n)}_1 + \cdots + \mathsf{c}^{\mathrm{U}(n)}_n$$

of $H^{*}_{\mathbb{Z}}(\mathrm{BU}(n))$ is called the total universal Chern class of $\mathrm{U}(n)$.

Remark 4.2.3

1. For $l \leq n$, one has

$$\left(\mathrm{B}j^{\mathrm{U}}_{l,n}\right)^* \mathsf{c}^{\mathrm{U}(n)}_k = \begin{cases} \mathsf{c}^{\mathrm{U}(l)}_k & 1 \leq k \leq l\,, \\ 0 & l < k \leq n\,. \end{cases} \tag{4.2.11}$$

   Indeed, if $k \leq l$, due to (4.2.8), the element $\left(\mathrm{B}j^{\mathrm{U}}_{l,n}\right)^* \mathsf{c}^{\mathrm{U}(n)}_k$ of $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(l))$ fulfils

$$\left(\mathrm{B}j^{\mathrm{U}}_{k,l}\right)^* \left(\left(\mathrm{B}j^{\mathrm{U}}_{l,n}\right)^* \mathsf{c}^{\mathrm{U}(n)}_k\right) = \mathsf{c}^{\mathrm{U}(k)}_k\,.$$

   Hence, by Theorem 4.2.1, it coincides with $\mathsf{c}^{\mathrm{U}(l)}_k$. If $k > l$, we may use (4.2.8) to write

$$\left(\mathrm{B}j^{\mathrm{U}}_{l,n}\right)^* \mathsf{c}^{\mathrm{U}(n)}_k = \left(\mathrm{B}j^{\mathrm{U}}_{l,k-1}\right)^* \circ \left(\mathrm{B}j^{\mathrm{U}}_{k-1,k}\right)^* \circ \left(\mathrm{B}j^{\mathrm{U}}_{k,n}\right)^* \left(\mathsf{c}^{\mathrm{U}(n)}_k\right).$$

   By exactness of the Gysin sequence (4.2.9),

$$\left(\mathrm{B}j^{\mathrm{U}}_{k-1,k}\right)^* \circ \left(\mathrm{B}j^{\mathrm{U}}_{k,n}\right)^* \left(\mathsf{c}^{\mathrm{U}(n)}_k\right) = \left(\mathrm{B}j^{\mathrm{U}}_{k-1,k}\right)^* \mathsf{c}^{\mathrm{U}(k)}_k = 0\,.$$

   In view of Theorem 4.2.1, equation (4.2.11) implies that $(\mathrm{B}j^{\mathrm{U}}_{l,n})^*$ is surjective and that its kernel coincides with the ideal in $H^{*}_{\mathbb{Z}}(\mathrm{BU}(n))$ generated by $\mathsf{c}^{\mathrm{U}(n)}_{l+1}, \ldots, \mathsf{c}^{\mathrm{U}(n)}_n$. In particular, $(\mathrm{B}j^{\mathrm{U}}_{k,l})^*$ is injective in degree less than $2l+2$, so that, for $k \leq l$, $\mathsf{c}^{\mathrm{U}(n)}_k$ is the only element of $H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n))$ satisfying (4.2.11).
2. Owing to the Universal Coefficient Theorem for cohomology in the form of Theorem 5.5.10 of [598],[5] Theorem 4.2.1 implies that $H^{*}_{\mathbb{R}}(\mathrm{BU}(n))$ is the polynomial ring over $\mathbb{R}$ in the Chern classes. and that $H^{*}_{\mathbb{Z}_2}(\mathrm{BU}(n))$ is the polynomial ring over $\mathbb{Z}_2$ in the mod 2 reductions of the Chern classes. ♦

The classes $\mathsf{c}^{\mathrm{U}(n)}_k$ and $\mathsf{c}^{\mathrm{U}(n)}$ define characteristic classes for principal $\mathrm{U}(n)$-bundles and complex vector bundles. The latter are denoted, respectively, by $\mathsf{c}_k(P)$, $\mathsf{c}(P)$, $\mathsf{c}_k(E)$ and $\mathsf{c}(E)$. By construction,

[5] The assumption made there that $H_k(X)$ be finitely generated for every k is met by all topological spaces of CW-homotopy type.

$$\mathsf{c}(P) = 1 + \mathsf{c}_1(P) + \cdots + \mathsf{c}_n(P)\,, \quad \mathsf{c}(E) = 1 + \mathsf{c}_1(E) + \cdots + \mathsf{c}_n(E)\,.$$

Moreover,

$$\mathsf{c}_k^{\mathrm{U}(n)} = \mathsf{c}_k\big(\mathrm{EU}(n)\big) = \mathsf{c}_k\left(\mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}^n\right)\,, \quad k = 1, \ldots, n\,.$$

Remark 4.2.4

1. The top Chern class $\mathsf{c}_n(E)$ of a complex vector bundle E of rank n coincides with the Euler class of the real vector bundle $E_{\mathbb{R}}$ obtained by field restriction and endowed with the induced orientation,

$$\mathsf{c}_n(E) = \mathsf{e}(E_{\mathbb{R}})\,. \tag{4.2.12}$$

   To see this, let B be the base space of E and let $f : B \to \mathrm{BU}(n)$ be a classifying mapping for E. By definition of $\mathsf{c}_n^{\mathrm{U}(n)}$,

$$\mathsf{c}_n(E) = f^*\mathsf{c}_n^{\mathrm{U}(n)} = f^*\mathsf{e}(E_n^{\mathrm{U}})\,.$$

   Let $F : E \to \mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}^n$ be the complex vector bundle morphism obtained by composing a vertical isomorphism $E \to f^*(\mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}^n)$ with the natural morphism associated with the pullback. The morphism F projects to f. By field restriction, it induces a real vector bundle morphism $E_{\mathbb{R}} \to E_n^{\mathrm{U}}$. Since the latter preserves the induced orientations, Proposition 4.1.12/3 yields $\mathsf{e}(E_{\mathbb{R}}) = f^*\mathsf{e}(E_n^{\mathrm{U}})$. This proves (4.2.12).
2. In view of Remark 4.1.6, it follows from point 1 that the top Chern class $\mathsf{c}_n(P)$ of a principal $\mathrm{U}(n)$-bundle P coincides with the Euler class of the real vector bundle obtained by field restriction from the complex vector bundle $P \times_{\mathrm{U}(n)} \mathbb{C}^n$ associated with P via the basic representation, endowed with the induced orientation:

$$\mathsf{c}_n(P) = \mathsf{e}\big(P \times_{\mathrm{U}(n)} \mathbb{C}^n_{\mathbb{R}}\big)\,. \tag{4.2.13}$$

3. Let P be a principal $\mathrm{U}(n)$-bundle over a manifold M. Via the natural surjective homomorphism $H^{2k}_{\mathbb{Z}}(M) \to \mathrm{Hom}(H_{2k}(M), \mathbb{Z})$, the Chern classes $\mathsf{c}_k(P)$ define homomorphisms $H_{2k}(M) \to \mathbb{Z}$. Given a set of generators $\{s_1, \ldots, s_r\}$ of $H_{2k}(M)$, the integers obtained by evaluating $\mathsf{c}_k(P)$ on the s_i are called the Chern indices of P and are denoted by $\mathfrak{c}_{k,1}(P), \ldots, \mathfrak{c}_{k,r}(P)$. Accordingly, one defines the Chern indices $\mathfrak{c}_{k,i}(E)$ of a complex vector bundle E. ♦

Next, we discuss the special unitary groups $\mathrm{SU}(n)$. Let $j_n^{\mathrm{SU,U}} : \mathrm{SU}(n) \to \mathrm{U}(n)$ denote the natural inclusion mapping. Since $\mathrm{SU}(n)$ is a normal subgroup of $\mathrm{U}(n)$,

$$\mathrm{B}j_n^{\mathrm{SU,U}} : \mathrm{BSU}(n) \to \mathrm{BU}(n)$$

is a principal bundle with structure group $\mathrm{U}(n)/\mathrm{SU}(n) \cong \mathrm{U}(1)$, cf. Proposition 3.7.5/1.

Theorem 4.2.5 (Integral cohomology of $\mathrm{BSU}(n)$) *One has* $\left(\mathrm{B}j_n^{\mathrm{SU,U}}\right)^*\mathsf{c}_1^{\mathrm{U}(n)} = 0$. *The ring* $H^*_{\mathbb{Z}}(\mathrm{BSU}(n))$ *is the polynomial ring over* $\mathbb{Z}$ *in the generators*

$$\mathsf{c}_k^{\mathrm{SU}(n)} := (\mathrm{B}j_n^{\mathrm{SU,U}})^*\mathsf{c}_k^{\mathrm{U}(n)}, \quad k = 2, \dots, n.$$

Proof Let $E_n^{\det}$ denote the real vector bundle obtained by field restriction from the associated vector bundle $\mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}$, where $\mathrm{U}(n)$ acts on $\mathbb{C}$ via the determinant, and endow $E_n^{\det}$ with the induced orientation.

We claim that the Euler class of $E_n^{\det}$ is given by $\mathsf{c}_1^{\mathrm{U}(n)}$. For $n = 1$, this holds trivially, because $E_1^{\det} = E_1^{\mathrm{U}}$ as oriented vector bundles. For $n > 1$, we realize $\mathrm{BU}(1)$ as $\mathrm{EU}(n)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ acts on $\mathrm{EU}(n)$ via $j_{1,n}^{\mathrm{U}}$. Then, $E_1^{\det} = \mathrm{EU}(n) \times_{\mathrm{U}(1)} \mathbb{C}_{\mathbb{R}}$, and the natural projection to classes $\mathrm{EU}(n) \times \mathbb{C}_{\mathbb{R}} \to E_n^{\det}$ descends to an orientation-preserving vector bundle morphism $E_1^{\det} \to E_n^{\det}$ which projects to $\mathrm{B}j_{1,n}^{\mathrm{U}}$ and whose fibre mappings are isomorphisms. Hence, Proposition 4.1.12/3 yields that $\left(\mathrm{B}j_{1,n}^{\mathrm{U}}\right)^*$ maps the Euler class of $E_n^{\det}$ to that of $E_1^{\det}$, that is, to $\mathsf{c}_1^{\mathrm{U}(1)}$. Then, Theorem 4.2.1 yields the assertion.

Next, similar to the proof of Theorem 4.2.1, we use the vertical isomorphism $\mathrm{BSU}(n) \cong \mathrm{S}E_n^{\det}$ provided by Example 1.2.4 to replace $\mathrm{S}E_n^{\det}$ by $\mathrm{BSU}(n)$ in the Gysin sequence of $E_n^{\det}$:

$$\cdots \xrightarrow{\varphi} H^l_{\mathbb{Z}}(\mathrm{BU}(n)) \xrightarrow{\cup\, \mathsf{c}_1^{\mathrm{U}(n)}} H^{l+2}_{\mathbb{Z}}(\mathrm{BU}(n)) \xrightarrow{(\mathrm{B}j_n^{\mathrm{SU,U}})^*} H^{l+2}_{\mathbb{Z}}(\mathrm{BSU}(n)) \xrightarrow{\varphi} H^{l+1}_{\mathbb{Z}}(\mathrm{BU}(n)) \to \cdots \tag{4.2.14}$$

For $l = 0$, exactness implies $\left(\mathrm{B}j_n^{\mathrm{SU,U}}\right)^*\mathsf{c}_1^{\mathrm{U}(n)} = 0$. Second, Theorem 4.2.1 yields that multiplication by $\mathsf{c}_1^{\mathrm{U}(n)}$ is injective. By exactness, then $\varphi = 0$ and, therefore, $(\mathrm{B}j_n^{\mathrm{SU,U}})^*$ is surjective in each degree. Third, Theorem 4.2.1 yields that the monomials of degree $l + 2$ in $\mathsf{c}_2^{\mathrm{U}(n)}, \dots, \mathsf{c}_n^{\mathrm{U}(n)}$ freely generate a subgroup of $H^{l+2}_{\mathbb{Z}}(\mathrm{BU}(n))$ which is complementary to $H^l_{\mathbb{Z}}(\mathrm{BU}(n)) \cup \mathsf{c}_1^{\mathrm{U}(n)}$. By exactness again, $(\mathrm{B}j_n^{\mathrm{SU,U}})^*$ maps that subgroup isomorphically onto $H^{l+2}_{\mathbb{Z}}(\mathrm{BSU}(n))$. This proves the theorem. ■

Definition 4.2.6 For $k = 2, \dots, n$, the element $\mathsf{c}_k^{\mathrm{SU}(n)}$ of $H^{2k}_{\mathbb{Z}}(\mathrm{BSU}(n))$ is called the k-th universal Chern class of $\mathrm{SU}(n)$. The element

$$\mathsf{c}^{\mathrm{SU}(n)} := 1 + \mathsf{c}_2^{\mathrm{SU}(n)} + \cdots + \mathsf{c}_n^{\mathrm{SU}(n)}$$

of $H^*_{\mathbb{Z}}(\mathrm{BSU}(n))$ is called the total universal Chern class of $\mathrm{SU}(n)$.

By construction,

$$\left(\mathrm{B}j_n^{\mathrm{SU,U}}\right)^*\mathsf{c}^{\mathrm{U}(n)} = \mathsf{c}^{\mathrm{SU}(n)}\,. \tag{4.2.15}$$

The Chern classes and the total Chern class of $\mathrm{SU}(n)$ define characteristic classes for principal $\mathrm{SU}(n)$-bundles, denoted by $\mathsf{c}_k(P)$ and $\mathsf{c}(P)$, respectively. By construction,

$$\mathsf{c}(P) = 1 + \mathsf{c}_2(P) + \cdots + \mathsf{c}_n(P)\,.$$

Moreover, $\mathsf{c}_k^{\mathrm{SU}(n)} = \mathsf{c}_k\big(\mathrm{ESU}(n)\big) = \mathsf{c}_k\left(\mathrm{ESU}(n) \times_{\mathrm{SU}(n)} \mathbb{C}^n\right)$ for $k = 2, \dots, n$. Let us add that, for convenience, we sometimes use the notation $\mathsf{c}_1^{\mathrm{SU}(n)} = 0$ and, for principal $\mathrm{SU}(n)$-bundles, $\mathsf{c}_1(P) = 0$.

Remark 4.2.7

1. According to Theorem 4.2.5, $(\mathrm{B}j_n^{\mathrm{SU,U}})^*$ is surjective and its kernel is the ideal in $H^*_{\mathbb{Z}}(\mathrm{BU}(n))$ generated by $\mathsf{c}_1^{\mathrm{U}(n)}$.
2. For $l \leq n$, let $j_{l,n}^{\mathrm{SU}} : \mathrm{SU}(l) \to \mathrm{SU}(n)$ denote the Lie subgroup embedding induced by (4.2.7). Then,
$$j_n^{\mathrm{SU,U}} \circ j_{l,n}^{\mathrm{SU}} = j_{l,n}^{\mathrm{U}} \circ j_l^{\mathrm{SU,U}}$$
and (4.2.11) remains valid if we replace $j_{l,n}^{\mathrm{U}}$ by $j_{l,n}^{\mathrm{SU}}$ and $\mathsf{c}_k^{\mathrm{U}(l)}$ by $\mathsf{c}_k^{\mathrm{SU}(l)}$. In view of Theorem 4.2.5, this implies that $(\mathrm{B}j_{l,n}^{\mathrm{SU}})^*$ is surjective and that its kernel is the ideal in $H^*_{\mathbb{Z}}(\mathrm{BSU}(n))$ generated by $\mathsf{c}_{l+1}^{\mathrm{SU}(n)}, \ldots, \mathsf{c}_n^{\mathrm{SU}(n)}$. In particular, $(\mathrm{B}j_{l,n}^{\mathrm{SU}})^*$ is injective in degree less than $2l+2$, so that, in this case, $\mathsf{c}_k^{\mathrm{SU}(n)}$ is the only element of $H^{2k}_{\mathbb{Z}}(\mathrm{BSU}(n))$ satisfying (4.2.11).
3. Point 2 of Remark 4.2.3 carries over to the present case in an obvious way. ♦

Corollary 4.2.8 (Obstructions to orientability in the complex case)

1. *Let P be a principal* $\mathrm{SU}(n)$*-bundle and let Q be the extension of P to the structure group* $\mathrm{U}(n)$. *Then,*
$$\mathsf{c}(Q) = \mathsf{c}(P)\,.$$
In particular, if a principal $\mathrm{U}(n)$*-bundle Q admits a reduction to the structure group* $\mathrm{SU}(n)$*, then* $\mathsf{c}_1(Q) = 0$.
2. *If a complex vector bundle E is orientable, then* $\mathsf{c}_1(E) = 0$.

In Sect. 4.8 we will prove that the vanishing of the first Chern class is also sufficient for a principal $\mathrm{U}(n)$-bundle to admit a reduction to the structure group $\mathrm{SU}(n)$ and for a complex vector bundle to be orientable, cf. Corollary 4.8.2.

Proof

1. Let f be a classifying mapping for P. According to Proposition 3.7.2/1, then $\mathrm{B}j_n^{\mathrm{SU,U}} \circ f$ is a classifying mapping for the extension Q. Hence, (4.2.15) yields
$$\mathsf{c}(Q) = f^* \circ \left(\mathrm{B}j_n^{\mathrm{SU,U}}\right)^* \left(\mathsf{c}^{\mathrm{U}(n)}\right) = f^* \mathsf{c}^{\mathrm{SU}(n)} = \mathsf{c}(P)\,.$$
If now Q is a principal $\mathrm{U}(n)$-bundle and P is a reduction of Q to the structure group $\mathrm{SU}(n)$, then Q is vertically isomorphic to the extension of P to the structure group $\mathrm{U}(n)$. Since vertically isomorphic principal bundles have the same characteristic classes, $\mathsf{c}(Q)$ coincides with the total Chern class of the extension. As was just shown, then $\mathsf{c}(Q) = \mathsf{c}(P)$ and, therefore, $\mathsf{c}_1(Q) = 0$.

2. This follows from point 1 by observing that if E is orientable, then the orthonormal frame bundle of E with respect to some auxiliary fibre metric admits a reduction to the structure group $\mathrm{SU}(n)$ (Exercise 4.2.1). ■

Next, we pass to the discussion of the compact symplectic groups $\mathrm{Sp}(n)$. The arguments are largely analogous to those for $\mathrm{U}(n)$. Let

$$E_n^{\mathrm{Sp}} = \left(\mathrm{ESp}(n) \times_{\mathrm{Sp}(n)} \mathbb{H}^n\right)_{\mathbb{R}},$$

where $\mathrm{Sp}(n)$ acts on $\mathbb{H}^n$ in the basic representation, and endow E_n^{Sp} with the induced orientation. Let $\mathsf{p}_n^{\mathrm{Sp}(n)} \in H_{\mathbb{Z}}^{4n}(\mathrm{BU}(n))$ denote the corresponding Euler class, that is,

$$\mathsf{p}_n^{\mathrm{Sp}(n)} = \mathsf{e}\left(E_n^{\mathrm{Sp}}\right).$$

For $0 < k \leq n$, let $j_{k,n}^{\mathrm{Sp}} : \mathrm{Sp}(k) \to \mathrm{Sp}(n)$ denote the Lie subgroup embedding induced by the linear subspace embedding

$$\mathbb{H}^k \to \mathbb{H}^n, \quad (q_1, \ldots, q_k) \mapsto (q_1, \ldots, q_k, 0, \ldots, 0). \tag{4.2.16}$$

By construction, for $0 < k \leq l \leq n$,

$$j_{l,n}^{\mathrm{Sp}} \circ j_{k,l}^{\mathrm{Sp}} = j_{k,n}^{\mathrm{Sp}}. \tag{4.2.17}$$

Theorem 4.2.9 (Integral cohomology of $\mathrm{BSp}(n)$) *For $k = 1, \ldots, n-1$, there exists a unique element $\mathsf{p}_k^{\mathrm{Sp}(n)}$ of $H_{\mathbb{Z}}^{4k}(\mathrm{BSp}(n))$ such that*

$$(\mathrm{B}j_{k,n}^{\mathrm{Sp}})^* \mathsf{p}_k^{\mathrm{Sp}(n)} = \mathsf{p}_k^{\mathrm{Sp}(k)}.$$

The ring $H_{\mathbb{Z}}^*(\mathrm{BSp}(n))$ is the polynomial ring over $\mathbb{Z}$ in the generators $\mathsf{p}_1^{\mathrm{Sp}(n)}, \ldots, \mathsf{p}_n^{\mathrm{Sp}(n)}$.

Proof By taking the real part of the standard scalar product on $\mathbb{H}^n$, we obtain a scalar product on the real vector space $\mathbb{H}_{\mathbb{R}}^n$ and thus a Riemannian fibre metric on the real vector bundle E_n^{Sp}. Then,

$$\mathrm{D}E_n^{\mathrm{Sp}} = \mathrm{ESp}(n) \times_{\mathrm{Sp}(n)} \mathrm{D}^{4n}, \quad \mathrm{S}E_n^{\mathrm{Sp}} = \mathrm{ESp}(n) \times_{\mathrm{Sp}(n)} \mathrm{S}^{4n-1},$$

where D^{4n} and S^{4n-1} stand for the unit disk and the unit sphere in $\mathbb{H}^n$. By an argument analogous to that for $\mathrm{U}(n)$, one can show that the fibre bundle $\mathrm{B}j_{n-1,n}^{\mathrm{Sp}} : \mathrm{BSp}(n-1) \to \mathrm{BSp}(n)$ is vertically isomorphic to $\mathrm{S}E_n^{\mathrm{Sp}}$ for all $n = 1, 2\ldots$, where $\mathrm{Sp}(0)$ is defined as the trivial group. Under this isomorphism, the Gysin sequence of E_n^{Sp} translates into the exact sequence

$$\cdots \xrightarrow{\varphi} H_{\mathbb{Z}}^l(\mathrm{BSp}(n)) \xrightarrow{\cup \mathsf{p}_n^{\mathrm{Sp}(n)}} H_{\mathbb{Z}}^{l+4n}(\mathrm{BSp}(n)) \xrightarrow{(\mathrm{B}j_{n-1,n}^{\mathrm{Sp}})^*} H_{\mathbb{Z}}^{l+4n}(\mathrm{BSp}(n-1)) \xrightarrow{\varphi} H_{\mathbb{Z}}^{l+1}(\mathrm{BSp}(n)) \to \cdots$$

where $l \in \mathbb{Z}$. Now, the assertion is proved by induction on n in a similar way as for $\mathrm{U}(n)$ (Exercise 4.2.2). ∎

Definition 4.2.10 (*Symplectic Pontryagin classes*) For $k = 1, \ldots, n$, the element $\mathsf{p}_k^{\mathrm{Sp}(n)}$ of $H_{\mathbb{Z}}^{4k}(\mathrm{BSp}(n))$ provided by Theorem 4.2.9 is called the k-th universal Pontryagin class of $\mathrm{Sp}(n)$ and the element

$$\mathsf{p}^{\mathrm{Sp}(n)} := 1 + \mathsf{p}_1^{\mathrm{Sp}(n)} + \cdots + \mathsf{p}_n^{\mathrm{Sp}(n)}$$

of $H_{\mathbb{Z}}^*(\mathrm{BSp}(n))$ is called the total universal Pontryagin class of $\mathrm{Sp}(n)$.

Remark 4.2.3 carries over in an obvious way to the present case (Exercise 4.2.3). The classes $\mathsf{p}_k^{\mathrm{Sp}(n)}$ and $\mathsf{p}^{\mathrm{Sp}(n)}$ define characteristic classes for principal $\mathrm{Sp}(n)$-bundles and quaternionic vector bundles. The latter are denoted, respectively, by $\mathsf{p}_k(P)$, $\mathsf{p}(P)$, $\mathsf{p}_k(E)$ and $\mathsf{p}(E)$. By construction,

$$\mathsf{p}(P) = 1 + \mathsf{p}_1(P) + \cdots + \mathsf{p}_n(P)\,, \quad \mathsf{p}(E) = 1 + \mathsf{p}_1(E) + \cdots + \mathsf{p}_n(E)\,.$$

By means of the natural homomorphism $H^{4k}_{\mathbb{Z}}(M) \to \mathrm{Hom}(H_{4k}(M), \mathbb{Z})$, for every principal $\mathrm{Sp}(n)$-bundle P and every quaternionic vector bundle one can define the Pontryagin indices $\mathfrak{p}_{k,i}(P)$ and $\mathfrak{p}_{k,i}(E)$, respectively, relative to a chosen set of generators of $H_{4k}(M)$.

Finally, we discuss the orthogonal groups $\mathrm{O}(n)$ and the special orthogonal groups $\mathrm{SO}(n)$. The standard action of $\mathrm{O}(n)$ on $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$, defines the associated real vector bundle

$$E_n^{\mathrm{O}} = \mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbb{R}^n\,.$$

Since E_n^{O} is universal for real vector bundles of rank n, it cannot be orientable, because otherwise every real vector bundle of rank n would be orientable. It is known that this is not true. An argument will be given below. Thus, E_n^{O} has an Euler class in $\mathbb{Z}_2$-cohomology only, and the Gysin sequence of E_n^{O} can provide information about the $\mathbb{Z}_2$-cohomology of $\mathrm{BO}(n)$ only. Nevertheless, Gysin sequences allow for deriving the integral cohomology of $H^*_{\mathbb{Z}}(\mathrm{BSO}(n))$ and $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$. The arguments involved are sophisticated though.

Let $\mathsf{w}_n^{\mathrm{O}(n)}$ denote the $\mathbb{Z}_2$-Euler class of E_n^{O}, that is,

$$\mathsf{w}_n^{\mathrm{O}(n)} = \mathsf{e}\left(E_n^{\mathrm{O}}\right) \in H^n_{\mathbb{Z}_2}(\mathrm{BO}(n))\,.$$

For $0 < k \leq n$, let $j^{\mathrm{O}}_{k,n} : \mathrm{O}(k) \to \mathrm{O}(n)$ denote the Lie subgroup embedding induced by the linear subspace embedding

$$\mathbb{R}^k \to \mathbb{R}^n\,, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)\,. \tag{4.2.18}$$

By construction, for $0 < k \leq l \leq n$,

$$j^{\mathrm{O}}_{l,n} \circ j^{\mathrm{O}}_{k,l} = j^{\mathrm{O}}_{k,n}\,. \tag{4.2.19}$$

Theorem 4.2.11 ($\mathbb{Z}_2$-cohomology of $\mathrm{BO}(n)$) *For $k = 1, \ldots, n-1$, there exists a unique element $\mathsf{w}_k^{\mathrm{O}(n)}$ of $H^k_{\mathbb{Z}_2}(\mathrm{BO}(n))$ such that*

$$(\mathrm{B}j^{\mathrm{O}}_{k,n})^* \mathsf{w}_k^{\mathrm{O}(n)} = \mathsf{w}_k^{\mathrm{O}(k)}\,.$$

The ring $H^*_{\mathbb{Z}_2}(\mathrm{BO}(n))$ is the polynomial ring over $\mathbb{Z}_2$ in the generators $\mathsf{w}_1^{\mathrm{O}(n)}, \ldots, \mathsf{w}_n^{\mathrm{O}(n)}$.

Proof The standard scalar product on $\mathbb{R}^n$ defines a Riemannian fibre metric on E_n^{O}. As in the case of $\mathrm{U}(n)$, one can show that the sphere bundle $j^{\mathrm{O}}_{n-1,n} : \mathrm{BO}(n-1) \to \mathrm{BO}(n)$

is vertically isomorphic to SE_n^{o} for $n = 1, 2, \dots$, where $\mathrm{O}(0)$ is defined as the trivial group. Under this isomorphism, the Gysin sequence in $\mathbb{Z}_2$-cohomology of E_n^{o} with the chosen Riemannian fibre metric translates into the exact sequence

$$\cdots \xrightarrow{\varphi} H^l_{\mathbb{Z}_2}(\mathrm{BO}(n)) \xrightarrow{\cup \mathsf{w}_n^{\mathrm{O}(n)}} H^{l+n}_{\mathbb{Z}_2}(\mathrm{BO}(n)) \xrightarrow{(\mathrm{B}j^{\mathrm{O}}_{n-1,n})^*} H^{l+n}_{\mathbb{Z}_2}(\mathrm{BO}(n-1)) \xrightarrow{\varphi} H^{l+1}_{\mathbb{Z}_2}(\mathrm{BO}(n)) \to \cdots$$

where $l \in \mathbb{Z}$. Now, the assertion is proved by induction on n in a similar way as for $\mathrm{U}(n)$ (Exercise 4.2.4). ■

Definition 4.2.12 (*Stiefel–Whitney classes*) For $k = 1, \dots, n$, the element $\mathsf{w}_k^{\mathrm{O}(n)}$ of $H^k_{\mathbb{Z}_2}(\mathrm{BO}(n))$ provided by Theorem 4.2.11 is called the k-th universal Stiefel–Whitney class of $\mathrm{O}(n)$ and the element

$$\mathsf{w}^{\mathrm{O}(n)} := 1 + \mathsf{w}_1^{\mathrm{O}(n)} + \cdots + \mathsf{w}_n^{\mathrm{O}(n)}$$

of $H^*_{\mathbb{Z}_2}(\mathrm{BO}(n))$ is called the total universal Stiefel–Whitney class of $\mathrm{O}(n)$.

The classes $\mathsf{w}_k^{\mathrm{O}(n)}$ and $\mathsf{w}^{\mathrm{O}(n)}$ define characteristic classes for principal $\mathrm{O}(n)$-bundles and real vector bundles, denoted by, respectively, $\mathsf{w}_k(P)$, $\mathsf{w}(P)$, $\mathsf{w}_k(E)$ and $\mathsf{w}(E)$. By construction,

$$\mathsf{w}(P) = 1 + \mathsf{w}_1(P) + \cdots + \mathsf{w}_n(P)\,, \quad \mathsf{w}(E) = 1 + \mathsf{w}_1(E) + \cdots + \mathsf{w}_n(E)\,.$$

Remark 4.2.13
By analogy with Remark 4.2.3, one can check the following (Exercise 4.2.5).

1. For $l \leq n$, one has
$$(\mathrm{B}j^{\mathrm{O}}_{l,n})^* \mathsf{w}_k^{\mathrm{O}(n)} = \begin{cases} \mathsf{w}_k^{\mathrm{O}(l)} & 1 \leq k \leq l\,, \\ 0 & l < k \leq n\,. \end{cases} \tag{4.2.20}$$
Moreover, $(\mathrm{B}j^{\mathrm{O}}_{l,n})^*$ is surjective and its kernel is the ideal in $H^*_{\mathbb{Z}_2}(\mathrm{BO}(n))$ generated by $\mathsf{w}_{l+1}^{\mathrm{O}(n)}, \dots, \mathsf{w}_n^{\mathrm{O}(n)}$. In particular, for $k \leq l$, $\mathsf{w}_k^{\mathrm{O}(n)}$ is the only element of $H^k_{\mathbb{Z}_2}(\mathrm{BO}(n))$ satisfying (4.2.20).
2. The top Stiefel–Whitney class $\mathsf{w}_n(E)$ of a real vector bundle E of rank n coincides with the $\mathbb{Z}_2$-Euler class of that vector bundle.
3. The top Stiefel–Whitney class $\mathsf{w}_n(P)$ of a principal $\mathrm{O}(n)$-bundle P coincides with the $\mathbb{Z}_2$-Euler class of the real vector bundle $P \times_{\mathrm{O}(n)} \mathbb{R}^n$ associated with P by means of the basic representation of $\mathrm{O}(n)$. ♦

Now, we turn to the discussion of the special orthogonal groups $\mathrm{SO}(n)$. We derive the $\mathbb{Z}_2$-cohomology of $\mathrm{BSO}(n)$ from that of $\mathrm{BO}(n)$ in complete analogy with the discussion for $\mathrm{SU}(n)$. Let $j_n^{\mathrm{SO,O}} : \mathrm{SO}(n) \to \mathrm{O}(n)$ denote the natural inclusion mapping. Since $\mathrm{SO}(n)$ is a normal subgroup of $\mathrm{O}(n)$, the fibre bundle

$$\mathrm{B}j_n^{\mathrm{SO,O}} : \mathrm{BSO}(n) \to \mathrm{BO}(n)$$

is a principal bundle with structure group $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathrm{O}(1) \cong \mathbb{Z}_2$.

Theorem 4.2.14 ($\mathbb{Z}_2$-cohomology of $\mathrm{BSO}(n)$) *One has* $\left(\mathrm{B}j_n^{\mathrm{SO,O}}\right)^* \mathsf{w}_1^{\mathrm{O}(n)} = 0$. *The ring* $H^*_{\mathbb{Z}_2}(\mathrm{BSO}(n))$ *is the polynomial ring over* $\mathbb{Z}_2$ *in the generators* $\mathsf{w}_k^{\mathrm{SO}(n)} := (\mathrm{B}j_n^{\mathrm{SO,O}})^* \mathsf{w}_k^{\mathrm{O}(n)}$, $k = 2, \ldots, n$.

As a consequence, the homomorphism $\left(\mathrm{B}j_n^{\mathrm{SO,O}}\right)^* : H^*_{\mathbb{Z}_2}(\mathrm{BO}(n)) \to H^*_{\mathbb{Z}_2}(\mathrm{BSO}(n))$ is surjective.

Proof The proof is completely analogous to that of Theorem 4.2.5. Starting with the real line bundle $\mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbb{R}$, with $\mathrm{O}(n)$ acting via the determinant, one just has to carry out the obvious modifications and forget about orientations. This leads to the exact sequence

$$\cdots \xrightarrow{\varphi} H^l_{\mathbb{Z}_2}(\mathrm{BO}(n)) \xrightarrow{\cup \mathsf{w}_1^{\mathrm{O}(n)}} H^{l+1}_{\mathbb{Z}_2}(\mathrm{BO}(n)) \xrightarrow{(\mathrm{B}j_n^{\mathrm{SO,O}})^*} H^{l+1}_{\mathbb{Z}_2}(\mathrm{BSO}(n)) \xrightarrow{\varphi} H^{l+1}_{\mathbb{Z}_2}(\mathrm{BO}(n)) \to \cdots$$

to which the rest of the argument can be adapted easily. ■

Definition 4.2.15 For $k = 2, \ldots, n$, the element $\mathsf{w}_k^{\mathrm{SO}(n)}$ of $H^k_{\mathbb{Z}_2}(\mathrm{BSO}(n))$ is called the k-th universal Stiefel–Whitney class of $\mathrm{SO}(n)$. The element

$$\mathsf{w}^{\mathrm{SO}(n)} := 1 + \mathsf{w}_2^{\mathrm{SO}(n)} + \cdots + \mathsf{w}_n^{\mathrm{SO}(n)}$$

of $H^*_{\mathbb{Z}_2}(\mathrm{BSO}(n))$ is called the total universal Stiefel–Whitney class of $\mathrm{SO}(n)$.

By construction,

$$\left(\mathrm{B}j_n^{\mathrm{SO,O}}\right)^* \mathsf{w}^{\mathrm{O}(n)} = \mathsf{w}^{\mathrm{SO}(n)} . \tag{4.2.21}$$

The classes $\mathsf{w}_k^{\mathrm{SO}(n)}$ and $\mathsf{w}^{\mathrm{SO}(n)}$ define characteristic classes for principal $\mathrm{SO}(n)$-bundles, denoted by $\mathsf{w}_k(P)$ and $\mathsf{w}(P)$, respectively. One has

$$\mathsf{w}(P) = 1 + \mathsf{w}_2(P) + \cdots + \mathsf{w}_n(P) .$$

Moreover,

$$\mathsf{w}_k^{\mathrm{SO}(n)} = \mathsf{w}_k\big(\mathrm{ESO}(n)\big) = \mathsf{w}_k\left(\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} \mathbb{R}^n\right) , \quad k = 2, \ldots, n .$$

As for the Chern classes, for convenience, we sometimes use the notation $\mathsf{w}_1^{\mathrm{SO}(n)} = 0$.

Remark 4.2.16

1. Point 1 of Remark 4.2.13 carries over to the case of $\mathrm{SO}(n)$ in an obvious way.
2. According to Theorem 4.2.14, $(\mathrm{B}j_n^{\mathrm{SO,O}})^*$ is surjective and its kernel is the ideal in $H^*_{\mathbb{Z}_2}(\mathrm{BO}(n))$ generated by $\mathsf{w}_1^{\mathrm{O}(n)}$. ♦

The proof of the following corollary of Theorem 4.2.14 is completely analogous to that of Corollary 4.2.17 and is left to the reader.

Corollary 4.2.17 (Obstructions to orientability in the real case)

1. *Let P be a principal* $\mathrm{SO}(n)$*-bundle and let Q be the extension of P to the structure group* $\mathrm{O}(n)$*. Then,*
$$\mathsf{w}(Q) = \mathsf{w}(P)\,.$$
In particular, if a principal $\mathrm{O}(n)$*-bundle Q admits a reduction to the structure group* $\mathrm{SO}(n)$*, then* $\mathsf{w}_1(Q) = 0$*.*
2. *If a real vector bundle E is orientable, then* $\mathsf{w}_1(E) = 0$*.* ■

In Sect. 4.8, we will prove that the vanishing of the first Stiefel–Whitney class is also sufficient for a principal $\mathrm{O}(n)$-bundle to admit a reduction to the structure group $\mathrm{SO}(n)$ and for a real vector bundle to be orientable, cf. Corollary 4.8.4.

Example 4.2.18 We determine the Chern class of the canonical $\mathrm{U}(1)$-bundle $P_n = \mathrm{S}_{\mathbb{C}}(1, n+1)$ over $\mathrm{G}_{\mathbb{C}}(1, n+1) = \mathbb{C}\mathrm{P}^n$, cf. Remark 1.1.21/3 and Example 1.1.24. Let L_n denote the tautological line bundle over $\mathbb{C}\mathrm{P}^n$, obtained by attaching to each point of $\mathbb{C}\mathrm{P}^n$ the subspace of $\mathbb{C}^{n+1}$ represented by this point. One can check that the mapping
$$P_n \times_{\mathrm{U}(1)} \mathbb{C} \to L_n\,, \quad [(\mathbf{z}, \zeta)] \mapsto \zeta\mathbf{z}\,,$$
is a vertical isomorphism. Hence, $\mathsf{c}(P_n) = \mathsf{c}(L_n)$. We show that $\mathsf{c}_1(L_n)$ is a generator of $H^2_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n)$. For that purpose, we write down the integral Gysin sequence of the realification $(L_n)_{\mathbb{R}}$ endowed with the induced orientation and with the Riemannian fibre metric induced from the standard scalar product on $\mathbb{C}^{n+1}_{\mathbb{R}}$,
$$\cdots \to H^k_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cup\,\mathsf{e}((L_n)_{\mathbb{R}})} H^{k+2}_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\pi_{\mathrm{S}}^*} H^{k+2}_{\mathbb{Z}}(\mathrm{S}L_n) \xrightarrow{\varphi} H^{k+1}_{R}(\mathbb{C}\mathrm{P}^n) \to \cdots$$
By Remark 4.2.4/1, we can replace $\mathsf{e}\big((L_n)_{\mathbb{R}}\big) = \mathsf{c}_1(L_n)$. Since
$$\mathrm{S}L_n = \{(p, \mathbf{z}) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1} : \mathbf{z} \in p\,,\ \|\mathbf{z}\| = 1\} = \{(p, \mathbf{z}) \in \mathbb{C}\mathrm{P}^n \times \mathrm{S}^{2n+1} : \mathbf{z} \in p\}\,,$$
we can define mutually inverse continuous mappings
$$\mathrm{S}L_n \to \mathrm{S}^{2n+1}\,, \quad (p, \mathbf{z}) \mapsto \mathbf{z}\,, \qquad \mathrm{S}^{2n+1} \to \mathrm{S}L_n\,, \quad \mathbf{z} \mapsto \big([\mathbf{z}], \mathbf{z}\big)\,.$$
That is, $\mathrm{S}L_n$ is homeomorphic to S^{2n+1}. Hence, for $k < 2n - 1$, the Gysin sequence splits into the pieces
$$0 \to H^k_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cup\,\mathsf{c}_1(L_n)} H^{k+2}_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n) \xrightarrow{\pi_{\mathrm{S}}^*} 0\,. \tag{4.2.22}$$
By setting $k = 0$, we obtain that $\mathsf{c}_1(L_n) = \mathsf{c}_1(P_n)$ is a generator of $H^2_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n)$. Thus, $H^*_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n)$ is the free Abelian group generated by $1, \mathsf{c}_1(L_n), \dots, \mathsf{c}_1(L_n)^n$. As a special case, we obtain that the first Chern class of the complex Hopf bundle is a generator of the second integral cohomology group of $\mathbb{C}\mathrm{P}^1 \cong \mathrm{S}^2$.

The arguments given for $\mathbb{C}\mathrm{P}^n$ can be adapted to $\mathbb{R}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$. This leads to the following results (Exercise 4.2.18).

1. The first Stiefel–Whitney class of the canonical O(1)-bundle $P_n^{\mathbb{R}}$ (canonical real line bundle $L_n^{\mathbb{R}}$) over $\mathbb{R}\mathrm{P}^n$ is a generator of $H^1_{\mathbb{Z}_2}(\mathbb{R}\mathrm{P}^n)$ and hence $H^*_{\mathbb{Z}_2}(\mathbb{R}\mathrm{P}^n)$ is the $\mathbb{Z}_2$-module freely generated by 1, $\mathsf{w}_1(L_n^{\mathbb{R}}), \ldots, \mathsf{w}_1(L_n^{\mathbb{R}})^n$. In particular, the first Stiefel–Whitney class of the real Hopf bundle is a generator of the first $\mathbb{Z}_2$-cohomology group of $\mathbb{R}\mathrm{P}^1 \cong \mathrm{S}^1$.
2. The first Pontryagin class of the canonical Sp(1)-bundle $P_n^{\mathbb{H}}$ (canonical quaternionic line bundle $L_n^{\mathbb{H}}$) over $\mathbb{H}\mathrm{P}^n$ is a generator of $H^1_{\mathbb{Z}}(\mathbb{H}\mathrm{P}^n)$ and hence $H^*_{\mathbb{Z}}(\mathbb{H}\mathrm{P}^n)$ is the free Abelian group generated by 1, $\mathsf{p}_1(L_n^{\mathbb{H}}), \ldots, \mathsf{p}_1(L_n^{\mathbb{H}})^n$. In particular, the first Pontryagin class of the quaternionic Hopf bundle is a generator of the fourth integral cohomology group of $\mathbb{H}\mathrm{P}^1 \cong \mathrm{S}^4$. ♦

To conclude this section, we give a survey of the integral cohomology of BO(n) and BSO(n). This was worked out independently in [107, 193]. We will confine ourselves to citing the result and adding a comment on how to derive it using Gysin sequences.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the integer part of x. Define

$$q_n := \lfloor \tfrac{n-1}{2} \rfloor\,, \quad K_n := \{1, \ldots, q_n\}\,, \qquad \bar{q}_n := \lfloor \tfrac{n}{2} \rfloor\,, \quad \bar{K}_n := \{\tfrac{1}{2}\} \cup \{1, \ldots, \bar{q}_n\}\,.$$

One type of generators of $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$ and $H^*_{\mathbb{Z}}(\mathrm{BSO}(n))$ is given by the Chern classes of U(n).

Definition 4.2.19 (*Pontryagin classes*) For $k = 1, \ldots, \bar{q}_n$, the element

$$\mathsf{p}_k^{\mathrm{O}(n)} := (-1)^k \left(\mathrm{B}j_n^{\mathrm{O,U}}\right)^* \mathsf{c}_{2k}^{\mathrm{U}(n)}$$

of $H^{4k}_{\mathbb{Z}}(\mathrm{BO}(n))$ is called the k-th universal Pontryagin class of O(n). The sum

$$\mathsf{p}^{\mathrm{O}(n)} := 1 + \mathsf{p}_1^{\mathrm{O}(n)} + \cdot + \mathsf{p}_{\bar{q}_n}^{\mathrm{O}(n)}$$

is called the total universal Pontryagin class of O(n). By analogy, one defines the Pontryagin classes of SO(n) via the embedding $j_n^{\mathrm{SO,U}}$.

The other type of generators is related to the Stiefel–Whitney classes. Recall that, for every topological space X, the exact sequence of coefficient groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces a long exact sequence in cohomology,

$$\cdots \longrightarrow H^k_{\mathbb{Z}}(X) \xrightarrow{2\cdot} H^k_{\mathbb{Z}}(X) \xrightarrow{\rho_2} H^k_{\mathbb{Z}_2}(X) \xrightarrow{\beta} H^{k+1}_{\mathbb{Z}}(X) \longrightarrow \cdots\,,$$

where ρ_2 denotes reduction modulo 2. The connecting homomorphism β is usually referred to as the (integral) Bockstein homomorphism.

Definition 4.2.20 (*Integral Stiefel–Whitney classes*) Given a nonempty subset $I \subset \bar{K}_n$, the element

$$\mathsf{W}_I^{\mathrm{O}(n)} := \beta\left(\prod_{l \in I} \mathsf{w}_{2l}^{\mathrm{O}(n)}\right)$$

of $H_{\mathbb{Z}}^{1+\sum_{l \in I} 2l}(\mathrm{BO}(n))$ is called the universal integral Stiefel–Whitney class of type I of $\mathrm{O}(n)$. By analogy, given a nonempty subset $I \subset K_n$, one defines the universal integral Stiefel–Whitney class of type I of $\mathrm{SO}(n)$, denoted by $\mathsf{W}_I^{\mathrm{SO}(n)}$.

For $\mathrm{SO}(n)$, there will be one further generator, the universal integral Euler class

$$\mathsf{e}^{\mathrm{SO}(n)} := \mathsf{e}\left(\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} \mathbb{R}^n\right) \in H_{\mathbb{Z}}^n(\mathrm{BSO}(n)) \,,$$

where the orientation is given fibrewise by the standard orientation of $\mathbb{R}^n$.

Remark 4.2.21

1. As a consequence of (4.2.11), for $l \leq n$, one has
$$\left(\mathrm{B}j_{l,n}^{\mathrm{O}}\right)^* \mathsf{p}_k^{\mathrm{O}(n)} = \begin{cases} \mathsf{p}_k^{\mathrm{O}(l)} & 1 \leq k \leq \bar{q}_l \,, \\ 0 & \bar{q}_l < k \leq \bar{q}_n \,, \end{cases} \tag{4.2.23}$$
and an analogous formula for $\mathrm{SO}(n)$. Moreover, by construction,
$$\left(\mathrm{B}j_n^{\mathrm{SO,O}}\right)^* \mathsf{p}^{\mathrm{O}(n)} = \mathsf{p}^{\mathrm{SO}(n)} \,. \tag{4.2.24}$$
2. By the naturality property[6] of the Bockstein homomorphism β, for $l \leq n$ and $I \subset \bar{K}_n$, one has
$$\left(\mathrm{B}j_{l,n}^{\mathrm{O}}\right)^* \mathsf{W}_I^{\mathrm{O}(n)} = \begin{cases} \mathsf{W}_I^{\mathrm{O}(l)} & I \subset \bar{K}_l \,, \\ 0 & I \not\subset \bar{K}_l \,, \end{cases} \tag{4.2.25}$$
and an analogous formula for $\mathrm{SO}(n)$. Using in addition (4.2.21), as well as the fact that for even n the mod 2 reduction of $\beta(\mathsf{w}_n^{\mathrm{O}(n)})$ is given by $\mathsf{w}_1^{\mathrm{O}(n)}\mathsf{w}_n^{\mathrm{O}(n)}$, cf. [598, p. 281],[7] we obtain
$$\left(\mathrm{B}j_n^{\mathrm{SO,O}}\right)^* \mathsf{W}_I^{\mathrm{O}(n)} = \begin{cases} \mathsf{W}_I^{\mathrm{SO}(n)} & I \subset K_n \\ 0 & I \not\subset K_n \end{cases} \tag{4.2.26}$$
for all $I \subset \bar{K}_n$.
3. By Proposition 4.1.12/1, one has $2\mathsf{e}^{\mathrm{SO}(n)} = 0$ if n is odd. ♦

[6]That is, β intertwines the homomorphism induced by a continuous mapping in $\mathbb{Z}_2$-cohomology with that induced in integral cohomology.

[7]Composition of mod 2 reduction with the integral Bockstein homomorphism yields the Steenrod square.

The universal Pontryagin and integral Stiefel–Whitney classes define characteristic classes for principal $\mathrm{O}(n)$-bundles, principal $\mathrm{SO}(n)$-bundles and real vector bundles. These are denoted, respectively, by $\mathsf{p}_k(P)$, $\mathsf{p}(P)$, $\mathsf{W}_I(P)$, $\mathsf{p}_k(E)$, $\mathsf{p}(E)$, and $\mathsf{W}_I(E)$. In addition, the universal Euler class defines a characteristic class for principal $\mathrm{SO}(n)$-bundles and for oriented real vector bundles, denoted by $\mathsf{e}(P)$ and $\mathsf{e}(E)$, respectively, where the latter is due to the fact that $\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} \mathbb{R}^n$ is universal for oriented real vector bundles.

Remark 4.2.22

1. In terms of the integral cohomology class $\mathsf{W}^{\mathrm{O}(n)}_{\{\frac{1}{2}\}} = \beta(\mathsf{w}^{\mathrm{O}(n)}_1)$, the necessary orientability condition of Corollary 4.2.17 reads as follows. If a principal $\mathrm{O}(n)$-bundle P admits a reduction to the subgroup $\mathrm{SO}(n)$, then $\mathsf{W}_{\{\frac{1}{2}\}}(P) = 0$. If a real vector bundle E is orientable, then $\mathsf{W}_{\{\frac{1}{2}\}}(E) = 0$.
2. By analogy with the structure groups $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, using the natural homomorphism $H^{4k}_{\mathbb{Z}}(M) \to \mathrm{Hom}(H_{4k}(M), \mathbb{Z})$, for every principal $\mathrm{O}(n)$-bundle P and every real vector bundle E one can define the Pontryagin indices $\mathfrak{p}_{k,i}(P)$ and $\mathfrak{p}_{k,i}(E)$, respectively, relative to a given set of generators of $H_{4k}(M)$. ♦

The following was proved independently in [107, 193].

Theorem 4.2.23 (Integral cohomology of $\mathrm{BSO}(n)$ and $\mathrm{BO}(n)$) *Let $n \geq 2$.*

1. *The ring $H^*_{\mathbb{Z}}(\mathrm{BSO}(n))$ is generated by $\mathsf{p}^{\mathrm{SO}(n)}_k$ with $k = 0, \dots, q_n$, by $\mathsf{W}^{\mathrm{SO}(n)}_I$ with $I \subset K_n$ nonempty and, in case n is even, by $\mathsf{e}^{\mathrm{SO}(n)}$. The subring generated by $\mathsf{p}^{\mathrm{SO}(n)}_1, \dots, \mathsf{p}^{\mathrm{SO}(n)}_{q_n}$ and, in case n is even, by $\mathsf{e}^{\mathrm{SO}(n)}$, is torsion-free.*
2. *The ring $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$ is generated by $\mathsf{p}^{\mathrm{O}(n)}_k$ with $k = 0, \dots, \bar{q}_n$ and by $\mathsf{W}^{\mathrm{O}(n)}_I$ with $I \subset \bar{K}_n$ nonempty. The subring generated by $\mathsf{p}^{\mathrm{SO}(n)}_1, \dots, \mathsf{p}^{\mathrm{SO}(n)}_{q_n}$ is torsion-free.* ■

For the corresponding torsion ideals and the free quotient rings, we read off the following.

Corollary 4.2.24

1. *The torsion ideals of $H^*_{\mathbb{Z}}(\mathrm{BSO}(n))$ and $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$ are generated by the corresponding integral Stiefel–Whitney classes. In particular, every torsion element has order 2.*
2. *The free quotient ring of $H^*_{\mathbb{Z}}(\mathrm{BSO}(n))$ is the polynomial ring over $\mathbb{Z}$ in the Pontryagin classes $\mathsf{p}^{\mathrm{SO}(n)}_1, \dots, \mathsf{p}^{\mathrm{SO}(n)}_{q_n}$ and, if n is even, the Euler class $\mathsf{e}^{\mathrm{SO}(n)}$. The free quotient ring of $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$ is the polynomial ring over $\mathbb{Z}$ in the Pontryagin classes $\mathsf{p}^{\mathrm{O}(n)}_1, \dots, \mathsf{p}^{\mathrm{O}(n)}_{\bar{q}_n}$.* ■

Remark 4.2.25

1. In addition, in [107, 193], the following was shown. For $I \subset \bar{K}_n$ and $i \in \bar{K}_n$, denote $I_i := I\backslash\{i\}$. For $I, J \subset \bar{K}_n$, let $I \underline{\cup} J := (I \cup J)\backslash(I \cap J)$ (exclusive 'or'). Put $\mathsf{W}_{\varnothing}^{\mathrm{SO}(n)} = \mathsf{W}_{\varnothing}^{\mathrm{O}(n)} = 0$ and $p_{\frac{1}{2}}^{\mathrm{O}(n)} := W_{\{\frac{1}{2}\}}^{\mathrm{O}(n)}$. The defining relations between the generators of $H_{\mathbb{Z}}^*(\mathrm{BSO}(n))$ are
$$\mathsf{W}_I^{\mathrm{SO}(n)}\mathsf{W}_J^{\mathrm{SO}(n)} = \sum_{i\in I}\left(\mathsf{W}_{\{i\}}^{\mathrm{SO}(n)}\mathsf{W}_{I_i\underline{\cup}J}^{\mathrm{SO}(n)}\prod_{j\in I_i\cap J}\mathsf{p}_j^{\mathrm{SO}(n)}\right) \quad \text{for all } I, J \subset K_n, I \neq \varnothing,$$
with the convention that $\prod_{j\in I_i\cap J}\mathsf{p}_j^{\mathrm{SO}(n)} = 1$ in case $I_i \cap J = \varnothing$. The defining relations between the generators of $H_{\mathbb{Z}}^*(\mathrm{BO}(n))$ are
$$W_I^{\mathrm{O}(n)}W_J^{\mathrm{O}(n)} = \sum_{i\in I}\left(W_{\{i\}}^{\mathrm{O}(n)}W_{I_i\underline{\cup}J}^{\mathrm{O}(n)}\prod_{j\in I_i\cap J}p_j^{\mathrm{O}(n)}\right) \quad \text{for all } I, J \subset \bar{K}_n, I \neq \varnothing,$$
holding for all n, and
$$W_{\{\frac{1}{2},\bar{q}_n\}\cup J}^{\mathrm{O}(n)} = W_{\{\bar{q}_n\}}^{\mathrm{O}(n)}W_J^{\mathrm{O}(n)}, \quad W_{\{\bar{q}_n\}}^{\mathrm{O}(n)}W_{\{\bar{q}_n\}\cup J}^{\mathrm{O}(n)} = p_{\bar{q}_n}^{\mathrm{O}(n)}W_{\{\frac{1}{2}\}\cup J}^{\mathrm{O}(n)} \quad \text{for all } J \subset K_n,$$
holding for even n only.
2. Theorem 4.2.23 and the relations given in point 1 can be proved by fairly elementary means, using the Gysin sequence with integral coefficients of the oriented universal vector bundle $\mathrm{ESO}(n) \times_{\mathrm{SO}(n)} \mathbb{R}^n$ to derive $H_{\mathbb{Z}}^*(\mathrm{BSO}(n))$ and the Gysin sequence with local coefficients for the (non-oriented) universal vector bundle $\mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbb{R}^n$ to derive $H_{\mathbb{Z}}^*(\mathrm{BO}(n))$ from $H_{\mathbb{Z}}^*(\mathrm{BSO}(n))$.
3. In view of the Universal Coefficient Theorem for cohomology in the form of Theorem 5.5.10 of [598], point 1 of Corollary 4.2.24 implies that it suffices to control the real and the $\mathbb{Z}_2$-valued cohomology of $\mathrm{BSO}(n)$ and $\mathrm{BO}(n)$. While the latter is given by Theorems 4.2.11 and 4.2.14, the former can be read off from point 2 of Corollary 4.2.24. Thus, $H_{\mathbb{R}}^*(\mathrm{BSO}(n))$ is the polynomial ring over $\mathbb{R}$ in the Pontryagin classes $\mathsf{p}_1^{\mathrm{SO}(n)}, \ldots, \mathsf{p}_{q_n}^{\mathrm{SO}(n)}$ and, if n is even, the Euler class $\mathsf{e}^{\mathrm{SO}(n)}$, and $H_{\mathbb{R}}^*(\mathrm{BO}(n))$ is the polynomial ring over $\mathbb{R}$ in the Pontryagin classes $\mathsf{p}_1^{\mathrm{O}(n)}, \ldots, \mathsf{p}_{\bar{q}_n}^{\mathrm{O}(n)}$.

◆

Exercises

4.2.1 Complete the proof of Corollary 4.2.8/2 by showing that if a complex vector bundle is orientable, then its orthonormal frame bundle with respect to some auxiliary fibre metric admits a reduction to the structure group $\mathrm{SU}(n)$. Prove a similar statement for real vector bundles and the structure group $\mathrm{SO}(n)$.

4.2.2 Complete the proof of Theorem 4.2.9 by adapting the induction argument given for $\mathrm{U}(n)$ in the proof of Theorem 4.2.1 to $\mathrm{Sp}(n)$.

4.2.3 Carry over the statements of Remark 4.2.3 to the case of the symplectic groups.

4.2.4 Complete the proof of Theorem 4.2.11 by adapting the induction argument given for $\mathrm{U}(n)$ in the proof of Theorem 4.2.1 to $\mathrm{O}(n)$.

4.2.5 Prove the statements of Remark 4.2.13.

4.2.6 Adapt the arguments given for $\mathbb{C}\mathrm{P}^n$ in Example 4.2.18 to $\mathbb{R}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ to prove points 1 and 2 in that example.

4.3 Whitney Sum Formula and Splitting Principle

We start with deriving the Whitney Sum Formula. This formula expresses the characteristic classes of the direct sum of vector bundles in terms of the characteristic classes of the constituents.

In the course of the discussion, we will use that the classifying space of a direct product of Lie groups $G_1 \times G_2$ can be realised by $\mathrm{B}G_1 \times \mathrm{B}G_2$. Recall that for elements $\alpha_i \in H^k_{\mathbb{Z}}(\mathrm{B}G_i)$, the cohomology cross product $\alpha_1 \times \alpha_2 \in H^{k+l}_{\mathbb{Z}}(\mathrm{B}G_1 \times \mathrm{B}G_2)$ is defined by

$$\alpha_1 \times \alpha_2 := (\mathrm{pr}_1^* \alpha_1) \cup (\mathrm{pr}_2^* \alpha_2) \,, \tag{4.3.1}$$

where $\mathrm{pr}_i : \mathrm{B}G_1 \times \mathrm{B}G_2 \to \mathrm{B}G_i$ for $i = 1, 2$ denotes the natural projection to the i-th factor. For further use, we note that

$$(\alpha_1 \times \alpha_2) \cup (\beta_1 \times \beta_2) = (-1)^{\deg(\alpha_2)\deg(\beta_1)} (\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2) \,. \tag{4.3.2}$$

Moreover, for the diagonal mapping $\Delta_B : B \to B \times B$,

$$\Delta_B^*(\alpha_1 \times \alpha_2) = \alpha_1 \cup \alpha_2 \,. \tag{4.3.3}$$

The Whitney Sum Formula will be a consequence of the following.

Theorem 4.3.1
For the standard blockwise embeddings

$$\begin{aligned} j_{\mathrm{O}} &: \mathrm{O}(n_1) \times \mathrm{O}(n_2) \to \mathrm{O}(n_1 + n_2) \,, \\ j_{\mathrm{U}} &: \mathrm{U}(n_1) \times \mathrm{U}(n_2) \to \mathrm{U}(n_1 + n_2) \,, \\ j_{\mathrm{Sp}} &: \mathrm{Sp}(n_1) \times \mathrm{Sp}(n_2) \to \mathrm{Sp}(n_1 + n_2) \,, \end{aligned}$$

one has

$$\begin{aligned} (\mathrm{B}j_{\mathrm{O}})^* \mathsf{w}^{\mathrm{O}(n_1+n_2)} &= \mathsf{w}^{\mathrm{O}(n_1)} \times \mathsf{w}^{\mathrm{O}(n_2)} \,, \\ (\mathrm{B}j_{\mathrm{U}})^* \mathsf{c}^{\mathrm{U}(n_1+n_2)} &= \mathsf{c}^{\mathrm{U}(n_1)} \times \mathsf{c}^{\mathrm{U}(n_2)} \,, \\ (\mathrm{B}j_{\mathrm{Sp}})^* \mathsf{p}^{\mathrm{Sp}(n_1+n_2)} &= \mathsf{p}^{\mathrm{Sp}(n_1)} \times \mathsf{p}^{\mathrm{Sp}(n_2)} \,. \end{aligned}$$

Analogous formulae hold for the special orthogonal and the special unitary groups (Exercise 4.3.2).

Proof To be definite, we give the proof for the unitary groups and leave the rest to the reader. Let us write $j \equiv j_{\mathrm{U}}$ and $n = n_1 + n_2$ and let us put $\mathsf{c}_k^{\mathrm{U}(n)} := 0$ for all $k > n$. We have to show that

$$(\mathrm{B}j)^* \mathsf{c}_k^{\mathrm{U}(n)} = \sum_{i=0}^{k} \mathsf{c}_i^{\mathrm{U}(n_1)} \times \mathsf{c}_{k-i}^{\mathrm{U}(n_2)} \tag{4.3.4}$$

for all $k = 0, \ldots, n$. For that purpose, we will fix k and let n run. That is, we will prove (4.3.4) by induction on n, starting with $n = k$. Thus, let k be chosen and let n_1, n_2 be such that $n = k$. Consider the pullback principal $\mathrm{U}(n)$-bundle $(\mathrm{B}j)^*\mathrm{EU}(n)$. By definition of $\mathrm{B}j$, it is vertically isomorphic to the associated principal $\mathrm{U}(n)$-bundle $P := \big(\mathrm{EU}(n_1) \times \mathrm{EU}(n_2)\big)^{[j]}$. By Proposition 1.2.8, this isomorphism induces a vertical isomorphism

$$P \times_{\mathrm{U}(n)} \mathbb{C}^n_{\mathbb{R}} \cong (\mathrm{B}j)^* E^{\mathrm{U}}_n \,.$$

On the other hand, by Proposition 1.6.7, $P \times_{\mathrm{U}(n)} \mathbb{C}^n_{\mathbb{R}}$ is vertically isomorphic to the associated vector bundle

$$\big(\mathrm{EU}(n_1) \times \mathrm{EU}(n_2)\big) \times_{\mathrm{U}(n_1) \times \mathrm{U}(n_2)} \mathbb{C}^n_{\mathbb{R}} \,,$$

where $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$ acts on $\mathbb{C}^n$ via the composition of j with the basic representation of $\mathrm{U}(n)$. One can check that this vector bundle, in turn, is vertically isomorphic to the direct sum $\big(\mathrm{pr}_1^* E^{\mathrm{U}}_{n_1}\big) \oplus \big(\mathrm{pr}_2^* E^{\mathrm{U}}_{n_2}\big)$ (Exercise 4.3.1). Putting all this together, we end up with a vertical isomorphism

$$\big(\mathrm{pr}_1^* E^{\mathrm{U}}_{n_1}\big) \oplus \big(\mathrm{pr}_2^* E^{\mathrm{U}}_{n_2}\big) \cong (\mathrm{B}j)^* E^{\mathrm{U}}_n \,. \tag{4.3.5}$$

In local trivializations induced from local trivializations of the corresponding principal bundles, this isomorphism is given fibrewise by the obvious identification $\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} \equiv \mathbb{R}^{2n}$. Hence, it preserves the orientations. Now, using points 3 and 4 of Proposition 4.1.12, for the Euler classes we find

$$\begin{aligned}
(\mathrm{B}j)^* \mathsf{e}\big(E^{\mathrm{U}}_n\big) &= \mathsf{e}\big((\mathrm{B}j)^* E^{\mathrm{U}}_n\big) \\
&= \mathsf{e}\big((\mathrm{pr}_1^* E^{\mathrm{U}}_{n_1}) \oplus (\mathrm{pr}_2^* E^{\mathrm{U}}_{n_2})\big) \\
&= \mathsf{e}\big(\mathrm{pr}_1^* E^{\mathrm{U}}_{n_1}\big)\, \mathsf{e}\big(\mathrm{pr}_2^* E^{\mathrm{U}}_{n_2}\big) \\
&= \big(\mathrm{pr}_1^* \mathsf{e}\big(E^{\mathrm{U}}_{n_1}\big)\big)\big(\mathrm{pr}_2^* \mathsf{e}\big(E^{\mathrm{U}}_{n_2}\big)\big) \\
&= \mathsf{e}\big(E^{\mathrm{U}}_{n_1}\big) \times \mathsf{e}\big(E^{\mathrm{U}}_{n_2}\big) \,.
\end{aligned}$$

This proves (4.3.4) for $k = n$.

Now, let n_1, n_2 be such that $n > k$ and assume that (4.3.4) holds for all m_1, m_2 such that $m_1 + m_2 < n$. Since, as a module over $\mathbb{Z}$, $H^l_{\mathbb{Z}}(\mathrm{BU}(n_2))$ is finitely and freely generated for all l, the Künneth Theorem for cohomology yields that the group homomorphism

$$H^*_{\mathbb{Z}}(\mathrm{BU}(n_1)) \otimes H^*_{\mathbb{Z}}(\mathrm{BU}(n_2)) \to H^*_{\mathbb{Z}}(\mathrm{BU}(n_1) \times \mathrm{BU}(n_2)) \tag{4.3.6}$$

defined by $\alpha_1 \otimes \alpha_2 \mapsto \alpha_1 \times \alpha_2$ is an isomorphism of Abelian groups. By restricting to degree $2k$, and by using that the cohomology of $\mathrm{BU}(m)$ is trivial in odd degree, we obtain an isomorphism

$$\bigoplus_{j=0}^{k} H^{2j}_{\mathbb{Z}}(\mathrm{BU}(n_1)) \otimes H^{2k-2j}_{\mathbb{Z}}(\mathrm{BU}(n_2)) \to H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n_1) \times \mathrm{BU}(n_2)) \,.$$

The inverse of this isomorphism combines with the natural projections associated with the direct sum to homomorphisms

$$p_j : H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n_1) \times \mathrm{BU}(n_2)) \to H^{2j}_{\mathbb{Z}}(\mathrm{BU}(n_1)) \otimes H^{2k-2j}_{\mathbb{Z}}(\mathrm{BU}(n_2)) \,.$$

Consider a given i with $0 \leq i \leq k$. Since $k < n = n_1 + n_2$, either $i < n_1$ or $k - i < n_2$. Without loss of generality, we give the argument for the first case and leave it to the reader to adapt this to the second case. Replacing n_1 by i in the above argument, we obtain homomorphisms

$$\tilde{p}_j : H^{2k}_{\mathbb{Z}}(\mathrm{BU}(i) \times \mathrm{BU}(n_2)) \to H^{2j}_{\mathbb{Z}}(\mathrm{BU}(i)) \otimes H^{2k-2j}_{\mathbb{Z}}(\mathrm{BU}(n_2)) \,.$$

Let

$$\tilde{j} : \mathrm{U}(i) \times \mathrm{U}(n_2) \to \mathrm{U}(i + n_2)$$

denote the standard blockwise embedding and let $\varphi : \mathrm{U}(i + n_2) \to \mathrm{U}(n)$ denote the embedding induced by the vector subspace embedding

$$\mathbb{C}^{i+n_2} \to \mathbb{C}^n \,, \quad (z_1, \ldots, z_{i+n_2}) \mapsto (z_1, \ldots, z_i, 0, \ldots, 0, z_{i+1}, \ldots, z_{i+n_2}) \,.$$

Then, the diagram of Lie group homomorphisms

$$\begin{tikzcd}
\mathrm{U}(i) \times \mathrm{U}(n_2) \arrow[r, "\tilde{j}"] \arrow[d, "j^{\mathrm{U}}_{i,n_1} \times \mathrm{id}"'] & \mathrm{U}(i + n_2) \arrow[d, "\varphi"] \\
\mathrm{U}(n_1) \times \mathrm{U}(n_2) \arrow[r, "j"] & \mathrm{U}(n)
\end{tikzcd}$$

commutes. It induces a commutative diagram

$$\begin{tikzcd}
H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n)) \arrow[r, "(\mathrm{B}j)^*"] \arrow[d, "(\mathrm{B}\varphi)^*"'] & H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n_1) \times \mathrm{BU}(n_2)) \arrow[d, "(\mathrm{B}j^{\mathrm{U}}_{i,n_1} \times \mathrm{id})^*"] \\
H^{2k}_{\mathbb{Z}}(\mathrm{BU}(i+n_2)) \arrow[r, "(\mathrm{B}\tilde{j})^*"] & H^{2k}_{\mathbb{Z}}(\mathrm{BU}(i) \times \mathrm{BU}(n_2))
\end{tikzcd}$$

Since φ and $j^{\mathrm{U}}_{i+n_2,n}$ differ by an inner automorphism of $\mathrm{U}(n)$, and since $\mathrm{U}(n)$ is connected, $\mathrm{B}\varphi$ and $\mathrm{B}j^{\mathrm{U}}_{i+n_2,n}$ are homotopic and hence induce the same homomorphism in cohomology. Using this and composing $(\mathrm{B}j)^*$ and $(\mathrm{B}\tilde{j})^*$ with p_i and $\tilde{p}_i$, respectively, we obtain the commutative diagram

$$\begin{tikzcd}
H^{2k}_{\mathbb{Z}}(\mathrm{BU}(n)) \arrow[r, "p_i \circ (\mathrm{B}j)^*"] \arrow[d, "(\mathrm{B}j^{\mathrm{U}}_{i+n_2,n})^*"'] & H^{2i}_{\mathbb{Z}}(\mathrm{BU}(n_1)) \otimes H^{2k-2i}_{\mathbb{Z}}(\mathrm{BU}(n_2)) \arrow[d, "(\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}"] \\
H^{2k}_{\mathbb{Z}}(\mathrm{BU}(i+n_2)) \arrow[r, "\tilde{p}_i \circ (B\tilde{j})^*"] & H^{2i}_{\mathbb{Z}}(\mathrm{BU}(i)) \otimes H^{2k-2i}_{\mathbb{Z}}(\mathrm{BU}(n_2))
\end{tikzcd}$$

Applying this to $\mathsf{c}^{\mathrm{U}(n)}_k$ and using (4.2.11), we find

$$\left((\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}\right) \left(p_i\left((\mathrm{B}j)^* \mathsf{c}^{\mathrm{U}(n)}_k\right)\right) = \tilde{p}_i \circ (\mathrm{B}\tilde{j})^* \left(\mathsf{c}^{\mathrm{U}(i+n_2)}_k\right).$$

Since $i + n_2 < n$, by the induction assumption, (4.3.4) holds with n_1 replaced by i and j replaced by $\tilde{j}$. Hence,

$$\left((\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}\right) \left(p_i\left((\mathrm{B}j)^* \mathsf{c}^{\mathrm{U}(n)}_k\right)\right) = \mathsf{c}^{\mathrm{U}(i)}_i \otimes \mathsf{c}^{\mathrm{U}(n_2)}_{k-i}.$$

On the other hand, by (4.2.11),

$$\left((\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}\right) \left(\mathsf{c}^{\mathrm{U}(n_1)}_i \otimes \mathsf{c}^{\mathrm{U}(n_2)}_{k-i}\right) = \mathsf{c}^{\mathrm{U}(i)}_i \otimes \mathsf{c}^{\mathrm{U}(n_2)}_{k-i}.$$

Thus,

$$\left((\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}\right) \left(p_i\left((\mathrm{B}j)^* \mathsf{c}^{\mathrm{U}(n)}_k\right)\right) = \left((\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}\right) \left(\mathsf{c}^{\mathrm{U}(n_1)}_i \otimes \mathsf{c}^{\mathrm{U}(n_2)}_{k-i}\right). \tag{4.3.7}$$

Since $i < n_1$, Theorem 4.2.1 yields that $(\mathrm{B}j^{\mathrm{U}}_{i,n_1})^*$ is injective on $H^{2i}_{\mathbb{Z}}(\mathrm{BU}(n_1))$. Hence, $(\mathrm{B}j^{\mathrm{U}}_{i,n_1})^* \otimes \mathrm{id}$ is injective on $H^{2i}_{\mathbb{Z}}(\mathrm{BU}(n_1)) \otimes H^{2k-2i}_{\mathbb{Z}}(\mathrm{BU}(n_2))$. Therefore, (4.3.7) implies

$$p_i\left((\mathrm{B}j)^* \mathsf{c}^{\mathrm{U}(n)}_k\right) = \mathsf{c}^{\mathrm{U}(n_1)}_i \otimes \mathsf{c}^{\mathrm{U}(n_2)}_{k-i}.$$

Since this holds for all $i = 0, \ldots, k$, and since the p_i sum up to the inverse of the isomorphism (4.3.6), formula (4.3.4) follows. This proves the theorem. ∎

Theorem 4.3.2 (Whitney Sum Formula) *For $\mathbb{K}$-vector bundles E_1 and E_2 over the same base space,*

$$\alpha(E_1 \oplus E_2) = \alpha(E_1)\alpha(E_2)\,,$$

where α stands for the total Stiefel–Whitney class w in case $\mathbb{K} = \mathbb{R}$, for the total Chern class c in case $\mathbb{K} = \mathbb{C}$ and for the total symplectic Pontryagin class p in case $\mathbb{K} = \mathbb{H}$.

Proof As before, we give the proof for the complex case. Let n_i be the rank of E_i and let $n = n_1 + n_2$. Choose auxiliary fibre metrics on E_1 and E_2. Their orthogonal direct sum defines a fibre metric on $E_1 \oplus E_2$. Let P_i and $P_\oplus$ denote the corresponding orthonormal frame bundles of E_i and $E_1 \oplus E_2$, respectively. P_i has structure group $\mathrm{U}(n_i)$ and $P_\oplus$ has structure group $\mathrm{U}(n)$. Choose classifying mappings $f_i : B \to \mathrm{BU}(n_i)$ for P_i and $f_\oplus : B \to \mathrm{BU}(n)$ for $P_\oplus$. By definition,

$$\mathsf{c}(E_i) = f_i^* \mathsf{c}^{\mathrm{U}(n_i)}\,, \quad \mathsf{c}(E_1 \oplus E_2) = f_\oplus^* \mathsf{c}^{\mathrm{U}(n)}\,. \tag{4.3.8}$$

Consider the principal $\big(\mathrm{U}(n_1) \times \mathrm{U}(n_2)\big)$-bundle $P_1 \times_B P_2$. It has the classifying mapping $(f_1 \times f_2) \circ \Delta$, where $\Delta : B \to B \times B$ is the diagonal mapping. By combining orthonormal frames in $(E_1)_b$ with orthonormal frames in $(E_2)_b$ to orthonormal frames in $(E_1 \oplus E_2)_b = (E_1)_b \oplus (E_2)_b$, we obtain a vertical morphism of principal bundles $P_1 \times_B P_2 \to P_\oplus$ with associated Lie group homomorphism given by the standard blockwise embedding $j : \mathrm{U}(n_1) \times \mathrm{U}(n_2) \to \mathrm{U}(n)$. Hence, Proposition 3.7.6 yields that $f_\oplus$ is homotopic to $\mathrm{B}j \circ (f_1 \times f_2) \circ \Delta$. Using this, together with formula (4.3.8) and Theorem 4.3.1, we find

$$\begin{aligned}
\mathsf{c}(E_1 \oplus E_2) &= f_\oplus^* \mathsf{c}^{\mathrm{U}(n)} \\
&= \Delta^* \circ (f_1 \times f_2)^* \circ \mathrm{B}j^* \big(\mathsf{c}^{\mathrm{U}(n)}\big) \\
&= \Delta^* \circ (f_1 \times f_2)^* \big(\mathsf{c}^{\mathrm{U}(n_1)} \times \mathsf{c}^{\mathrm{U}(n_2)}\big) \\
&= \Delta^* \big((f_1^* \mathsf{c}^{\mathrm{U}(n_1)}) \times (f_2^* \mathsf{c}^{\mathrm{U}(n_2)})\big) \\
&= \Delta^* \big(\mathsf{c}(E_1) \times \mathsf{c}(E_2)\big) \\
&= \mathsf{c}(E_1)\mathsf{c}(E_2)\,.
\end{aligned}$$

■

Recall that two $\mathbb{K}$-vector bundles E_1, E_2 over a topological space B are said to be stably equivalent if there exist non-negative integers r_1, r_2 such that $E_1 \oplus (B \times \mathbb{K}^{r_1})$ is vertically isomorphic to $E_2 \oplus (B \times \mathbb{K}^{r_2})$.

Corollary 4.3.3 *Stably equivalent real (complex, quaternionic) vector bundles have the same Stiefel–Whitney (Chern, symplectic Pontryagin) classes.*

Proof We give the argument for the complex case. Let E_1 and E_2 be complex vector bundles over B. If the vector bundles $E_1 \oplus (B \times \mathbb{C}^{r_1})$ and $E_2 \oplus (B \times \mathbb{C}^{r_2})$ are vertically isomorphic, they have the same Chern class. By Remark 4.1.6/2, we have $\mathsf{c}(M \times \mathbb{C}^{r_i}) = 1$. Hence, the Whitney Sum Formula implies

$$\mathsf{c}\big(E_i \oplus (B \times \mathbb{C}^{r_i})\big) = \mathsf{c}(E_i)\,, \quad i = 1, 2\,.$$

This yields the assertion. ■

Corollary 4.3.3 implies that the Chern (Stiefel–Whitney, Pontryagin) classes yield invariants in complex (real, quaternionic) K-theory.

Another important consequence of the Whitney Sum Formula is the following. Let $\sigma_k(x_1, \ldots, x_n)$ denote the elementary symmetric polynomial of order k in the indeterminates $x_1, \ldots, x_n$, that is,

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}\,.$$

Corollary 4.3.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let $L_1, \ldots, L_n$ be $\mathbb{K}$-line bundles over a topological space B. Then,*

$$\alpha_k\,(L_1 \oplus \cdots \oplus L_n) = \sigma_k\big(\alpha_1(L_1), \ldots, \alpha_1(L_n)\big)\,,$$

where $\alpha = \mathsf{w}$ in case $\mathbb{K} = \mathbb{R}$, $\alpha = \mathsf{c}$ in case $\mathbb{K} = \mathbb{C}$ and $\alpha = \mathsf{p}$ in case $\mathbb{K} = \mathbb{H}$.

Proof Let $E := L_1 \oplus \cdots \oplus L_n$. In the complex case, by the Whitney Sum Formula,

$$\mathsf{c}(E) = \prod_{i=1}^{n} \mathsf{c}(L_i)\,.$$

By plugging in $\mathsf{c}(L_i) = 1 + \mathsf{c}_1(L_i)$ and expanding the product, we obtain

$$\mathsf{c}(E) = 1 + \sigma_1\big(\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)\big) + \cdots + \sigma_n\big(\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)\big).$$

The real and the quaternionic case are analogous. ■

The characteristic classes $\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)$ are referred to as the Chern roots of E. By analogy, one speaks of the Stiefel–Whitney roots of E in the real case and the Pontryagin roots of E in the quaternionic case. Thus, Corollary 4.3.4 states that if a complex vector bundle splits into a sum of line bundles, its Chern classes are given by the elementary symmetric polynomials in the first Chern classes of its factors, and that analogous statements hold for real and quaternionic vector bundles.

Behind Corollary 4.3.4, there is a relation between the corresponding universal characteristic classes, which we now derive from Theorem 4.3.1. We show that by iterated application of this theorem, we can embed $H^*_{\mathbb{Z}}(\mathrm{BU}(n))$ into $H^*_{\mathbb{Z}}(\mathrm{BU}(1)^n)$, $H^*_{\mathbb{Z}_2}(\mathrm{BO}(n))$ into $H^*_{\mathbb{Z}_2}(\mathrm{BO}(1)^n)$ and $H^*_{\mathbb{Z}}(\mathrm{BSp}(n))$ into $H^*_{\mathbb{Z}}(\mathrm{BSp}(1)^n)$. By the Künneth Theorem for cohomology, $H^*_{\mathbb{Z}}(\mathrm{BU}(1)^n)$ is the polynomial ring over $\mathbb{Z}$ in the generators

$$\mathsf{c}_1^{\mathrm{U}(1)} \times 1 \times \cdots \times 1,\; 1 \times \mathsf{c}_1^{\mathrm{U}(1)} \times 1 \times \cdots \times 1, \ldots,\; 1 \times \cdots \times 1 \times \mathsf{c}_1^{\mathrm{U}(1)}.$$

This ring contains the symmetric polynomials as a subring. Using (4.3.1), the generators can be rewritten in terms of the natural projections $\mathrm{pr}_k : \mathrm{U}(1)^n \to \mathrm{U}(1)$ as

$$\mathsf{c}_1^{\mathrm{U}(1)} \times 1 \times \cdots \times 1 = (\mathrm{B}\,\mathrm{pr}_1)^* \mathsf{c}_1^{\mathrm{U}(1)}, \ \ldots, 1 \times \cdots \times 1 \times \mathsf{c}_1^{\mathrm{U}(1)} = (\mathrm{B}\,\mathrm{pr}_n)^* \mathsf{c}_1^{\mathrm{U}(1)}.$$

Similar statements hold for $H^*_{\mathbb{Z}_2}(\mathrm{BO}(1)^n)$ and $H^*_{\mathbb{Z}}(\mathrm{BSp}(1)^n)$.

Proposition 4.3.5 *For the standard diagonal embeddings*

$$j_n^{\mathrm{O}} : \mathrm{O}(1)^n \to \mathrm{O}(n)\,, \qquad j_n^{\mathrm{U}} : \mathrm{U}(1)^n \to \mathrm{U}(n)\,, \qquad j_n^{\mathrm{Sp}} : \mathrm{Sp}(1)^n \to \mathrm{Sp}(n)\,,$$

one has

$$\begin{aligned}
\left(\mathrm{B}j_n^{\mathrm{O}}\right)^* \mathsf{w}_k^{\mathrm{O}(n)} &= \sigma_k\left((\mathrm{B}\,\mathrm{pr}_1)^* \mathsf{w}_1^{\mathrm{O}(1)}, \ldots, (\mathrm{B}\,\mathrm{pr}_n)^* \mathsf{w}_1^{\mathrm{O}(1)}\right)\,,\\
\left(\mathrm{B}j_n^{\mathrm{U}}\right)^* \mathsf{c}_k^{\mathrm{U}(n)} &= \sigma_k\left((\mathrm{B}\,\mathrm{pr}_1)^* \mathsf{c}_1^{\mathrm{U}(1)}, \ldots, (\mathrm{B}\,\mathrm{pr}_n)^* \mathsf{c}_1^{\mathrm{U}(1)}\right)\,,\\
\left(\mathrm{B}j_n^{\mathrm{Sp}}\right)^* \mathsf{p}_k^{\mathrm{Sp}(n)} &= \sigma_k\left((\mathrm{B}\,\mathrm{pr}_1)^* \mathsf{p}_1^{\mathrm{Sp}(1)}, \ldots, (\mathrm{B}\,\mathrm{pr}_n)^* \mathsf{p}_1^{\mathrm{Sp}(1)}\right)\,.
\end{aligned}$$

In particular, the homomorphisms $(\mathrm{B}j_n^{\mathrm{O}})^*$, $(\mathrm{B}j_n^{\mathrm{U}})^*$ *and* $(\mathrm{B}j_n^{\mathrm{Sp}})^*$ *are injective and their images are the subrings of symmetric polynomials.*

Proof As usual, we give the argument for the complex case and leave the other cases to the reader. By iterated application of Theorem 4.3.1, we obtain

$$\left(\mathrm{B}j_n^{\mathrm{U}}\right)^* \mathsf{c}^{\mathrm{U}(n)} = \mathsf{c}^{\mathrm{U}(1)} \times \cdots \times \mathsf{c}^{\mathrm{U}(1)}\,.$$

By plugging in $\mathsf{c}^{\mathrm{U}(1)} = 1 + \mathsf{c}_1^{\mathrm{U}(1)}$ and evaluating the product in degree k, we find that $\left(\mathrm{B}j_n^{\mathrm{U}}\right)^* \mathsf{c}_k^{\mathrm{U}(n)}$ equals the sum over all cross products having a factor $\mathsf{c}_1^{\mathrm{U}(1)}$ in k places and a factor 1 in $n-k$ places. By (4.3.2), this sum coincides with

$$\sigma_k\left(\mathsf{c}_1^{\mathrm{U}(1)} \times 1 \times \cdots \times 1, \ldots, 1 \times \cdots \times 1 \times \mathsf{c}_1^{\mathrm{U}(1)}\right).$$

Rewriting the generators in terms of the natural projections pr_k, we obtain the asserted formula. Finally, since the ring of symmetric polynomials in n indeterminates with coefficients in $\mathbb{Z}$ coincides with the polynomial ring generated over $\mathbb{Z}$ by the elementary symmetric polynomials [399, Sect. IV.6], it follows that $(\mathrm{B}j_n^{\mathrm{U}})^*$ is injective and that its image is the subring of $H^*_{\mathbb{Z}}(\mathrm{BU}(1)^n)$ of symmetric polynomials. ∎

In view of Corollary 4.1.4 and the fact that a principal G-bundle P admits a reduction Q to a Lie subgroup $j : H \to G$ iff it is vertically isomorphic to $Q^{[j]}$, Proposition 4.3.5 entails the following.

Corollary 4.3.6 *If P is a principal* $\mathrm{U}_{\mathbb{K}}(n)$*-bundle which admits a reduction Q to the subgroup* $\mathrm{U}_{\mathbb{K}}(1)^n$*, then*

$$\alpha_k(P) = \sigma_k\left(\alpha_1\left(Q^{[\mathrm{pr}_1]}\right), \ldots, \alpha_1\left(Q^{[\mathrm{pr}_n]}\right)\right),$$

where $\alpha = \mathsf{w}$ *for* $\mathbb{K} = \mathbb{R}$, $\alpha = \mathsf{c}$ *for* $\mathbb{K} = \mathbb{C}$ *and* $\alpha = \mathsf{p}$ *for* $\mathbb{K} = \mathbb{H}$. ■

Taking up the terminology for vector bundles, in case of the structure group $\mathrm{U}(n)$, the characteristic classes $\mathsf{c}_1\big(Q^{[\mathrm{pr}_1]}\big), \dots, \mathsf{c}_1\big(Q^{[\mathrm{pr}_n]}\big)$ of a reduction Q are referred to as the Chern roots of P. By analogy, one speaks of the Stiefel–Whitney roots of P in case of the structure group $\mathrm{O}(n)$ and the Pontryagin roots of P in case of the structure group $\mathrm{Sp}(n)$.

Next, we prove that the situation of Corollaries 4.3.4 and 4.3.6 can be achieved for every vector bundle and every principal bundle with structure group $\mathrm{O}(n)$, $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$ by passing to an appropriate pullback bundle. This result is known as the Splitting Principle. We treat the case of principal bundles first.

Theorem 4.3.7 (Splitting Principle for principal bundles) *Let* $G = \mathrm{O}(n)$, $\mathrm{U}(n)$ *or* $\mathrm{Sp}(n)$ *and let* H *denote, respectively, the subgroup* $\mathrm{O}(1)^{\mathrm{n}}$, $\mathrm{U}(1)^n$ *or* $\mathrm{Sp}(1)^n$. *Let* P *be a principal* G*-bundle over a topological space* B *and let* $\rho : P/H \to B$ *denote the induced projection. Let* $R = \mathbb{Z}_2$ *for* $G = \mathrm{O}(n)$ *and* $R = \mathbb{Z}$ *for* $G = \mathrm{U}(n)$ *or* $\mathrm{Sp}(n)$.

1. *The principal* G*-bundle* ρ^*P *over* P/H *admits a reduction to the subgroup* H.
2. *The induced homomorphism* $\rho^* : H^*_R(B) \to H^*_R(P/H)$ *is injective.*

Proof 1. Let $\mathrm{pr} : P \to P/H$ denote the natural projection to classes. One can check that the mapping

$$P \to \rho^* P\,, \quad p \mapsto \big(\mathrm{pr}(p), p\big)\,,$$

is well defined and yields a reduction of ρ^*P to the subgroup H.

2. The proof boils down to another application of the Leray–Hirsch Theorem. To be definite, we give it for $G = \mathrm{U}(n)$. Let Ψ denote the action of $\mathrm{U}(n)$ on P. Define

$$Q_0 := P/\big(\{1\} \times \mathrm{U}(n-1)\big)\,, \quad Y_0 := P/\big(\mathrm{U}(1) \times \mathrm{U}(n-1)\big)\,.$$

Clearly, Q_0 is a principal $\mathrm{U}(1)$-bundle over Y_0 and Y_0 is a fibre bundle over B with typical fibre

$$\mathrm{U}(n)/\big(\mathrm{U}(1) \times \mathrm{U}(n-1)\big) \cong \mathbb{C}\mathrm{P}^{n-1}\,.$$

Let $\rho_0 : Y_0 \to B$ denote the induced projection. Choose $p \in P$ and let $m = \pi(p)$. The mapping $\Psi_p : \mathrm{U}(n) \to P$ induced by Ψ is equivariant with respect to the action of $\mathrm{U}(1) \times \mathrm{U}(n-1)$ on $\mathrm{U}(n)$ by right translation and thus descends to a mapping

$$j : \mathbb{C}\mathrm{P}^{n-1} \to Y_0$$

of $\mathbb{C}\mathrm{P}^{n-1}$ onto the fibre $(Y_0)_m$. Consider the induced principal $\mathrm{U}(1)$-bundle j^*Q_0 over $\mathbb{C}\mathrm{P}^{n-1}$. One can check that the mapping

$$\mathrm{U}(n) \to \mathbb{C}\mathrm{P}^{n-1} \times Q_0\,, \quad a \mapsto \big([a], [\Psi_a(p)]\big)\,, \tag{4.3.9}$$

induces a vertical isomorphism from the canonical $\mathrm{U}(1)$-bundle over $\mathbb{C}\mathrm{P}^{n-1}$, which has $\mathrm{U}(n)/\big(\{1\} \times \mathrm{U}(n-1)\big) \cong \mathrm{S}^{2n-1}$ as its bundle space,[8] onto $j^* Q_0$ (Exercise 4.3.3). According to Example 4.2.18, then the cohomology classes

$$1, \mathsf{c}_1(j^* Q_0), \ldots, \mathsf{c}_1(j^* Q_0)^{n-1}$$

form a free basis of $H^*_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^{n-1})$ as a $\mathbb{Z}$-module. Since $\mathsf{c}_1(j^* Q_0) = j^* \mathsf{c}_1(Q_0)$, the Leray–Hirsch Theorem 4.1.7 implies that the cohomology classes

$$1, \mathsf{c}_1(Q_0), \ldots, \mathsf{c}_1(Q_0)^{n-1}$$

form a free basis of $H^*_{\mathbb{Z}}(Y_0)$ as a module over $H^*_{\mathbb{Z}}(B)$. In particular, the mapping $H^*_{\mathbb{Z}}(B) \to H^*_{\mathbb{Z}}(Y_0)$ given by $\alpha \mapsto \alpha \cdot 1$ is injective. Since $\alpha \cdot 1 = \rho_0^* \alpha$, this means that the induced homomorphism $\rho_0^* : H^*_{\mathbb{Z}}(B) \to H^*_{\mathbb{Z}}(Y_0)$ is injective.

Now, in the above argument, we replace the principal $\mathrm{U}(n)$-bundle P over B by the principal $\mathrm{U}(n-1)$-bundle $P_1 := P/\big(\mathrm{U}(1) \times \{\mathbb{1}_{n-1}\}\big)$ over Y_0. This yields a fibre bundle over Y_0 with bundle space

$$Y_1 := P_1/\big(\mathrm{U}(1) \times \mathrm{U}(n-2)\big) \equiv P/\big(\mathrm{U}(1)^2 \times \mathrm{U}(n-2)\big)$$

and typical fibre $\mathbb{C}\mathrm{P}^{n-2}$, whose projection $\rho_1 : Y_1 \to Y_0$ induces an injection $\rho_1^* : H^*_{\mathbb{Z}}(Y_0) \to H^*_{\mathbb{Z}}(Y_1)$. Iterating this, we finally arrive at a bundle projection

$$\rho_{n-2} : Y_{n-2} \equiv P/\mathrm{U}(1)^n \to Y_{n-3} \equiv P/\big(\mathrm{U}(1)^{n-2} \times \mathrm{U}(2)\big)$$

with fibre $\mathbb{C}\mathrm{P}^1$, inducing an injection

$$\rho_{n-2}^* : H^*_{\mathbb{Z}}(Y_{n-3}) \to H^*_{\mathbb{Z}}(Y_{n-2}) \equiv H^*_{\mathbb{Z}}(P/\mathrm{U}(1)^n)\,.$$

Since $\rho_0 \circ \cdots \circ \rho_{n-2} = \rho$, this proves point 2. ■

From the Splitting Principle for principal bundles we can derive the Splitting Principle for vector bundles.

Corollary 4.3.8 (Splitting Principle for vector bundles) *Let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $R = \mathbb{Z}_2$ for $\mathbb{K} = \mathbb{R}$ and $R = \mathbb{Z}$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. For every $\mathbb{K}$-vector bundle E over a topological space B, there exists a fibre bundle $\rho : Y \to B$ such that*

1. *$\rho^* E$ is vertically isomorphic to a direct sum of line bundles,*
2. *the induced homomorphism $\rho^* : H^*_R(B) \to H^*_R(Y)$ is injective.*

Proof As before, to be definite, we give the proof for $\mathbb{K} = \mathbb{C}$. Let n denote the rank of E. Choose a fibre metric on E and consider the corresponding orthonormal frame bundle $O(E)$, which is a principal $\mathrm{U}(n)$-bundle over B. Define $Y := O(E)/\mathrm{U}(1)^n$ and

[8] In fact, this is the Stiefel bundle $\mathrm{S}_{\mathbb{C}}(1, n) \to \mathrm{G}_{\mathbb{C}}(1, n)$.

let $\rho : Y \to B$ denote the induced projection. Then, Y is a fibre bundle over B, with typical fibre $\mathrm{U}(n)/\mathrm{U}(1)^n$. Point 2 of Theorem 4.3.7 yields point 2 of the corollary. By point 1 of that theorem, $\rho^* O(E)$ admits a reduction Q to the subgroup $\mathrm{U}(1)^n$. Then, on the one hand, using Propositions 1.6.7 and 1.2.5/2 and Theorem 3.6.8, we obtain the vertical isomorphisms

$$Q \times_{\mathrm{U}(1)^n} \mathbb{C}^n \cong \big(\rho^* O(E)\big) \times_{\mathrm{U}(n)} \mathbb{C}^n \cong \rho^*\big(O(E) \times_{\mathrm{U}(n)} \mathbb{C}^n\big) \cong \rho^* E\,.$$

On the other hand,

$$Q \times_{\mathrm{U}(1)^n} \mathbb{C}^n \cong \big(Q \times_{\mathrm{U}(1)^n} \mathbb{C}_1\big) \oplus \cdots \oplus \big(Q \times_{\mathrm{U}(1)^n} \mathbb{C}_n\big)\,,$$

where $\mathrm{U}(1)^n$ acts on $\mathbb{C}_i$ via multiplication by the i-th entry. ■

Remark 4.3.9 According to the proof of Corollary 4.3.8, if E has rank n, the fibre bundle $\rho : Y \to B$ of Corollary 4.3.8 can be chosen to have typical fibre $\mathrm{O}(n)/\mathrm{O}(1)^n$ in case $\mathbb{K} = \mathbb{R}$, $\mathrm{U}(n)/\mathrm{U}(1)^n$ in case $\mathbb{K} = \mathbb{C}$ and $\mathrm{Sp}(n)/\mathrm{Sp}(1)^n$ in case $\mathbb{K} = \mathbb{H}$. ♦

The Splitting Principle implies that for proving an algebraic relation between the Chern (Stiefel–Whitney, Pontryagin) classes of complex (real, quaternionic) vector bundles, it suffices to prove this relation under the assumption that all bundles involved are sums of line bundles. Let us illustrate this by deriving a formula for the total Chern class of a tensor product of complex vector bundles.

Define a polynomial $T_{n,m}$ in the real variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ by

$$T_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m) := \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i + y_j)\,. \tag{4.3.10}$$

Since $T_{n,m}$ is symmetric under separate permutations of the x_i and the y_j, it can be written in the form

$$\begin{aligned} &T_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ &\quad = P_{n,m}\big(\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_n(x_1, \ldots, x_n)\,,\ \sigma_1(y_1, \ldots, y_m), \ldots, \sigma_m(y_1, \ldots, y_m)\big) \end{aligned}$$

with a unique polynomial in $n + m$ variables $P_{n,m}$. For the explicit form of $P_{n,m}$, see Remark 4.3.12.

Proposition 4.3.10 *For complex vector bundles E of rank n and F of rank m over a topological space B, one has*

$$\mathsf{c}(E \otimes F) = P_{n,m}\big(\mathsf{c}_1(E), \ldots, \mathsf{c}_n(E), \mathsf{c}_1(F), \ldots, \mathsf{c}_m(F)\big)\,.$$

Proof By the Splitting Principle, it suffices to prove the assertion under the assumption that $E = \bigoplus_{i=1}^n L_i$ and $F = \bigoplus_{j=1}^m K_j$ for appropriate line bundles L_i and K_j. According to Corollary 4.3.4, then

$$\mathsf{c}_k(E) = \sigma_k\big(\mathsf{c}_1(L_1), \dots, \mathsf{c}_1(L_n)\big)\,, \quad \mathsf{c}_k(F) = \sigma_k\big(\mathsf{c}_1(K_1), \dots, \mathsf{c}_1(K_m)\big)\,.$$

Thus, we have to show that

$$\mathsf{c}(E \otimes F) = T_{n,m}\big(\mathsf{c}_1(L_1), \dots, \mathsf{c}_1(L_n), \mathsf{c}_1(K_1), \dots, \mathsf{c}_1(K_m)\big)\,.$$

By the Whitney Sum Formula,

$$\mathsf{c}(E \otimes F) = \prod_{i=1}^{n} \prod_{j=1}^{m} \mathsf{c}(L_i \otimes K_j)\,.$$

Hence, the proof boils down to showing that for arbitrary line bundles L and K, one has

$$\mathsf{c}_1(L \otimes K) = \mathsf{c}_1(L) + \mathsf{c}_1(K)\,. \tag{4.3.11}$$

To prove this, we use that $L \otimes K$ can be written as an associated vector bundle as follows. We may assume that L and K are associated with principal $\mathrm{U}(1)$-bundles P and Q, respectively, via the basic representation of $\mathrm{U}(1)$ on $\mathbb{C}$. Consider the fibre product $P \times_B Q$. This is a principal $(\mathrm{U}(1) \times \mathrm{U}(1))$-bundle over B. Since $\mathrm{U}(1)$ is Abelian, the multiplication mapping $\mu : \mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1)$ is a group homomorphism. Hence, we can form the associated principal $\mathrm{U}(1)$-bundle $(P \times_B Q)^{[\mu]}$ and, in turn, the associated line bundle

$$E = \big((P \times_B Q)^{[\mu]}\big) \times_{\mathrm{U}(1)} \mathbb{C}\,,$$

where $\mathrm{U}(1)$ acts on $\mathbb{C}$ in the basic representation. We leave it to the reader to show that the mapping $(P \times \mathbb{C}) \times_B (Q \times \mathbb{C}) \to \big((P \times_B Q) \times \mathrm{U}(1)\big) \times \mathbb{C}$ defined by

$$\big((p, z), (q, w)\big) \mapsto \Big(\big((p, q), 1\big), zw\Big) \tag{4.3.12}$$

descends to a vertical vector bundle isomorphism $L \otimes K \to E$ (Exercise 4.3.4). It follows that

$$\mathsf{c}_1(L \otimes K) = \mathsf{c}_1\left((P \times_B Q)^{[\mu]}\right)\,.$$

According to Remark 3.4.22 and Proposition 3.7.2/1, if P and Q have classifying mappings $f, g : B \to \mathrm{BU}(1)$, respectively, then $(P \times_B Q)^{[\mu]}$ has classifying mapping $\mathrm{B}\mu \circ (f \times g) \circ \Delta$. Hence,

$$\mathsf{c}_1(L \otimes K) = \Delta^* \circ (f^* \times g^*) \circ (\mathrm{B}\mu)^*\big(\mathsf{c}_1^{\mathrm{U}(1)}\big)\,. \tag{4.3.13}$$

An easy computation yields (Exercise 4.3.4)

$$(\mathrm{B}\mu)^* \mathsf{c}_1^{\mathrm{U}(1)} = \mathsf{c}_1^{\mathrm{U}(1)} \times 1 + 1 \times \mathsf{c}_1^{\mathrm{U}(1)}\,. \tag{4.3.14}$$

Plugging this into (4.3.13), we obtain (4.3.11). ■

From the proof we extract the formula for the Chern class of the tensor product of complex line bundles L_1 and L_2 over B,

$$\mathsf{c}(L_1 \otimes L_2) = 1 + \mathsf{c}_1(L_1) + \mathsf{c}_1(L_2)\,. \tag{4.3.15}$$

In combination with the Splitting Principle, this formula allows for computing the Chern class of the dual vector bundle. To formulate the result, define the conjugate universal Chern classes and the conjugate total universal Chern class by, respectively,

$$\overline{\mathsf{c}}_k^{\mathrm{U}(n)} := (-1)^k \mathsf{c}_k^{\mathrm{U}(n)}\,, \quad \overline{\mathsf{c}}^{\mathrm{U}(n)} := 1 + \overline{\mathsf{c}}_1^{\mathrm{U}(n)} + \cdots + \overline{\mathsf{c}}_n^{\mathrm{U}(n)}\,. \tag{4.3.16}$$

There correspond the conjugate Chern classes of principal $\mathrm{U}(n)$-bundles and of complex vector bundles.

Corollary 4.3.11 *For the dual bundle E^* of a complex vector bundle E, one has*

$$\mathsf{c}(E^*) = \overline{\mathsf{c}}(E)\,.$$

Proof By the Splitting Principle, it suffices to prove the assertion for the case where E is a sum of line bundles, $E = L_1 \oplus \cdots \oplus L_n$. Then, $E^* = L_1^* \oplus \cdots \oplus L_n^*$. By the Whitney Sum Formula,

$$\mathsf{c}(E) = \big(1 + \mathsf{c}_1(L_1)\big) \cdots \big(1 + \mathsf{c}_1(L_n)\big)\,, \quad \mathsf{c}(E^*) = \big(1 + \mathsf{c}_1(L_1^*)\big) \cdots \big(1 + \mathsf{c}_1(L_n^*)\big)\,.$$

To compute $\mathsf{c}_1(L_i^*)$, we observe that $L_i \otimes L_i^* \cong \mathrm{End}(L_i)$ and that $\mathrm{End}(L_i)$ is trivial, because the identity homomorphisms of the fibres of L_i combine to a global nonzero section. Hence, $\mathsf{c}_1(L_i \otimes L_i^*) = 0$ and (4.3.15) implies

$$\mathsf{c}_1(L_i^*) = -\mathsf{c}_1(L_i)\,, \quad i = 1, \ldots, n\,.$$

Thus, the Chern classes of E and E^* built from an even number of factors $\mathsf{c}_1(L_i)$ coincide and those built from an odd number have opposite sign. ■

Remark 4.3.12 In concrete situations, the polynomial $P_{n,m}$ may be read off directly from $T_{n,m}$ by expanding the product and expressing everything in terms of elementary symmetric polynomials. For example, for $m = 1$, one finds

$$T_{n,1}(x_1, \ldots, x_n, y) = \prod_{i=1}^{n} \big((1 + y) + x_i\big) = \sum_{k=0}^{n} \sigma_k(x_1, \ldots, x_n)(1 + y)^k\,,$$

from which we read off

$$P_{n,1}(a_1, \ldots, a_n, b_1) = \sum_{k=0}^{n} a_k (1 + b_1)^k\,. \tag{4.3.17}$$

Hence, for a complex vector bundle E of rank n over B and a complex line bundle L over B, we obtain

$$\mathsf{c}(E \otimes L) = \sum_{k=0}^{n} \mathsf{c}_k(E)\mathsf{c}(L)^k .$$

By a similar argument one finds that the first and the second Chern classes of $E \otimes F$ are given by

$$\mathsf{c}_1(E \otimes F) = m\,\mathsf{c}_1(E) + n\,\mathsf{c}_1(F) , \tag{4.3.18}$$

$$\begin{aligned}\mathsf{c}_2(E \otimes F) = m\,\mathsf{c}_2(E) + n\,\mathsf{c}_2(F) + \tbinom{m}{2}\mathsf{c}_1(E)^2 + \tbinom{n}{2}\mathsf{c}_1(F)^2 \\ + (mn-1)\mathsf{c}_1(E)\mathsf{c}_1(F) ,\end{aligned} \tag{4.3.19}$$

where n and m denote the ranks of E and F, respectively (Exercise 4.3.5).

For general n and m, the polynomial $P_{n,m}$ can be expressed in terms of Schur functions, see Example 5 in Sect. 1.4 of [418]. ♦

Example 4.3.13 As an application, we consider a principal $\mathrm{SU}(n)$-bundle P and determine the second Chern class of the complexification of the adjoint bundle $\mathrm{Ad}(P) = P \times_{\mathrm{SU}(n)} \mathfrak{su}(n)$. We have

$$\big(\mathrm{Ad}(P)\big)_{\mathbb{C}} = P \times_{\mathrm{SU}(n)} \mathfrak{sl}(n, \mathbb{C}) ,$$

where the action of $\mathrm{SU}(n)$ on $\mathfrak{sl}(n, \mathbb{C})$ may be viewed as being induced from the representation of $\mathrm{SU}(2)$ on the vector space $\mathrm{End}(\mathbb{C}^n)$ defined by conjugation. The natural isomorphism $\mathrm{End}(\mathbb{C}^n) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$ intertwines this representation with the tensor product of the basic representation of $\mathrm{SU}(n)$ with its dual representation. Hence, this natural isomorphism embeds $\mathfrak{sl}(n, \mathbb{C})$ as an invariant subspace of codimension 1 in $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$. By complete reducibility, the representation of $\mathrm{SU}(n)$ on $\mathfrak{sl}(n, \mathbb{C})$ thus differs from that on $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ by taking the direct sum with a one-dimensional representation. Since the latter is necessarily trivial, it follows that $\big(\mathrm{Ad}(P)\big)_{\mathbb{C}}$ differs from $P \times_{\mathrm{SU}(n)} \big(\mathbb{C}^n \otimes (\mathbb{C}^n)^*\big)$ by taking the direct sum with a trivial line bundle. By Corollary 4.3.3, the two bundles have the same Chern classes then. Now, $P \times_{\mathrm{SU}(n)} \big(\mathbb{C}^n \otimes (\mathbb{C}^n)^*\big)$ is vertically isomorphic to $E \otimes E^*$, where $E = P \times_{\mathrm{SU}(n)} \mathbb{C}^n$ with $\mathrm{SU}(n)$ acting in the basic representation. Thus, formula (4.3.19), Corollary 4.3.11 and the identity $\mathsf{c}(E) = \mathsf{c}(P)$ imply

$$\mathsf{c}_2\big(\mathrm{Ad}(P)_{\mathbb{C}}\big) = 2n\mathsf{c}_2(P) \tag{4.3.20}$$

and hence

$$\mathsf{p}_1\big(\mathrm{Ad}(P)\big) = -2n\mathsf{c}_2(P) . \tag{4.3.21}$$

♦

Exercises

4.3.1 Complete the proof of Theorem 4.3.1 by showing that the mapping

$$\left(\mathrm{pr}_1^* E_{n_1}^{\mathrm{U}}\right) \oplus \left(\mathrm{pr}_2^* E_{n_2}^{\mathrm{U}}\right) \to \left(\mathrm{EU}(n_1) \times \mathrm{EU}(n_2)\right) \times_{\mathrm{U}(n_1)\times\mathrm{U}(n_2)} \mathbb{C}_{\mathbb{R}}^n$$

defined by

$$\left((x_1, x_2), \left([(y_1, \mathbf{z}_1)], [(y_2, \mathbf{z}_2)]\right)\right) \mapsto \left[\left((y_1, y_2), (\mathbf{z}_1, \mathbf{z}_2)\right)\right],$$

where $\mathbf{z}_i \in \mathbb{C}^{n_i}$, $x_i \in \mathrm{BU}(n_i)$ and $y_i \in \mathrm{EU}(n_i)$ in the fibre over x_i, $i = 1, 2$, is a vertical vector bundle isomorphism.

4.3.2 Use the formulae for $(\mathrm{B}j_{\mathrm{O}})^*$ and $(\mathrm{B}j_{\mathrm{U}})^*$ given in Theorem 4.3.1 to calculate $(\mathrm{B}j_{\mathrm{SO}})^*$ and $(\mathrm{B}j_{\mathrm{SU}})^*$ for the standard blockwise embeddings

$$j_{\mathrm{SO}} : \mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \to \mathrm{SO}(n_1 + n_2),$$
$$j_{\mathrm{SU}} : \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \to \mathrm{SU}(n_1 + n_2).$$

4.3.3 Show that the mapping (4.3.9) induces a vertical isomorphism from the canonical $\mathrm{U}(1)$-bundle over $\mathbb{C}\mathrm{P}^{n-1}$ onto the principal $\mathrm{U}(1)$-bundle $j^* Q_0$ defined in the proof of Theorem 4.3.7.

4.3.4 Complete the proof of Proposition 4.3.10 by showing that the mapping (4.3.12) descends to a vertical vector bundle isomorphism from $L \otimes K$ to E and by proving formula (4.3.14).

4.3.5 Prove the formulae for the first and the second Chern class of a tensor product given in (4.3.18) and (4.3.19) by expressing the contributions of first and second order in the polynomial $T_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m)$ defined in (4.3.10) in terms of elementary symmetric polynomials.

4.4 Field Restriction and Field Extension

First, we analyze how the Chern classes behave under complex conjugation $z \mapsto \bar{z}$. For $a \in \mathrm{M}_n(\mathbb{C})$, let $\bar{a}$ denote the matrix obtained from a by taking the complex conjugate of every entry. The mapping

$$\kappa : \mathrm{U}(n) \to \mathrm{U}(n), \quad \kappa(a) := \bar{a},$$

is a Lie group isomorphism. Given a complex vector bundle E, we may redefine the multiplication by scalars as

$$z \cdot y := \bar{z} y, \quad z \in \mathbb{C}, \ y \in E.$$

With this new multiplication and the original fibrewise additive structure, E is a complex vector bundle of the same rank. It is called the conjugate vector bundle of E and is denoted by $\overline{E}$. Let us point out the following. While the real vector bundles $E_{\mathbb{R}}$ and $\overline{E}_{\mathbb{R}}$ obtained by field restriction from E and $\overline{E}$, respectively, are identical, their induced orientations coincide only if the rank of E is even, and are opposite otherwise. The reason is that the induced orientation of $E_{\mathbb{R}}$ is defined by ordered local frames of the form $(e_1, \mathrm{i}e_1, \ldots, e_n, \mathrm{i}e_n)$, whereas that of $\overline{E}_{\mathbb{R}}$ is defined by ordered local frames of the form

$$(e_1, \mathrm{i} \cdot e_1, \ldots, e_n, \mathrm{i} \cdot e_n) = (e_1, -\mathrm{i}e_1, \ldots, e_n, -\mathrm{i}e_n)\,,$$

where in both cases, $(e_1, \ldots, e_n)$ is an ordered local frame in E (and hence in $\overline{E}$). Recall that $\bar{\mathsf{c}}$ denotes the conjugate Chern class, cf. formula (4.3.16).

Proposition 4.4.1 (Complex conjugation)

1. *One has* $(\mathrm{B}\kappa)^*\mathsf{c}^{\mathrm{U}(n)} = \bar{\mathsf{c}}^{\mathrm{U}(n)}$.
2. *For every complex vector bundle E, one has* $\mathsf{c}(\overline{E}) = \bar{\mathsf{c}}(E)$.

Proof 1. By definition, $\mathrm{B}\kappa : \mathrm{BU}(k) \to \mathrm{BU}(k)$ is the classifying mapping of the associated principal $\mathrm{U}(k)$-bundle $P := \mathrm{EU}(k) \times_{\mathrm{U}(k)} \mathrm{U}(k)$, where $\mathrm{U}(k)$ acts on itself by left translation via κ. Hence,

$$(\mathrm{B}\kappa)^*\mathsf{c}_k^{\mathrm{U}(k)} = \mathsf{c}_k(P)\,.$$

By Remark 4.2.4/1,

$$\mathsf{c}_k(P) = \mathsf{e}(E_{\mathbb{R}})\,.$$

Here, $E_{\mathbb{R}}$ denotes the oriented real vector bundle induced by the complex vector bundle $E := P \times_{\mathrm{U}(k)} \mathbb{C}^k$ with $\mathrm{U}(k)$ acting on $\mathbb{C}^k$ in the basic representation. We leave it to the reader to check that the mapping

$$F : E_{\mathbb{R}} \to E_k^{\mathrm{U}}\,, \quad F\big([(\,[(y, a)]\,,\, \mathbf{z}\,)]\big) := [(y, \bar{a}\,\bar{\mathbf{z}})]\,,$$

is well defined and that it yields a vertical real vector bundle isomorphism. If $(e_1, \ldots, e_k)$ is an ordered local frame in E, then $\big(F(e_1), \ldots, F(e_k)\big)$ is an ordered local frame in E_k^{U} and

$$\big(F(e_1), F(\mathrm{i}e_1), \ldots, F(e_k), F(\mathrm{i}e_k)\big) = \big(F(e_1), -\mathrm{i}F(e_1), \ldots, F(e_k), -\mathrm{i}F(e_k)\big)\,.$$

It follows that F preserves the orientations iff k is even. Hence,

$$\mathsf{e}(E_{\mathbb{R}}) = (-1)^k \mathsf{e}(E_k^{\mathrm{U}})$$

and thus

$$(\mathrm{B}\kappa)^*\mathsf{c}_k^{\mathrm{U}(k)} = (-1)^k \mathsf{c}_k^{\mathrm{U}(k)} = \bar{\mathsf{c}}_k^{\mathrm{U}(k)}\,. \tag{4.4.1}$$

Putting $k = n$, we obtain the assertion for the top Chern class $\mathsf{c}_n^{\mathrm{U}(n)}$. For the classes $\mathsf{c}_k^{\mathrm{U}(n)}$ with $k < n$, we use $\kappa \circ j_{k,n}^{\mathrm{U}} = j_{k,n}^{\mathrm{U}} \circ \kappa$ and (4.4.1) to obtain

$$\left(\mathrm{B}j_{k,n}^{\mathrm{U}}\right)^* \circ (\mathrm{B}\kappa)^*\left(\mathsf{c}_k^{\mathrm{U}(n)}\right) = (\mathrm{B}\kappa)^* \circ \left(\mathrm{B}j_{k,n}^{\mathrm{U}}\right)^*\left(\mathsf{c}_k^{\mathrm{U}(n)}\right) = (\mathrm{B}\kappa)^*\mathsf{c}_k^{\mathrm{U}(k)} = \overline{\mathsf{c}}_k^{\mathrm{U}(k)}.$$

Then, the assertion follows from Theorem 4.2.1.

2. Choose an auxiliary fibre metric h on E. Composition of h with subsequent complex conjugation yields a fibre metric $\overline{\mathsf{h}}$ on $\overline{E}$. An h-orthonormal frame in E is, at the same time, an $\overline{\mathsf{h}}$-orthonormal frame in $\overline{E}$. Hence, as a set, $O(\overline{E})$ coincides with $O(E)$, and the identical mapping defines a vertical isomorphism of principal $\mathrm{U}(n)$-bundles with Lie group homomorphism κ. Then, Corollary 4.1.4 and point 1 imply $\mathsf{c}\big(O(\overline{E})\big) = \overline{\mathsf{c}}\big(O(E)\big)$. This yields the assertion. For an alternative proof, see Exercise 4.4.1. ■

If E is a real vector bundle, the complex vector bundles $\overline{E_{\mathbb{C}}}$ and $E_{\mathbb{C}}$ are vertically isomorphic via (A.12). Hence, point 2 of Proposition 4.4.1 implies the following.

Corollary 4.4.2 *For a real vector bundle E one has* $2\mathsf{c}_{2k+1}(E_{\mathbb{C}}) = 0$. ■

Remark 4.4.3 Comparing point 2 of Proposition 4.4.1 with Corollary 4.3.11, we see that $\mathsf{c}(\overline{E}) = \mathsf{c}(E^*)$. This is not surprising, because for every Hermitean fibre metric h on E, the mapping $\overline{E} \to E^*$ defined by assigning to $y \in \overline{E}_m$ the linear functional on E_m given by $y' \mapsto \mathsf{h}(y, y')$ is a vertical isomorphism of complex vector bundles. In fact, one may use this argument to deduce either one of the two assertions from the other one. ♦

Now, we turn to the discussion of the relations between real, complex and quaternionic characteristic classes. In effect, this amounts to calculating the homomorphisms induced in cohomology by the classifying mappings of the Lie subgroup embeddings

$$j_n^{\mathrm{U,O}} : \mathrm{U}(n) \to \mathrm{O}(2n)\,, \quad j_n^{\mathrm{Sp,O}} : \mathrm{Sp}(n) \to \mathrm{O}(4n)\,, \quad j_n^{\mathrm{Sp,U}} : \mathrm{Sp}(n) \to \mathrm{U}(2n) \tag{4.4.2}$$

defined by field restriction and the isomorphisms (4.2.1)–(4.2.3), and by the classifying mappings of the Lie subgroup embeddings

$$j_n^{\mathrm{O,U}} : \mathrm{O}(n) \to \mathrm{U}(n)\,, \quad j_n^{\mathrm{O,Sp}} : \mathrm{O}(n) \to \mathrm{Sp}(n)\,, \quad j_n^{\mathrm{U,Sp}} : \mathrm{U}(n) \to \mathrm{Sp}(n) \tag{4.4.3}$$

defined by field extension. For the conventions we use and for some standard facts about field restriction and field extension needed below, we refer to Appendix A.

We start with the case of field restriction, that is, with the homomorphisms induced by the classifying mappings of the embeddings (4.4.2). For $G = \mathrm{O}(n)$ and $G = \mathrm{Sp}(n)$, define the conjugate universal Pontryagin classes and the conjugate total universal Pontryagin class by, respectively,

$$\hat{\mathsf{p}}^G_k := (-1)^k \mathsf{p}^G_k\,, \quad k = 1, \ldots, \overline{q}_n \text{ or } n, \textit{ respectively},$$
$$\hat{\mathsf{p}}^G := 1 + \hat{\mathsf{p}}^G_1 + \hat{\mathsf{p}}^G_2 + \cdots = 1 - \mathsf{p}^G_1 + \mathsf{p}^G_2 - \cdots\,.$$

There correspond the conjugate Pontryagin classes of principal $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$-bundles and of real or quaternionic vector bundles. Recall that ρ_2 denotes reduction modulo 2.

Proposition 4.4.4 *For $n = 1, 2, 3, \ldots$, one has*

$$\big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \mathsf{w}^{\mathrm{O}(2n)} = \rho_2\big(\mathsf{c}^{\mathrm{U}(n)}\big)\,, \qquad \big(\mathrm{B}j^{\mathrm{Sp,O}}_n\big)^* \mathsf{w}^{\mathrm{O}(4n)} = \rho_2\big(\mathsf{p}^{\mathrm{Sp}(n)}\big)\,, \tag{4.4.4}$$
$$\big(\mathrm{B}j^{\mathrm{Sp,U}}_n\big)^* \mathsf{c}^{\mathrm{U}(2n)} = \hat{\mathsf{p}}^{\mathrm{Sp}(n)}\,, \tag{4.4.5}$$
$$\big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \hat{\mathsf{p}}^{\mathrm{O}(2n)} = \mathsf{c}^{\mathrm{U}(n)}\overline{\mathsf{c}}^{\mathrm{U}(n)}\,, \qquad \big(\mathrm{B}j^{\mathrm{Sp,O}}_n\big)^* \mathsf{p}^{\mathrm{O}(4n)} = \big(\mathsf{p}^{\mathrm{Sp}(n)}\big)^2\,, \tag{4.4.6}$$
$$\big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \mathsf{W}^{\mathrm{O}(2n)}_I = 0\,, \qquad \big(\mathrm{B}j^{\mathrm{Sp,O}}_n\big)^* \mathsf{W}^{\mathrm{O}(4n)}_I = 0\,, \tag{4.4.7}$$
$$\big(\mathrm{B}j^{\mathrm{U,SO}}_n\big)^* \mathsf{e}^{\mathrm{SO}(2n)} = \mathsf{c}^{\mathrm{U}(n)}_n\,, \qquad \big(\mathrm{B}j^{\mathrm{Sp,SO}}_n\big)^* \mathsf{e}^{\mathrm{SO}(4n)} = \hat{\mathsf{p}}^{\mathrm{Sp}(n)}_n\,. \tag{4.4.8}$$

Since $j^{\mathrm{U,O}}_n(\mathrm{U}(n)) \subset \mathrm{SO}(2n)$ and $j^{\mathrm{Sp,U}}_n(\mathrm{Sp}(n)) \subset \mathrm{SU}(2n)$, there follow analogous formulae with $\mathrm{O}(n)$ replaced by $\mathrm{SO}(n)$ and/or $\mathrm{U}(n)$ replaced by $\mathrm{SU}(n)$.

Proof To prove the first formula in (4.4.4), we have to show that for $k = 1, \ldots, n$,

$$\big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \mathsf{w}^{\mathrm{O}(2n)}_{2k-1} = 0\,, \quad \big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \mathsf{w}^{\mathrm{O}(2n)}_{2k} = \rho_2\big(\mathsf{c}^{\mathrm{U}(n)}_k\big)\,.$$

The first formula is due to the fact that the integral cohomology of $\mathrm{BU}(n)$ vanishes in odd degree. To prove the second formula, we realize $\mathrm{EU}(k)$ as $\mathrm{EO}(2k)$, with $\mathrm{U}(k)$ acting via $j^{\mathrm{U,O}}_k$, and view E^{U}_k as the real vector bundle obtained from the complex vector bundle $\mathrm{EO}(2k) \times_{\mathrm{U}(k)} \mathbb{C}^k$ by field restriction. Then, by taking the direct product of the identical mapping of $\mathrm{EO}(2k)$ with the real vector space isomorphism $\mathbb{C}^k \to \mathbb{R}^{2k}$ given by (4.2.1) and passing to quotients, we obtain a real vector bundle morphism $F : E^{\mathrm{U}}_k \to E^{\mathrm{O}}_{2k}$ which projects to $\mathrm{B}j^{\mathrm{U,O}}_k$ and whose fibre mappings are isomorphisms. Hence, point 3 of Proposition 4.1.12 yields that $\big(\mathrm{B}j^{\mathrm{U,O}}_k\big)^*$ maps the $\mathbb{Z}_2$-Euler class $\mathsf{w}^{\mathrm{O}(2k)}_{2k}$ of $E^{\mathbb{R}}_{2k}$ to the $\mathbb{Z}_2$-Euler class of E^{U}_k. By point 2 of that proposition, the latter is given by the mod 2-reduction of the integral Euler class of E^{U}_k. Thus,

$$\big(\mathrm{B}j^{\mathrm{U,O}}_k\big)^* \left(\mathsf{w}^{\mathrm{O}(2k)}_{2k}\right) = \rho_2\left(\mathsf{c}^{\mathrm{U}(k)}_k\right)\,. \tag{4.4.9}$$

Putting $k = n$, we obtain the assertion for the top classes. For $k < n$, using

$$j^{\mathrm{U,O}}_n \circ j^{\mathrm{U}}_{k,n} = j^{\mathrm{O}}_{2k,2n} \circ j^{\mathrm{U,O}}_k$$

and Theorem 4.2.11, we find

$$\big(\mathrm{B}j^{\mathrm{U}}_{k,n}\big)^* \left(\big(\mathrm{B}j^{\mathrm{U,O}}_n\big)^* \mathsf{w}^{\mathrm{O}(2n)}_{2k}\right) = \big(\mathrm{B}j^{\mathrm{U,O}}_k\big)^* \left(\big(\mathrm{B}j^{\mathrm{O}}_{2k,2n}\big)^* \mathsf{w}^{\mathrm{O}(2n)}_{2k}\right) = \big(\mathrm{B}j^{\mathrm{U,O}}_k\big)^* \mathsf{w}^{\mathrm{O}(2k)}_{2k}\,.$$

By (4.4.9), the right hand side equals $\rho_2\big(\mathsf{c}_k^{\mathrm{U}(k)}\big)$. Now, the assertion for $k < n$ follows from Theorem 4.2.1. The proof of the second formula in (4.4.4) is completely analogous to that for $j_n^{\mathrm{U,O}}$ and is therefore left to the reader.

To prove (4.4.5), we have to show that for $k = 1, \ldots, n$,

$$\big(\mathrm{B}j_n^{\mathrm{Sp,U}}\big)^*\mathsf{c}_{2k-1}^{\mathrm{U}(2n)} = 0\,, \quad \big(\mathrm{B}j_n^{\mathrm{Sp,U}}\big)^*\mathsf{c}_{2k}^{\mathrm{U}(2n)} = (-1)^k\mathsf{p}_k^{\mathrm{Sp}(n)}\,.$$

The first formula is due to the fact that the integral cohomology of $\mathrm{BSp}(n)$ vanishes in degree $4k-2$. The proof of the second formula is similar to that for $j_n^{\mathrm{U,O}}$, except for the fact that we have to keep track of the orientations here. By analogy with $j_n^{\mathrm{U,O}}$, we use the real vector space isomorphism $\mathbb{H}^k \to \mathbb{C}^{2k}$ given by (4.2.3) to construct a real vector bundle morphism $F : E_k^{\mathrm{Sp}} \to E_{2k}^{\mathrm{U}}$ which projects to $\mathrm{B}j_k^{\mathrm{Sp,U}}$ and whose fibre mappings are isomorphisms. F preserves the orientations iff so does the isomorphism $\mathbb{R}^{4k} \to \mathbb{H}^k \to \mathbb{C}^{2k} \to \mathbb{R}^{4k}$ defined by composition of the isomorphisms (4.2.2), (4.2.3) and (4.2.1). According to (4.2.4), this isomorphism is given by

$$(x_1, \ldots, x_{4k}) \mapsto (x_1, x_2, x_3, -x_4, \ldots, x_{4k-3}, x_{4k-2}, x_{4k-1}, -x_{4k})\,.$$

Hence, F preserves the orientations if k is even, and Proposition 4.1.12/3 yields

$$\big(\mathrm{B}j_k^{\mathrm{Sp,U}}\big)^*\mathsf{c}_{2k}^{\mathrm{U}(2k)} = (-1)^k\mathsf{p}_k^{\mathrm{Sp}(k)}\,.$$

This proves the assertion for the top classes. The case $k < n$ then follows by the same argument as for $j_n^{\mathrm{U,O}}$.

To prove the first formula in (4.4.6), we recall that, by definition of $\mathsf{p}_k^{\mathrm{O}(2n)}$,

$$\big(\mathrm{B}j_n^{\mathrm{U,O}}\big)^*\hat{\mathsf{p}}_k^{\mathrm{O}(2n)} = \big(\mathrm{B}(j_{2n}^{\mathrm{O,U}} \circ j_n^{\mathrm{U,O}})\big)^*\mathsf{c}_{2k}^{\mathrm{U}(2n)}\,.$$

One can check that there exists $b \in \mathrm{U}(2n)$ such that

$$j_{2n}^{\mathrm{O,U}} \circ j_n^{\mathrm{U,O}} = \mathrm{C}_b \circ j_{\mathrm{U}} \circ (\mathrm{id}_{\mathrm{U}(n)} \times \kappa) \circ \Delta_{\mathrm{U}(n)}$$

with the diagonal mapping $\Delta_{\mathrm{U}(n)} : \mathrm{U}(n) \to \mathrm{U}(n) \times \mathrm{U}(n)$, the complex conjugation mapping $\kappa : \mathrm{U}(n) \to \mathrm{U}(n)$, the standard blockwise embedding $j_{\mathrm{U}} : \mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(2n)$ and the inner automorphism $\mathrm{C}_b : \mathrm{U}(2n) \to \mathrm{U}(2n)$ defined by b (Exercise 4.4.2). By points 1 and 2 of Proposition 3.7.4, then $\mathrm{B}(j_{2n}^{\mathrm{O,U}} \circ j_n^{\mathrm{U,O}})$ is homotopic to the mapping $B\big(j_{\mathrm{U}} \circ (\mathrm{id}_{\mathrm{U}(n)} \times \kappa) \circ \Delta_{\mathrm{U}(n)}\big)$. By Proposition 3.7.7, then

$$\big(\mathrm{B}j_n^{\mathrm{U,O}}\big)^*\hat{\mathsf{p}}_k^{\mathrm{O}(2n)} = \Delta_{\mathrm{BU}(n)}^* \circ \big(\mathrm{B}\,\mathrm{id}_{\mathrm{U}(n)} \times \mathrm{B}\kappa\big)^* \circ \big(Bj_{\mathrm{U}}\big)^*\big(\mathsf{c}_{2k}^{\mathrm{U}(2n)}\big)\,.$$

Using Theorem 4.3.1, Proposition 4.4.1 and (4.3.3), for the right hand side we obtain $\big(\mathsf{c}^{\mathrm{U}(n)}\bar{\mathsf{c}}^{\mathrm{U}(n)}\big)_{2k}$. Since by (4.4.15), $\mathsf{c}^{\mathrm{U}(n)}\bar{\mathsf{c}}^{\mathrm{U}(n)}$ has contributions in degree 0 modulo 4 only, summation over k yields $\mathsf{c}^{\mathrm{U}(n)}\bar{\mathsf{c}}^{\mathrm{U}(n)}$.

To prove the second formula in (4.4.6), we use $j_n^{\mathrm{Sp,O}} = j_{2n}^{\mathrm{U,O}} \circ j_n^{\mathrm{Sp,U}}$ and the first formula to obtain $\left(\mathrm{B}j_n^{\mathrm{Sp,O}}\right)^* \hat{\mathsf{p}}^{\mathrm{O}(4n)} = \left(\mathrm{B}j_n^{\mathrm{Sp,U}}\right)^* \left(\mathsf{c}^{\mathrm{U}(2n)} \bar{\mathsf{c}}^{\mathrm{U}(2n)}\right)$. By (4.4.5), we have $\left(\mathrm{B}j_n^{\mathrm{Sp,U}}\right)^* \mathsf{c}^{\mathrm{U}(2n)} = \left(\mathrm{B}j_n^{\mathrm{Sp,U}}\right)^* \bar{\mathsf{c}}^{\mathrm{U}(2n)} = \hat{\mathsf{p}}^{\mathrm{Sp}(n)}$. Hence,

$$\left(\mathrm{B}j_n^{\mathrm{Sp,O}}\right)^* \hat{\mathsf{p}}^{\mathrm{O}(4n)} = \left(\hat{\mathsf{p}}^{\mathrm{Sp}(n)}\right)^2 .$$

A direct calculation shows that $\left(\hat{\mathsf{p}}^{\mathrm{Sp}(n)}\right)^2_{4k} = (-1)^k \left(\mathsf{p}^{\mathrm{Sp}(n)}\right)^2_{4k}$ for all k. This yields the assertion.

Formula (4.4.7) is due to $H^*_{\mathbb{Z}}(\mathrm{BU}(n))$ and $H^*_{\mathbb{Z}}(\mathrm{BSp}(n))$ having no torsion.

Finally, to prove (4.4.8), by analogy with the proof of (4.4.4), we construct a vector bundle morphism $F : E_n^{\mathrm{U}} \to E_{2n}^{\mathrm{SO}}$ covering $\mathrm{B}j_n^{\mathrm{U,SO}}$ whose fibre mappings are induced by the inverse of the isomorphism $\mathbb{R}^{2n} \to \mathbb{C}^n$ given by (A.1). Then, F preserves the orientations and hence Proposition 4.1.12/3 yields

$$\left(\mathrm{B}j_n^{\mathrm{U,SO}}\right)^* \mathsf{e}^{\mathrm{SO}(2n)} = \left(\mathrm{B}j_n^{\mathrm{U,SO}}\right)^* \mathsf{e}\left(E_{2n}^{\mathrm{SO}}\right) = \mathsf{e}\left(E_n^{\mathrm{U}}\right) = \mathsf{c}_n^{\mathrm{U}(n)} .$$

The second formula then follows by means of (4.4.5). ■

In view of Corollary 4.1.4, Proposition 4.4.4 implies the following.

Corollary 4.4.5

1. *For the principal* $\mathrm{SO}(2n)$*-bundle* Q *obtained from a principal* $\mathrm{U}(n)$*-bundle* P *by extension of the structure group via* $j_n^{\mathrm{U,SO}}$*, one has*

$$\mathsf{w}(Q) = \rho_2\big(\mathsf{c}(P)\big) , \quad \hat{\mathsf{p}}(Q) = \mathsf{c}(P)\bar{\mathsf{c}}(P) , \quad \mathsf{e}(Q) = \mathsf{c}_n(P) .$$

2. *For the principal* $\mathrm{SO}(4n)$*-bundle* Q *obtained from a principal* $\mathrm{Sp}(n)$*-bundle* P *by extension of the structure group via* $j_n^{\mathrm{Sp,SO}}$*, one has*

$$\mathsf{w}(Q) = \rho_2\big(\mathsf{p}(P)\big) , \quad \mathsf{p}(Q) = \mathsf{p}(P)^2 , \quad \mathsf{e}(Q) = \hat{\mathsf{p}}_n(P) .$$

3. *For the principal* $\mathrm{SU}(2n)$*-bundle* Q *obtained from a principal* $\mathrm{Sp}(n)$*-bundle* P *by extension of the structure group via* $j_n^{\mathrm{Sp,SU}}$*, one has* $\mathsf{c}(Q) = \hat{\mathsf{p}}(P)$ *.* ■

From Corollary 4.4.5, we read off the following obstructions to the existence of bundle reductions.

Corollary 4.4.6 *For a principal* $\mathrm{SO}(2n)$*-bundle to admit a reduction to the subgroup* $\mathrm{U}(n)$*, its Stiefel–Whitney classes must vanish in odd degree. For a principal* $\mathrm{SO}(4n)$*-bundle to admit a reduction to the subgroup* $\mathrm{Sp}(n)$*, its Stiefel–Whitney classes must vanish in any degree not divisible by* 4*. For a principal* $\mathrm{SU}(2n)$*-bundle to admit a reduction to the subgroup* $\mathrm{Sp}(n)$*, its Chern classes must vanish in degrees* 2 *mod* 4*.* ■

For vector bundles, Proposition 4.4.4 implies the following.

Corollary 4.4.7

1. *For the real vector bundle $E_{\mathbb{R}}$ obtained from a complex vector bundle E by field restriction, one has*

$$\mathsf{w}(E_{\mathbb{R}}) = \rho_2\big(\mathsf{c}(E)\big)\,, \quad \hat{\mathsf{p}}(E_{\mathbb{R}}) = \mathsf{c}(E)\bar{\mathsf{c}}(E)\,, \quad \mathsf{e}(E_{\mathbb{R}}) = \mathsf{c}_{\mathrm{top}}(E)\,.$$

2. *For the real vector bundle $E_{\mathbb{R}}$ obtained from a quaternionic vector bundle E by field restriction, one has*

$$\mathsf{w}(E_{\mathbb{R}}) = \rho_2\big(\mathsf{p}(E)\big)\,, \quad \mathsf{p}(E_{\mathbb{R}}) = \mathsf{p}(E)^2\,, \quad \mathsf{e}(E_{\mathbb{R}}) = \hat{\mathsf{p}}_{\mathrm{top}}(E)\,.$$

3. *For the complex vector bundle $E_{\mathbb{C}}$ obtained from a quaternionic vector bundle E by field restriction, one has* $\mathsf{c}(E_{\mathbb{C}}) = \hat{\mathsf{p}}(E)$.

Proof We give the proof for point 1. The other points are analogous. Choose an auxiliary fibre metric on E and consider the induced fibre metric on $E_{\mathbb{R}}$, defined fibrewise by (A.9). According to Lemma A.1/2, there exists a vertical morphism of principal bundles $O(E) \to O(E_{\mathbb{R}})$ with Lie group homomorphism $j_n^{\mathrm{U,O}} : \mathrm{U}(n) \to \mathrm{O}(2n)$. Hence, Corollary 4.1.4 yields

$$\left(\left(\mathrm{B}j_n^{\mathrm{U,O}}\right)^*\alpha\right)\big(O(E)\big) = \alpha\big(O(E_{\mathbb{R}})\big)\,, \quad \alpha = \mathsf{w}, \mathsf{p}, \mathsf{e}\,,$$

and the assertion follows from Proposition 4.4.4 and Remark 4.1.6/1. ■

From Corollary 4.4.7, we read off the following obstructions to the existence of complex or quaternionic structures.

Corollary 4.4.8 *For a real vector bundle to admit a complex (quaternionic) structure, its Stiefel–Whitney classes must vanish in odd degree (any degree not divisible by* 4*).*[9] *For a complex vector bundle to admit a quaternionic structure, its Chern classes must vanish in degrees* 2 *mod* 4. ■

Now, we turn to the discussion of the relations between Chern, Pontryagin and Stiefel–Whitney classes which arise by field extension. That is, we calculate the homomorphisms induced by the classifying mappings of the embeddings (4.4.3). Denote

$$\mathsf{W}^{\mathrm{O}(n)} := \mathsf{W}^{\mathrm{O}(n)}_{\{1\}} + \cdots + \mathsf{W}^{\mathrm{O}(n)}_{\{\bar{q}_n\}}\,.$$

Proposition 4.4.9 *For* $n = 1, 2, 3, \ldots,$ *one has*

$$\rho_2\left(\left(\mathrm{B}j_n^{\mathrm{O,U}}\right)^*\mathsf{c}^{\mathrm{U}(n)}\right) = \left(\mathsf{w}^{\mathrm{O}(n)}\right)^2\,, \tag{4.4.10}$$

[9] In view of Corollary 4.2.17/2, the vanishing of w_1 follows also from the fact that a real vector bundle admitting a complex or quaternionic structure is necessarily orientable.

$$\big(\mathrm{B}j_n^{\mathrm{O,U}}\big)^*\mathsf{c}^{\mathrm{U}(n)} = \hat{\mathsf{p}}^{\mathrm{O}(n)} + \mathsf{W}^{\mathrm{O}(n)}_{\{\frac{1}{2}\}}\mathsf{p}^{\mathrm{O}(n)} + \big(\mathsf{W}^{\mathrm{O}(n)}\big)^2\,, \tag{4.4.11}$$

$$\rho_2\left(\big(\mathrm{B}j_n^{\mathrm{O,Sp}}\big)^*\mathsf{p}^{\mathrm{Sp}(n)}\right) = \big(\mathsf{w}^{\mathrm{O}(n)}\big)^4\,, \tag{4.4.12}$$

$$\big(\mathrm{B}j_n^{\mathrm{O,Sp}}\big)^*\mathsf{p}^{\mathrm{Sp}(n)} = \big(\hat{\mathsf{p}}^{\mathrm{O}(n)}\big)^2 + \big(\mathsf{W}^{\mathrm{O}(n)}_{\{\frac{1}{2}\}}\big)^2\big(\mathsf{p}^{\mathrm{O}(n)}\big)^2 + \big(\mathsf{W}^{\mathrm{O}(n)}\big)^4\,, \tag{4.4.13}$$

$$\big(\mathrm{B}j_n^{\mathrm{U,Sp}}\big)^*\mathsf{p}^{\mathrm{Sp}(n)} = \mathsf{c}^{\mathrm{U}(n)}\overline{\mathsf{c}}^{\mathrm{U}(n)}\,. \tag{4.4.14}$$

By an explicit calculation, one may convince oneself that

$$\big(\mathsf{w}^{\mathrm{O}(n)}\big)^2 = 1 + \big(\mathsf{w}_1^{\mathrm{O}(n)}\big)^2 + \big(\mathsf{w}_2^{\mathrm{O}(n)}\big)^2 + \cdots\,,$$
$$\big(\mathsf{w}^{\mathrm{O}(n)}\big)^4 = 1 + \big(\mathsf{w}_1^{\mathrm{O}(n)}\big)^4 + \big(\mathsf{w}_4^{\mathrm{O}(n)}\big)^4 + \cdots\,,$$

and that $\mathsf{c}(E)\overline{\mathsf{c}}(E)$ has contributions in degrees 0 modulo 4 only, given by

$$\big(\mathsf{c}^{\mathrm{U}(n)}\overline{\mathsf{c}}^{\mathrm{U}(n)}\big)_{2k} = \sum_{l=0}^{k}\mathsf{c}_{2l}^{\mathrm{U}(n)}\mathsf{c}_{2(k-l)}^{\mathrm{U}(n)} - \sum_{l=1}^{k}\mathsf{c}_{2l-1}^{\mathrm{U}(n)}\mathsf{c}_{2(k-l)+1}^{\mathrm{U}(n)}\,. \tag{4.4.15}$$

In particular, the contributions to the right hand sides of (4.4.10), (4.4.12) and (4.4.14) do indeed vanish in the degrees required by the corresponding left hand sides.

Proof To prove (4.4.10), we check that there exists $b \in \mathrm{O}(2n)$ such that

$$j_n^{\mathrm{U,O}} \circ j_n^{\mathrm{O,U}} = \mathrm{C}_b \circ j_{\mathrm{O}} \circ \varDelta_{\mathrm{O}(n)}$$

with the diagonal mapping $\varDelta_{\mathrm{O}(n)} : \mathrm{O}(n) \to \mathrm{O}(n) \times \mathrm{O}(n)$ and the standard blockwise embedding $j_{\mathrm{O}} : \mathrm{O}(n) \times \mathrm{O}(n) \to \mathrm{O}(2n)$. By the same argument as in the proof of formula (4.4.6) in Proposition 4.4.4, this implies

$$(\mathrm{B}j_n^{\mathrm{O,U}})^* \circ (\mathrm{B}j_n^{\mathrm{U,O}})^*\mathsf{w}^{\mathrm{O}(2n)} = \big(\mathsf{w}^{\mathrm{O}(n)}\big)^2\,.$$

Now, the assertion follows from the first formula in (4.4.4). A similar argument applies to (4.4.12) and (4.4.14), where for the latter, we have to check that

$$j_n^{\mathrm{Sp,U}} \circ j_n^{\mathrm{U,Sp}} = \mathrm{C}_b \circ j_{\mathrm{U}} \circ (\mathrm{id}_{\mathrm{U}(n)} \times \kappa) \circ \varDelta_{\mathrm{U}(n)}$$

with the complex conjugation mapping $\kappa : \mathrm{U}(n) \to \mathrm{U}(n)$ (Exercise 4.4.2).

Now, consider (4.4.11). In degree $4k$, this reproduces the definition of the Pontryagin classes. In degree $4k+2$, it reads

$$\big(\mathrm{B}j_n^{\mathrm{O,U}}\big)^*\mathsf{c}_{2k+1}^{\mathrm{U}(n)} = \begin{cases} \mathsf{W}^{\mathrm{O}(n)}_{\{\frac{1}{2}\}} & k = 0\,, \\ \mathsf{W}^{\mathrm{O}(n)}_{\{\frac{1}{2}\}}\mathsf{p}_k^{\mathrm{O}(n)} + \big(\mathsf{W}^{\mathrm{O}(n)}_{\{k\}}\big)^2 & 0 < k \leq \bar{q}_n\,. \end{cases} \tag{4.4.16}$$

According to Theorem 4.2.23, $H^{4k+2}_{\mathbb{Z}}(\mathrm{BO}(n))$ consists of torsion elements of order 2. Therefore, it suffices to check (4.4.16) under reduction mod 2. The latter can be verified by an easy computation using (4.4.10) and the fact that $\rho_2 \circ \beta$ is the Steenrod square and thus fulfils [598, p. 281]

$$\rho_2 \circ \beta\left(\mathsf{w}_k^{\mathrm{O}(n)}\right) = \begin{cases} \left(\mathsf{w}_1^{\mathrm{O}(n)}\right)^2 & k = 1\,, \\ 0 & 1 < k \leq n \text{ odd,} \\ \mathsf{w}_{k+1}^{\mathrm{O}(n)} + \mathsf{w}_1^{\mathrm{O}(n)}\mathsf{w}_k^{\mathrm{O}(n)} & 1 < k < n \text{ even,} \\ \mathsf{w}_1^{\mathrm{O}(n)}\mathsf{w}_n^{\mathrm{O}(n)} & k = n \text{ even.} \end{cases} \tag{4.4.17}$$

Finally, to prove (4.4.13), we use that (4.4.14) implies

$$\left(\mathrm{B}j_n^{\mathrm{O,Sp}}\right)^* \mathsf{p}^{\mathrm{Sp}(n)} = \left(\mathrm{B}j_n^{\mathrm{O,U}}\right)^* \left(\mathsf{c}^{\mathrm{U}(n)}\bar{\mathsf{c}}^{\mathrm{U}(n)}\right)$$

and apply (4.4.11). ■

Remark 4.4.10 Formula (4.4.16) may be interpreted as an extension of the definition of the Pontryagin classes. Accordingly, the classes on the right hand side of (4.4.16) are sometimes referred to as the torsion Pontryagin classes [622]. ♦

In view of Corollary 4.1.4, Proposition 4.4.9 implies the following.

Corollary 4.4.11

1. *For the principal* $\mathrm{U}(n)$*-bundle Q obtained from a principal* $\mathrm{O}(n)$*-bundle P by extension of the structure group via $j_n^{\mathrm{O,U}}$, one has*

$$\rho_2\big(\mathsf{c}(Q)\big) = \mathsf{w}(P)^2\,, \quad \mathsf{c}(Q) = \hat{\mathsf{p}}(P) + \mathsf{W}_{\{\frac{1}{2}\}}(P)\mathsf{p}(P) + \mathsf{W}(P)^2\,.$$

2. *For the principal* $\mathrm{Sp}(n)$*-bundle Q obtained from a principal* $\mathrm{O}(n)$*-bundle P by extension of the structure group via $j_n^{\mathrm{O,Sp}}$, one has*

$$\rho_2\big(\mathsf{p}(Q)\big) = \mathsf{w}(P)^4\,, \quad \mathsf{p}(Q) = \hat{\mathsf{p}}(P)^2 + \mathsf{W}_{\{\frac{1}{2}\}}(P)^2\mathsf{p}(P)^2 + \mathsf{W}(P)^4\,.$$

3. *For the principal* $\mathrm{Sp}(n)$*-bundle Q obtained from a principal* $\mathrm{U}(n)$*-bundle P by extension of the structure group via $j_n^{\mathrm{U,Sp}}$, one has* $\mathsf{p}(Q) = \mathsf{c}(P)\bar{\mathsf{c}}(P)$. ■

For vector bundles, Proposition 4.4.9 implies the following.

Corollary 4.4.12

1. *For the complex vector bundle $E_{\mathbb{C}}$ obtained from a real vector bundle E by field extension, one has*

$$\rho_2\big(\mathsf{c}(E_{\mathbb{C}})\big) = \mathsf{w}(E)^2\,, \quad \mathsf{c}(E_{\mathbb{C}}) = \hat{\mathsf{p}}(E) + \mathsf{W}_{\{\frac{1}{2}\}}(E)\mathsf{p}(E) + \mathsf{W}(E)^2\,.$$

2. *For the quaternionic vector bundle $E_{\mathbb{H}}$ obtained from a real vector bundle E by field extension, one has*

$$\rho_2\big(\mathsf{p}(E_{\mathbb{H}})\big) = \mathsf{w}(E)^4\,, \quad \mathsf{p}(E_{\mathbb{H}}) = \hat{\mathsf{p}}(E)^2 + \mathsf{W}_{\{\frac{1}{2}\}}(E)^2\mathsf{p}(E)^2 + \mathsf{W}(E)^4\,.$$

3. *For the quaternionic vector bundle $E_{\mathbb{H}}$ obtained from a complex vector bundle E by field extension, one has* $\mathsf{p}(E_{\mathbb{H}}) = \mathsf{c}(E)\bar{\mathsf{c}}(E)$.

Proof We give the argument for point 1. Choose an auxiliary Riemannian fibre metric h on E and let $\mathsf{h}_{\mathbb{C}}$ be the induced Hermitean fibre metric on $E_{\mathbb{C}}$, defined by (A.13). According to Lemma A.2/2, there exists a vertical principal bundle morphism $O(E) \to O(E_{\mathbb{C}})$ with Lie group homomorphism $j_n^{\mathrm{O,U}}$. Now, the rest of the proof is analogous to that of Corollary 4.4.7. ■

Combining point 1 of Corollary 4.4.12 with the Whitney Sum Formula for the Chern class of complex vector bundles, we obtain a Whitney Sum Formula for the Pontryagin class of real vector bundles.

Corollary 4.4.13 *For real vector bundles E_1 and E_2 over M, one has*

$$\begin{aligned}&\mathsf{p}(E_1 \oplus E_2)\\&\quad = \mathsf{p}(E_1)\mathsf{p}(E_2) + \Big(\mathsf{W}_{\{\frac{1}{2}\}}(E_1)\mathsf{p}(E_1) + \mathsf{W}(E_1)^2\Big)\Big(\mathsf{W}_{\{\frac{1}{2}\}}(E_2)\mathsf{p}(E_2) + \mathsf{W}(E_2)^2\Big)\end{aligned} \tag{4.4.18}$$

Proof According to Corollary 4.4.12/1, Theorem 4.3.2 implies

$$\begin{aligned}&\hat{p}(E_1 \oplus E_2) + \mathsf{W}_{\{\frac{1}{2}\}}(E_1 \oplus E_2)\mathsf{p}(E_1 \oplus E_2) + \mathsf{W}(E_1 \oplus E_2)^2\\&\quad = \Big(\hat{\mathsf{p}}(E_1) + \mathsf{W}_{\{\frac{1}{2}\}}(E_1)\mathsf{p}(E_1) + \mathsf{W}(E_1)^2\Big)\Big(\hat{\mathsf{p}}(E_2) + \mathsf{W}_{\{\frac{1}{2}\}}(E_2)\mathsf{p}(E_2) + \mathsf{W}(E_2)^2\Big)\end{aligned}$$

Taking this equality in degree 0 mod 4 and changing signs in degree 4 mod 8, we obtain the assertion. ■

Remark 4.4.14 In the case where E_1 and E_2 are orientable, according to Remark 4.2.22/1, the Whitney Sum Formula (4.4.18) reads

$$\mathsf{p}(E_1 \oplus E_2) = \mathsf{p}(E_1)\mathsf{p}(E_2) + \mathsf{W}(E_1)^2\mathsf{W}(E_2)^2\,. \tag{4.4.19}$$

In the general case, by passing to real coefficients, from (4.4.18) we obtain

$$\mathsf{p}(E_1 \oplus E_2) = \mathsf{p}(E_1)\mathsf{p}(E_2) \ \text{ in } \ H^*_{\mathbb{R}}(M)\,. \tag{4.4.20}$$

Alternatively, this can be read off directly from the Whitney Sum Formula for the Chern class with real coefficients as follows. The argument proving point 1 of Corollary 4.4.12 shows that $\hat{\mathsf{p}}_k(E) = \mathsf{c}_{2k}(E_{\mathbb{C}})$ in $H^*_{\mathbb{Z}}(M)$. Combining this with Corollary

4.4.2, we obtain $\hat{\mathsf{p}}(E) = \mathsf{c}(E_{\mathbb{C}})$ in $H^*_{\mathbb{R}}(M)$. Hence, the Whitney Sum Formula for c yields $\hat{\mathsf{p}}(E_1 \oplus E_2) = \hat{\mathsf{p}}(E_1)\hat{\mathsf{p}}(E_2)$, which entails (4.4.20).

Let us add that this argument may be complemented as follows to provide an alternative proof of (4.4.18), which uses computations in real and $\mathbb{Z}_2$-valued cohomology only. Since every torsion element of $H^*_{\mathbb{Z}}(\mathrm{BO}(n))$ has order 2, in addition to (4.4.20), it suffices to prove (4.4.18) under reduction mod 2. Using the first formula in Corollary 4.4.12/1, from

$$\hat{\mathsf{p}}_k(E_1 \oplus E_2) = \mathsf{c}_{2k}\big((E_1)_{\mathbb{C}} \oplus (E_2)_{\mathbb{C}}\big) = \big[\mathsf{c}\big((E_1)_{\mathbb{C}}\big)\mathsf{c}\big((E_2)_{\mathbb{C}}\big)\big]_{4k} ,$$

one obtains

$$\rho_2\big(\mathsf{p}(E_1 \oplus E_2)\big) = \rho_2\big(\mathsf{p}(E_1)\mathsf{p}(E_2)\big) + \mathsf{w}_{\mathrm{odd}}(E_1)^2\mathsf{w}_{\mathrm{odd}}(E_2)^2 , \tag{4.4.21}$$

where $\mathsf{w}_{\mathrm{odd}}(E) = \mathsf{w}_1(E) + \mathsf{w}_3(E) + \cdots$ (Exercise 4.4.3). This is the mod 2 reduction of (4.4.18), indeed. ♦

Finally, we find the following.

Corollary 4.4.15 *Stably equivalent real vector bundles have the same Pontryagin and integral Stiefel–Whitney classes.*

Proof For the Pontryagin classes, this follows from the corresponding statement about the Chern classes in Corollary 4.3.3 by taking the second formula in point 1 of Corollary 4.4.12 in degree $4k$. For the integral Stiefel–Whitney classes, it follows from the corresponding statement about the ordinary Stiefel–Whitney classes in the same corollary and naturality of the Bockstein homomorphism. ■

Exercises

4.4.1 Let P be a principal $\mathrm{U}(n)$-bundle and take $E = P \times_{\mathrm{U}(n)} \mathbb{C}^n$ with $\mathrm{U}(n)$ acting in the basic representation. Show that $\overline{E}$ is vertically isomorphic to the complex vector bundle associated via the basic representation with $P \times_{\mathrm{U}(n)} \mathrm{U}(n)$, where $\mathrm{U}(n)$ acts on itself by left translation via $\kappa : \mathrm{U}(n) \to \mathrm{U}(n)$, $\kappa(a) = \overline{a}$. Use this and Proposition 1.2.8/3 to prove point 2 of Proposition 4.4.1.

4.4.2 Show that there exists $b \in \mathrm{U}(2n)$ such that for all $a \in \mathrm{U}(n)$,

$$j_{2n}^{\mathrm{O,U}} \circ j_n^{\mathrm{U,O}}(a) = b \begin{bmatrix} a & 0 \\ 0 & \overline{a} \end{bmatrix} b^{-1} .$$

Prove a similar statement for the composition $j_n^{\mathrm{Sp,U}} \circ j_n^{\mathrm{U,Sp}}$. (This complements the proofs of formula (4.4.6) in Proposition 4.4.4 and formula (4.4.14) in Proposition 4.4.9.)

4.4.3 Prove formula (4.4.21) and show that it coincides with the mod 2 reduction of the Whitney Sum Formula (4.4.18) for the Pontryagin class.

4.5 Characteristic Classes for Manifolds

Via the tangent bundle, the characteristic classes for real or complex vector bundles define characteristic classes for manifolds. If M is a smooth real manifold of dimension n, the Stiefel–Whitney classes of M are defined by

$$\mathsf{w}_i(M) := \mathsf{w}_i(\mathrm{T}M)\,, \quad i = 1, \ldots, n\,,$$

and the Pontryagin classes of M are defined by

$$\mathsf{p}_i(M) := \mathsf{p}_i(\mathrm{T}M)\,, \quad i = 1, \ldots, \bar{q}_n = \lfloor \tfrac{n}{2} \rfloor\,.$$

If M, and thus $\mathrm{T}M$, is oriented, we can define the Euler class by

$$\mathsf{e}(M) := \mathsf{e}(\mathrm{T}M)\,.$$

By summing over the Stiefel–Whitney and Pontryagin classes, we obtain the total Stiefel–Whitney class $\mathsf{w}(M)$ and the total Pontryagin class $\mathsf{p}(M)$, respectively.

If the tangent bundle of M carries an additional structure, like a complex or a quaternionic structure, one can define further characteristic classes and apply the appropriate relations of the previous section. In particular, if $\dim(M) = 2n$ and if $\mathrm{T}M$ carries a complex structure, and thus M is an almost complex manifold, we can define the Chern classes of M by

$$\mathsf{c}_i(M) := \mathsf{c}_i(\mathrm{T}M)\,, \quad i = 1, \ldots, n\,,$$

where $\mathrm{T}M$ is viewed as a complex vector bundle. Then, Propositions 4.4.4 and 4.4.9 yield

$$\mathsf{w}_{2i-1}(M) = 0\,, \quad \mathsf{w}_{2i}(M) = \rho_2\big(\mathsf{c}_i(M)\big)\,, \quad i = 1, \ldots, n\,.$$

In particular, this applies if M is a complex manifold of complex dimension n.

Analogously, if $\dim(M) = 4n$ and if $\mathrm{T}M$ carries a quaternionic structure, we can define the symplectic Pontryagin classes of M by

$$\mathsf{p}_i^{\mathrm{Sp}}(M) := \mathsf{p}_i(\mathrm{T}M)\,, \quad i = 1, \ldots, n\,,$$

where $\mathrm{T}M$ is viewed as a quaternionic vector bundle. Here, for $i = 1, \ldots, n$ and $d = 1, 2, 3$, Propositions 4.4.4 and 4.4.9 yield

$$\mathsf{w}_{4i-d}(M) = 0\,, \quad \mathsf{w}_{4i}(M) = \rho_2\big(\mathsf{p}_i^{\mathrm{Sp}}(M)\big)\,.$$

Example 4.5.1 For a parallelizable manifold, $\mathrm{T}M$ is trivial and hence $\mathsf{w}_k(M) = 0$ and $\mathsf{p}_k(M) = 0$ for all $k > 0$. This applies in particular to Lie groups. ♦

Example 4.5.2 Consider $M = \mathrm{S}^n$, realized as the unit sphere in $\mathbb{R}^{n+1}$. Recall that the tangent space of S^n at $\mathbf{x}$ can be realized as the subspace of $\mathbb{R}^{n+1}$ orthogonal to $\mathbf{x}$. Thus, by attaching to $\mathbf{x} \in \mathrm{S}^n$ the subspace of $\mathbb{R}^{n+1}$ spanned by $\mathbf{x}$, we obtain a realization of the normal bundle NS^n of the submanifold $\mathrm{S}^n \subset \mathbb{R}^{n+1}$. This bundle is trivial, because by assigning to each point $\mathbf{x} \in \mathrm{S}^n$ the vector $\mathbf{x} \in \mathrm{N}_{\mathbf{x}}\mathrm{S}^n$, we obtain a nowhere vanishing section. Hence, TS^n is stably equivalent to the trivial vector bundle $\mathrm{TS}^n \oplus \mathrm{NS}^n \cong (\mathrm{T}\mathbb{R}^{n+1})_{\restriction \mathrm{S}^n} = \mathrm{S}^n \times \mathbb{R}^{n+1}$. By Corollaries 4.3.3 and 4.4.15, then $\mathsf{w}_k(\mathrm{S}^n) = 0$ and $\mathsf{p}_k(\mathrm{S}^n) = 0$ for all $k > 0$. ♦

Example 4.5.3 We determine the characteristic classes of $M = \mathbb{K}\mathrm{P}^n$. First, we compute the Chern classes of $\mathbb{C}\mathrm{P}^n$. For that purpose, recall that we may view $\mathbb{C}\mathrm{P}^n$ both as the manifold of one-dimensional subspaces of $\mathbb{C}^{n+1}$ and as the quotient manifold of the action of $\mathrm{U}(1)$ on the submanifold $\mathrm{S}^{2n+1} \subset \mathbb{C}^{n+1}$ of unit vectors. Moreover, recall that the tangent space of S^{2n+1} at $\mathbf{x}$ can be realized as the real subspace of $\mathbb{C}^{n+1}$ orthogonal to $\mathbf{x}$ with respect to the scalar product (A.9) induced on the real vector space $\mathbb{C}^{n+1}_{\mathbb{R}}$ by the standard scalar product on $\mathbb{C}^{n+1}$.

We start with deriving a description of the tangent bundle $\mathrm{T}(\mathbb{C}\mathrm{P}^n)$. Let L_n denote the tautological line bundle over $\mathbb{C}\mathrm{P}^n$, viewed as a vertical vector subbundle of the trivial complex vector bundle $\mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}$. Let E be the vector subbundle of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}$ given by the orthogonal complements of the fibres of L_n with respect to the standard complex scalar product on $\mathbb{C}^{n+1}$. Let $p \in \mathbb{C}\mathrm{P}^n$ and let $\lambda : (L_n)_p \to E_p$ be a linear mapping. Choose an element $\mathbf{x}$ of the subspace p such that $\|\mathbf{x}\| = 1$. Then, $\mathbf{x} \in \mathrm{S}^{2n+1} \subset \mathbb{C}^{n+1}$ and $\lambda(\mathbf{x}) \in \mathrm{T}_{\mathbf{x}}\mathrm{S}^{2n+1}$, because orthogonality in $\mathbb{C}^{n+1}$ implies orthogonality in $\mathbb{C}^{n+1}_{\mathbb{R}}$, and so $E_p \subset \mathrm{T}_{\mathbf{x}}\mathrm{S}^{2n+1}$. Let $\mathrm{pr} : \mathrm{S}^{2n+1} \to \mathbb{C}\mathrm{P}^n$ denote the natural projection to $\mathrm{U}(1)$-orbits. Then, $\mathrm{pr}' \circ \lambda(\mathbf{x}) \in \mathrm{T}_p\mathbb{C}\mathrm{P}^n$. If $\mathbf{y}$ is another element of the subspace p with $\|\mathbf{y}\| = 1$, there exists $\alpha \in \mathrm{U}(1)$ such that $\mathbf{y} = \alpha\mathbf{x}$. Then, by linearity of λ,

$$\mathrm{pr}' \circ \lambda(\mathbf{y}) = \mathrm{pr}'\big(\alpha\lambda(\mathbf{x})\big) = \mathrm{pr}'\big(\lambda(\mathbf{x})\big)\,.$$

Hence, the assignment of $\mathrm{pr}' \circ \lambda(\mathbf{x})$ to λ defines a mapping

$$\Phi : \mathrm{Hom}(L_n, E) \to \mathrm{T}(\mathbb{C}\mathrm{P}^n)$$

and this mapping is a vertical complex vector bundle morphism. It is not hard to see that E_p together with the value at $\mathbf{x}$ of the Killing vector field of $\mathrm{i} \in \mathfrak{u}(1)$ span $\mathrm{T}_{\mathbf{x}}\mathrm{S}^{2n+1}$ over the reals. Hence, Φ is fibrewise surjective. By counting dimensions, we then find that Φ is a vertical isomorphism of complex vector bundles.

Now, since $E \oplus L_n = \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}$, we have

$$\mathrm{Hom}(L_n, E) \oplus \mathrm{End}(L_n) \cong \mathrm{Hom}(L_n, E \oplus L_n) = \mathrm{Hom}(L_n, \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}) \cong \bigoplus_{k=1}^{n+1} L_n^*\,.$$

Since the endomorphism bundle of a line bundle is always trivial, because the identical mappings of the fibres define a nowhere vanishing section, it follows that $\mathrm{T}(\mathbb{C}\mathrm{P}^n)$ is stably equivalent to the $(n+1)$-fold direct sum of the dual bundle L_n^*. As a result, the Whitney Sum Formula and Corollaries 4.3.3 and 4.3.11 yield

$$\mathsf{c}(\mathbb{C}\mathrm{P}^n) = \big(1 - \mathsf{c}_1(L_n)\big)^{n+1} . \tag{4.5.1}$$

Finally, by Example 4.2.18, $\mathsf{c}_1(L_n)$ is a generator of $H^2_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n)$ and hence a ring generator of $H^*_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^n)$. Thus, if we choose $\alpha = -\mathsf{c}_1(L_n) \equiv \mathsf{c}_1(L_n^*)$ as a generator, then (4.5.1) reads

$$\mathsf{c}(\mathbb{C}\mathrm{P}^n) = (1 + \alpha)^{n+1} . \tag{4.5.2}$$

Since α has degree 2 and $\mathbb{C}\mathrm{P}^n$ has dimension $2n$, the highest order term of the right hand side is $(n+1)\alpha^n$ and not α^{n+1}.

We leave it to the reader to adapt the arguments given for $\mathbb{C}\mathrm{P}^n$ to $\mathbb{R}\mathrm{P}^n$ (Exercise 4.5.1). As a result,

$$\mathsf{w}(\mathbb{R}\mathrm{P}^n) = (1 + \alpha)^{n+1} , \tag{4.5.3}$$

where α is the first Stiefel–Whitney class of the canonical (real) line bundle over $\mathbb{R}\mathrm{P}^n$. According to Example 4.2.18, α is a generator of $H^1_{\mathbb{Z}_2}(\mathbb{R}\mathrm{P}^n)$ and hence a ring generator of $H^*_{\mathbb{Z}_2}(\mathbb{R}\mathrm{P}^n)$.

For $\mathbb{H}\mathrm{P}^n$, the argument is slightly different. This has to do with the fact that the linear mappings between quaternionic vector spaces form a real vector space only. This applies in particular to the dual space, although the latter may be endowed with a natural left quaternionic vector space structure. By analogy with the complex case, we take the tautological quaternionic line bundle over $\mathbb{H}\mathrm{P}^n$ and construct the quaternionic orthogonal complement E together with the mapping $\Phi : \mathrm{Hom}(L_n, E) \to \mathrm{T}(\mathbb{H}\mathrm{P}^n)$. As already mentioned, here $\mathrm{Hom}(L_n, E)$ is just a real vector bundle and Φ is an isomorphism of real vector bundles. Accordingly, $\mathrm{T}(\mathbb{H}\mathrm{P}^n)$ is stably equivalent to the sum of real vector bundles $\bigoplus_{k=1}^{n+1} L_n^*$. Then, Corollary 4.4.15 implies

$$\mathsf{p}(\mathrm{T}(\mathbb{H}\mathrm{P}^n)) = \mathsf{p}(L_n^*)^{n+1} .$$

Using that L_n^* is vertically isomorphic to the real vector bundle obtained from L_n by field restriction,[10] as well as Corollary 4.4.7/2, we obtain

$$\mathsf{p}(\mathrm{T}(\mathbb{H}\mathrm{P}^n)) = \big(1 + \mathsf{p}_1((L_n)_{\mathbb{R}})\big)^{n+1} = \big(1 + 2\mathsf{p}_1(L_n)\big)^{n+1} . \tag{4.5.4}$$

Thus,

$$\mathsf{p}(\mathbb{H}\mathrm{P}^n) = (1 + \alpha)^{n+1} , \tag{4.5.5}$$

where α is the first Pontryagin class of the real vector bundle obtained from the tautological (quaternionic) line bundle over $\mathbb{H}\mathrm{P}^n$ by field restriction, or twice the first Pontryagin class of the tautological line bundle itself. According to Example 4.2.18, the latter generates $H^*_{\mathbb{Z}}(\mathbb{H}\mathrm{P}^n)$. ♦

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[10] Every quaternionic Hermitean fibre metric on L_n provides an isomorphism.

Remark 4.5.4 In case $n = 1$, the tautological line bundle L_1 over $\mathbb{K}\mathrm{P}^1$ is associated with the $\mathbb{K}$-Hopf bundle, which we denote by $P_{\mathbb{K}}$ here. We use this and the results of Example 4.5.3 to compute the first Chern index $\mathfrak{c}_1(P_{\mathbb{C}})$ of the complex Hopf bundle $P_{\mathbb{C}}$ and the first Pontryagin index of the quaternionic Hopf bundle $P_{\mathbb{H}}$. First, consider the complex Hopf bundle. Here, the base space is $\mathbb{C}\mathrm{P}^1$. The second homology group $H_2(\mathbb{C}\mathrm{P}^1)$ is generated by a single element, which may be chosen to be represented by the diffeomorphism $s : \mathrm{S}^2 \to \mathbb{C}\mathrm{P}^1$ defined in Remark 1.1.21/3. To compute $\mathfrak{c}_1(P_{\mathbb{C}})$, we have to evaluate the integral cohomology class $\mathsf{c}(P_{\mathbb{C}}) = \mathsf{c}(L_1)$ on $[s]$. On the one hand, according to Example 4.2.18, the class $\mathsf{c}_1(L_1)$ generates $H^2_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^1)$ and hence the corresponding homomorphism $H_2(\mathbb{C}\mathrm{P}^1) \to \mathbb{Z}$ generates $\mathrm{Hom}(H_2(\mathbb{C}\mathrm{P}^1), \mathbb{Z})$. Therefore, $\langle \mathsf{c}_1(P_{\mathbb{C}}), [s]\rangle = \pm 1$. On the other hand, by (4.5.1), we have $\mathsf{c}_1(\mathrm{T}(\mathbb{C}\mathrm{P}^1)) = -2\mathsf{c}_1(L_1)$. Since $\mathsf{c}_1(\mathrm{T}(\mathbb{C}\mathrm{P}^1))$ is the top Chern class of the complex vector bundle $\mathrm{T}(\mathbb{C}\mathrm{P}^1)$, according to Remark 4.2.4/1, it coincides with the integral Euler class of the oriented real vector bundle obtained by realification. Since the pullback of this orientation under s coincides with the standard orientation of S^2 defined by the outward coorientation as a submanifold of $\mathbb{R}^3$, we conclude that $\langle \mathsf{c}_1(\mathrm{T}(\mathbb{C}\mathrm{P}^1)), [s]\rangle$ is positive. As a result,

$$\mathfrak{c}_1(P_{\mathbb{C}}) = \langle \mathsf{c}_1(P_{\mathbb{C}}), [s]\rangle = -1\,.$$

The argument for the quaternionic Hopf bundle $P_{\mathbb{H}}$ is similar. We choose the generator of $H_4(\mathbb{H}\mathrm{P}^1)$ to be represented by the diffeomorphism $s : \mathrm{S}^4 \to \mathbb{H}\mathrm{P}^1$ defined in (B.1). In contrast to the complex case, this diffeomorphism reverses the natural orientation of $\mathbb{H}\mathrm{P}^1$ defined by the quaternionic structure on $\mathrm{T}(\mathbb{H}\mathrm{P}^1)$. The reason behind is that the latter is inherited from multiplication of elements of $\mathbb{H}^2$ by conjugate quaternions from the left. However, the sign we pick up here cancels against the different sign in (4.5.4), so that, in the end, we obtain an analogous result,

$$\mathfrak{p}_1(P_{\mathbb{H}}) = \langle \mathsf{p}_1(P_{\mathbb{H}}), [s]\rangle = -1\,.$$

For the Chern indices, this yields $\mathfrak{c}_1(P_{\mathbb{H}}) = 0$ and $\mathfrak{c}_2(P_{\mathbb{H}}) = 1$. ♦

Exercises

4.5.1 Adapt the arguments given for $\mathbb{C}\mathrm{P}^n$ in Example 4.5.3 to $\mathbb{R}\mathrm{P}^n$ to prove (4.5.3).

4.6 The Weil Homomorphism

In the present section, we give a geometric description of characteristic classes using connection theory. This will be accomplished via the Weil homomorphism, which allows for constructing characteristic classes in de Rham cohomology from polynomial invariants of the structure group. Necessarily, we have to restrict attention to smooth principal bundles $P(M, G)$.

The Weil homomorphism will be defined on the algebra $\mathrm{Pol}_G(\mathfrak{g})$ of real-valued Ad-invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. Recall that a function $\xi : \mathfrak{g} \to \mathbb{R}$

is said to be polynomial if it can be written as a polynomial in the expansion coefficients of its argument with respect to some basis in $\mathfrak{g}$. That is, relative to a basis $\{\mathfrak{t}_a\}$ in $\mathfrak{g}$, the function ξ is a sum of functions ξ_k of the form

$$\xi_k(A) = \xi_k(A^a \mathfrak{t}_a) = \xi_{a_1 \ldots a_k} A^{a_1} \ldots A^{a_k} \tag{4.6.1}$$

(summation convention) with $\xi_{a_1 \ldots a_k} \in \mathbb{R}$. Here, $A = A^a \mathfrak{t}_a$. The system $\xi_{a_1 \ldots a_k}$ may be assumed to be symmetric under permutation of indices. It is then uniquely determined by ξ and transforms like a tensor under a change of basis. Clearly, the Ad-invariant polynomial functions form a subalgebra of $C^\infty(\mathfrak{g})$, denoted by $\mathrm{Pol}_G(\mathfrak{g})$. As a vector space,

$$\mathrm{Pol}_G(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathrm{Pol}_G^k(\mathfrak{g}) \, ,$$

where $\mathrm{Pol}_G^k(\mathfrak{g}) \subset \mathrm{Pol}_G(\mathfrak{g})$ denotes the subspace of homogeneous polynomial functions of order k.

To construct the Weil homomorphism, we have to turn homogeneous polynomials into symmetric multilinear forms. Let $\mathrm{Sym}_G^k(\mathfrak{g})$ denote the vector space of real-valued symmetric k-linear forms on $\mathfrak{g}$ which are invariant under the adjoint action of G and let

$$\mathrm{Sym}_G(\mathfrak{g}) := \bigoplus_{i=0}^{\infty} \mathrm{Sym}_G^k(\mathfrak{g}) \, .$$

With the product defined on homogeneous elements f of order k and g of order l by

$$\begin{aligned}(f \cdot g)&(A_1, \ldots, A_{k+l}) \\ &:= \frac{1}{k!l!} \sum_{\pi \in \mathrm{S}_{k+l}} f\left(A_{\pi(1)}, \ldots, A_{\pi(k)}\right) g\left(A_{\pi(k+1)}, \ldots, A_{\pi(k+l)}\right) \, ,\end{aligned} \tag{4.6.2}$$

$\mathrm{Sym}_G(\mathfrak{g})$ is an infinite dimensional real associative algebra.[11] Every $f \in \mathrm{Sym}_G^k(\mathfrak{g})$ defines an element $\hat{f}$ of $\mathrm{Pol}_G(\mathfrak{g})$ by

$$\hat{f}(A) := \frac{1}{k!} f(A, \ldots, A) \, . \tag{4.6.3}$$

It is easy to see that the assignment $f \mapsto \hat{f}$ extends to a homomorphism of algebras from $\mathrm{Sym}_G(\mathfrak{g})$ to $\mathrm{Pol}_G(\mathfrak{g})$ (Exercise 4.6.1). This homomorphism is referred to as the polarization homomorphism. One has the polarization formula

[11]The factor $\frac{1}{k!l!}$ in this definition is dictated by our choice of the wedge product of differential forms, see formula (2.4.17) in Part I. In many textbooks, the coefficient in (4.6.2) is $\frac{1}{(k+l)!}$ which corresponds to the other common choice of the wedge product. These different conventions lead to different combinatorial factors on the way, but the final formulae for the Chern classes will be the same. We will comment on this at the end of this section in Remark 4.6.10/2.

$$f(A_1, \ldots, A_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} \hat{f}\,(t_1 A_1 + \cdots + t_k A_k)\ , \tag{4.6.4}$$

which holds in general for all symmetric k-linear forms on $\mathfrak{g}$, invariant or not (Exercise 4.6.2). In other words, (4.6.4) states that $f(A_1, \ldots, A_k)$ coincides with the coefficient of the monomial $t_1 \cdots t_k$ in the expansion of $\hat{f}\,(t_1 A_1 + \cdots + t_k A_k)$ as a polynomial in the indeterminates t_i.

Lemma 4.6.1 *The polarization homomorphism is an isomorphism.*

Proof Injectivity follows at once from the polarization formula (4.6.4). To prove surjectivity, let $\xi \in \mathrm{Pol}_G(\mathfrak{g})$ be homogeneous of degree k. Choose a basis and write ξ in the form (4.6.1) with symmetric coefficients $\xi_{a_1 \ldots a_k}$. Define a k-linear form on $\mathfrak{g}$ by

$$f(A_1, \ldots, A_k) := k!\, \xi_{a_1 \ldots a_k} A_1^{a_1} \ldots A_k^{a_k}\ .$$

This form is symmetric and fulfils $\hat{f} = \xi$. Finally, by the polarization formula (4.6.4), invariance of ξ implies invariance of f. ■

The inverse of polarization is referred to as multilinearization. Given $\xi \in \mathrm{Pol}_G(\mathfrak{g})$, the multilinearization of ξ will be denoted by $\check{\xi}$. By (4.6.4),

$$\check{\xi}(A_1, \ldots, A_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k}\, \xi\,(t_1 A_1 + \cdots + t_k A_k)\ . \tag{4.6.5}$$

The further analysis uses invariant horizontal forms on $P(M, G)$. Such forms constitute a subalgebra of $\Omega^*(P)$, denoted by $\Omega^*_{G,\mathrm{hor}}(P)$. They are related to forms on M as follows.

Lemma 4.6.2 *Let P be a principal G-bundle over M with projection π.*

1. *The homomorphism π^* maps $\Omega^*(M)$ isomorphically onto $\Omega^*_{G,\mathrm{hor}}(P)$.*
2. *For all $\alpha \in \Omega^*_{G,\mathrm{hor}}(P)$, one has $\mathrm{d}\alpha \in \Omega^*_{G,\mathrm{hor}}(P)$.*
3. *For all $\alpha \in \Omega^*_{G,\mathrm{hor}}(P)$ and all connections ω on P, one has $\mathrm{D}_\omega \alpha = \mathrm{d}\alpha$.*

Proof 1. Since π is a surjective submersion, π^* is injective. To see that π^* maps $\Omega^*(M)$ onto all of $\Omega^*_{G,\mathrm{hor}}(P)$, let $\alpha \in \Omega^*_{G,\mathrm{hor}}(P)$ be given. It suffices to give the argument under the assumption that α is a 1-form. Choose a covering of M by local sections s_i in P over U_i and consider the local k-forms $s_i^* \alpha$ on M. Let $\rho_{ij} : U_i \cap U_j \to G$ denote the transition mappings. They fulfil $s_j(m) = \Psi_{\rho_{ij}(m)}\big(s_i(m)\big)$ for all $m \in U_i \cap U_j$. Thus,

$$\big(s_j\big)'_m = (\Psi_a)'_{s_i(m)} \circ (s_i)'_m + \big(\Psi_{s_i(m)}\big)'_a \circ \big(\rho_{ij}\big)'_m\ ,$$

where $a = \rho_{ij}(m)$. Since the second term in this formula is vertical, horizontality and invariance of α yield

$$\left(s_j^*\alpha\right)_m(X) = \alpha_{s_j(m)}\left(s_j' X\right) = \alpha_{\Psi_a \circ s_i(m)}\left(\Psi_a' \circ s_i'(X)\right) = \left(s_i^*\alpha\right)_m(X)$$

for all $X \in \mathrm{T}_m M$. Hence, the local forms $s_i^*\alpha$ combine to a global form $\hat{\alpha}$ on M. It remains to show that $\pi^*\hat{\alpha} = \alpha$. For given i, let $\kappa_i : \pi^{-1}(U_i) \to G$ be the mapping defined by (1.1.1). Then, for all $p \in \pi^{-1}(U_i)$, one has $p = \Psi_{\kappa_i(p)} \circ s_i \circ \pi(p)$ and hence

$$(s_i)'_{\pi(p)} \circ \pi'_p = (\Psi_b)'_p + \left(\Psi_p\right)'_b \circ (\mathrm{inv}_G \circ \kappa_i)'_p \,,$$

where $\mathrm{inv}_G : G \to G$ denotes the inversion mapping and $b = \kappa_i(p)^{-1}$. Since the second term is vertical, for $Y \in \mathrm{T}_p P$ we obtain

$$(\pi^*\hat{\alpha})_p(Y) = \alpha_{s_i \circ \pi(p)}\left(s_i' \circ \pi'(Y)\right) = \alpha_{\Psi_b(p)}\left(\Psi_b' Y\right) = \alpha_p(Y) \,,$$

as asserted.

2. This follows from point 1 and the fact that the exterior differential commutes with taking pullbacks.

3. By point 2, the form $\mathrm{d}\alpha$ is horizontal. Hence, for $p \in P$ and $Y_0, \ldots, Y_k \in \mathrm{T}_p P$, we find $(\mathrm{D}_\omega \alpha)(Y_0, \ldots, Y_k) = \mathrm{d}\alpha\left(\mathrm{hor}_\omega Y_0, \cdots, \mathrm{hor}_\omega Y_k\right) = \mathrm{d}\alpha(Y_0, \ldots, Y_k)$. ■

Now, let $\alpha \in \Omega^2(P, \mathfrak{g})$. Using multilinearization, we can assign to every $\xi \in \mathrm{Pol}^k_G(\mathfrak{g})$ a $2k$-form $h_\alpha(\xi)$ on P by

$$\begin{aligned} &h_\alpha(\xi)\,(X_1, \ldots, X_{2k}) \\ &\quad := \frac{1}{k!} \sum_{\rho \in \mathrm{S}_{2k}} \mathrm{sign}(\rho)\, \check{\xi}\left(\alpha\left(X_{\rho(1)}, X_{\rho(2)}\right), \ldots, \alpha\left(X_{\rho(2k-1)}, X_{\rho(2k)}\right)\right) . \end{aligned} \tag{4.6.6}$$

The assignment $\xi \mapsto h_\alpha(\xi)$ extends to a linear mapping $h_\alpha : \mathrm{Pol}_G(\mathfrak{g}) \to \Omega^*(P)$.

Remark 4.6.3 Let $\{\mathfrak{t}_a\}$ be a basis in $\mathfrak{g}$ and let $\alpha^a \in \Omega^2(P)$ denote the corresponding coefficient 2-forms, defined by $\alpha(Y_1, Y_2) = \alpha^a(Y_1, Y_2)\,\mathfrak{t}_a$ for all $p \in P$ and $Y_1, Y_2 \in \mathrm{T}_p P$. By plugging this expansion into the definition of $h_\alpha(p)$ for $\xi \in \mathrm{Pol}^k_G(\mathfrak{g})$, we obtain

$$h_\alpha(\xi) = 2^k\, \xi_{a_1,\ldots,a_k}\, \alpha^{a_1} \wedge \cdots \wedge \alpha^{a_k} \,, \tag{4.6.7}$$

where $\xi_{a_1,\ldots,a_k}$ are the symmetric coefficients of ξ defined by (4.6.1). This implies

$$h_\alpha(\xi) = 2^k \xi^\wedge(\alpha) \,, \tag{4.6.8}$$

where $\xi^\wedge$ means that all products in the polynomial ξ are replaced by the exterior product. ♦

Lemma 4.6.4 *Let $\alpha \in \Omega^2(P, \mathfrak{g})$.*

1. *The mapping $h_\alpha : \mathrm{Pol}_G(\mathfrak{g}) \to \Omega^*(P)$ is an algebra homomorphism, that is,*

$$h_\alpha(\xi\zeta) = h_\alpha(\xi) \wedge h_\alpha(\zeta) \quad \textit{for all} \;\; \xi, \zeta \in \mathrm{Pol}_G(\mathfrak{g}) \,.$$

2. *For all* $\xi \in \mathrm{Pol}_G(\mathfrak{g})$*, the following holds true.*

   a. *If* α *is of type* Ad*, then* $h_\alpha(\xi)$ *is invariant.*
   b. *If* α *is horizontal, then* $h_\alpha(\xi)$ *is horizontal.*
   c. *If* Ω *is the curvature form of a connection, then* $h_\Omega(\xi)$ *is closed.*

3. *If* $F : Q \to P$ *is a morphism of principal* G*-bundles, then, for all* $\xi \in \mathrm{Pol}_G(\mathfrak{g})$*,*

$$F^* h_\alpha(\xi) = h_{F^*\alpha}(\xi) \,.$$

Proof Points 2b and 3 are immediate. Point 2a follows from point 3 by using that α is of type Ad and ξ is invariant. Point 1 is straightforward (Exercise 4.6.3). It remains to prove point 2c. Assume that ξ is homogeneous of degree k. If Ω is the curvature form of a connection ω, it is horizontal and of type Ad. Then, points 2a and 2b imply that $h_\Omega(\xi)$ is horizontal and invariant. Hence, Lemma 4.6.2/3 yields

$$\mathrm{d}\big(h_\Omega(\xi)\big) = \mathrm{D}_\omega\big(h_\Omega(\xi)\big) \,.$$

Choose a basis $\{\mathfrak{t}_a\}$ in $\mathfrak{g}$ and decompose $h_\Omega(\xi)$ according to (4.6.7). Then,

$$\mathrm{D}_\omega\big(h_\Omega(\xi)\big) = 2^k \xi_{a_1,\dots,a_k}\, \mathrm{D}_\omega\big(\Omega^{a_1} \wedge \cdots \wedge \Omega^{a_k}\big) \,.$$

Since the forms Ω^a are horizontal,

$$\mathrm{D}_\omega\big(\Omega^{a_1} \wedge \cdots \wedge \Omega^{a_k}\big) = \big(\mathrm{D}_\omega \Omega^{a_1}\big) \wedge \cdots \wedge \Omega^{a_k} + \cdots + \Omega^{a_1} \wedge \cdots \wedge \big(\mathrm{D}_\omega \Omega^{a_k}\big) \,.$$

By the Bianchi identity, $\big(\mathrm{D}_\omega \Omega^a\big)\mathfrak{t}_a = \mathrm{D}_\omega \Omega = 0$ and hence $\mathrm{D}_\omega \Omega^a = 0$ for all a. ■

In view of points 2a and 2b of Lemma 4.6.4, if α is horizontal of type Ad, we may compose h_α with the inverse of the isomorphism $\pi^* : \Omega^*(M) \to \Omega^*_{G,\mathrm{hor}}(P)$, provided by Lemma 4.6.2/1, thus obtaining an algebra homomorphism

$$\hat{h}_\alpha : \mathrm{Pol}_G(\mathfrak{g}) \to \Omega^*(M) \,.$$

By construction, $\hat{h}_\alpha$ is determined by

$$\pi^* \circ \hat{h}_\alpha = h_\alpha \,. \tag{4.6.9}$$

By point 2c of Lemma 4.6.4, if Ω is the curvature form of a connection, then $\hat{h}_\Omega$ takes values in the closed forms on M and thus induces a homomorphism

$$\mathfrak{w}_P : \mathrm{Pol}_G(\mathfrak{g}) \to H^*_{\mathrm{dR}}(M) \,, \quad \mathfrak{w}_P(\xi) := \big[\hat{h}_\Omega(\xi)\big] \,. \tag{4.6.10}$$

Lemma 4.6.5 *If* Ω_0 *and* Ω_1 *are the curvature forms of connections on* P*, then* $\hat{h}_{\Omega_1}(\xi) - \hat{h}_{\Omega_0}(\xi)$ *is exact for all* $\xi \in \mathrm{Pol}_G(\mathfrak{g})$*.*

Proof Let ω_0 and ω_1 be connection forms on P and assume that ξ is homogeneous of degree k. Define $\beta := \omega_1 - \omega_0$ and $\omega_t := \omega_0 + t\beta$. Then, β is horizontal of type Ad and ω_t is a connection form for all t. Let Ω_t denote the curvature of ω_t. Choose a basis $\{\mathfrak{t}_a\}$ in $\mathfrak{g}$ and let β^a and Ω_t^a denote the corresponding coefficient forms. Define[12]

$$\phi_t := 2^k k\, \xi_{a_1,\dots,a_k} \beta^{a_1} \wedge \Omega_t^{a_2} \wedge \cdots \wedge \Omega_t^{a_k} \quad \text{and} \quad \phi := \int_0^1 \phi_t \, \mathrm{d}t \,,$$

where $\xi_{a_1,\dots,a_k}$ denote the symmetric coefficients of ξ defined by (4.6.1). We claim that ϕ is a potential for $\hat{h}_{\Omega_1}(\xi) - \hat{h}_{\Omega_0}(\xi)$. One has

$$\mathrm{d}\phi = \int_0^1 \mathrm{d}\phi_t \; \mathrm{d}t \,. \tag{4.6.11}$$

Since β and Ω_t are horizontal and of type Ad, so is ϕ_t. Hence, by Lemma 4.6.2/3, $\mathrm{d}\phi_t = \mathrm{D}_{\omega_t}\phi_t$. By horizontality,

$$\begin{aligned} \mathrm{D}_{\omega_t}\phi_t = 2^k k\, \xi_{a_1,\dots,a_k} \Big(& \big(\mathrm{D}_{\omega_t}\beta^{a_1}\big) \wedge \Omega_t^{a_2} \wedge \cdots \wedge \Omega_t^{a_k} \\ & + \beta^{a_1} \wedge \big(\mathrm{D}_{\omega_t}\Omega_t^{a_2}\big) \wedge \cdots \wedge \Omega_t^{a_k} + \cdots + \beta^{a_1} \wedge \Omega_t^{a_2} \wedge \cdots \wedge \big(\mathrm{D}_{\omega_t}\Omega_t^{a_k}\big)\Big). \end{aligned}$$

By the Bianchi identity, $\mathrm{D}_{\omega_t}\Omega_t^a = 0$ for all a. Thus,

$$\mathrm{d}\phi_t = \mathrm{D}_{\omega_t}\phi_t = 2^k k\, \xi_{a_1,\dots,a_k} \big(\mathrm{D}_{\omega_t}\beta^{a_1}\big) \wedge \Omega_t^{a_2} \wedge \cdots \wedge \Omega_t^{a_k} \,. \tag{4.6.12}$$

Using $\beta = \frac{\mathrm{d}}{\mathrm{d}t}\omega_t$ and the Structure Equation (1.4.9), we find

$$\big(\mathrm{D}_{\omega_t}\beta^a\big)\mathfrak{t}_a = \mathrm{D}_{\omega_t}\beta = \mathrm{d}\beta + [\omega_t, \beta] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}\omega_t + \frac{1}{2}[\omega_t, \omega_t]\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Omega_t$$

and hence $\mathrm{D}_{\omega_t}\beta^a = \frac{\mathrm{d}}{\mathrm{d}t}\Omega_t^a$. Plugging this into (4.6.12) and using (4.6.7), we obtain

$$\mathrm{d}\phi_t = 2^k k\, \xi_{a_1,\dots,a_k} \left(\frac{\mathrm{d}}{\mathrm{d}t}\Omega_t^{a_1}\right) \wedge \Omega_t^{a_2} \wedge \cdots \wedge \Omega_t^{a_k} = \frac{\mathrm{d}}{\mathrm{d}t} h_{\Omega_t}(\xi) \,.$$

Consequently, (4.6.11) yields $\mathrm{d}\phi = h_{\Omega_1}(\xi) - h_{\Omega_0}(\xi)$ and hence the assertion. ■

As a result, the homomorphism (4.6.10) depends on the principal bundle P only and not on the specific connection whose curvature form is used in the definition.

Definition 4.6.6 The homomorphism $\mathfrak{w}_P$ is called the Weil homomorphism of P.

---

[12] See Remark 4.1.10/1 in Part I for the definitions of the integral and the derivative of a 1-parameter family of differential forms and for the corresponding calculus.

Let us study how the Weil homomorphism behaves under bundle morphisms. For a Lie group homomorphism $\lambda : G \to H$, let $\mathrm{d}\lambda : \mathfrak{g} \to \mathfrak{h}$ denote the induced homomorphism of Lie algebras. It is elementary to check that $(\mathrm{d}\lambda)^*$ maps $\mathrm{Pol}_H(\mathfrak{h})$ to $\mathrm{Pol}_G(\mathfrak{g})$.

Proposition 4.6.7 *Let P and Q be principal bundles over M and N with structure groups G and H, respectively, and let $\vartheta : P \to Q$ be a morphism with Lie group homomorphism $\lambda : G \to H$ and projection $f : M \to N$. Then,*

$$\mathfrak{w}_P \circ (\mathrm{d}\lambda)^* = f^* \circ \mathfrak{w}_Q .$$

Proof The morphism ϑ can be written as the composition of the vertical morphism

$$\Phi : P \to f^*Q , \quad \Phi(p) := \big(\pi_P(p), \vartheta(p)\big)$$

whose Lie group homomorphism is given by λ with the natural principal H-bundle morphism $F : f^*Q \to Q$ covering f. It suffices to prove that $\mathfrak{w}_P \circ (\mathrm{d}\lambda)^* = \mathfrak{w}_{f^*Q}$ and $\mathfrak{w}_{f^*Q} = f^* \circ \mathfrak{w}_Q$.

To prove the first formula, let ω be a connection form on P. By Proposition 1.3.13, ω induces a connection $\tilde{\omega}$ on f^*Q such that $\Phi^*\tilde{\omega} = \mathrm{d}\lambda \circ \omega$. Then, by Remark 1.4.10/2,

$$\Phi^*\tilde{\Omega} = \mathrm{d}\lambda \circ \Omega .$$

Using this, for $\xi \in \mathrm{Pol}^k_G(\mathfrak{h})$ and $X_1, \dots, X_{2k} \in \mathfrak{X}(P)$, we obtain

$$\begin{aligned}
\Phi^*\big(h_{\tilde{\Omega}}(\xi)\big)(X_1, \dots, X_{2k}) &= h_{\tilde{\Omega}}(\xi)(\Phi' \circ X_1, \dots, \Phi' \circ X_{2k}) \\
&= \frac{1}{k!} \sum_{\pi \in \mathrm{S}_k} \mathrm{sign}(\pi) \check{\xi} \left(\Phi^*\tilde{\Omega}\left(X_{\pi(1)}, X_{\pi(2)}\right), \dots\right) \\
&= \frac{1}{k!} \sum_{\pi \in \mathrm{S}_k} \mathrm{sign}(\pi) \check{\xi} \left(\mathrm{d}\lambda \circ \Omega\left(X_{\pi(1)}, X_{\pi(2)}\right), \dots\right) \\
&= \big(h_\Omega \circ (\mathrm{d}\lambda)^*(\xi)\big)(X_1, \dots, X_{2k}) .
\end{aligned}$$

Thus, $\Phi^*\big(h_{\tilde{\Omega}}(\xi)\big) = h_\Omega \circ (\mathrm{d}\lambda)^*(\xi)$. Since Φ is vertical, formula (4.6.9) implies

$$\pi_P^*\left(\hat{h}_\Omega \circ (\mathrm{d}\lambda)^*(\xi)\right) = \Phi^* \circ \pi_P^*\big(\hat{h}_{\tilde{\Omega}}(\xi)\big) = \pi_P^*\big(\hat{h}_{\tilde{\Omega}}(\xi)\big) .$$

It follows that $\hat{h}_\Omega \circ (\mathrm{d}\lambda)^*(\xi) = \hat{h}_{\tilde{\Omega}}(\xi)$ and hence $\mathfrak{w}_P \circ (\mathrm{d}\lambda)^*(\xi) = \mathfrak{w}_{f^*Q}(\xi)$.

To see that $\mathfrak{w}_{f^*Q} = f^* \circ \mathfrak{w}_Q$, let ω be a connection form on Q and let Ω be its curvature. By Corollary 1.3.16, $F^*\omega$ is a connection form on f^*Q and by Remark 1.4.10/2, the curvature of this connection form is given by $F^*\Omega$. By Lemma 4.6.4/3, for $\xi \in \mathrm{Pol}_H(\mathfrak{h})$, then $h_{F^*\Omega}(\xi) = F^*\big(h_\Omega(\xi)\big)$. Using (4.6.9) and $\pi_Q \circ F = f \circ \pi_{f^*Q}$, we thus obtain

$$\pi^*_{f^*Q}\left(\hat{h}_{F^*\Omega}(\xi)\right) = F^* \circ \pi^*_Q\left(\hat{h}_\Omega(\xi)\right) = \pi^*_{f^*Q} \circ f^*\left(h_\Omega(\xi)\right) .$$

This implies $\hat{h}_{F^*\Omega}(\xi) = f^*\left(\hat{h}_\Omega(\xi)\right)$ and hence $\mathfrak{w}_{f^*Q}(\xi) = f^*\mathfrak{w}_Q(\xi)$. ■

Corollary 4.6.8

1. *Vertically isomorphic principal G-bundles define the same Weil homomorphism.*
2. *For every $\xi \in \mathrm{Pol}_G(\mathfrak{g})$, the assignment of $\mathfrak{w}_P(\xi)$ to P defines a characteristic class for principal G-bundles with values in the de Rham cohomology.*
3. *If P is a principal G-bundle and $\lambda : G \to H$ is a Lie group homomorphism, then*

$$\mathfrak{w}_{P^{[\lambda]}} = \mathfrak{w}_P \circ (\mathrm{d}\lambda)^* .$$

Proof Point 1 is immediate.

2. For a principal G-bundle P over M, write $\alpha(P) := \mathfrak{w}_P(\xi)$. If $f : N \to M$ is a smooth mapping, we have a natural morphism $F : f^*P \to P$ covering f. Hence, Proposition 4.6.7 yields $\mathfrak{w}_{f^*P}(\xi) = f^*\big(\mathfrak{w}_P(\xi)\big)$ and thus $\alpha(f^*P) = f^*\big(\alpha(P)\big)$.

3. This follows by observing that the mapping $\iota_{\mathbb{1}} : P \to P^{[\lambda]}$ defined by $\iota_{\mathbb{1}}(p) := [(p, \mathbb{1})]$, together with the Lie group homomorphism λ, provides a vertical morphism of principal bundles over M. ■

Recall that in Sect. 4.2 we have constructed characteristic classes in singular cohomology for the classical compact Lie groups. We are now going to analyze how these are related to the characteristic classes in de Rham cohomology provided by the Weil homomorphism. We start with briefly recalling the relation between de Rham cohomology and singular cohomology, cf. Sect. 4.3 of Part I.

By the de Rham Theorem [104, Sect. V.9], there exists an isomorphism between the de Rham cohomology ring $H^*_{\mathrm{dR}}(M)$ and the singular cohomology ring with real coefficients $H^*_{\mathbb{R}}(M)$. This isomorphism is referred to as the de Rham isomorphism. It is obtained as follows. Let $C^\infty_k(M)$ denote the free Abelian group generated by smooth k-simplices. Together with the ordinary boundary operator, the groups $C^\infty_k(M)$ form a chain complex. Recall that $H^k_{\mathbb{R}}(M)$ may be thought of as being the (co)homology groups of the corresponding cochain complex $\mathrm{Hom}(C^\infty_k(M), \mathbb{R})$. Let δ denote the coboundary homomorphism. Given a k-form α on M, we may define a homomorphism $\hat{\alpha} : C^\infty_k(M) \to \mathbb{R}$ by assigning to each smooth simplex $\sigma : \Delta^k \to M$ the integral

$$\hat{\alpha}(\sigma) := \int_{\Delta^k} \sigma^*\alpha .$$

By linearity of pullback and integration, the assignment $\alpha \mapsto \hat{\alpha}$ defines a group homomorphism $\Omega^k(M) \to \mathrm{Hom}(C^\infty_k(M), \mathbb{R})$. By Stokes' Theorem, one has $\widehat{\mathrm{d}\alpha} = \delta\hat{\alpha}$. It follows that the mapping $\alpha \mapsto \hat{\alpha}$ induces a group homomorphism $H^k_{\mathrm{dR}}(M) \to H^k_{\mathbb{R}}(M)$. This homomorphism is the de Rham isomorphism in degree k. One can show that the induced group isomorphism $H^*_{\mathrm{dR}}(M) \to H^*_{\mathbb{R}}(M)$ is in fact a ring isomorphism, see [652, Theorem 5.45].

Thus, by means of composition with the de Rham isomorphism, we may view the Weil homomorphism as a mapping

$$\mathfrak{w}_P : \mathrm{Pol}_G(\mathfrak{g}) \to H^*_{\mathbb{R}}(M) .$$

We do not distinguish in notation between these viewpoints.

Next, we determine a system of generators of the algebra $\mathrm{Pol}_G(\mathfrak{g})$ for the classical compact Lie groups. For that purpose, we consider a maximal Abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$. We denote the normalizer and the centralizer of $\mathfrak{t}$ in G by

$$\mathrm{N}_G(\mathfrak{t}) = \{a \in G : \mathrm{Ad}(a)\mathfrak{t} \subset \mathfrak{t}\} , \quad \mathrm{C}_G(\mathfrak{t}) = \{a \in G : \mathrm{Ad}(a)_{\restriction \mathfrak{t}} = \mathrm{id}_{\mathfrak{t}}\} ,$$

respectively, and define

$$W := \mathrm{N}_G(\mathfrak{t})/\mathrm{C}_G(\mathfrak{t}) .$$

Since $\mathrm{C}_G(\mathfrak{t})$ is a normal subgroup of $\mathrm{N}_G(\mathfrak{t})$, W is a group. It is called the Weyl group of $\mathfrak{g}$. The adjoint representation induces an action of W on $\mathfrak{t}$. Let $\mathrm{Pol}_W(\mathfrak{t})$ denote the algebra of polynomial functions on $\mathfrak{t}$ which are invariant under the action of W. In the theory of compact Lie groups[13] it is shown that W is finite and that the mapping

$$\mu : G \times \mathfrak{t} \to \mathfrak{g} , \quad \mu(a, B) := \mathrm{Ad}(a)B , \tag{4.6.13}$$

is a surjective submersion. The latter implies, in particular, that any two maximal Abelian subalgebras are conjugate to one another under $\mathrm{Ad}(G)$. As a consequence, W does not depend on the choice of $\mathfrak{t}$. Another consequence is that every orbit of the adjoint representation in $\mathfrak{g}$ intersects $\mathfrak{t}$, because, obviously, every element of $\mathfrak{g}$ is contained in a maximal Abelian subalgebra. It is furthermore shown that any two elements of $\mathfrak{t}$ belong to the same G-orbit in $\mathfrak{g}$ iff they belong to the same W-orbit in $\mathfrak{t}$. Thus, more precisely, each orbit of the adjoint representation intersects $\mathfrak{t}$ in a W-orbit. It follows that restriction to $\mathfrak{t}$ defines a homomorphism $\mathrm{Pol}_G(\mathfrak{g}) \to \mathrm{Pol}_W(\mathfrak{t})$.

Lemma 4.6.9 *The restriction homomorphism* $\mathrm{Pol}_G(\mathfrak{g}) \to \mathrm{Pol}_W(\mathfrak{t})$ *is an isomorphism.*

Proof To prove injectivity, let $p_1, p_2 \in \mathrm{Pol}_G(\mathfrak{g})$ be such that $p_{1\restriction \mathfrak{t}} = p_{2\restriction \mathfrak{t}}$ and let $A \in \mathfrak{g}$. Since the orbit of A under the adjoint representation intersects $\mathfrak{t}$, there exist $B \in \mathfrak{t}$ and $a \in G$ such that $A = \mathrm{Ad}(a)B$. By Ad-invariance,

$$p_i(A) = p_i\big(Ad(a)B\big) = p_i(B) , \quad i = 1, 2 .$$

Since $p_1(B) = p_2(B)$, we conclude $p_1(A) = p_2(A)$ and hence $p_1 = p_2$.

[13] A standard reference is [105]. The arguments for the classical compact Lie groups are elementary though, see the discussion below.

To prove surjectivity, let $q \in \mathrm{Pol}_W(\mathfrak{t})$ be given. Since for each $A \in \mathfrak{g}$, the orbit of A under the adjoint representation of G intersects $\mathfrak{t}$ in a W-orbit, there exists $B \in \mathfrak{t}$ such that $A = \mathrm{Ad}(a)B$ for some $a \in G$ and any two such B are mapped to one another by an element of W. Since q is W-invariant, we can define a mapping $p : \mathfrak{g} \to \mathbb{R}$ by $p(A) = q(B)$. By construction, p is invariant. It remains to show that p is polynomial. We may assume that q is homogeneous of degree k. Then, so is p. Since (4.6.13) is a surjective submersion and since submersions admit local sections, for every $A \in \mathfrak{g}$, there exists an open neighbourhood U of A and a smooth mapping $s : U \to G \times \mathfrak{g}$ such that composition of (4.6.13) with s yields id_U. Hence, on U, p coincides with $s^* \circ \mathrm{pr}_{\mathfrak{t}}^*(q)$, where $\mathrm{pr}_{\mathfrak{t}} : G \times \mathfrak{t} \to \mathfrak{t}$ denotes the natural projection to the second factor. This shows that p is smooth. Now, we choose a basis $\{\mathrm{e}_a\}$ in $\mathfrak{g}$ and consider the corresponding partial derivatives, given by

$$\frac{\partial p}{\partial A^a}(A) := \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} p(A + t\,\mathrm{e}_a)\,.$$

One can check that the functions

$$\frac{\partial}{\partial A^{a_1}} \cdots \frac{\partial}{\partial A^{a_l}} p$$

are homogeneous of degree $k - l$ for $l \leq k$ and that they vanish for $l > k$ (Exercise 4.6.4). It follows that p coincides with its k-th order Taylor polynomial centered at the origin. Thus, p is polynomial. ■

Now, we are going to determine $\mathrm{Pol}_G(\mathfrak{g})$ for the classical compact Lie groups explicitly. We start with the case $G = \mathrm{U}(n)$. A maximal Abelian subalgebra $\mathfrak{t}_{\mathrm{U}} \subset \mathfrak{u}(n)$ is given by the subalgebra of diagonal matrices. Since every element of $\mathfrak{u}(n)$ is skew-adjoint, it admits an orthonormal eigenbasis. Hence, it is conjugate under the adjoint representation to an element of $\mathfrak{t}_{\mathrm{U}}$. Since the spectrum of an element of $\mathfrak{u}(n)$ is invariant under the adjoint representation, if two elements of $\mathfrak{t}_{\mathrm{U}}$ are conjugate under the adjoint representation, they have the same eigenvalues and hence they differ by a permutation of entries. In particular, the normalizer $\mathrm{N}_{\mathrm{U}(n)}(\mathfrak{t}_{\mathrm{U}})$ acts on $\mathfrak{t}_{\mathrm{U}}$ by permutation of entries. Since every permutation can be represented in this way, the Weyl group W_{U} coincides with S_n. Thus, in the present example we see explicitly that the Weyl group is finite and that every orbit of the adjoint representation intersects a maximal Abelian subalgebra in a Weyl group orbit.

Let $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n]$ denote the algebra of symmetric polynomials with real coefficients in the real variables $x_1, \ldots, x_n$. Since the elements of $\mathfrak{t}_{\mathrm{U}}$ are skew-adjoint, they have purely imaginary entries. Hence, every $p \in \mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n]$ defines a W_{U}-invariant polynomial function p^{U} on $\mathfrak{t}_{\mathrm{U}}$ by[14]

[14] We will see below that the normalization factor 4π will make the cohomology classes obtained via the Weil homomorphism match the Chern classes. In many textbooks, the factor is 2π. This will be explained in Remark 4.6.10.

$$p^{\mathrm{U}}(A) := p\left(\frac{\mathrm{i}}{4\pi}A_{11}, \ldots, \frac{\mathrm{i}}{4\pi}A_{nn}\right), \tag{4.6.14}$$

and the assignment $p \mapsto p^{\mathrm{U}}$ yields an algebra isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n] \cong \mathrm{Pol}_{W_{\mathrm{U}}}(\mathfrak{t}_{\mathrm{U}})$. By Lemma 4.6.9, the W_{U}-invariant polynomial functions p^{U} extend to Ad-invariant polynomial functions on $\mathfrak{u}(n)$, denoted by the same symbol, and the assignment $p \mapsto p^{\mathrm{U}}$ defines an isomorphism of algebras $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n] \cong \mathrm{Pol}_{\mathrm{U}(n)}(\mathfrak{u}(n))$. Clearly,

$$p^{\mathrm{U}}(A) = p(\lambda_1, \ldots, \lambda_n), \tag{4.6.15}$$

where λ_j are the eigenvalues of $\frac{\mathrm{i}}{4\pi}A$, counted with multiplicity.

Since the algebra $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n]$ is generated by the elementary symmetric polynomials $\sigma_0, \ldots, \sigma_n$, the algebra $\mathrm{Pol}_{\mathrm{U}(n)}(\mathfrak{u}(n))$ is generated by the corresponding invariant polynomial functions $\sigma_0^{\mathrm{U}}, \ldots, \sigma_n^{\mathrm{U}}$ defined by (4.6.15). Thus, in order to control the Weil homomorphism for a given principal $\mathrm{U}(n)$-bundle P over M, it suffices to know (the cohomology classes of) the forms $\hat{h}_{\Omega}(\sigma_k^{\mathrm{U}})$ for the curvature form Ω of some connection on P. To compute these classes, we recall that the eigenvalues λ_j of $\frac{\mathrm{i}}{4\pi}A$ are the zeros of the characteristic polynomial

$$\chi_{\frac{\mathrm{i}}{4\pi}A}(\lambda) = \det\left(\lambda \mathbb{1}_n - \frac{\mathrm{i}}{4\pi}A\right).$$

Thus, the characteristic polynomial has the factor decomposition $\prod_{j=1}^{n}(\lambda - \lambda_j)$. Expansion yields

$$\chi_{\frac{\mathrm{i}}{4\pi}A}(\lambda) = \sum_{k=0}^{n}(-1)^k \sigma_k(\lambda_1, \ldots, \lambda_n)\,\lambda^{n-k} = \sum_{k=0}^{n}(-1)^k \sigma_k^{\mathrm{U}}(A)\,\lambda^{n-k}. \tag{4.6.16}$$

On the other hand, one can check that the characteristic polynomial of an arbitrary n-dimensional complex square matrix C satisfies

$$\chi_C(\lambda) = \sum_{k=0}^{n}(-1)^k \operatorname{tr}\left(\bigwedge^k C\right)\lambda^{n-k}, \tag{4.6.17}$$

where $\bigwedge^k C : \bigwedge^k \mathbb{C}^n \to \bigwedge^k \mathbb{C}^n$ denotes the endomorphism induced by C via

$$\left(\bigwedge^k C\right)(\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_k) = (C\mathbf{z}_1) \wedge \cdots \wedge (C\mathbf{z}_k)$$

for all $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \mathbb{C}^n$ (Exercise 4.6.5). Comparing (4.6.16) with (4.6.17), we thus read off

$$\sigma_k^{\mathrm{U}}(A) = \left(\frac{\mathrm{i}}{4\pi}\right)^k \operatorname{tr}\left(\bigwedge^k A\right).$$

One can further check that $\operatorname{tr}\left(\bigwedge^k C\right) = D_k(C)$, where D_k denotes the polynomial function on complex square matrices defined by

$$D_k(C) = \frac{1}{k!}\det\begin{bmatrix} \operatorname{tr}(C) & k-1 & 0 & \cdots & 0 \\ \operatorname{tr}(C^2) & \operatorname{tr}(C) & k-2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \operatorname{tr}(C^{k-1}) & \cdots & \operatorname{tr}(C^2) & \operatorname{tr}(C) & 1 \\ \operatorname{tr}(C^k) & \operatorname{tr}(C^{k-1}) & \cdots & \operatorname{tr}(C^2) & \operatorname{tr}(C) \end{bmatrix}. \tag{4.6.18}$$

Finally, using (4.6.8), we obtain

$$h_\Omega(\sigma_k^{\mathrm{U}}) = 2^k(\sigma_k^{\mathrm{U}})^\wedge(\Omega) = 2^k\left(\frac{\mathrm{i}}{4\pi}\right)^k D_k^\wedge(\Omega)\,,$$

that is,

$$h_\Omega(\sigma_k^{\mathrm{U}}) = \left(\frac{\mathrm{i}}{2\pi}\right)^k D_k^\wedge(\Omega) = D_k^\wedge\left(\frac{\mathrm{i}}{2\pi}\Omega\right). \tag{4.6.19}$$

In particular, we have

$$h_\Omega(\sigma_0^{\mathrm{U}}) = 1\,, \tag{4.6.20}$$

$$h_\Omega(\sigma_1^{\mathrm{U}}) = \frac{\mathrm{i}}{2\pi}\operatorname{tr}(\Omega)\,, \tag{4.6.21}$$

$$h_\Omega(\sigma_2^{\mathrm{U}}) = \frac{1}{8\pi^2}\left(\operatorname{tr}(\Omega\wedge\Omega) - \operatorname{tr}(\Omega)\wedge\operatorname{tr}(\Omega)\right). \tag{4.6.22}$$

Here, $\Omega\wedge\cdots\wedge\Omega$ denotes the exterior product of $\mathfrak{gl}(n,\mathbb{C})$-valued forms induced by the associative matrix product, cf. Remark 1.4.8/1. We obtain the same formulae for $\hat{h}_\Omega(\sigma_k^{\mathrm{U}})$ by viewing the right hand sides as forms on M.

Remark 4.6.10

1. According to (4.6.19), the Weil homomorphism is formally given by plugging in the matrix elements of $\frac{\mathrm{i}\Omega}{2\pi}$ relative to some local frame in $\mathrm{Ad}(P)$, viewed as local 2-forms, into the polynomial given and replacing all products by wedge products. Therefore, it is common to write

$$\sigma_k\left(\frac{\mathrm{i}}{2\pi}\Omega\right) \equiv \hat{h}_\Omega(\sigma_k^{\mathrm{U}})\,,\quad k = 0,\ldots,n\,,$$

   although the actual scaling factor is 4π and not 2π, because of the factor 2^k one acquires by rewriting products as wedge products, cf. (4.6.8).
2. If for the wedge product of a k-form and an l-form on M one uses the convention to multiply by a factor $\frac{1}{(k+l)!}$ instead of $\frac{1}{k!l!}$, as is done for example in [383], the construction of the Weil homomorphism has to be modified as follows.

a. The product of a k-linear form with an l-linear form on $\mathfrak{g}$ is defined with a factor $\frac{1}{(k+l)!}$.
b. Polarization reads $\hat{\xi}(A) = \xi(A, \ldots, A)$.
c. For a polynomial function ξ of order k on $\mathfrak{g}$, the form $h_\alpha(\xi)$ is defined with a factor $\frac{1}{(2k)!}$.
d. The polynomial function p^{U} on $\mathfrak{g}$ induced from $p \in \mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_n]$ is defined by $p(\lambda_1, \ldots, \lambda_n)$ where λ_i are the eigenvalues of $\frac{\mathrm{i}}{2\pi}A$.

Under these modifications, the mapping h_α is a homomorphism for every 2-form α and formulae (4.6.19)–(4.6.22) hold true. ♦

Now, we can compare the cohomology classes $\mathfrak{w}_P(\sigma_k^{\mathrm{U}})$ with the Chern classes of P.

Theorem 4.6.11 *For every principal* $\mathrm{U}(n)$*-bundle* P *over* M *and every* $k = 0, \ldots, n$*, there holds* $\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = \mathsf{c}_k(P)$ *under the de Rham isomorphism.*

Proof Clearly, the assertion holds for $k = 0$, so that we may assume $k \geq 1$. Our proof is along the lines of [451]. We proceed by showing the following.

1. The assertion holds if it holds for all smooth principal $\mathrm{U}(1)$-bundles.
2. The assertion holds for all smooth principal $\mathrm{U}(1)$-bundles if it holds for the complex Hopf bundle.
3. The assertion holds for the complex Hopf bundle.

1. Assume that the assertion holds for all principal $\mathrm{U}(1)$-bundles. First, we use the Splitting Principle to argue that we may restrict attention to bundles P admitting a reduction to $\mathrm{U}(1)^n$. Indeed, given an arbitrary P, let $\rho : P/\mathrm{U}(1)^n \to M$ denote the induced projection. Since $\mathfrak{w}_P(\sigma_k^{\mathrm{U}})$ and c_k are characteristic classes, one has $\mathfrak{w}_{\rho^*P}(\sigma_k^{\mathrm{U}}) = \rho^*\mathfrak{w}_P(\sigma_k^{\mathrm{U}})$ and $\mathsf{c}_k(\rho^*P) = \rho^*\mathsf{c}_k(P)$. By Theorem 4.3.7, then $\mathfrak{w}_{\rho^*P}(\sigma_k^{\mathrm{U}}) = \mathsf{c}_k(\rho^*P)$ implies $\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = \mathsf{c}_k(P)$ and ρ^*P admits a reduction to $\mathrm{U}(1)^n$. This shows that without loss of generality we may assume that P itself admits a reduction to $\mathrm{U}(1)^n$.

Now, if Q is a reduction of P to $\mathrm{U}(1)^n$, then P is vertically isomorphic to $Q^{[j]}$, where $j : \mathrm{U}(1)^n \to \mathrm{U}(n)$ denotes the natural inclusion mapping. By Corollary 4.6.8/3, then

$$\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = \mathfrak{w}_Q\big((\mathrm{d}j)^*\sigma_k^{\mathrm{U}}\big).$$

We have

$$(\mathrm{d}j)^*\sigma_k^{\mathrm{U}} = \sigma_k\big(\mathrm{pr}_1^*\,\sigma_1^{\mathrm{U}}, \ldots, \mathrm{pr}_n^*\,\sigma_1^{\mathrm{U}}\big),$$

where $\mathrm{pr}_i : \mathrm{U}(1)^n \to \mathrm{U}(1)$ denotes projection to the i-th factor and σ_1^{U} on the right hand side is defined on $\mathfrak{u}(1)$. Using that $\mathfrak{w}_Q$ is a homomorphism, we thus obtain

$$\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = \sigma_k\big(\mathfrak{w}_Q(\mathrm{pr}_1^*\,\sigma_1^{\mathrm{U}}), \ldots, \mathfrak{w}_Q(\mathrm{pr}_n^*\,\sigma_1^{\mathrm{U}})\big). \tag{4.6.23}$$

Applying Corollary 4.6.8/3 once again, for $i = 1, \dots, n$, we find

$$\mathfrak{w}_Q(\mathrm{pr}_i^* \, \sigma_1^{\mathrm{U}}) = \mathfrak{w}_{Q^{[\mathrm{pr}_i]}}(\sigma_1^{\mathrm{U}}) \,.$$

Since $Q^{[\mathrm{pr}_i]}$ is a principal $\mathrm{U}(1)$-bundle, by assumption,

$$\mathfrak{w}_{Q^{[\mathrm{pr}_i]}}(\sigma_1^{\mathrm{U}}) = \mathsf{c}_1 \left(Q^{[\mathrm{pr}_i]}\right)$$

under the de Rham isomorphism. Let $f : M \to \mathrm{BU}(1)^n$ be a classifying mapping for Q. By Corollary 3.7.3, then $\mathrm{B\,pr}_i \circ f$ is a classifying mapping for $Q^{[\mathrm{pr}_i]}$. Hence,

$$\mathfrak{w}_{Q^{[\mathrm{pr}_i]}}(\sigma_1^{\mathrm{U}}) = f^* \circ (\mathrm{B\,pr}_i)^* \mathsf{c}_1^{\mathrm{U}(1)} \,.$$

Plugging this into (4.6.23), we obtain

$$\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = f^* \left(\sigma_k \left((\mathrm{B\,pr}_1)^* \mathsf{c}_1^{\mathrm{U}(1)}, \dots, (\mathrm{B\,pr}_n)^* \mathsf{c}_1^{\mathrm{U}(1)}\right)\right) \,.$$

By Proposition 4.3.5, then

$$\mathfrak{w}_P(\sigma_k^{\mathrm{U}}) = f^* \circ (\mathrm{B}j)^* \left(\mathsf{c}_k^{\mathrm{U}(n)}\right) \,.$$

Since $\mathrm{B}j \circ f$ is a classifying mapping for $Q^{[j]}$ and $Q^{[j]}$ is vertically isomorphic to P, we finally obtain the assertion.

2. Let P_n denote the canonical $\mathrm{U}(1)$-bundle over $\mathbb{C}\mathrm{P}^n$ (Stiefel bundle), cf. Remark 1.1.25 and Example 4.2.18. Recall that P_1 is the complex Hopf bundle and that P_n is $(n-1)$-universal, cf. Theorem 3.4.10. Thus, it suffices to prove that if the assertion holds for P_1, then it holds for P_n for all n. Let Ω_n denote the curvature form of the canonical connection on P_n, cf. Example 1.3.20.

The standard embedding of $\mathbb{C}^2$ into $\mathbb{C}^{n+1}$ induces a morphism of principal $\mathrm{U}(1)$-bundles $F : P_1 \to P_n$ covering the standard embedding $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$. Composition of f with the mapping

$$s : \mathrm{D}^2 \to \mathrm{S}^2 \cong \mathbb{C}\mathrm{P}^1 \,, \quad s(z) := \left(z, \sqrt{1 - |z|^2}\right) \,, \tag{4.6.24}$$

yields the 2-cell of the standard cell complex structure of $\mathbb{C}\mathrm{P}^n$, which is obtained by successively attaching to $\mathbb{C}\mathrm{P}^i = \{[(z_0, \cdots, z_i, 0, \cdots, 0)] : \mathbf{z} \in \mathrm{S}^{2i+1}\} \subset \mathbb{C}\mathrm{P}^n$ the $2(i+1)$-cell given by the mapping

$$\mathrm{D}^{2(i+1)} \to \mathbb{C}\mathrm{P}^n \,, \quad \mathbf{z} \mapsto \left[\left(z_0, \cdots, z_i, \sqrt{1 - \|\mathbf{z}\|^2}, 0, \cdots, 0\right)\right] \,,$$

see for example [104, Example IV.8.9]. Thus, the homology class $[f \circ s]$ represents a generator of the singular homology group $H_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$. Under the identification $H^2_{\mathbb{R}}(\mathbb{C}\mathrm{P}^n) \cong \mathrm{Hom}(H_2(\mathbb{C}\mathrm{P}^n), \mathbb{R})$ provided by the Universal Coefficient Theorem, the de Rham isomorphism maps the cohomology class $\mathfrak{w}_{P_n}(\sigma_1^{\mathrm{U}})$ to the homomorphism

$H_2(\mathbb{C}\mathrm{P}^n) \to \mathbb{R}$ which assigns to this generator the value

$$\int_{\mathrm{D}^2} s^* f^* \hat{h}_{\Omega_n}(\sigma_1^{\mathrm{U}}) = \int_{\mathbb{C}\mathrm{P}^1} f^* \hat{h}_{\Omega_n}(\sigma_1^{\mathrm{U}}) \,.$$

Here, we have used that s preserves the orientations (Exercise 4.6.6). Thus, what we have to show is

$$\int_{\mathbb{C}\mathrm{P}^1} f^* \hat{h}_{\Omega_n}(\sigma_1^{\mathrm{U}}) = \langle \mathsf{c}_1(P_n), [f \circ s] \rangle \,, \tag{4.6.25}$$

where $\mathsf{c}_1(P_n)$ stands for the corresponding homomorphism $H_2(\mathbb{C}\mathrm{P}^n) \to \mathbb{R}$. By Lemma 4.6.4/3 and formula (4.6.9), $f^* \hat{h}_{\Omega_n}(\sigma_1^{\mathrm{U}}) = \hat{h}_{F^*\Omega_n}(\sigma_1^{\mathrm{U}})$. Via the vertical isomorphism $P_1 \to f^* P_n$ provided by F, the form $F^*\Omega_n$ corresponds to Ω_1. Hence, for the left hand side of (4.6.25), we may write

$$\int_{\mathbb{C}\mathrm{P}^1} \hat{h}_{\Omega_1}(\sigma_1^{\mathrm{U}}) \,.$$

Since P_1 and $f^* P_n$ are vertically isomorphic, the right hand side of (4.6.25) can be rewritten as $\langle \mathsf{c}_1(P_1), [s] \rangle$. Thus, (4.6.25) holds for all n if

$$\int_{\mathbb{C}\mathrm{P}^1} \hat{h}_{\Omega_1}(\sigma_1^{\mathrm{U}}) = \langle \mathsf{c}_1(P_1), [s] \rangle \,, \tag{4.6.26}$$

that is, if it holds for $n = 1$.

3. It remains to prove (4.6.26). By Remark 4.5.4, evaluation of the right hand side yields

$$\langle \mathsf{c}_1(P_1), [s] \rangle = -1 \,.$$

To compute the left hand side of (4.6.26), we identify $\mathbb{C}\mathrm{P}^1$ with S^2 via the diffeomorphism of Remark 1.1.21/3 and the bundle manifold of P_1 with S^3 via the natural diffeomorphism $\mathrm{S}^3 \to \mathrm{S}_{\mathbb{C}}(2, 1)$ induced from the embedding $\mathrm{S}^3 \to \mathbb{C}^2$. Under these identifications, the bundle projection is given by the mapping

$$\mathrm{S}^3 \to \mathrm{S}^2 \,, \quad \mathbf{z} \mapsto \Big(\mathrm{Re}(2\overline{z_1} z_2), \mathrm{Im}(2\overline{z_1} z_2), \big(|z_1|^2 - |z_2|^2\big)^2 \Big) \tag{4.6.27}$$

and the canonical connection form is given by $\omega_1 = \overline{z_1}\mathrm{d}z_1 + \overline{z_2}\mathrm{d}z_2$, cf. Example 1.3.22. For the curvature, we obtain

$$\Omega_1 = \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_1 + \mathrm{d}\overline{z_2} \wedge \mathrm{d}z_2 \,.$$

Define coordinates ϑ, φ, χ on $\mathrm{S}^3 \subset \mathbb{C}^2$ by

$$z_1 = \mathrm{e}^{\mathrm{i}/2(\chi - \varphi)} \cos\frac{\vartheta}{2} \,, \quad z_2 = \mathrm{e}^{\mathrm{i}/2(\chi + \varphi)} \sin\frac{\vartheta}{2} \quad \text{(Euler angles)}$$

and coordinates θ, ϕ on $\mathrm{S}^2 \subset \mathbb{R}^3$ by

$$x_1 = \sin\theta\cos\phi\,, \quad x_2 = \sin\theta\sin\phi\,, \quad x_3 = \cos\theta \quad \text{(spherical coordinates).}$$

We leave it to the reader to check that, in these coordinates,

(a) the fibres are parameterized by χ,
(b) the bundle projection (4.6.27) is given plainly by $(\vartheta, \varphi, \chi) \mapsto (\theta, \phi) \equiv (\vartheta, \varphi)$,
(c) $\hat{h}_{\Omega_1}\big(\sigma_1^{\mathrm{U}}\big) = -\frac{1}{4\pi}\sin\vartheta\, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$.

Points (b) and (c) yield

$$\hat{h}_{\Omega_1}\big(\sigma_1^{\mathrm{U}}\big) = -\frac{1}{4\pi}\sin\vartheta\, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -\frac{1}{4\pi}\mathsf{v}_{\mathrm{S}^2}$$

with the natural volume form $\mathsf{v}_{\mathrm{S}^2}$ on S^2. Thus,

$$\int_{\mathbb{C}\mathrm{P}^1} \hat{h}_{\Omega_1}(\sigma_1^{\mathrm{U}}) = -\frac{1}{4\pi}\int_{\mathrm{S}^2} \mathsf{v}_{\mathrm{S}^2} = -1\,.$$

This proves (4.6.26) and thus completes the proof of the theorem. ■

Corollary 4.6.12 *Let P be a principal* $\mathrm{U}(n)$*-bundle over a manifold M. The Chern indices $\mathsf{c}_{k,i}(P)$ of P relative to a chosen set of generators $\{s_i\}$ of $H_{2k}(M)$ are given by*

$$\mathsf{c}_{k,i}(P) = \int_{s_i} \mathfrak{w}_P(\sigma_k^{\mathrm{U}})\,.$$

Proof We have the commutative diagram

$$\begin{tikzcd}
H^{2k}_{\mathbb{Z}}(M) \arrow[r] \arrow[d] & H^{2k}_{\mathbb{Z}}(M) \otimes \mathbb{R} \arrow[r, "\cong"] & H^{2k}_{\mathbb{R}}(M) \arrow[d, "\cong"] \\
\mathrm{Hom}\big(H_{2k}(M), \mathbb{Z}\big) \arrow[rr] & & \mathrm{Hom}\big(H_{2k}(M), \mathbb{R}\big)
\end{tikzcd}$$

where the first upper horizontal arrow sends $[\alpha]$ to $[\alpha] \otimes 1$, the lower horizontal arrow is defined by composition with the natural inclusion mapping $\mathbb{Z} \subset \mathbb{R}$ and the vertical arrows are given by the natural homomorpisms. Theorem 4.6.11 implies that integration of $\mathfrak{w}_P(\sigma_k^{\mathrm{U}})$ over a closed $2k$-cycle s in M yields the same result as evaluation of $\mathsf{c}_k(P)$, viewed via the homomorphism $H^{2k}_{\mathbb{Z}}(M) \to H^{2k}_{\mathbb{R}}(M) \cong \mathrm{Hom}(H_{2k}(M), \mathbb{R})$ as a homomorphism $H_{2k}(M) \to \mathbb{R}$, on the homology class $[s]$. According to the diagram, this is the same as evaluating the homomorphism $H_{2k}(M) \to \mathbb{Z}$ defined by $\mathsf{c}_k(P)$ via the left vertical arrow on $[s]$. ■

Next, we discuss the groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$. We start with $\mathrm{O}(n)$. If $n = 2l$, a maximal Abelian subalgebra $\mathfrak{t}_{\mathrm{o}}$ is given by the block diagonal matrices with blocks

$$\begin{bmatrix} 0 & x_i \\ -x_i & 0 \end{bmatrix}, \quad i = 1, \ldots, l\,.$$

The Weyl group W_{o} is generated by the permutations of the blocks and by the operations of taking the transpose of individual blocks. Hence, every $p \in \mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_l]$ defines a W_{o}-invariant polynomial function p^{o} on $\mathfrak{t}_{\mathrm{o}}$ by

$$p^{\mathrm{o}}\left(\mathrm{diag}\left(\begin{bmatrix} 0 & x_1 \\ -x_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & x_l \\ -x_l & 0 \end{bmatrix}\right)\right) := p\left(\left(\frac{x_1}{4\pi}\right)^2, \ldots, \left(\frac{x_l}{4\pi}\right)^2\right),$$

and the assignment $p \mapsto p^{\mathrm{o}}$ defines an isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_l] \cong \mathrm{Pol}_{W_{\mathrm{o}}}(\mathfrak{t}_{\mathrm{o}})$. By analogy with the case of $\mathrm{U}(n)$, the W_{o}-invariant polynomial function p^{o} extends to an Ad-invariant polynomial function on $\mathfrak{o}(n)$, denoted by the same symbol and given by

$$p^{\mathrm{o}}(A) := p\,(x_1, \ldots, x_l)\;, \tag{4.6.28}$$

where $\mathrm{i}x_1, -\mathrm{i}x_1, \ldots, \mathrm{i}x_l, -\mathrm{i}x_l$ are the eigenvalues of $\frac{1}{4\pi}A$, counted with multiplicities.[15] The assignment $p \mapsto p^{\mathrm{o}}$ defines an algebra isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_l] \cong \mathrm{Pol}_{\mathrm{O}(n)}(\mathfrak{o}(n))$. Consequently, $\mathrm{Pol}_{\mathrm{O}(n)}(\mathfrak{o}(n))$ is generated by $\sigma_0^{\mathrm{o}}, \ldots, \sigma_l^{\mathrm{o}}$.

In case $n = 2l+1$, the induced homomorphism $\mathrm{d}j^{\mathrm{o}}_{2l,2l+1} : \mathfrak{o}(2l) \to \mathfrak{o}(2l+1)$ embeds the maximal Abelian subalgebra $\mathfrak{t}_{\mathrm{o}}$ of $\mathfrak{o}(2l)$ into $\mathfrak{o}(2l+1)$ as a maximal Abelian subalgebra of $\mathfrak{o}(2l+1)$. Moreover, it translates the Weyl group actions into one another. As a consequence, pullback by $\mathrm{d}j^{\mathrm{o}}_{2l,2l+1}$ defines an algebra isomorphism $\mathrm{Pol}_{\mathrm{O}(2l+1)}\mathfrak{o}(2l+1) \cong \mathrm{Pol}_{\mathrm{O}(2l)}\mathfrak{o}(2l)$. Via this isomorphism and the construction for $\mathrm{O}(2l)$, each $p \in \mathrm{Sym}_{\mathbb{R}}[x_1, \ldots, x_l]$ defines an element p^{o} of $\mathrm{Pol}_{\mathrm{O}(2l+1)}\mathfrak{o}(2l+1)$. As for $n = 2l$, the assignment $p \mapsto p^{\mathrm{o}}$ defines an algebra isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \ldots x_l] \cong \mathrm{Pol}_{\mathrm{O}(2l+1)}\mathfrak{o}(2l+1)$ and thus $\mathrm{Pol}_{\mathrm{O}(2l+1)}\mathfrak{o}(2l+1)$ is generated by $\sigma_0^{\mathrm{o}}, \ldots, \sigma_l^{\mathrm{o}}$.

Now, consider the group $\mathrm{SO}(n)$. Since $\mathfrak{so}(n) = \mathfrak{o}(n)$, we may use the same maximal Abelian subalgebra, $\mathfrak{t}_{\mathrm{o}}$. In case $n = 2l+1$, the group $\mathrm{O}(2l+1)$ is generated by its center and $\mathrm{SO}(2l+1)$. Hence, in this case, there is no difference in the adjoint actions of $\mathrm{O}(2l+1)$ and $\mathrm{SO}(2l+1)$ and thus there is no difference in the Weyl groups, $W_{\mathrm{so}} = W_{\mathrm{o}}$. Therefore,

$$\mathrm{Pol}_{\mathrm{SO}(2l+1)}\mathfrak{so}(2l+1) = \mathrm{Pol}_{\mathrm{O}(2l+1)}\mathfrak{o}(2l+1)\,.$$

In case $n = 2l$, however, W_{so} is generated by the permutations of the blocks and by the operations of simultaneously taking the transpose of two distinct blocks. Therefore, $W_{\mathrm{so}} \subset W_{\mathrm{o}}$ and hence $\mathrm{Pol}_{W_{\mathrm{o}}}(\mathfrak{t}_{\mathrm{o}})$ is a subalgebra of $\mathrm{Pol}_{W_{\mathrm{so}}}(\mathfrak{t}_{\mathrm{o}})$ and $\mathrm{Pol}_{\mathrm{O}(2l)}\big(\mathfrak{o}(2l)\big)$ is a subalgebra of $\mathrm{Pol}_{\mathrm{SO}(2l)}\big(\mathfrak{so}(2l)\big)$. As a matter of fact, $\mathrm{Pol}_{W_{\mathrm{so}}}(\mathfrak{t}_{\mathrm{o}})$ is generated by the subalgebra $\mathrm{Pol}_{W_{\mathrm{o}}}(\mathfrak{t}_{\mathrm{o}})$ and the polynomial function

$$\varepsilon\left(\mathrm{diag}\left(\begin{bmatrix} 0 & x_1 \\ -x_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & x_l \\ -x_l & 0 \end{bmatrix}\right)\right) := \frac{x_1}{4\pi} \cdots \frac{x_l}{4\pi}\,, \tag{4.6.29}$$

[15] Since A is real and the eigenvalues are purely imaginary, they come in conjugate pairs.

which is W_{SO}-invariant but not W_{O}-invariant. Hence, $\mathrm{Pol}_{\mathrm{SO}(2l)}\big(\mathfrak{so}(2l)\big)$ is generated by $\sigma_0^{\mathrm{O}}, \dots, \sigma_l^{\mathrm{O}}$ and the Ad-invariant extension of (4.6.29), which we denote by the same symbol. Thus, for $A \in \mathfrak{so}(2l)$, $\varepsilon(A)$ is given by the right hand side of (4.6.29), where A is conjugate under $\mathrm{SO}(n)$ to the block diagonal matrix on the left hand side of this equation. Note that the overall sign of the product of the x_i is fixed by requiring conjugacy under $\mathrm{SO}(n)$ rather than $\mathrm{O}(n)$. Note further that ε is related to the Pfaffian[16] $\mathrm{pf} : \mathfrak{so}(2l) \to \mathbb{R}$ by

$$\varepsilon(A) = \mathrm{pf}\left(\frac{A}{4\pi}\right) = \frac{\mathrm{pf}(A)}{(4\pi)^l}\,.$$

Let us compare σ_k^{O} with the pullback of the functions σ_k^{U} under the Lie algebra embedding $\mathrm{d}j_n^{\mathrm{O,U}} : \mathfrak{o}(n) \to \mathfrak{u}(n)$ induced by $j_n^{\mathrm{O,U}}$. Recall that q_n and $\bar{q}_n$ denote the integer part of $\frac{n-1}{2}$ and $\frac{n}{2}$, respectively.

Lemma 4.6.13 *For $k = 0, \dots, \bar{q}_n$, one has*

$$(\mathrm{d}j_n^{\mathrm{O,U}})^* \sigma_{2k+1}^{\mathrm{U}} = 0\,, \quad (\mathrm{d}j_n^{\mathrm{O,U}})^* \sigma_{2k}^{\mathrm{U}} = (-1)^k\, \sigma_k^{\mathrm{O}}\,.$$

Proof It suffices to consider the case $n = 2l$, because the case $n = 2l + 1$ follows by the identity $j_{2l,2l+1}^{\mathrm{U}} \circ j_{2l}^{\mathrm{O,U}} = j_{2l+1}^{\mathrm{O,U}} \circ j_{2l,2l+1}^{\mathrm{O}}$. Let $A \in \mathfrak{o}(2l)$ and assume that A has eigenvalues $\mathrm{i}x_1, -\mathrm{i}x_1, \dots, \mathrm{i}x_l, -\mathrm{i}x_l$. Then,

$$\big((\mathrm{d}j_n^{\mathrm{O,U}})^* \sigma_k^{\mathrm{U}}\big)(A) = \sigma_k^{\mathrm{U}}(A) = \frac{1}{(4\pi)^k}\sigma_k(-x_1, x_1, \dots, -x_l, x_l)\,.$$

To the sum $\sigma_k(-x_1, x_1, \dots, -x_l, x_l)$, only the terms containing all x_i in even order contribute, because all other terms appear pairwise with opposite signs. Hence, $(\mathrm{d}j_n^{\mathrm{O,U}})^* \sigma_{2k+1}^{\mathrm{U}}(A) = 0$ and

$$(\mathrm{d}j_n^{\mathrm{O,U}})^* \sigma_{2k}^{\mathrm{U}}(A) = \frac{1}{(4\pi)^{2k}}\sigma_k(-x_1^2, \dots, -x_l^2) = (-1)^k \sigma_k\left(\left(\frac{x_1}{4\pi}\right)^2, \dots, \left(\frac{x_l}{4\pi}\right)^2\right)\,.$$

The right hand side coincides with $(-1)^k \sigma_k^{\mathrm{O}}(A)$. ∎

Theorem 4.6.14 *Under the de Rham isomorphism, for every principal bundle P with structure group* $\mathrm{O}(n)$ *or* $\mathrm{SO}(n)$*, one has*

$$\mathfrak{w}_P(\sigma_k^{\mathrm{O}}) = \mathsf{p}_k(P)\,, \quad k = 0, \dots, \bar{q}_n\,.$$

If n is even and the structure group is $\mathrm{SO}(n)$*, in addition, one has,*

$$\mathfrak{w}_P\,(\varepsilon) = \mathsf{e}(P)\,.$$

[16] By definition, $\mathrm{pf}(A) = \frac{1}{2^l l!}\sum_{\sigma \in \mathrm{S}_{2l}} \prod_{i=1}^l A_{\sigma(2i-1),\sigma(2i)}$.

Proof Denote $j = j_n^{\mathrm{O,U}}$ or $j_n^{\mathrm{SO,U}}$, respectively. By Theorem 4.6.11, under the de Rham isomorphism,

$$\mathsf{p}_k(P) = (-1)^k \mathsf{c}_{2k}\left(P^{[j]}\right) = (-1)^k \mathfrak{w}_{P^{[j]}}(\sigma_{2k}^{\mathrm{U}}) .$$

By Corollary 4.6.8/3, the right hand side coincides with $(-1)^k \mathfrak{w}_P\big((\mathrm{d}j)^* \sigma_{2k}^{\mathrm{U}}\big)$. According to Lemma 4.6.13, this equals $\mathfrak{w}_P(\sigma_k^{\mathrm{O}})$.

For the assertion about the Euler class in the case where the structure group is $\mathrm{SO}(2l)$, one may give an argument which is essentially analogous to that for the Chern classes in the proof of Theorem 4.6.11. Let us sketch this.

By Theorem 3.4.10, it suffices to prove the assertion under the assumption that P is a Stiefel bundle. Thus, take $P = \mathrm{S}_{\mathbb{R}}(2l, m)$ and $M = \mathrm{G}_{\mathbb{R}}(2l, m)$ for some m. By embedding $\mathrm{U}(l)$ via $j_l^{\mathrm{U,SO}}$ into $\mathrm{SO}(2l)$, we can form the quotient manifold $P/\mathrm{U}(l)$. It has the structure of a locally trivial fibre bundle over M with typical fibre $\mathrm{SO}(2l)/\mathrm{U}(l)$. Let

$$f : P/\mathrm{U}(l) \to M \tag{4.6.30}$$

be the induced projection. One can show that the homomorphism induced in cohomology with real coefficients, $f^* : H_{\mathbb{R}}^*(M) \to H_{\mathbb{R}}^*(P/\mathrm{U}(l))$, is injective. For example, according to Theorem 4.2 and Lemma 4.5 in [452], this follows from the fact that the Serre spectral sequence of the fibre bundle (4.6.30) collapses which, in turn, is due to the fact that the cohomology with real coefficients of both the base $M = \mathrm{G}_{\mathbb{R}}(2l, m)$ and the fibre $\mathrm{SO}(2l)/\mathrm{U}(l)$ vanish in odd degree, see [90, 621] and [452, Theorem 6.11], respectively. Thus, it suffices to prove the assertion for the principal $\mathrm{SO}(2l)$-bundle f^*P over $P/\mathrm{U}(l)$. Since this bundle admits a reduction to the subgroup $\mathrm{U}(l)$, we conclude that it suffices to prove the assertion for all principal $\mathrm{SO}(2l)$-bundles which admit a reduction to the subgroup $\mathrm{U}(l)$.

Thus, denote $\iota = j_l^{\mathrm{U,SO}}$ and let P be a principal $\mathrm{SO}(2l)$-bundle such that $P = Q^{[\iota]}$ for some principal $\mathrm{U}(l)$-bundle Q. By Corollary 4.6.8/3,

$$\mathfrak{w}_P(\varepsilon) = \mathfrak{w}_Q\big((\mathrm{d}\iota)^* \varepsilon\big) .$$

Using (A.6), for $x_1, \ldots, x_l \in \mathbb{R}$ we compute

$$\begin{aligned}
(\mathrm{d}\iota)^* \varepsilon\big(\mathrm{diag}(\mathrm{i}x_1, \ldots, \mathrm{i}x_l)\big) &= \varepsilon\left(\mathrm{diag}\left(\begin{bmatrix} 0 & -x_1 \\ x_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -x_l \\ x_l & 0 \end{bmatrix}\right)\right) \\
&= \frac{1}{(4\pi)^l}(-x_1)\cdots(-x_l) \\
&= \sigma_l^{\mathrm{U}}\big(\mathrm{diag}(\mathrm{i}x_1, \ldots, \mathrm{i}x_l)\big) .
\end{aligned}$$

It follows that $\mathfrak{w}_P(\varepsilon) = \mathfrak{w}_Q(\sigma_l^{\mathrm{U}})$. By Theorem 4.6.11, $\mathfrak{w}_Q(\sigma_l^{\mathrm{U}}) = \mathsf{c}_l(Q)$. By Proposition 3.7.2/1 and formula (4.4.8), the latter equals $\mathsf{e}(P)$. ■

By analogy with Corollary 4.6.12, from Theorem 4.6.14, we obtain the following.

Corollary 4.6.15 *Let P be a principal* $\mathrm{O}(n)$*-bundle over a manifold M. The Pontryagin indices $\mathfrak{p}_{k,i}(P)$ of P relative to a chosen set of generators $\{s_i\}$ of $H_{4k}(M)$ are given by*

$$\mathfrak{p}_{k,i}(P) = \int_{s_i} \mathfrak{w}_P(\sigma_k^{\mathrm{O}}) .$$

■

Finally, let us discuss the case of $G = \mathrm{Sp}(n)$. Elements of $\mathfrak{sp}(n)$ are skew-adjoint quaternionic matrices of dimension n, where taking the adjoint means taking the transpose of the matrix and the quaternionic conjugate of every entry. Hence, the entries of diagonal elements of $\mathfrak{sp}(n)$ are linear combinations of the quaternionic units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. To obtain a maximal Abelian subalgebra $\mathfrak{t}_{\mathrm{Sp}}$ we have to stick to one of these. Let us choose

$$\mathfrak{t}_{\mathrm{Sp}} = \left\{ \mathrm{diag}(x_1\mathbf{i}, \dots, x_n\mathbf{i}) : x_1, \dots, x_n \in \mathbb{R} \right\} .$$

The Weyl group W_{Sp} is generated by the permutations of the entries and the operations of taking the quaternionic conjugate of an individual entry, which amounts to multiplying one of the x_i by -1. Hence, every $p \in \mathrm{Sym}_{\mathbb{R}}[x_1, \dots, x_n]$ defines an element p^{Sp} of $\mathrm{Pol}_{W_{\mathrm{Sp}}} \mathfrak{t}_{\mathrm{Sp}}$ by

$$p^{\mathrm{Sp}}\big(\mathrm{diag}(x_1\mathbf{i}, \dots, x_n\mathbf{i})\big) = p\left(\left(\frac{x_1}{4\pi}\right)^2, \dots, \left(\frac{x_n}{4\pi}\right)^2\right) ,$$

and the assignment $p \mapsto p^{\mathrm{Sp}}$ yields an isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \dots, x_n] \cong \mathrm{Pol}_{W_{\mathrm{Sp}}}(\mathfrak{t}_{\mathrm{Sp}})$. By Ad-invariant extension, we obtain an element p^{Sp} of $\mathrm{Pol}_{\mathrm{Sp}(n)}(\mathfrak{sp}(n))$, and the assignment $p \mapsto p^{\mathrm{Sp}}$ defines an isomorphism $\mathrm{Sym}_{\mathbb{R}}[x_1, \dots, x_n] \cong \mathrm{Pol}_{\mathrm{Sp}(n)}(\mathfrak{sp}(n))$. As a consequence, $\mathrm{Pol}_{\mathrm{Sp}(n)}(\mathfrak{sp}(n))$ is generated by $\sigma_0^{\mathrm{Sp}}, \dots, \sigma_n^{\mathrm{Sp}}$. By the same argument as for $\mathrm{O}(n)$ in the proof of Lemma 4.6.13, we obtain

$$(\mathrm{d}j_n^{\mathrm{Sp},\mathrm{U}})^* \sigma_{2k+1}^{\mathrm{U}} = 0 , \quad (\mathrm{d}j_n^{\mathrm{Sp},\mathrm{U}})^* \sigma_{2k}^{\mathrm{U}} = (-1)^k \sigma_k^{\mathrm{Sp}} .$$

A similar calculation as in the proof of Theorem 4.6.14 then yields the following.

Theorem 4.6.16 *For every principal* $\mathrm{Sp}(n)$*-bundle P and every $k = 0, \dots, n$, one has $\mathfrak{w}_P(\sigma_k^{\mathrm{Sp}}) = \mathsf{p}_k(P)$ under the de Rham isomorphism.* ■

By analogy with Corollary 4.6.12, Theorem 4.6.16 implies the following.

Corollary 4.6.17 *Let P be a principal* $\mathrm{Sp}(n)$*-bundle over a manifold M. The Pontryagin indices $\mathfrak{p}_{k,i}(P)$ of P relative to a chosen set of generators $\{s_i\}$ of $H_{4k}(M)$ are given by*

$$\mathfrak{p}_{k,i}(P) = \int_{s_i} \mathfrak{w}_P(\sigma_k^{\mathrm{Sp}}) .$$

■

To conclude this section, we carry over the above concepts to vector bundles. The Weil homomorphism for principal bundles induces a Weil homomorphism for vector bundles as follows. Given a $\mathbb{K}$-vector bundle E of rank n over M, choose a fibre metric on E and let $O(E)$ denote the corresponding orthonormal frame bundle. This is a principal bundle with structure group $G = \mathrm{O}(n)$ in case $\mathbb{K} = \mathbb{R}$, $G = \mathrm{U}(n)$ in case $\mathbb{K} = \mathbb{C}$ and $G = \mathrm{Sp}(n)$ in case $\mathbb{K} = \mathbb{H}$. Thus, we can define

$$\mathfrak{w}_E := \mathfrak{w}_{O(E)} : \mathrm{Sym}_G\big(\mathfrak{g}\big) \to H^*_{\mathrm{dR}}(M)\,.$$

If $\mathbb{K} = \mathbb{R}$ and E is orientable, $O(E)$ admits a reduction $O_+(E)$ to the subgroup $G = \mathrm{SO}(n)$. In this case, we can define the oriented Weil homomorphism of E by

$$\tilde{w}_E := \mathfrak{w}_{O_+(E)} : \mathrm{Sym}_{\mathrm{SO}(n)}\big(\mathfrak{so}(n)\big) \to H^*_{\mathrm{dR}}(M)\,.$$

Now, Theorems 4.6.11, 4.6.14 and 4.6.16 imply that under the de Rham isomorphism, for $\mathbb{K} = \mathbb{C}, \mathbb{R}, \mathbb{H}$, one has

- $\mathfrak{w}_E(\sigma_k^{\mathrm{U}}) = \mathsf{c}_k(E)$ for $k = 0, \ldots, n$,
- $\mathfrak{w}_E(\sigma_k^{\mathrm{O}}) = \mathsf{p}_k(E)$ for $k = 0, \ldots, \bar{q}_n$,
- $\mathfrak{w}_E(\sigma_k^{\mathrm{Sp}}) = \mathsf{p}_k(E)$ for $k = 0, \ldots, n$.

Moreover, Theorem 4.6.14 implies that in case $\mathbb{K} = \mathbb{R}$, if E is orientable and has even rank n, one has, in addition

$$\tilde{w}_E(\varepsilon) = \mathsf{e}(E)\,. \tag{4.6.31}$$

More generally, a part of the construction of the Weil homomorphism carries over to vector bundles. This allows for defining genuine characteristic classes of vector bundles, notably the twisted Chern character of a graded vector bundle and the relative Chern character of a graded Dirac bundle, to be discussed in Sect. 5.8. For a real vector bundle E, let $\mathrm{Pol}(E)$ and $\mathrm{Sym}(E)$ denote the algebra bundles whose fibres over $m \in M$ are, respectively, the algebra of real polynomial functions on E_m and the algebra generated by the real symmetric multilinear forms on E_m. Fibrewise polarization and multilinearization define mutually inverse vertical algebra bundle isomorphisms

$$\hat{\ } : \mathrm{Sym}(E) \to \mathrm{Pol}(E) \quad \text{and} \quad \check{\ } : \mathrm{Pol}(E) \to \mathrm{Sym}(E)\,. \tag{4.6.32}$$

The spaces of sections in $\mathrm{Pol}(E)$ and $\mathrm{Sym}(E)$ form real algebras with respect to pointwise multiplication

$$(f \cdot g)(m) := f(m) \cdot g(m)\,,$$

where on the right hand side the product is taken in the corresponding fibre over m, that is, in $\mathrm{Pol}(E_m)$ and in $\mathrm{Sym}(E_m)$, respectively. The vertical isomorphisms

(4.6.32) defined by fibrewise polarization and multilinearization induce mutually inverse algebra isomorphisms

$$\hat{}\,:\varGamma^\infty\big(\mathrm{Sym}(E)\big)\to\varGamma^\infty\big(\mathrm{Pol}(E)\big)\quad\text{and}\quad\check{}\,:\varGamma^\infty\big(\mathrm{Pol}(E)\big)\to\varGamma^\infty\big(\mathrm{Sym}(E)\big)\,.$$

Given $\alpha\in\varOmega^2(M,E)$, we can define a mapping

$$h_\alpha:\varGamma^\infty\big(\mathrm{Pol}(E)\big)\to\varOmega^*(M)$$

by assigning to $\kappa\in\varGamma^\infty\big(\mathrm{Pol}^k(E)\big)$ the $2k$-form

$$\begin{aligned}\big(h_\alpha(\kappa)\big)&(X_1,\dots,X_{2k})\\ &:=\frac{1}{k!}\sum_{\pi\in\mathrm{S}_{2k}}\mathrm{sign}(\pi)\check{\kappa}\big(\alpha(X_{\pi(1)},X_{\pi(2)}),\dots,\alpha(X_{\pi(2k-1)},X_{\pi(2k)})\big)\,.\end{aligned}\tag{4.6.33}$$

Note that the construction directly produces forms on M, so there is no need to project here.

Lemma 4.6.18

1. *The mapping* (4.6.33) *is a homomorphism of algebras.*
2. *Let* $\varPhi:E_1\to E_2$ *be a vertical vector bundle morphism and let* $\alpha\in\varOmega^2(M,E_1)$ *and* $q\in\mathrm{Pol}(E_2)$. *Then,* $\varPhi\circ\alpha\in\varOmega^2(E_2)$, $q\circ\varPhi\in\mathrm{Pol}(E_1)$ *and*

$$h_{\varPhi\circ\alpha}(q)=h_\alpha(q\circ\varPhi).$$

Proof Exercise 4.6.7. ∎

Generally, the forms $h_\alpha(f)$ need not be closed and hence they need not represent cohomology classes. Thus, if one wants to construct a characteristic class this way, one has to ensure closedness separately. Let us discuss a special situation where closedness is granted. In what follows, for a complex Hermitean vector bundle E, let $\mathfrak{u}(E)\subset\mathrm{End}(E)$ denote the vertical subbundle of skew-adjoint endomorphisms. Let P be a principal bundle over M with compact structure group G and let σ be a unitary representation of G on a finite-dimensional complex Hilbert space V. Then, $P\times_G V$ is a complex Hermitean vector bundle over M. We apply the construction of forms $h_\alpha(\kappa)$ just explained to the real vector bundle $E=\mathfrak{u}(P\times_G V)$. Recall from Remark 1.2.9/2 that $\mathfrak{u}(P\times_G V)$ is naturally vertically isomorphic to $P\times_G\mathfrak{u}(V)$, where G acts on $\mathfrak{u}(V)$ via the induced representation

$$(a,A)\mapsto\sigma(a)\circ A\circ\sigma(a^{-1})\,.\tag{4.6.34}$$

The isomorphism is given by

$$\varPhi:P\times_G\mathfrak{u}(V)\to\mathfrak{u}(P\times_G V)\,,\quad\varPhi([(p,A)]):=\iota_p\circ A\circ\iota_p^{-1}\,.\tag{4.6.35}$$

Let $\mathrm{Pol}_G\big(\mathfrak{u}(V)\big) \subset \mathrm{Pol}\big(\mathfrak{u}(V)\big)$ and $\mathrm{Sym}_G\big(\mathfrak{u}(V)\big) \subset \mathrm{Sym}\big(\mathfrak{u}(V)\big)$ denote the subalgebras consisting of elements invariant under the induced representation (4.6.34). Under the identification (4.6.35), every $\kappa \in \mathrm{Pol}^k_G\big(\mathfrak{u}(V)\big)$ defines a section $\underline{\kappa}$ in $\mathrm{Pol}^k\big(\mathfrak{u}(P \times_G V)\big)$ by

$$\underline{\kappa}_m\big(\Phi([p, A])\big) := \kappa(A)\,, \quad m \in M\,, \tag{4.6.36}$$

where p is some point in P_m. This extends to an algebra homomorphism

$$\mathrm{Pol}_G\big(\mathfrak{u}(V)\big) \to \Gamma^\infty\left(\mathrm{Pol}\big(\mathfrak{u}(P \times_G V)\big)\right)\,.$$

Lemma 4.6.19 *Let P be a principal G-bundle over M and let (V, σ) be a unitary representation of G. Let Ω be the curvature of a connection on P and let $\mathsf{R} \in \Omega^2\big(M, \mathfrak{u}(P \times_G V)\big)$ be the curvature endomorphism form of the corresponding connection induced on $P \times_G V$. Then,*

$$h_{\mathsf{R}}(\underline{\kappa}) = \hat{h}_\Omega\big((\mathrm{d}\sigma)^*\kappa\big) \tag{4.6.37}$$

for all $\kappa \in \mathrm{Pol}_G\big(\mathfrak{u}(V)\big)$. In particular, $\mathrm{d}h_{\mathsf{R}}(\underline{\kappa}) = 0$.

Proof Since σ is unitary, the induced representation $\mathrm{d}\sigma$ takes values in $\mathfrak{u}(V)$. Moreover, $\mathrm{d}\sigma$ is equivariant with respect to the adjoint representation of G on $\mathfrak{g}$ and the representation (4.6.34). Hence, if $\kappa \in \mathrm{Pol}_G\big(\mathfrak{u}(V)\big)$, then $(\mathrm{d}\sigma)^*\kappa \in \mathrm{Pol}_G(\mathfrak{g})$, so that $h_\Omega\big((\mathrm{d}\sigma)^*\kappa\big)$ is well defined. For the proof, we assume that κ is homogeneous of degree k. By definition of R, cf. (1.5.13), for $p \in P$ and $X_1, X_2 \in \mathrm{T}_pP$, one has

$$\mathsf{R}_{\pi(p)}(\pi' X_1, \pi' X_2) = \Phi\big([p, \sigma'(\Omega)_p(X_1, X_2)]\big)\,.$$

Hence, for $m \in M$, $p \in P_m$, $Y_1, \dots, Y_{2k} \in \mathrm{T}_mM$ and $X_1, \dots, X_{2k} \in \mathrm{T}_pP$ such that $\pi' X_i = Y_i$, we find

$$\big(h_{\mathsf{R}}(\underline{\kappa})\big)_m(Y_1, \dots, Y_{2k}) = \big(h_\Omega(\kappa)\big)_p(X_1, \dots, X_{2k})\,.$$

By plugging in $h_\Omega(\kappa) = \pi^*\hat{h}_\Omega(\kappa)$, we obtain the assertion. ■

As a result, in the present situation, the assignment $\kappa \mapsto h_{\mathsf{R}}(\underline{\kappa})$ defines a homomorphism

$$\mathrm{Pol}_G\big(\mathfrak{u}(V)\big) \to H^*_{\mathrm{dR}}(M)\,.$$

Now, consider the special situation where E is a Hermitean $\mathbb{K}$-vector bundle of rank n over M. For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, let G denote, respectively, the Lie group $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ and let $\mathfrak{g}$ denote the corresponding Lie algebra. Recall that we have a natural vertical isomorphism $E \cong O(E) \times_G \mathbb{K}^n$ and hence a vertical isomorphism

$$\mathfrak{u}(E) \cong \mathfrak{u}\big(O(E) \times_G \mathbb{K}^n\big)\,. \tag{4.6.38}$$

Corollary 4.6.20 *Let E be a $\mathbb{K}$-vector bundle of rank n over M endowed with a positive definite fibre metric and let R be the curvature endomorphism form of a compatible connection ∇. If a section κ in* $\mathrm{Pol}(\mathfrak{u}(E))$ *corresponds under the natural vertical isomorphism* (4.6.38) *to the section $\underline{\kappa}'$ defined by some $\kappa' \in \mathrm{Pol}_G(\mathfrak{g})$, then $h_{\mathsf{R}}(\kappa)$ is closed and represents $\mathfrak{w}_E(\kappa')$.*

Proof Denote the vertical isomorphism $E \to O(E) \times_G \mathbb{K}^n$ by Ψ. That κ corresponds to $\underline{\kappa}'$ under Ψ means

$$\kappa_m(A) = \underline{\kappa}'_m(\Psi_m \circ A \circ \Psi_m^{-1}), \quad m \in M, \quad A \in \big(\mathfrak{u}(E)\big)_m. \tag{4.6.39}$$

Via Ψ, the connection ∇ on E induces a connection ∇' in $O(E) \times_G \mathbb{K}^n$. One has

$$\nabla'_X s' = \Psi \circ \big(\nabla_X(\Psi^{-1} \circ s')\big), \quad s' \in \Gamma^\infty\big(O(E) \times_G \mathbb{K}^n\big), \quad X \in \mathfrak{X}(M).$$

For the curvature endomorphism form R' of ∇', this implies

$$\mathsf{R}'(X, Y) = \Psi \circ \mathsf{R}(X, Y) \circ \Psi^{-1}, \quad X, Y \in \mathfrak{X}(M). \tag{4.6.40}$$

In turn, ∇' corresponds to a connection on $O(E)$ with curvature form Ω'. By Lemma 4.6.19, we have $h_{\mathsf{R}'}(\underline{\kappa}') = \hat{h}_{\Omega'}(\kappa')$, hence $h_{\mathsf{R}'}(\underline{\kappa}')$ represents $\mathfrak{w}_E(\kappa')$. On the other hand, (4.6.39) and (4.6.40) imply $h_{\mathsf{R}'}(\underline{\kappa}') = h_{\mathsf{R}}(\kappa)$. ■

Remark 4.6.21 For a given complex vector bundle E of rank n and some chosen Hermitean fibre metric on E, consider the section σ_k^E in $\mathrm{Pol}\big(\mathfrak{u}(E)\big)$ given by

$$(\sigma_k^E)_m(A) := \sigma_k\left(\frac{\mathrm{i}\lambda_1}{4\pi}, \ldots, \frac{\mathrm{i}\lambda_n}{4\pi}\right), \quad A \in \mathfrak{u}(E_m), \quad m \in M,$$

where λ_i are the eigenvalues of A, counted with multiplicities. Under the isomorphism (4.6.38), σ_k^E corresponds to the section $\underline{\sigma_k}^{\mathrm{U}}$ induced by the Ad-invariant polynomial σ_k^{U} on $\mathfrak{u}(n)$ defined by (4.6.15). Hence, Theorem 4.6.11 and Corollary 4.6.20 imply that, under the de Rham isomorphism, $\mathsf{c}_k(E)$ is represented by the form $h_{\mathsf{R}}(\sigma_k^E)$, where R is the curvature endomorphism form of some connection on E compatible with the fibre metric. Finally, we observe that the computation yielding the trace formulae (4.6.19)–(4.6.22) for the forms representing the Chern classes of a principal bundle carries over to $h_{\mathsf{R}}(\sigma_k^E)$. As a result, we obtain the corresponding formulae for $\mathsf{c}_k(E)$ with Ω replaced by R.

We leave it to the reader to derive analogous statements for the Pontryagin classes of real and quaternionic vector bundles and for the Euler class of an oriented real vector bundle. For the latter, one finds that under the de Rham isomorphism, $\mathsf{e}(E)$ is represented by the form $h_{\mathsf{R}}(\varepsilon^E)$, where ε^E denotes the section in $\mathrm{Pol}\big(\mathfrak{o}(E)\big)$ defined by

$$\varepsilon_m^E(A) := \mathrm{pf}\left(\frac{A}{4\pi}\right), \quad A \in \mathfrak{o}(E_m), \quad m \in M.$$

♦

Example 4.6.22 We derive a trace formula for the first Pontryagin class of the adjoint bundle $\mathrm{Ad}(P)$ associated with a given principal bundle P with compact structure group G over some manifold M. Recall that, by definition, $\mathsf{p}_1(\mathrm{Ad}(P)) = -\mathsf{c}_2(\mathrm{Ad}(P)_{\mathbb{C}})$. By choosing a G-invariant scalar product on $\mathfrak{g}_{\mathbb{C}}$, we can turn the complexification $\mathrm{Ad}(P)_{\mathbb{C}} = P \times_G \mathfrak{g}_{\mathbb{C}}$ into a Hermitean vector bundle. Let Ω be the curvature of some compatible connection on P and let $\mathsf{R} \in \Omega^2\big(M, \mathrm{End}(\mathrm{Ad}(P))\big)$ be the corresponding curvature endomorphism form induced on $\mathrm{Ad}(P)$. By prolongation to the complexification, R defines a form $\mathsf{R}^{\mathbb{C}} \in \Omega^2\big(M, \mathrm{End}(\mathrm{Ad}(P)_{\mathbb{C}})\big)$ and the latter is the curvature endomorphism form induced by Ω on $\mathrm{Ad}(P)_{\mathbb{C}}$. Thus, by Remark 4.6.21 and (4.6.22),

$$\mathsf{c}_2\big(\mathrm{Ad}(P)_{\mathbb{C}}\big) = \frac{1}{8\pi^2}\,\mathrm{tr}_{\mathrm{Ad}(P)_{\mathbb{C}}}(\mathsf{R}^{\mathbb{C}} \wedge \mathsf{R}^{\mathbb{C}}).$$

Obviously, $\mathrm{tr}_{\mathrm{Ad}(P)_{\mathbb{C}}}(\mathsf{R}^{\mathbb{C}} \wedge \mathsf{R}^{\mathbb{C}}) = \mathrm{tr}_{\mathrm{Ad}(P)}(\mathsf{R} \wedge \mathsf{R})$. Computation of the pullback of $\mathrm{tr}_{\mathrm{Ad}(P)}(\mathsf{R} \wedge \mathsf{R})\big)$ under the projection of P yields $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\Omega \wedge \mathrm{ad}\Omega)$. As a result, we may write

$$\mathsf{p}_1(\mathrm{Ad}(P)) = -\frac{1}{8\pi^2}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\Omega \wedge \mathrm{ad}\Omega), \tag{4.6.41}$$

where the right hand side is viewed as a form on M. ♦

Exercises

4.6.1 Check that the mapping $\mathrm{Sym}_G\big(\mathfrak{g}\big) \to \mathrm{Pol}_G(\mathfrak{g}), f \mapsto \hat{f}$, defined by (4.6.3) satisfies $\widehat{fg} = \hat{f}\hat{g}$.

4.6.2 Prove the polarization formula (4.6.4).

4.6.3 Prove that the mapping h_α defined by (4.6.6) is a homomorphism, cf. point 1 of Proposition 4.6.4.

4.6.4 Let f be a smooth function in n real variables which is homogeneous of degree k. Show that the functions

$$\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}} f$$

are homogeneous of degree $k - l$ for $l \leq k$ and that they vanish for $l > k$.

4.6.5 Prove formulae (4.6.16) and (4.6.18) for all n-dimensional complex square matrices C.

4.6.6 Verify that the mapping s defined in (4.6.24) preserves the orientations.

4.6.7 Prove Lemma 4.6.18.

4.7 Genera

The discussion in the previous section carries over without change from polynomials to formal power series. For a vector space V, let $\mathrm{FPS}(V)$ denote the vector space of formal power series on V. For a vector bundle E, let $\mathrm{FPS}(E)$ denote the vector bundle whose fibre at $m \in M$ is given by $\mathrm{FPS}(E_m)$.

Given a principal G-bundle P over M, the Weil homomorphism $\mathfrak{w}_P$ extends to a homomorphism

$$\mathfrak{w}_P : \mathrm{FPS}_G(\mathfrak{g}) \to H^*_{\mathrm{dR}}(M) .$$

Similarly, given a complex vector bundle E, the Weil homomorphism of E extends to a homomorphism

$$\mathfrak{w}_E : \mathrm{FPS}_{\mathrm{U}(n)}(\mathfrak{u}(n)) \to H^*_{\mathrm{dR}}(M) ,$$

and a similar statement holds for real and quaternionic vector bundles. More generally, given $\alpha \in \Omega^2(M, E)$, the homomorphism h_α defined by (4.6.33) extends to a homomorphism

$$h_\alpha : \Gamma^\infty\big(\mathrm{FPS}(E)\big) \to \Omega^*(M) .$$

This gives rise to genera of vector bundles. Let us explain this in detail for the case of complex vector bundles. Assume that we are given a formal power series

$$q(x) = \sum_l a_l x^l$$

in one real variable x with real coefficients a_l and constant term $a_0 = 1$. This series defines a symmetric formal power series in n real variables $x_1, \dots, x_n$ by

$$q(x_1, \dots, x_n) := q(x_1) \cdots q(x_n) . \tag{4.7.1}$$

Being symmetric, the latter defines an element q^{U} of $\mathrm{FPS}_{\mathrm{U}(n)}\mathfrak{u}(n)$ by (4.6.15). Then,

$$q^{\mathrm{U}}(A) = q(\lambda_1) \cdots q(\lambda_n) , \tag{4.7.2}$$

where $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of $\frac{\mathrm{i}A}{4\pi}$.

Given a complex vector bundle E of rank n over a manifold M, we can define

$$\gamma(E) := \mathfrak{w}_E\left(q^{\mathrm{U}}\right) \in H^*_{\mathrm{dR}}(M) .$$

The class $\gamma(E)$ is called the genus of the complex vector bundle E defined by the formal power series q, or the q-genus of E for short. In the spirit of Remark 4.6.10, we write

$$\gamma(E) = q\left(\frac{\mathrm{i}\Omega}{2\pi}\right) ,$$

where Ω is the curvature form of some connection on E.

Let us express γ in terms of the Chern classes. Being symmetric, every homogeneous component q_k of q can be expressed as a polynomial in the elementary symmetric polynomials,

$$q_k(x_1, \dots, x_n) = K_k\big(\sigma_1(x_1, \dots, x_n), \dots, \sigma_n(x_1, \dots, x_n)\big) . \tag{4.7.3}$$

The following argument shows that the polynomials K_k do not depend on the number of independent variables n. For clarity, let us display this number by writing $q_k^{(n)}$ and $K_k^{(n)}$. For $l < n$, we have

$$q_k^{(n)}(x_1, \dots, x_l, 0, \dots, 0) = q_k^{(l)}(x_1, \dots, x_l)$$

and

$$\sigma_i(x_1, \dots, x_l, 0, \dots, 0) = \begin{cases} \sigma_i(x_1, \dots, x_l) & i \leq l\,, \\ 0 & i > l\,. \end{cases}$$

Hence,

$$\begin{aligned} q_k^{(l)}(x_1, \dots, x_l) &= q_k^{(n)}(x_1, \dots, x_l, 0, \dots, 0) \\ &= K_k^{(n)}\big(\sigma_1(x_1, \dots, x_l, 0, \dots, 0), \dots, \sigma_n(x_1, \dots, x_l, 0, \dots, 0)\big) \\ &= K_k^{(n)}\big(\sigma_1(x_1, \dots, x_l), \dots, \sigma_l(x_1, \dots, x_l), 0, \dots, 0\big) . \end{aligned}$$

This shows that $K_k^{(l)}$ can be obtained from $K_k^{(n)}$ by setting the last $n - l$ entries to 0. Thus, if we extend the notation of elementary symmetric polynomials in n variables to arbitrary order by setting $\sigma_i(x_1, \dots, x_n) = 0$ for $i > n$, then $K_k^{(l)} = K_k^{(n)}$, as asserted. Since K_k can depend on σ_l with $l \leq k$ only, irrespective of the number of independent variables we thus have $q_k = K_k(\sigma_1, \dots, \sigma_k)$ and hence

$$q_k^{\mathrm{U}} = K_k(\sigma_1^{\mathrm{U}}, \dots, \sigma_k^{\mathrm{U}})\,, \quad k = 0, 1, 2, \dots .$$

Applying the Weil homomorphism and using Theorem 4.6.11, we obtain that

$$\gamma(E) = 1 + \gamma_1(E) + \gamma_2(E) + \cdots \tag{4.7.4}$$

with

$$\gamma_k(E) = K_k\big(\mathsf{c}_1(E), \dots, \mathsf{c}_k(E)\big) \in H^{2k}_{\mathrm{dR}}(M) \tag{4.7.5}$$

under the de Rham isomorphism. That is, in effect, $\gamma_k(E)$ is obtained by replacing the elementary symmetric polynomials in K_k by the Chern classes.

By analogy, the formal power series q defines a genus for real vector bundles and a genus for quaternionic vector bundles. In the above construction, we just replace q_k^{U} by q_k^{O} and q_k^{Sp}, respectively. Thus, for a real or quaternionic vector bundle E, the genus defined by q is given by (4.7.4) with

$$\gamma_k(E) = K_k\big(\mathsf{p}_1(E), \dots, \mathsf{p}_k(E)\big) \in H^{4k}_{\mathrm{dR}}(M)\,. \tag{4.7.6}$$

According to (4.7.5) and (4.7.6), sometimes, the q-genus for complex vector bundles is referred to as the Chern q-genus and the q-genus for real or quaternionic vector bundles is referred to as the Pontryagin q-genus.

Proposition 4.7.1 *Let q be a formal power series in one real variable with constant coefficient* 1 *and let γ be the corresponding genus for $\mathbb{K}$-vector bundles, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

1. *The assignment $E \mapsto \gamma(E)$ defines a characteristic class for vector bundles.*
2. *For $\mathbb{K}$-vector bundles E_1, E_2 over M, one has $\gamma(E_1 \oplus E_2) = \gamma(E_1)\gamma(E_2)$.*
3. *If E has rank* 1*, then $\gamma(E) = 1$ in the real case, $\gamma(E) = q\big(\mathsf{c}_1(E)\big)$ in the complex case and $\gamma(E) = q\big(\mathsf{p}_1(E)\big)$ in the quaternionic case.*

Proof 1. This follows from Corollary 4.6.8/2.

2. First, we show that if $a_1, \dots, a_k$ and $b_1, \dots, b_k$ are independent variables, and if

$$c_k := \sum_{i+j=k} a_i b_j\,,$$

then

$$K_k(c_1, \dots, c_k) = \sum_{i+j=k} K_i(a_1, \dots, a_i) K_j(b_1, \dots, b_j)\,. \tag{4.7.7}$$

By uniqueness of the polynomials K_k, it suffices to check this for a_i and b_i being the elementary symmetric polynomials in the independent variables $x_1, \dots, x_k$ and $y_1, \dots, y_k$, respectively. Clearly, then $c_i = \sigma_i(x_1, \dots, x_k, y_1, \dots, y_k)$. Hence,

$$K_k(c_1, \dots, c_k) = q_k(x_1, \dots, x_k, y_1, \dots, y_k)\,.$$

Since $q(x_1, \dots, x_k, y_1, \dots, y_k) = q(x_1, \dots, x_k)q(y_1, \dots, y_k)$, we obtain

$$\begin{aligned} q_k(x_1, \dots, x_k, y_1, \dots, y_k) &= \sum_{i+j=k} q_i(x_1, \dots, x_k)\, q_j(y_1, \dots, y_k) \\ &= \sum_{i+j=k} K_i(a_1, \dots, a_i) K_j(b_1, \dots, b_j)\,. \end{aligned}$$

This proves (4.7.7). Now, we use this to prove the assertion. In the complex case, we plug in $\mathsf{c}_i(E_1)$ for a_i and $\mathsf{c}_i(E_2)$ for b_i. Then, $K_i(a_1, \dots, a_i) = \gamma_i(E_1)$ and $K_j(b_1, \dots, b_j) = \gamma_j(E_2)$. Moreover, by the Whitney Sum Formula, $c_k = \mathsf{c}_k(E_1 \oplus E_2)$ and hence $K_k(c_1, \dots, c_k) = \gamma_k(E_1 \oplus E_2)$. Thus, (4.7.7) yields the assertion. The quaternionic case and the real case are analogous, cf. (4.4.20) for the latter.

3. The real case is obvious, because $\mathsf{p}(E) = 1$ if E has rank 1. Consider the complex case. The quaternionic case is analogous. For one variable x, one has $\sigma_1(x) = x$ and $\sigma_k(x) = 0$ for all k > 1. Hence,

$$q(x) = 1 + K_1(x) + K_2(x, 0) + \cdots .$$

Plugging in $\mathsf{c}_1(E)$ for x, we obtain

$$q\big(\mathsf{c}_1(E)\big) = 1 + K_1\big(\mathsf{c}_1(E)\big) + K_2\big(\mathsf{c}_1(E), 0\big) + \cdots .$$

Since $\mathsf{c}_k(E) = 0$ for all $k > 1$, the right hand side equals $\gamma(E)$. ■

Remark 4.7.2

1. In view of the Splitting Principle (Theorem 4.3.7), in the complex and the quaternionic case, γ is completely determined by points 2 and 3 of Proposition 4.7.1.
2. For each k, the k-th q-genus γ_k for complex vector bundles is the characteristic class defined by

$$\gamma_k^{\mathrm{U}(n)} := K_k\left(\mathsf{c}_1^{\mathrm{U}(n)}, \dots, \mathsf{c}_k^{\mathrm{U}(n)}\right) \in H^{2k}_{\mathbb{R}}(\mathrm{BU}(n)),$$

where, as usual, $\mathsf{c}_i^{\mathrm{U}(n)} = 0$ in case $i > n$. One may call $\gamma_k^{\mathrm{U}(n)}$ the k-th total genus of $\mathrm{U}(n)$ defined by q. Clearly, the family $\gamma_k^{\mathrm{U}(n)}$, $k = 0, 1, 2, \dots$ does not define an element of $H^*_{\mathbb{R}}(\mathrm{BU}(n))$ unless q is just a polynomial. Similarly, the k-th genus γ_k for real or quaternionic vector bundles is the characteristic class defined by

$$\gamma_k^{\mathrm{O}(n)} := K_k\left(\mathsf{p}_1^{\mathrm{O}(n)}, \dots, \mathsf{p}_k^{\mathrm{O}(n)}\right) \in H^{4k}_{\mathbb{R}}(\mathrm{BO}(n)),$$
$$\gamma_k^{\mathrm{Sp}(n)} := K_k\left(\mathsf{p}_1^{\mathrm{Sp}(n)}, \dots, \mathsf{p}_k^{\mathrm{Sp}(n)}\right) \in H^{4k}_{\mathbb{R}}(\mathrm{BSp}(n)),$$

respectively. ♦

The following genera will appear in Sects. 5.8 and 5.9.

Example 4.7.3 (*Genera of vector bundles*)

1. The Todd genus is the genus of complex vector bundles defined by the Taylor series of the function

$$f(x) = \frac{x}{1 - \mathrm{e}^{-x}}$$

about $x = 0$. One has

$$q(x) = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{2k}}{(2k)!} x^k ,$$

where B_l are the Bernoulli numbers, given by

$$B_0 = 1\,, \quad B_2 = \frac{1}{6}\,, \quad B_4 = -\frac{1}{30}\,, \quad B_6 = \frac{1}{42}\,, \quad \text{etc.}$$

The first terms are

$$q(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^4}{720} + \cdots .$$

By expressing the k-th order term of $q(x_1, \ldots, x_n)$ in terms of $\sigma_1, \ldots, \sigma_k$, we obtain

$$K_1(\sigma_1) = \frac{\sigma_1}{2}\,, \quad K_2(\sigma_1, \sigma_2) = \frac{\sigma_2 + \sigma_1^2}{12}\,, \quad K_3(\sigma_1, \sigma_2, \sigma_3) = \frac{\sigma_1\sigma_2}{24}\,, \quad \ldots .$$

Thus, writing $T \equiv \gamma$, for a complex vector bundle E we read off

$$T_1(E) = \frac{\mathsf{c}_1(E)}{2}\,, \tag{4.7.8}$$

$$T_2(E) = \frac{\mathsf{c}_2(E) + \mathsf{c}_1(E)^2}{12}\,, \tag{4.7.9}$$

$$T_3(E) = \frac{\mathsf{c}_1(E)\mathsf{c}_2(E)}{24}\,. \tag{4.7.10}$$

The Todd genus occurs in the Riemann–Roch Theorem 5.9.8.

2. The L-genus is the genus of real vector bundles defined by the Taylor series of the function

$$f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$$

about $x = 0$. One has

$$q(x) = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!}\, x^k = 1 + \frac{x}{3} - \frac{x^2}{45} + \frac{2x^3}{945} + \cdots ,$$

which leads to

$$K_1(\sigma_1) = \frac{\sigma_1}{3}\,,$$

$$K_2(\sigma_1, \sigma_2) = \frac{7\sigma_2 - \sigma_1^2}{45}\,,$$

$$K_3(\sigma_1, \sigma_2, \sigma_3) = \frac{62\sigma_3 - 13\sigma_1\sigma_2 + 2\sigma_1^3}{945}\,,$$

etc. Thus, writing $L \equiv \gamma$, for a real vector bundle E we read off

$$L_1(E) = \frac{\mathsf{p}_1(E)}{3}\,, \tag{4.7.11}$$

$$L_2(E) = \frac{7\mathsf{p}_2(E) - \mathsf{p}_1(E)^2}{45}\,, \tag{4.7.12}$$

$$L_3(E) = \frac{62\mathsf{p}_3(E) - 13\mathsf{p}_1(E)\mathsf{p}_2(E) + 2\mathsf{p}_1(E)^3}{945}\,, \tag{4.7.13}$$

etc. The L-genus appears in the Hirzebruch Signature Theorem 5.9.6.

3. The $\hat{A}$-genus is the genus of real vector bundles defined by the Taylor series of the analytic function

$$f(x) = \frac{\sqrt{x}/2}{\sinh\left(\sqrt{x}/2\right)} \tag{4.7.14}$$

about $x = 0$. One has

$$q(x) = \sum_{k=0}^{\infty} \frac{1 - 2^{2k-1}B_{2k}}{2^{2k-1}(2k)!}\, x^k = 1 - \frac{x}{24} + \frac{7x^2}{5760} + \cdots\,,$$

which leads to

$$K_1(\sigma_1) = -\frac{\sigma_1}{24}\,,$$
$$K_2(\sigma_1, \sigma_2) = \frac{-4\sigma_2 + 7\sigma_1^2}{5760}\,,$$

etc. Thus, writing $\hat{A} \equiv \gamma$, for a real vector bundle E we read off

$$\hat{A}_1(E) = -\frac{\mathsf{p}_1(E)}{24}\,, \tag{4.7.15}$$

$$\hat{A}_2(E) = \frac{-4\mathsf{p}_2(E) + 7\mathsf{p}_1(E)^2}{5760}\,, \tag{4.7.16}$$

etc. The $\hat{A}$-genus appears in the Atiyah–Singer Index Theorem 5.8.14. ♦

Via the tangent bundle or its complexification, the genera for vector bundles define genera for manifolds. The latter will play a role in the discussion of the Atiyah–Singer Index Theorem and its applications in Sects. 5.8 and 5.9, as well as in the discussion of the instanton moduli space in Sect. 6.5. In what follows, we derive the properties needed there.

Let us start with expressing the $\hat{A}$-genus of a real vector bundle E of even rank $n = 2l$ in terms of a determinant. Let q be the Taylor series of the analytic function (4.7.14). By construction, for $A \in \mathfrak{o}(2l)$,

$$q^{\mathrm{O}}(A) = q(x_1^2) \cdots q(x_l^2)\,,$$

where $\mathrm{i}x_1, -\mathrm{i}x_1, \ldots, \mathrm{i}x_l, -\mathrm{i}x_l$ are the eigenvalues of $A/(4\pi)$. We may assume $x_i \geq 0$. Hence,

$$\begin{aligned}
q^{\mathrm{O}}(A) &= \frac{\sqrt{x_1^2}/2}{\sinh\left(\sqrt{x_1^2}/2\right)} \cdots \frac{\sqrt{x_l^2}/2}{\sinh\left(\sqrt{x_l^2}/2\right)} \\
&= \frac{x_1/2}{\sinh(x_1/2)} \cdots \frac{x_l/2}{\sinh(x_l/2)} \\
&= \left(\frac{-x_1/2}{\sinh(-x_1/2)} \frac{x_1/2}{\sinh(x_1/2)} \cdots \frac{-x_l/2}{\sinh(-x_l/2)} \frac{x_l/2}{\sinh(x_l/2)}\right)^{\frac{1}{2}} \\
&= \det{}^{\frac{1}{2}}\left(\frac{\frac{\mathrm{i}A}{8\pi}}{\sinh\left(\frac{\mathrm{i}A}{8\pi}\right)}\right),
\end{aligned} \tag{4.7.17}$$

where the matrix under the determinant is defined by plugging $\frac{\mathrm{i}A}{8\pi}$ as an argument into the Taylor series about $y = 0$ of the analytic function $y \mapsto \frac{y}{\sinh y}$. As a consequence, for a given real vector bundle E, we may write symbolically

$$\hat{A}(E) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\mathrm{i}\Omega}{4\pi}}{\sinh\left(\frac{\mathrm{i}\Omega}{4\pi}\right)}\right) \tag{4.7.18}$$

with Ω being the curvature of some connection on E, and with the convention that the right hand side is obtained by formally plugging $\frac{\mathrm{i}\Omega}{4\pi}$ into the polynomial $\det^{1/2}(x/\sinh(x))$ and replacing all products by wedge products, cf. Remark 4.6.10.

Next, let us discuss the Chern character. The formal series q^{U}, q^{O} and q^{Sp} can be assigned to an arbitrary symmetric formal power series in several variables. For example, given a formal power series q in one variable, instead of taking the product (4.7.1) to produce a formal power series in several variables, one may as well take the sum $q(x_1) + \cdots + q(x_n)$. This way, one may produce, for example, the series

$$\chi(x_1, \ldots, x_n) = \mathrm{e}^{x_1} + \cdots + \mathrm{e}^{x_n}.$$

The corresponding Ad-invariant formal power series χ^{U} on $\mathfrak{u}(n)$ is given by

$$\chi^{\mathrm{U}}(A) = \operatorname{tr}\left(\exp\left(\frac{\mathrm{i}A}{4\pi}\right)\right), \quad A \in \mathfrak{u}(n). \tag{4.7.19}$$

It defines the Chern character for principal $\mathrm{U}(n)$-bundles P and for complex vector bundles E of rank n,

$$\operatorname{ch}(P) := \mathfrak{w}_P(\chi^{\mathrm{U}}), \quad \operatorname{ch}(E) := \mathfrak{w}_E(\chi^{\mathrm{U}}).$$

According to (4.7.19) and Remark 4.6.10, we write

$$\operatorname{ch}(P) = \operatorname{tr}\left(\exp\left(\frac{\mathrm{i}\Omega}{2\pi}\right)\right), \tag{4.7.20}$$

where Ω is the curvature of some connection on P.

Remark 4.7.4 The Chern character of a complex vector bundle E can be expressed directly in terms of a connection on E as follows. Choose a fibre metric on E and a compatible connection ∇ and let R denote its curvature endomorphism form. Consider the section q in $\operatorname{FPS}\big(\mathfrak{u}(E)\big)$ defined by

$$q_m(A) := \operatorname{tr}\left(\mathrm{e}^{\frac{\mathrm{i}A}{4\pi}}\right), \quad A \in \big(\mathfrak{u}(E)\big)_m .$$

Via the homomorphism $h_{\mathsf{R}} : \Gamma^\infty\big(\operatorname{FPS}(\mathfrak{u}(E))\big) \to \Omega^*(M)$ defined by (4.6.37), it renders a form $h_{\mathsf{R}}(q)$ on M. A discussion analogous to Remark 4.6.21 yields that this form represents $\operatorname{ch}(E)$. Therefore, we can write

$$\operatorname{ch}(E) = \operatorname{tr}\left(\exp\left(\frac{\mathrm{i}\mathsf{R}}{2\pi}\right)\right). \tag{4.7.21}$$

♦

Since $\operatorname{ch}(E)$ is a characteristic class, it can be expressed as a polynomial in the Chern classes. To find this polynomial, we have to rewrite the power sums $x_1^k + \cdots + x_n^k$ in terms of the elementary symmetric polynomials σ_l. This leads to

$$\chi = \sum_{k=0}^{\infty} \frac{1}{k!} P_k^\chi\big(\sigma_1, \ldots, \sigma_k\big) \tag{4.7.22}$$

with the polynomials

$$P_k^\chi(y_1, \ldots, y_k) = (-1)^k \sum_{l_1,\ldots,l_k} \frac{k(l_1 + \cdots + l_k - 1)!}{l_1! \cdots l_k!} (-y_1)^{l_1} \cdots (-y_n)^{l_k}, \tag{4.7.23}$$

where the sum runs over all sequences $l_1, \ldots, l_k$ of non-negative integers such that

$$l_1 + 2l_2 + \cdots + kl_k = k ,$$

see [437] or Example 8 in Sect. I.2 of [418]. As a result, we obtain

$$\operatorname{ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} P_k^\chi\big(\mathsf{c}_1(E), \ldots, \mathsf{c}_k(E)\big). \tag{4.7.24}$$

In low orders, instead of using the general formula (4.7.23), it is easier to read off P_k directly from (4.7.22). This way one finds (Exercise 4.7.2), for example, that in case $\dim(M) \leq 7$,

$$\mathrm{ch}(E) = n + \mathsf{c}_1(E) + \frac{1}{2}\mathsf{c}_1(E)^2 - \mathsf{c}_2(E) + \frac{1}{6}\mathsf{c}_1(E)^3 - \frac{1}{2}\mathsf{c}_1(E)\mathsf{c}_2(E) + \frac{1}{2}\mathsf{c}_3(E)\,. \tag{4.7.25}$$

Lemma 4.7.5 *For complex line bundles $L_1, \ldots, L_n$ over a manifold M,*

$$\mathrm{ch}(L_1 \oplus \cdots \oplus L_n) = \mathrm{e}^{\mathsf{c}_1(L_1)} + \cdots + \mathrm{e}^{\mathsf{c}_1(L_n)}\,.$$

Proof Denote $E := L_1 \oplus \cdots \oplus L_n$ and $x_i := \mathsf{c}_1(L_i)$. By (4.7.24) and Corollary 4.3.4,

$$\begin{aligned}
\mathrm{ch}(E) &= \sum_{k=0}^{\infty} \frac{1}{k!} P_k^{\chi}\big(\mathsf{c}_1(E), \ldots, \mathsf{c}_k(E)\big) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} P_k^{\chi}\Big(\sigma_1\big(\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)\big), \ldots, \sigma_k\big(\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)\big)\Big)\,.
\end{aligned}$$

Now, (4.7.22) yields $\mathrm{ch}(E) = \chi\big(\mathsf{c}_1(L_1), \ldots, \mathsf{c}_1(L_n)\big)$ and hence the assertion. ■

Proposition 4.7.6 *For complex vector bundles E_1 and E_2 over a manifold M,*

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)\,, \tag{4.7.26}$$

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\,\mathrm{ch}(E_2)\,. \tag{4.7.27}$$

Proof By the Splitting Principle, it suffices to prove the assertions under the assumption that E_1 and E_2 split into sums of line bundles,

$$E_1 = \bigoplus_{i=1}^{n_1} L_{1i}\,, \quad E_2 = \bigoplus_{i=1}^{n_2} L_{2i}\,.$$

In this case, (4.7.26) is an immediate consequence of Lemma 4.7.5. Moreover, then

$$E_1 \otimes E_2 = \bigoplus_{i=1}^{n_1} \bigoplus_{j=1}^{n_2} L_{1i} \otimes L_{2j}$$

and the lemma implies

$$\mathrm{ch}\,(E_1 \otimes E_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mathrm{e}^{\mathsf{c}_1(L_{1i} \otimes L_{2j})}\,. \tag{4.7.28}$$

Since each $L_{1i} \otimes L_{2j}$ is a line bundle, (4.3.15) yields

$$\mathrm{e}^{\mathsf{c}_1(L_{1i}\otimes L_{2j})} = \mathrm{e}^{\mathsf{c}_1(L_{1i})+\mathsf{c}_1(L_{2j})} = \mathrm{e}^{\mathsf{c}_1(L_{1i})}\mathrm{e}^{\mathsf{c}_1(L_{2j})} ,$$

where we have used that the ring multiplication in $H^*_{\mathbb{R}}(M)$ is commutative in even degree. Plugging this into (4.7.28) and using the lemma once again, we obtain the assertion. ■

Exercises

4.7.1 Compute the genus γ of real, complex or quaternionic vector bundles defined by the polynomial $q(x) = 1 + x$.

4.7.2 Express the Chern character of a complex vector bundle E over a manifold of dimension $\dim(M) \leq 7$ in terms of the Chern classes (formula (4.7.25)).

4.8 The Postnikov Tower and Bundle Classification

Recall from Sect. 3.4 that, given a Lie group G, principal G-bundles over a manifold M are classified up to vertical isomorphisms by homotopy classes of mappings from M to the classifying space $\mathrm{B}G$. In this section, we explain how to extract information about principal G-bundles over manifolds of low dimension from an approximation of $\mathrm{B}G$ by means of Eilenberg–MacLane spaces. The necessary facts about these spaces are collected in Appendix G. Let us just state here that for every Abelian group A and every positive integer k, there exists a CW-complex having homotopy group A in dimension k and trivial homotopy groups in all other dimensions. This CW-complex is unique up to homotopy equivalence and is referred to as the Eilenberg–MacLane space $K(A, k)$.

First, we discuss two cases where $\mathrm{B}G$ happens to coincide with an Eilenberg–MacLane space, so that no approximation is needed. The first case is that of structure group $\mathrm{U}(1)$.

Theorem 4.8.1 *The assignment $P \to \mathsf{c}_1(P)$ induces a bijection from the set of vertical isomorphism classes of principal* $\mathrm{U}(1)$*-bundles over a manifold M onto the cohomology group $H^2_{\mathbb{Z}}(M)$.*

Proof Since the only nontrivial homotopy group of $\mathrm{U}(1)$ is $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, from the exact homotopy sequence of the universal $\mathrm{U}(1)$-bundle bundle we read off that the only nontrivial homotopy group of $\mathrm{BU}(1)$ is $\pi_2(\mathrm{BU}(1)) = \mathbb{Z}$. Thus, $\mathrm{BU}(1) = K(\mathbb{Z}, 2)$ and (G.1) implies that for every manifold M we have a bijection

$$[M, \mathrm{BU}(1)] \to H^2_{\mathbb{Z}}(M) , \quad f \mapsto f^*\gamma ,$$

where γ is a characteristic element[17] of $H^2_{\mathbb{Z}}(K(\mathbb{Z},2))$. Since $\mathsf{c}_1^{\mathrm{U}(1)}$ is a generator and thus is characteristic, we may choose it for γ. This proves the theorem. ■

Theorem 4.8.1 allows for complementing Corollary 4.2.8 on the orientability of complex vector bundles.

Corollary 4.8.2 *A principal* $\mathrm{U}(n)$*-bundle* P *admits a reduction to the structure group* $\mathrm{SU}(n)$ *iff* $\mathsf{c}_1(P)=0$. *A complex vector bundle* E *is orientable iff* $\mathsf{c}_1(E)=0$.

Proof We give the argument for principal bundles. By Corollary 1.6.5, a reduction of P to $\mathrm{SU}(n)$ exists iff the associated bundle $Q := P \times_{\mathrm{U}(n)} \mathrm{U}(n)/\mathrm{SU}(n)$ is trivial. Combining the embedding $j \equiv j^{\mathrm{U}}_{1,n} : \mathrm{U}(1) \to \mathrm{U}(n)$ with the natural projection to classes $p : \mathrm{U}(n) \to \mathrm{U}(n)/\mathrm{SU}(n)$, we obtain an isomorphism $\varphi = p \circ j : \mathrm{U}(1) \to \mathrm{U}(n)/\mathrm{SU}(n)$, which we use to view Q as a $\mathrm{U}(1)$-bundle. Then, by Theorem 4.8.1, Q is trivial iff $\mathsf{c}_1(Q)=0$.

It remains to show that $\mathsf{c}_1(Q)=\mathsf{c}_1(P)$. If $f : M \to \mathrm{BU}(n)$ is a classifying mapping for P, then $\mathrm{B}(\varphi^{-1}\circ p)\circ f$ is a classifying mapping for Q. Thus,

$$\mathsf{c}_1(Q) = f^* \circ \mathrm{B}(\varphi^{-1}\circ p)^* \mathsf{c}_1^{\mathrm{U}(1)} . \tag{4.8.1}$$

Since $\varphi^{-1}\circ p \circ j = \mathrm{id}_{\mathrm{U}(1)}$, we have $(\mathrm{B}j)^* \circ (\mathrm{B}(\varphi^{-1}\circ p))^* \mathsf{c}_1^{\mathrm{U}(1)} = \mathsf{c}_1^{\mathrm{U}(1)}$ and hence, by Theorem 4.2.1, $(\mathrm{B}(\varphi^{-1}\circ p))^*\mathsf{c}_1^{\mathrm{U}(1)} = \mathsf{c}_1^{\mathrm{U}(n)}$. Thus, (4.8.1) yields $\mathsf{c}_1(Q)=\mathsf{c}_1(P)$, as asserted. ■

The second case where $\mathrm{B}G$ is an Eilenberg–MacLane space is that of structure group $\mathbb{Z}_g$, where $g = 2, 3, \dots$. According to Example 3.4.17/2, the action of $\mathbb{Z}_g$ as a subgroup of $\mathrm{U}(1)$ on S^∞ turns S^∞ into the universal principal $\mathbb{Z}_g$-bundle over L^∞_g, the infinite lens space of order g. Thus, L^∞_g may be taken as the classifying space $\mathrm{B}\mathbb{Z}_g$. On the other hand, since S^∞ is weakly contractible and the only nontrivial homotopy group of $\mathbb{Z}_g$ is $\pi_0(\mathbb{Z}_g)=\mathbb{Z}_g$, from the exact homotopy sequence of the bundle $\mathrm{S}^\infty \to \mathrm{L}^\infty_g$ we read off that L^∞_g is a model of the Eilenberg–MacLane space $K(\mathbb{Z}_g,1)$. Accordingly, $H^1_{\mathbb{Z}_g}(\mathrm{L}^\infty_g)$ is generated by a single element δ_g and this element is characteristic. We will denote the corresponding $\mathbb{Z}_g$-valued characteristic class for principal $\mathbb{Z}_g$-bundles by the same symbol. A similar argument as in the proof of Theorem 4.8.1 yields the following.

Theorem 4.8.3 *The assignment* $P \to \delta_g(P)$ *induces a bijection from the vertical isomorphism classes of principal* $\mathbb{Z}_g$*-bundles over a manifold* M *onto* $H^1_{\mathbb{Z}_g}(M)$. ■

In case $g=2$, we have $\mathbb{Z}_2 = \mathrm{O}(1)$ and δ_2 is just the Stiefel–Whitney class $\mathsf{w}_1^{\mathrm{O}(1)}$. Thus, by analogy with Corollary 4.8.2, Theorem 4.8.3 allows for complementing Corollary 4.2.17 on orientability of real vector bundles (Exercise 4.8.1).

Corollary 4.8.4 *A principal* $\mathrm{O}(n)$*-bundle* P *admits a reduction to* $\mathrm{SO}(n)$ *iff* $\mathsf{w}_1(P)=0$. *A real vector bundle* E *is orientable iff* $\mathsf{w}_1(E)=0$. ■

[17] An element of $H^k_A(K(A,k))$ is called characteristic if under the bijection $H^k_A(K(A,k)) \cong \mathrm{Hom}\big(H_k(K(A,k)),A\big)$ it corresponds to an isomorphism $H_k(K(A,k)) \to A$.

$$
\begin{tikzcd}
 & \vdots \\
 & Y_3 \arrow[d, "q_2"] \\
 & Y_2 \arrow[d, "q_1"] \\
Y \arrow[r, "y_1"] \arrow[ur, "y_2"] \arrow[uur, "y_3"] & Y_1
\end{tikzcd}
$$

Fig. 4.1 The Postnikov tower of a pathwise connected CW-complex

As a consequence, every simply connected manifold is orientable.

Now, we turn to the general situation where $\mathrm{B}G$ is not just an Eilenberg–MacLane space, so that an approximation makes sense. In our presentation, we follow [287]. Recall that a continuous mapping $f : X \to Y$ of topological spaces is called an n-equivalence if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and a surjection for $k = n$. An ∞-equivalence is the same as a weak homotopy equivalence.

Theorem 4.8.5 (Postnikov tower) *Let Y be a pathwise connected CW-complex. For $n = 1, 2, \ldots,$ there exist CW-complexes Y_n and continuous mappings $y_n : Y \to Y_n$ and $q_n : Y_{n+1} \to Y_n$ such that, for every n,*

1. *y_n is an n-equivalence,*
2. *$q_n \circ y_{n+1} = y_n$,*
3. *Y_1 is contractible and $\pi_k(Y_n) = 0$ for $k \geq n$.*

The assertion can be summarized by saying that one has an infinite commutative diagram as shown in Fig. 4.1, with the spaces Y_n being CW-complexes having property 3.

Proof Let n be given. Since Y is a CW-complex, $\pi_n(Y)$ is finitely generated. Hence, there exists a finite number of mappings

$$f_i : (\mathrm{S}^n, \mathbf{e}_1) \to (Y, y_0)$$

whose homotopy classes generate $\pi_n(Y)$. We use these mappings as attaching mappings for $(n+1)$-cells to construct from Y a CW-complex X_1. Consider the natural inclusion mapping $j : Y \to X_1$. Being cellular, by the Cellular Approximation Theorem,[18] the induced homomorphism $j_* : \pi_k(Y) \to \pi_k(X_1)$ depends on the restriction of j to the $(k+1)$-skeletons $Y^{(k+1)} \to X_1^{(k+1)}$ only. For $k < n$, we have $Y^{(k+1)} = X_1^{(k+1)}$ and hence j_* is an isomorphism here. For $k = n$, by the Cellular Approximation Theorem, up to homotopy, every mapping $f : \mathrm{S}^n \to X_1$ may be chosen to take values in

[18] Every continuous mapping between CW-complexes is homotopic to a cellular mapping [287, Theorem 4.8].

$X_1^{(n)} = Y^{(n)}$, which implies that j_* is surjective here. Thus, j is an n-equivalence. In particular, $\pi_n(X_1)$ is generated by the homotopy classes of the mappings $j \circ f_i$. Since through the cells attached, the latter are homotopic to the constant mapping at y_0, we have $\pi_n(X_1) = 0$. Now, we repeat the procedure with Y replaced by X_1 and n replaced by $n+1$ to embed X_1 via an $(n+1)$-equivalence into a CW-complex X_2 with $\pi_{n+1}(X_2) = 0$. Iterating this, we finally obtain a CW-complex Y_n which contains Y as a subcomplex such that the natural inclusion mapping $y_n : Y \to Y_n$ is an n-equivalence and $\pi_k(Y) = 0$ for all $k \geq n$.

To see that Y_1 is contractible, we observe that due to $\pi_k(Y_1) = 0$ for all k, the constant mapping $Y_1 \to *$ is a weak homotopy equivalence. Since Y_1 is a CW-complex, the Whitehead Theorem [598] yields that this mapping is in fact a homotopy equivalence. Hence, Y_1 is contractible, indeed.

It remains to construct the mappings q_n. Since y_{n+1} is the natural inclusion mapping of the subcomplex Y of Y_{n+1}, the mapping q_n must be the extension of $y_n : Y \to Y_n$ to the ambient complex Y_{n+1}. Since $Y_{n+1} \backslash Y$ consists of cells of dimension $n+2$ and larger, whereas $\pi_k(Y_n) = 0$ for all $k \geq n$, such an extension exists and can be chosen to be cellular (Exercise 4.8.2). ■

Remark 4.8.6 From the construction in the proof we read off that the CW-complexes Y_n can be chosen so that $Y_n^{(k)} = Y^{(k)}$ for all $k \leq n$. ♦

While the cellular construction of the spaces Y_n is elementary, it requires concrete knowledge of the cell structure of Y and is therefore hardly manageable in the case where Y is a classifying space of a Lie group. On the other hand, in order to use the Postnikov tower for bundle classification, there is no need to know the spaces Y_n and the mappings q_n in detail. If one replaces the spaces Y_n by homotopy equivalent spaces and redefines the mappings y_n and q_n appropriately, then y_n is still an n-equivalence and the relations $q_n \circ y_{n+1} = y_n$ continue to hold up to homotopy. Thus, it is sufficient to know the homotopy types of the spaces Y_n. The following theorem provides information on that.

Recall that a CW-complex Y is said to be simple if it is pathwise connected and if the natural action[19] of $\pi_1(Y)$ on $\pi_k(Y)$ is trivial for all k.

Theorem 4.8.7 (Postnikov tower for simple CW-complexes) *Let Y be a simple CW-complex. For $n = 1, 2, 3, \ldots$, the CW-complex Y_{n+1} provided by Theorem 4.8.5 is weakly homotopy equivalent to the total space of the pullback of the path-loop fibration over the Eilenberg–MacLane space $K(\pi_n(Y), n+1)$ under some continuous mapping $\theta_n : Y_n \to K(\pi_n(Y), n+1)$.*

Let us add that according to the exact homotopy sequence of the path-loop fibration over $K(\pi_n(Y), n+1)$, the homotopy fibre of this fibration is a $K(\pi_n(Y), n)$. Moreover, since the homotopy type of the total space of a pullback fibration depends on the homotopy class of the mapping only [287, Proposition 4.62], it suffices to determine the homotopy classes of the mappings θ_n.

[19] Explained prior to Proposition 3.2.9.

Proof By passing to the mapping cylinder[20] of q_n, which is homotopy equivalent to Y_n, we may assume that Y_{n+1} is a subcomplex of Y_n and that q_n is the natural inclusion mapping.

First, we prove that

$$\pi_k(Y_n, Y_{n+1}) = \begin{cases} \pi_n(Y) & k = n+1\,, \\ 0 & k \neq n+1\,. \end{cases} \tag{4.8.2}$$

Consider the exact homotopy sequence (3.2.4) of the pair (Y_n, Y_{n+1}),

$$\cdots \to \pi_k(Y_{n+1}) \xrightarrow{q_{n*}} \pi_k(Y_n) \to \pi_k(Y_n, Y_{n+1}) \xrightarrow{\partial} \pi_{k-1}(Y_{n+1}) \xrightarrow{q_{n*}} \pi_{k-1}(Y_n) \to \cdots$$

with the connecting homomorphism ∂. By Remark 4.8.6, for $k < n$, we have $Y_n^{(k+1)} = Y^{(k+1)} = Y_{n+1}^{(k+1)}$. Hence, in this case, the Cellular Approximation Theorem yields that both q_{n*} in this sequence are isomorphisms. By exactness, then $\pi_k(Y_n, Y_{n+1}) = 0$. For $k = n$, the right q_{n*} is still an isomorphism, but now $\pi_k(Y_n) = 0$. Hence, still, $\pi_n(X_n, X_{n+1}) = 0$. For $k \geq n+1$, we have $\pi_k(Y_n) = \pi_{k-1}(Y_n) = 0$ and thus ∂ is an isomorphism here. Putting $k = n+1$, we obtain $\pi_{n+1}(Y_n, Y_{n+1}) = \pi_n(Y_{n+1}) = \pi_n(Y)$, because y_{n+1} is an $(n+1)$-equivalence. For all higher k, we obtain $\pi_k(Y_n, Y_{n+1}) = \pi_{k-1}(Y_{n+1}) = 0$. This proves (4.8.2).

Next, we show that

$$\pi_k(Y_n/Y_{n+1}) = \begin{cases} \pi_n(Y) & k = n+1\,, \\ 0 & k \leq n\,. \end{cases} \tag{4.8.3}$$

That $\pi_k(Y_n/Y_{n+1})$ is trivial for all $k \leq n$ follows from the fact that due to $Y_n^{(n)} = Y^{(n)} = Y_{n+1}^{(n)}$, the quotient space Y_n/Y_{n+1} consists of cells of dimension $n+1$ and higher only. As a consequence, the absolute Hurewicz Theorem yields

$$\pi_{n+1}(Y_n/Y_{n+1}) \cong H_{n+1}(Y_n/Y_{n+1})\,.$$

On the other hand, ∂ is equivariant with respect to the actions of $\pi_1(Y_{n+1})$ on $\pi_{n+1}(Y_n, Y_{n+1})$ and $\pi_n(Y_{n+1})$. Since y_{n+1} is an n-equivalence, $\pi_1(Y_{n+1}) = \pi_1(Y)$ and $\pi_n(Y_{n+1}) = \pi_n(Y)$. It follows that simplicity of Y implies that the action of $\pi_1(Y_{n+1})$ on $\pi_{n+1}(Y_n, Y_{n+1})$ is trivial. Hence, in view of (4.8.2), the relative Hurewicz Theorem[21] yields $\pi_{n+1}(Y_n, Y_{n+1}) \cong H_{n+1}(Y_n, Y_{n+1})$ and thus $H_{n+1}(Y_n, Y_{n+1}) \cong \pi_n(Y)$. Since $H_{n+1}(Y_n/Y_{n+1}) \cong H_{n+1}(Y_n, Y_{n+1})$, this proves (4.8.3).

Now, by the procedure of attaching cells used in the proof of Theorem 4.8.5 to construct Y_n from Y, we can construct a CW-complex K_n from Y_n/Y_{n+1} which has trivial homotopy groups in dimension $n+2$ and larger and is thus a model of the

[20]The mapping cylinder of $f : X \to Y$ is the quotient space of $(X \times I) \sqcup Y$ obtained by identifying each pair $(x, 1) \in X \times I$ with the point $f(x) \in Y$.

[21]See for example [287, Theorem 4.37].

Eilenberg–MacLane space $K(\pi_n(Y), n+1)$. As a base point, we choose the point $k_0 \in Y_n/Y_{n+1} \subset K_n$ to which Y_{n+1} is contracted. Define

$$\theta_n : Y_n \to Y_n/Y_{n+1} \to K_n ,$$

where the first mapping is the natural projection to classes and the second mapping is the natural inclusion mapping. Our next aim is to show that Y_{n+1} is weakly homotopy equivalent to the homotopy fibre of θ_n.

According to Proposition 3.2.16, θ_n decomposes as

$$\theta_n : Y_n \xrightarrow{j} E \xrightarrow{p} K_n ,$$

where j is a homotopy equivalence and p is a fibration. Since, by construction, $\theta_n \circ q_n$ sends Y_{n+1} to k_0, the composition $j \circ q_n$ sends Y_{n+1} to the fibre $F = p^{-1}(k_0)$. The exact homotopy sequences of the pairs (E, F) and (Y_n, Y_{n+1}) combine to the following commutative diagram with exact rows and with the vertical arrows given by inclusion:

$$\begin{tikzcd}
\pi_{k+1}(Y_n) \arrow[r] \arrow[d] & \pi_{k+1}(Y_n, Y_{n+1}) \arrow[r] \arrow[d] & \pi_k(Y_{n+1}) \arrow[r] \arrow[d] & \pi_k(Y_n) \arrow[r] \arrow[d] & \pi_k(Y_n, Y_{n+1}) \arrow[d] \\
\pi_{k+1}(E) \arrow[r] & \pi_{k+1}(E, F) \arrow[r] & \pi_k(F) \arrow[r] & \pi_k(E) \arrow[r] & \pi_k(E, F)
\end{tikzcd}$$

Since Y_n is a strong deformation retract of E, the first and the fourth vertical arrow are isomorphisms for all k. By Lemma 3.2.7, $\pi_k(E, F) = \pi_k(K_n)$. Comparing this with (4.8.2), we find that the second and the fifth vertical arrow are isomorphisms for all k, too. Now, the Five Lemma[22] implies that the central vertical arrow is an isomorphism for all k. This proves that Y_{n+1} is weakly homotopy equivalent to F, indeed.

Finally, we prove that F is homeomorphic to the total space of the pullback fibration $\theta_n^* \mathrm{P}K_n$. We have

$$\theta_n^* \mathrm{P}K_n = \{(y, \gamma) \in Y_{n+1} \times C(I, K_n) : \gamma(0) = k_0, \gamma(1) = \theta_n(y)\} ,$$

where $C(I, K_n)$ carries the compact-open topology. On the other hand, from the proof of Proposition 3.2.16 we read off

$$F = \{(y, \gamma) \in Y_{n+1} \times C(I, K_n) : \gamma(0) = \theta_n(y), \gamma(1) = k_0\} .$$

Thus, the assignment $(y, \gamma) \mapsto (y, \gamma^{-1})$ yields a bijection between $\theta_n^* \mathrm{P}K_n$ and F. Since the mapping $\gamma \mapsto \gamma^{-1}$ is continuous relative to the compact-open topology, this bijection is a homeomorphism. This completes the proof of Theorem 4.8.7. ∎

[22] If the diagram is commutative, if the rows are exact and if the vertical arrows except for that in the middle are isomorphisms, then the arrow in the middle is an isomorphism, too.

As an application, we use Theorems 4.8.5 and 4.8.7 to classify principal $\mathrm{U}(n)$-bundles over manifolds of dimension $\dim M \leq 4$. This argument belongs to Avis and Isham [43].

Denote $Y = \mathrm{BU}(n)$. Since $\pi_1(\mathrm{BU}(n)) = \pi_0(\mathrm{U}(n)) = 0$, the space Y is trivially simple and the assumption of Theorem 4.8.7 is fulfilled. We use the option to replace the spaces Y_n provided by Theorem 4.8.5 by homotopy equivalent spaces, cf. the discussion prior to Theorem 4.8.7.

At stage 1, Y_1 is contractible and may thus be replaced by $Y_1 = *$.

At stage 2, Y_2 is weakly homotopy equivalent to the total space of the pullback of the path loop fibration over $K(\pi_1(Y), 2)$ under a mapping $\theta_1 : Y_1 \to K(\pi_1(Y), 2)$. Since $Y_1 = *$, Y_2 coincides with the corresponding homotopy fibre, which is a $K(\pi_1(Y), 1)$. Since $\pi_1(Y) = 0$, we obtain that Y_2 is weakly homotopy equivalent to $*$. Being a CW-complex, it is then homotopy equivalent to $*$ and may thus be replaced by $Y_2 = *$.

At stage 3, Y_3 is weakly homotopy equivalent to the total space of the pullback of the path loop fibration over $K(\pi_2(Y), 3)$ under a mapping $\theta_2 : Y_2 \to K(\pi_2(Y), 3)$. Since $Y_2 = *$ and $\pi_2(Y) = \pi_1(\mathrm{U}(n)) = \mathbb{Z}$, we obtain that Y_3 is weakly homotopy equivalent to $K(\mathbb{Z}, 2)$. Thinking of $K(\mathbb{Z}, 2)$ as being realized by a CW-complex, it follows that Y_3 is homotopy equivalent to $K(\mathbb{Z}, 2)$ and thus may be replaced by $Y_3 = K(\mathbb{Z}, 2)$.

At stage 4, Y_4 is weakly homotopy equivalent to the total space of the pullback of the path loop fibration over $K(\pi_3(Y), 4)$ under a mapping $\theta_3 : Y_3 \to K(\pi_3(Y), 4)$. Since $\pi_3(Y) = \pi_2(\mathrm{U}(n)) = 0$, we have $K(\pi_3(Y), 4) = *$. Thus, Y_4 is weakly homotopy equivalent to Y_3 and may thus be replaced by $Y_4 = Y_3 = K(\mathbb{Z}, 2)$.

At stage 5, Y_5 is weakly homotopy equivalent to the total space of the pullback of the path loop fibration over $K(\pi_4(Y), 5)$ under a mapping $\theta_4 : Y_4 \to K(\pi_4(Y), 5)$. Since $\pi_4(Y) = \pi_3(\mathrm{U}(n)) = \mathbb{Z}$ and $Y_4 = Y_3 = K(\mathbb{Z}, 2)$, we have $\theta_4 : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 5)$. Thus, we have to determine $\big[K(\mathbb{Z}, 2), K(\mathbb{Z}, 5)\big]$. According to (G.1), we have a bijection

$$\big[K(\mathbb{Z}, 2), K(\mathbb{Z}, 5)\big] = H^5_{\mathbb{Z}}(K(\mathbb{Z}, 2)) .$$

According to Appendix G, $K(\mathbb{Z}, 2)$ may be realized as $\mathbb{C}\mathrm{P}^\infty$ and thus has trivial cohomology groups in odd dimension. It follows that θ_4 is homotopic to a constant mapping and thus Y_5 is weakly homotopy equivalent to the direct product of $K(\mathbb{Z}, 2)$ with the homotopy fibre, which is a $K(\mathbb{Z}, 4)$. Thus, realizing $K(\mathbb{Z}, 4)$ as a CW-complex, we finally may replace

$$Y_5 = K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) . \tag{4.8.4}$$

Let us use this to classify principal $\mathrm{U}(n)$-bundles over a manifold M of dimension 4 or less. By the very definition of the classifying space $\mathrm{BU}(n)$, vertical isomorphism classes of such bundles are in bijective correspondence with homotopy classes $[M, \mathrm{BU}(n)]$. Composition with the 5-equivalence y_5 yields a bijection $[M, \mathrm{BU}(n)] \to [M, Y_5]$. Using in addition (4.8.4), Corollary 3.1.3 and, once again, (G.1), we obtain a bijection

$$[M, \mathrm{BU}(n)] \to H^2_{\mathbb{Z}}(M) \times H^4_{\mathbb{Z}}(M)\,, \quad f \mapsto \big((\mathrm{pr}_1 \circ y_5 \circ f)^* \gamma_2, (\mathrm{pr}_2 \circ y_5 \circ f)^* \gamma_4\big)\,,$$

where γ_k is a generator of $H^k_{\mathbb{Z}}(K(\mathbb{Z}, k))$. Since y_5 is a 5-equivalence, $(\mathrm{pr}_1 \circ y_5)^* \gamma_2$ is a generator of $H^2_{\mathbb{Z}}(\mathrm{BU}(n))$ and $(\mathrm{pr}_2 \circ y_5)^* \gamma_4$ is a generator of $H^4_{\mathbb{Z}}(\mathrm{BU}(n))$. By possibly redefining γ_2 and γ_4, we can achieve that $(\mathrm{pr}_1 \circ y_5)^* \gamma_2 = \mathsf{c}_1^{\mathrm{U}(n)}$ and $(\mathrm{pr}_2 \circ y_5)^* \gamma_4 = \mathsf{c}_2^{\mathrm{U}(n)}$. As a consequence, we obtain a bijection

$$[M, \mathrm{BU}(n)] \to H^2_{\mathbb{Z}}(M) \times H^4_{\mathbb{Z}}(M)\,, \quad f \mapsto \big(f^* \mathsf{c}_1^{\mathrm{U}(n)}, f^* \mathsf{c}_2^{\mathrm{U}(n)}\big)\ .$$

This translates into the following classification result.

Theorem 4.8.8 *For a manifold M of dimension ≤ 4, the assignment*

$$P \mapsto \big(\mathsf{c}_1(P), \mathsf{c}_2(P)\big)$$

induces a bijection from the set of vertical isomorphism classes of principal $\mathrm{U}(n)$-*bundles over M onto the direct product $H^2_{\mathbb{Z}}(M) \times H^4_{\mathbb{Z}}(M)$.* ■

As an immediate consequence, the assignment $P \mapsto \mathsf{c}_2(P)$ induces a bijection from the set of vertical isomorphism classes of principal $\mathrm{SU}(n)$-bundles over M onto $H^4_{\mathbb{Z}}(M)$. Clearly, Theorem 4.8.8 carries over to complex vector bundles over M.

In Sect. 8.6, we will present another application of the approximation method described here.

Exercises

4.8.1 Adapt the proof of Corollary 4.8.2 to the situation of Corollary 4.8.4.

4.8.2 Prove the following statement, known as the Extension Lemma. Let X, Y be CW-complexes and let $A \subset X$ be a subcomplex. If $\pi_k(Y) = 0$ for every k such that $X \backslash A$ contains $(k+1)$-cells, then every cellular mapping $A \to Y$ admits a cellular extension.

Chapter 5
Clifford Algebras, Spin Structures and Dirac Operators

In this chapter we study the theory of Dirac operators in a systematic way. In Sects. 5.1–5.3, we present the algebraic basics: we discuss the theory of Clifford algebras and spinor groups, together with their representations. Next, in Sect. 5.4, we study spin- and Spin^c-structures on Riemannian manifolds including a number of relevant examples. The basic geometric structure of this chapter is that of a Dirac bundle, that is, a Riemannian (or Hermitean) Clifford module bundle over a (pseudo-)Riemannian manifold endowed with a Clifford connection. Associated with a Dirac bundle, one has a natural first order differential operator acting on sections of that bundle, called the Dirac operator. In Sect. 5.5, all these structures are discussed in a systematic way. In Dirac operator theory, one of the basic technical ingredients are Weitzenboeck type formulae. These will be derived in Sect. 5.6. Next, in Sect. 5.7, we give a short introduction to the theory of elliptic differential operators in the context of Sobolev spaces. We prove that the Dirac operator and its square are elliptic and Fredholm and we give a proof of the Hodge Decomposition Theorem. Finally, we discuss the classical elliptic complexes. Section 5.8 is devoted to the Atiyah–Singer Index Theorem. We give a complete proof of this theorem via the heat kernel method using Getzler rescaling. We also discuss the generalization of this theorem to families of Dirac operators in some detail, but we do not give a proof for that case. Finally, in Sect. 5.9, we outline how the index theorems for the classical elliptic complexes follow from the general Atiyah–Singer Index Theorem. For the Gauß–Bonnet Theorem we give a full proof.

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5.1 Clifford Algebras

Let us consider a finite-dimensional vector space[1] V over a commutative field $\mathbb{K}$ of characteristic zero endowed with a quadratic form q. The pair (V, q) will be called a quadratic space. Let

$$\mathscr{T}(V) := \bigoplus_{k=0}^{\infty} \Big(\bigotimes^{k} V \Big)$$

be the tensor algebra over V and let $\mathscr{I}_{\mathsf{q}}(V)$ be the two-sided ideal in $\mathscr{T}(V)$ generated by elements of the form $\{v \otimes v - \mathsf{q}(v)1\}$ where $v \in V$.

Definition 5.1.1 The Clifford algebra $Cl(V, \mathsf{q})$ of the quadratic space (V, q) is the quotient algebra defined by

$$Cl(V, \mathsf{q}) := \mathscr{T}(V)/\mathscr{I}_{\mathsf{q}}(V) . \tag{5.1.1}$$

By this definition, the canonical projection $\rho : \mathscr{T}(V) \to Cl(V, \mathsf{q})$ is an algebra homomorphism endowing $Cl(V, \mathsf{q})$ with the structure of an associative algebra with unit. First, note that the inclusion $\mathbb{K} \to \mathscr{T}(V)$ obviously descends to an inclusion $\mathbb{K} \to Cl(V, \mathsf{q})$. Next, since the elements of V generate $\mathscr{T}(V)$ multiplicatively, they also generate $Cl(V, \mathsf{q})$. Moreover, there is a natural linear mapping $j : V \to Cl(V, \mathsf{q})$ given by the restriction of ρ to the vector subspace $V \subset \mathscr{T}(V)$. By construction, j is injective and fulfils

$$j(v)^2 = \mathsf{q}(v)1 , \quad v \in V . \tag{5.1.2}$$

Therefore, we may view V as a linear subspace of $Cl(V, \mathsf{q})$.[2] By (5.1.2), $Cl(V, 0)$ coincides as an algebra with the exterior algebra $\bigwedge^* V$. Since the characteristic of $\mathbb{K}$ is by assumption different from 2,

$$j(u) \cdot j(v) + j(v) \cdot j(u) = 2\eta(u, v) , \quad u, v \in V , \tag{5.1.3}$$

where $2\eta(u, v) = \mathsf{q}(u + v) - \mathsf{q}(u) - \mathsf{q}(v)$ is the unique symmetric bilinear form obtained by polarizing q.

The Clifford algebra has the following universal property.

Proposition 5.1.2 (Universal property) *Let $F : V \to \mathfrak{A}$ be a linear mapping into a unital associative $\mathbb{K}$-algebra fulfilling*

$$F(v)^2 = \mathsf{q}(v)1 , \quad v \in V . \tag{5.1.4}$$

[1] Most of the statements of this section hold for infinite-dimensional V as well, see e.g. [407].

[2] If there will be no danger of confusion, we will sometimes omit the symbol j.

Then, F extends to a unique $\mathbb{K}$-algebra homomorphism $\hat{F} : Cl(V, \mathsf{q}) \to \mathfrak{A}$ fulfilling $F = \hat{F} \circ j$.

By analogy, in case $\mathbb{K} = \mathbb{C}$, every anti-linear mapping extends uniquely to an anti-homomorphism.

Proof Every linear mapping $F : V \to \mathfrak{A}$ extends to a unique algebra homomorphism $\tilde{F} : \mathscr{T}(V) \to \mathfrak{A}$ and, by (5.1.4), $\tilde{F}$ vanishes identically on $\mathscr{I}_{\mathsf{q}}(V)$. Thus, $\tilde{F}$ descends to a homomorphism $\hat{F} : Cl(V, \mathsf{q}) \to \mathfrak{A}$ fulfilling

$$\hat{F} \circ j(v) = \hat{F} \circ \rho(v) = \tilde{F}(v) = F(v)\,, \quad v \in V\,.$$

This property implies the uniqueness of $\hat{F}$, because it uniquely determines $\hat{F}$ on the set $j(V)$ generating $Cl(V, \mathsf{q})$. ■

Corollary 5.1.3 *For a quadratic space (V, q), the Clifford algebra $Cl(V, \mathsf{q})$ is unique up to an isomorphism. That is, any unital associative $\mathbb{K}$-algebra $\mathfrak{B}$ such that*

(a) *there exists a linear mapping $i : V \to \mathfrak{B}$,*
(b) *for a unital associative $\mathbb{K}$-algebra $\mathfrak{A}$, any linear mapping $F : V \to \mathfrak{A}$ fulfilling (5.1.4) extends to a unique algebra homomorphism $\hat{F} : \mathfrak{B} \to \mathfrak{A}$ fulfilling $F = \hat{F} \circ i$,*

is isomorphic to $Cl(V, \mathsf{q})$.

Proof For simplicity, let us denote $\mathfrak{C} = Cl(V, \mathsf{q})$. Putting $\mathfrak{A} = \mathfrak{C}$ and $F = j$, we conclude that j extends to a unique homomorphism $\hat{j} : \mathfrak{B} \to \mathfrak{C}$ fulfilling $j = \hat{j} \circ i$. By the same arguments, i extends to a unique homomorphism $\hat{i} : \mathfrak{C} \to \mathfrak{B}$ fulfilling $i = \hat{i} \circ j$. Thus, we obtain

$$i = (\hat{i} \circ \hat{j}) \circ i\,, \quad j = (\hat{j} \circ \hat{i}) \circ j\,,$$

that is, the restrictions of $\hat{i} \circ \hat{j} : \mathfrak{B} \to \mathfrak{B}$ and $\hat{j} \circ \hat{i} : \mathfrak{C} \to \mathfrak{C}$ to $i(V)$ and $j(V)$, respectively, coincide with the restrictions of the identical mappings $\mathrm{id}_{\mathfrak{B}}$ and $\mathrm{id}_{\mathfrak{C}}$. Thus, again by the uniqueness property of extensions, $\hat{i} \circ \hat{j} = \mathrm{id}_{\mathfrak{B}}$ and $\hat{j} \circ \hat{i} = \mathrm{id}_{\mathfrak{C}}$ showing that $\hat{j} : \mathfrak{B} \to \mathfrak{C}$ is an algebra isomorphism. ■

We note that the properties (a) and (b) in Corollary 5.1.3 may be taken as an axiomatic definition of the Clifford algebra. Each of the subsequent propositions of this section is a consequence of the universal property.

Proposition 5.1.4 (Parity automorphism) *Every Clifford algebra $Cl(V, \mathsf{q})$ admits a unique involutive automorphism induced from the linear mapping*

$$F : V \to Cl(V, \mathsf{q})\,, \quad F(v) := -j(v)\,.$$

Proof By definition of F, for every $v \in V$, we have $F(v)^2 = (-j(v))^2 = \mathsf{q}(v) \cdot 1$. Thus, there exists a unique algebra homomorphism $\mathsf{p} : Cl(V, \mathsf{q}) \to Cl(V, \mathsf{q})$ such that $\mathsf{p} \circ j(v) = -j(v)$. Since, for any $v \in V$,

$$\mathsf{p} \circ \mathsf{p} \circ j(v) = -\mathsf{p} \circ j(v) = j(v) ,$$

p^2 is the identity on the generating set $j(V)$ and, therefore, on the whole of $Cl(V, \mathsf{q})$. In particular, p is bijective. ■

The element $\mathsf{p} \in \mathrm{Aut}(Cl(V, \mathsf{q}))$ will be called the parity automorphism of $Cl(V, \mathsf{q})$.

Given an algebra $\mathfrak{A}$, by definition, the opposite algebra $\mathfrak{A}^{\mathrm{T}}$ is the unique algebra which coincides with $\mathfrak{A}$ as a vector space and whose multiplication $*$ is induced from the multiplication $\cdot$ of $\mathfrak{A}$ by reversing the order. Thus, the identical mapping yields an isomorphism of algebras $\varphi : \mathfrak{A} \to \mathfrak{A}^{\mathrm{T}}$ fulfilling

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b) = b \cdot a .$$

Proposition 5.1.5 (Canonical anti-automorphism) *Any Clifford algebra $Cl(V, \mathsf{q})$ admits a unique involutive anti-automorphism induced from the linear mapping*

$$F : V \to Cl(V, \mathsf{q})^{\mathrm{T}} , \quad F(v) := \varphi \circ j(v) .$$

Proof Let $*$ be the multiplication in $Cl(V, \mathsf{q})^{\mathrm{T}}$. Since, for every $v \in V$,

$$F(v) * F(v) = \varphi(j(v)) * \varphi(j(v)) = j(v) \cdot j(v) = \mathsf{q}(v) \cdot 1 ,$$

there exists an algebra homomorphism $\hat{F} : Cl(V, \mathsf{q}) \to Cl(V, \mathsf{q})^{\mathrm{T}}$ fulfilling $\hat{F} \circ j = F$. Then,

$$\mathrm{t} := \varphi^{-1} \circ \hat{F} : Cl(V, \mathsf{q}) \to Cl(V, \mathsf{q})$$

fulfils $\mathrm{t} \circ j = j$ and, thus, it is an involution. Moreover, for any $a, b \in Cl(V, \mathsf{q})$,

$$\mathrm{t}(a \cdot b) = \varphi^{-1} \circ \hat{F}(a \cdot b) = \varphi^{-1}(\hat{F}(a) * \hat{F}(b)) = (\varphi^{-1} \circ \hat{F}(b)) \cdot (\varphi^{-1} \circ \hat{F}(a)) ,$$

that is, $\mathrm{t}(a \cdot b) = \mathrm{t}(b) \cdot \mathrm{t}(a)$ showing that t is an anti-automorphism. ■

The mapping t will be called the canonical anti-automorphism. Occasionally, we will write $\mathrm{t}(a) \equiv a^{\mathrm{T}}$.

Remark 5.1.6

1. The parity automorphism p induces a $\mathbb{Z}_2$-grading of $Cl(V, \mathsf{q})$. Indeed, since $\mathsf{p}^2 = \mathrm{id}$, we may decompose the Clifford algebra into an even and an odd part:

$$Cl(V, \mathsf{q}) = Cl^0(V, \mathsf{q}) \oplus Cl^1(V, \mathsf{q}) , \tag{5.1.5}$$

where $Cl^k(V, \mathsf{q}) = \{a \in Cl(V, \mathsf{q}) : \mathsf{p}(a) = (-1)^k a\}$, $k = 0, 1$, are the eigenspaces of p corresponding to the eigenvalues ± 1. Clearly,

$$Cl^k(V, \mathsf{q}) \cdot Cl^l(V, \mathsf{q}) \subset Cl^{k+l}(V, \mathsf{q}),$$

where the indices are taken modulo 2. In particular, $Cl^0(V, \mathsf{q})$ is a subalgebra.

2. The canonical anti-automorphism t is obtained more directly as follows. The tensor algebra carries a unique involutive anti-automorphism given by $v_1 \otimes \ldots \otimes v_r \mapsto v_r \otimes \ldots \otimes v_1$. Note that this mapping coincides with the identity on $V \subset \mathscr{T}(V)$ and that it leaves the ideal $\mathscr{I}_{\mathsf{q}}(V)$ invariant. Thus, it descends to an anti-automorphism of $Cl(V, \mathsf{q})$ which coincides with t on $j(V)$ and, thus, on the whole of $Cl(V, \mathsf{q})$.
3. Clearly, $\mathsf{p} \circ \mathrm{t} = \mathrm{t} \circ \mathsf{p}$. ♦

Proposition 5.1.7 *The Clifford algebra of the direct sum* $(V_1 \oplus V_2, \mathsf{q}_1 \oplus \mathsf{q}_2)$ *of two quadratic spaces* (V_1, q_1) *and* (V_2, q_2) *is isomorphic to the* $\mathbb{Z}_2$*-graded tensor product*[3] *of their Clifford algebras,*

$$Cl(V_1 \oplus V_2, \mathsf{q}_1 \oplus \mathsf{q}_2) \cong Cl(V_1, \mathsf{q}_1) \hat{\otimes} Cl(V_2, \mathsf{q}_2).$$

Proof Consider the linear mapping

$$F : V_1 \oplus V_2 \to Cl(V_1, \mathsf{q}_1) \hat{\otimes} Cl(V_2, \mathsf{q}_2), \quad F(v_1, v_2) := j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2).$$

Then, omitting the symbols j_1 and j_2, we calculate

$$(F(v_1, v_2))^2 = (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2^2 = (\mathsf{q}_1(v_1) + \mathsf{q}_2(v_2)) \cdot (1 \otimes 1).$$

That is, $(F(v_1, v_2))^2 = (\mathsf{q}_1 \oplus \mathsf{q}_2)(v_1, v_2) \cdot 1$ and, thus, there exists a unique algebra homomorphism $\hat{F} : Cl(V_1 \oplus V_2, \mathsf{q}_1 \oplus \mathsf{q}_2) \to Cl(V_1, \mathsf{q}_1) \hat{\otimes} Cl(V_2, \mathsf{q}_2)$. Clearly, $\hat{F}$ is surjective, because its image is a subalgebra containing $Cl(V_1, \mathsf{q}_1) \otimes 1$ and $1 \otimes Cl(V_2, \mathsf{q}_2)$. It is also injective, because it is one-to-one on elements of $V_1 \oplus V_2$ generating $Cl(V_1 \oplus V_2, \mathsf{q}_1 \oplus \mathsf{q}_2)$. ■

This proposition implies the following.

Corollary 5.1.8 *Let* (V, q) *be an n-dimensional quadratic* $\mathbb{K}$*-vector space. Then, the vector space* $Cl(V, \mathsf{q})$ *has dimension* 2^n.

---

[3]Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathbb{Z}_2$-graded unital $\mathbb{K}$-algebras with decompositions $\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ and $\mathfrak{B} = \mathfrak{B}^0 \oplus \mathfrak{B}^1$. Then, $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is the $\mathbb{Z}_2$-graded algebra whose even and odd parts are given by

$$(\mathfrak{A} \hat{\otimes} \mathfrak{B})^0 := (\mathfrak{A}^0 \otimes \mathfrak{B}^0) \oplus (\mathfrak{A}^1 \otimes \mathfrak{B}^1), \quad (\mathfrak{A} \hat{\otimes} \mathfrak{B})^1 := (\mathfrak{A}^1 \otimes \mathfrak{B}^0) \oplus (\mathfrak{A}^0 \otimes \mathfrak{B}^1),$$

and whose multiplication law reads as follows:

$$(a \otimes b^j) \cdot (a^i \otimes b) := (-1)^{ij} (a \cdot a^i) \otimes (b \cdot b^j), \quad a \in \mathfrak{A},\ b \in \mathfrak{B},\ a^i \in \mathfrak{A}^i,\ b^j \in \mathfrak{B}^j.$$

Proof By a classical Theorem of Lagrange, every finite-dimensional bilinear form can be diagonalized. Thus, the quadratic form q may be viewed as the sum of n one-dimensional quadratic forms, $\mathsf{q} = \mathsf{q}_1 \oplus \ldots \oplus \mathsf{q}_n$. Clearly, the tensor algebra of a 1-dimensional vector space W coincides with the polynomial ring generated by one element, $\mathscr{T}(W) = \mathbb{K}[a]$. Thus, the Clifford algebras of the 1-dimensional quadratic forms q_i on $\mathbb{K}$ are given by

$$Cl(V_i, \mathsf{q}_i) \cong \mathbb{K}[a]/\left\{a^2 - \mathsf{q}_i(v)1\right\}, \quad j(v) = a,$$

that is, they are 2-dimensional. Now, Proposition 5.1.7 implies the assertion. ■

Remark 5.1.9 (*Basis of a Clifford algebra*) Let (V, q) be an n-dimensional quadratic space. Since $V \subset Cl(V, \mathsf{q})$ generates $Cl(V, \mathsf{q})$ multiplicatively, any basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of V generates $Cl(V, \mathsf{q})$ multiplicatively as well. Viewed as elements[4] of the Clifford algebra, the elements $\mathbf{e}_i$ are subject to the following relations:

$$\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_i = 2\eta(\mathbf{e}_i, \mathbf{e}_j). \tag{5.1.6}$$

Here, η is the bilinear form of q, cf. formula (5.1.3). Thus, the 2^n elements

$$1, \mathbf{e}_{i_1} \cdot \ldots \cdot \mathbf{e}_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n, \; 1 \leq k \leq n,$$

span $Cl(V, \mathsf{q})$ and, by Corollary 5.1.8, they form a vector space basis. In conclusion, the relation (5.1.6) is defining for $Cl(V, \mathsf{q})$. ♦

Given a q-orthogonal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of V, by the above remark, the mapping

$$1 \mapsto 1, \quad \mathbf{e}_{i_1} \cdot \ldots \cdot \mathbf{e}_{i_k} \mapsto \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \tag{5.1.7}$$

yields a vector space isomorphism $Cl(V, \mathsf{q}) \cong \bigwedge V$, where $\bigwedge V$ denotes the exterior algebra over V. We show that this isomorphism does not depend on the choice of a basis. For that purpose, recall from Remark 2.7.9 the contraction mapping $\iota : V^* \to \mathrm{End}(\bigwedge V)$ and the operation of exterior multiplication ε.

Proposition 5.1.10 *As vector spaces, the Clifford algebra $Cl(V, \mathsf{q})$ and the exterior algebra $\bigwedge V$ are canonically isomorphic.*

Proof Consider the mapping

$$F : V \to \mathrm{End}(\textstyle\bigwedge V), \quad F(v)\alpha := v \wedge \alpha + \eta(v) \lrcorner \alpha, \tag{5.1.8}$$

where η is viewed as a mapping $\eta : V \to V^*$. One easily shows (Exercise 5.1.1):

$$F(v)^2\alpha = \mathsf{q}(v)\alpha, \quad \alpha \in \textstyle\bigwedge V. \tag{5.1.9}$$

[4] We omit the mapping j.

Thus, the universal property implies the existence of an algebra homomorphism $\hat{F} : Cl(V, \mathsf{q}) \to \mathrm{End}(\bigwedge V)$ which, composed with the evaluation mapping at the identity $1_\wedge$ of $\bigwedge V$, yields a linear mapping

$$\sigma : Cl(V, \mathsf{q}) \to \bigwedge V\,, \quad \sigma(a) := \hat{F}(a)\big(1_\wedge\big)\,. \tag{5.1.10}$$

It is easy to check (Exercise 5.1.1) that for a chosen orthogonal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of V, this mapping coincides with the mapping (5.1.7). Thus, σ is an isomorphism of vector spaces. ■

Remark 5.1.11

1. The isomorphism σ will be called the symbol mapping. Via σ, the parity automorphism p and the canonical anti-automorphism t are transported to $\bigwedge V$ in an obvious way. The inverse $\mathsf{c} : \bigwedge V \to Cl(V, \mathsf{q})$ of σ will be referred to as the quantization mapping. By (5.1.7), for a chosen q-orthogonal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of V, it is given by $\mathsf{c}(1) = 1$ and

$$\mathsf{c}(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) = \mathbf{e}_{i_1} \cdot \ldots \cdot \mathbf{e}_{i_k}\,, \quad 1 \le i_1 < \ldots < i_k \le n\,, \;\; 1 \le k \le n\,. \tag{5.1.11}$$

   In particular, we see that the $\mathbb{Z}$-grading of $\bigwedge V$ defined by the form degree corresponds to the vector space $\mathbb{Z}$-grading of $Cl(V, \mathsf{q})$ inherited from the tensor algebra.
2. The Clifford algebra has a natural increasing filtration $Cl(V, \mathsf{q}) = \bigcup_i Cl_i(V, \mathsf{q})$ defined by

$$Cl_i(V, \mathsf{q}) = \bigoplus_{k=0}^{i} \mathsf{c}(\textstyle\bigwedge^k V)\,,$$

   see [72] and [407] for further details.
3. For $\mathsf{q} = 0$, c and σ are algebra isomorphisms. ♦

In the remainder of this section, we study the special case of the real vector space $V = \mathbb{R}^{r+s}$, endowed with the pseudo-Euclidean quadratic form

$$\mathsf{q}(\mathbf{x}) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2\,, \tag{5.1.12}$$

where $\mathbf{x} = (x_1, \ldots, x_{r+s})$ in the standard basis of $\mathbb{R}^{r+s}$. In the sequel, this quadratic space will be also denoted by $\mathbb{R}^{r,s}$. For the corresponding Clifford algebra, we write $Cl_{r,s}$ and we call it the pseudo-orthogonal Clifford algebra of type (r, s). In particular, we put $Cl_n := Cl_{n,0}$ and $Cl_n^* := Cl_{0,n}$.

Remark 5.1.12 By Remark 5.1.9, $Cl_{r,s}$ is multiplicatively generated by any q-orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_{r+s}$ of $\mathbb{R}^{r+s}$ under the relations

$$\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_i = \begin{cases} 2\delta_{ij} & \text{for} \quad i \leq r\,, \\ -2\delta_{ij} & \text{for} \quad i > r\,. \end{cases} \tag{5.1.13}$$

♦

Let $\mathbb{K}(n)$ denote the algebra of $n \times n$-matrices with entries in $\mathbb{K}$. This is a real algebra for $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and a complex one for $\mathbb{K} = \mathbb{C}$.

Example 5.1.13 (*Low-dimensional Clifford algebras $Cl_{r,s}$*) By the universal property, an associative algebra $\mathfrak{A}$ of dimension 2^{r+s} is isomorphic to $Cl_{r,s}$ if there exists a linear mapping $F : \mathbb{R}^{r,s} \to \mathfrak{A}$ fulfilling

$$F(\mathbf{e}_i) \cdot F(\mathbf{e}_j) + F(\mathbf{e}_j) \cdot F(\mathbf{e}_i) = \begin{cases} 2\delta_{ij} & \text{for} \quad i \leq r \\ -2\delta_{ij} & \text{for} \quad i > r \end{cases}$$

Thus, to present $Cl_{r,s}$ explicitly as matrix algebras, it is enough to find such a mapping. This way, for the cases $r + s \leq 2$, one obtains the isomorphisms

$$Cl_{0,1} = \mathbb{C}\,, \quad Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}\,, \quad Cl_{0,2} = \mathbb{H}\,, \quad Cl_{2,0} = \mathbb{R}(2)\,, \quad Cl_{1,1} = \mathbb{R}(2)\,, \tag{5.1.14}$$

with F given as follows:

$$Cl_{0,1} : \; F(\mathbf{e}_1) = i\,, \quad Cl_{1,0} : \; F(\mathbf{e}_1) = (1, -1)\,,$$

$$Cl_{0,2} : \quad F(\mathbf{e}_1) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad F(\mathbf{e}_2) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

$$Cl_{1,1} : \quad F(\mathbf{e}_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F(\mathbf{e}_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$Cl_{2,0} : \quad F(\mathbf{e}_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The reader can check the condition (5.1.4) in each case (Exercise 5.1.2). ♦

Together with the Clifford algebras $Cl_{r,s}$, let us consider their complexifications $Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 5.1.14 *Let (V, q) be a real quadratic space and let $(V_{\mathbb{C}}, \mathsf{q}_{\mathbb{C}})$ be its complexification.*[5] *Then, the following isomorphism of complex algebras holds:*

$$Cl(V_{\mathbb{C}}, \mathsf{q}_{\mathbb{C}}) \cong Cl(V, \mathsf{q}) \otimes_{\mathbb{R}} \mathbb{C}\,. \tag{5.1.15}$$

[5] Here, $\mathsf{q}_{\mathbb{C}}(v \otimes z) = z^2 \mathsf{q}(v)$.

Proof Consider the mapping

$$F : V \otimes_{\mathbb{R}} \mathbb{C} \to Cl(V, \mathsf{q}) \otimes_{\mathbb{R}} \mathbb{C}\,, \quad F(v \otimes z) := j(v) \otimes z\,.$$

Then,

$$F(v \otimes z)^2 = j(v)^2 \otimes z^2 = \mathsf{q}(v)z^2 \cdot 1 \otimes 1 = \mathsf{q}(v \otimes z) \cdot 1\,,$$

and, thus, the universal property yields the assertion. ■

We denote $Cl_n^c := Cl(\mathbb{C}^n, \mathsf{q}_{\mathbb{C}})$. Since every non-degenerate quadratic form q over $\mathbb{C}$ can be written in some orthonormal basis as $\mathsf{q}(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2$, we have

$$Cl_n^c \cong Cl_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong Cl_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \ldots \cong Cl_{0,n} \otimes_{\mathbb{R}} \mathbb{C}\,, \tag{5.1.16}$$

that is, all $Cl_{r,s}$ are real forms of Cl_{r+s}^c.

The first of the following two propositions allows for an explicit calculation of the algebras $Cl_{r,s}$ as matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the second one is useful in representation theory.

Proposition 5.1.15 *For the pseudo-orthogonal Clifford algebras, the following isomorphisms hold:*

$$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}\,, \tag{5.1.17}$$

$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}\,, \tag{5.1.18}$$

$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}\,. \tag{5.1.19}$$

Proof We give the proof of the isomorphism (5.1.17). The proof of the remaining two assertions is similar and is, therefore, left to the reader (Exercise 5.1.3). Let $\mathbf{e}_1, \ldots, \mathbf{e}_{n+2}$ be a q-orthonormal basis of $\mathbb{R}^{0,n+2}$ generating $Cl_{0,n+2}$. Then, the first n of these vectors generate the algebras $Cl_{0,n}$ and $Cl_{n,0}$. Viewed as generators of $Cl_{n,0}$, they are denoted by $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$. We define

$$F : \mathbb{R}^{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}\,, \quad F(\mathbf{e}_i) := 1 \otimes \mathbf{e}_i\,, \; F(\mathbf{e}_k) := \mathbf{e}'_{k-2} \otimes \mathbf{e}_1 \cdot \mathbf{e}_2\,,$$

for $i = 1, 2$ and $3 \leq k \leq n + 2$. We calculate, for $i = 1, 2$,

$$F(\mathbf{e}_i)^2 = (1 \otimes \mathbf{e}_i) \cdot (1 \otimes \mathbf{e}_i) = 1 \otimes \mathbf{e}_i^2 = -1\,,$$

and, for $3 \leq k \leq n + 2$,

$$F(\mathbf{e}_k)^2 = (\mathbf{e}'_{k-2} \otimes \mathbf{e}_1 \cdot \mathbf{e}_2) \cdot (\mathbf{e}'_{k-2} \otimes \mathbf{e}_1 \cdot \mathbf{e}_2) = (\mathbf{e}'_{k-2})^2 \otimes \mathbf{e}_1 \cdot \mathbf{e}_2 \cdot \mathbf{e}_1 \cdot \mathbf{e}_2\,.$$

Since $(\mathbf{e}'_{k-2})^2 = 1$ and $\mathbf{e}_1 \cdot \mathbf{e}_2 \cdot \mathbf{e}_1 \cdot \mathbf{e}_2 = -\mathbf{e}_1^2 \cdot \mathbf{e}_2^2 = -1$, we get $F(\mathbf{e}_k)^2 = -1$. Thus, by the universal property, there exists an algebra homomorphism

$$\hat{F} : Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}$$

fulfilling $\hat{F} \circ j = F$, which is obviously surjective. By Corollary 5.1.8, $\dim Cl_{0,n+2} = \dim(Cl_{n,0} \otimes Cl_{0,2})$ and, thus, $\hat{F}$ is an isomorphism. ■

Proposition 5.1.16 *The following isomorphism holds:*

$$Cl^0_{r+1,s} \cong Cl_{s,r} \,. \tag{5.1.20}$$

Proof Let q and $\tilde{\mathsf{q}}$ be the quadratic forms in $Cl_{r+1,s}$ and $Cl_{s,r}$, respectively. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{r+s+1}$ be a q-orthonormal basis of $\mathbb{R}^{r+s+1}$ fulfilling $\mathsf{q}(\mathbf{e}_i) = 1$ for $1 \le i \le r+1$ and $\mathsf{q}(\mathbf{e}_i) = -1$ for $i > r$. Let $\mathbb{R}^{r+s} = \operatorname{span}\{\mathbf{e}_i : i \neq r+1\}$. We define

$$F : \mathbb{R}^{r+s} \to Cl^0_{r+1,s} \,, \quad F(\mathbf{e}_i) := \mathbf{e}_{r+1}\mathbf{e}_i$$

for $i \neq r+1$. Let $\mathbf{x} = \sum_{i \neq r+1} x_i \mathbf{e}_i \in \mathbb{R}^{r+s}$. Using $\mathbf{e}^2_{r+1} = +1$ and $\mathbf{e}_{r+1}\mathbf{e}_i = -\mathbf{e}_i\mathbf{e}_{r+1}$ for any $i \neq r+1$, we calculate

$$F(\mathbf{x})^2 = \sum_{i,j} x_i x_j \mathbf{e}_{r+1}\mathbf{e}_i\mathbf{e}_{r+1}\mathbf{e}_j = -\sum_{i,j} x_i x_j \mathbf{e}_i\mathbf{e}_j = -\mathsf{q}(\mathbf{x})1 = \tilde{\mathsf{q}}(\mathbf{x})1 \,.$$

Thus, by the universal property, there exists an algebra homomorphism $\hat{F} : Cl_{s,r} \to Cl^0_{r+1,s}$ which is easily seen to be an isomorphism. ■

Remark 5.1.17 Similarly, as in Proposition 5.1.16, one shows (Exercise 5.1.4)

$$Cl^0_{r,s+1} \cong Cl_{r,s} \,. \tag{5.1.21}$$

We conclude that $Cl^0_{r,s}$ and $Cl^0_{s,r}$ are isomorphic. ♦

Remark 5.1.18 (*Classification of pseudo-orthogonal Clifford algebras*) Recall the following elementary isomorphisms:

$$\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm) \,, \quad \mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(n) \,, \quad \mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H}(n) \,, \tag{5.1.22}$$

and

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \,, \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \,, \quad \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4) \,. \tag{5.1.23}$$

Now, using Proposition 5.1.15, together with (5.1.16) and the above isomorphisms, one may calculate iteratively all Clifford algebras $Cl_{r,s}$, starting from the isomorphisms given in (5.1.14). On the way, one finds the following periodicity isomorphisms (Exercise 5.1.5), which make the classification table finite:

$$Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0} \,, \quad Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8} \,. \tag{5.1.24}$$

For the final result, we refer the reader to Table 2 in Sect. 1.4 of [407]. ♦

As a simple consequence of the above discussion, we obtain the following.

Proposition 5.1.19 *The following isomorphisms hold:*

$$Cl^c_{n+2} \cong Cl^c_n \otimes_{\mathbb{C}} \mathbb{C}(2)\,, \quad Cl^c_{2n} \cong \mathbb{C}(2^n)\,, \quad Cl^c_{2n+1} \cong \mathbb{C}(2^n) \oplus \mathbb{C}(2^n)\,. \tag{5.1.25}$$

Proof By (5.1.16) and (5.1.17),

$$Cl^c_{n+2} \cong Cl_{0,n+2} \otimes_{\mathbb{R}} \mathbb{C} \cong (Cl_{n,0} \otimes Cl_{0,2}) \otimes_{\mathbb{R}} \mathbb{C} \cong (Cl_{n,0} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl_{0,2} \otimes_{\mathbb{R}} \mathbb{C})$$

and, thus, $Cl^c_{n+2} \cong Cl^c_n \otimes_{\mathbb{C}} Cl^c_2$. In particular, by (5.1.14), we obtain

$$Cl^c_2 \cong Cl_{2,0} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)\,,$$

and, thus, $Cl^c_{n+2} \cong Cl^c_n \otimes_{\mathbb{C}} \mathbb{C}(2)$ indeed. Now, again by (5.1.14), we have $Cl^c_1 \cong \mathbb{C} \oplus \mathbb{C}$. Using this, together with $Cl^c_2 \cong \mathbb{C}(2)$, and iterating $Cl^c_{n+2} \cong Cl^c_n \otimes_{\mathbb{C}} \mathbb{C}(2)$ we obtain

$$Cl^c_{2n} \cong \bigotimes^n \mathbb{C}(2) \cong \mathrm{End}\left(\bigotimes^n \mathbb{C}^2\right) \cong \mathrm{End}\left(\mathbb{C}^{2^n}\right),$$

and

$$Cl^c_{2n+1} \cong \left(\bigotimes^n \mathbb{C}(2)\right) \oplus \left(\bigotimes^n \mathbb{C}(2)\right) \cong \mathrm{End}\left(\mathbb{C}^{2^n}\right) \oplus \mathrm{End}\left(\mathbb{C}^{2^n}\right).$$

■

We note that the formulae contained in (5.1.25) in fact yield representations of Cl^c_n by endomorphisms on a complex vector space. These will be systematically studied in Sect. 5.3.

Remark 5.1.20 An explicit formula for the first of the isomorphisms in (5.1.25) can be easily deduced from the proof of Proposition 5.1.15. For later use, we provide explicit formulae for the second and the third one in terms of generators $\mathbf{e}_j$ fulfilling (5.1.13), see also [59]. Given $\mathbb{R}^{r,s}$, we denote

$$W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \tau(j) = \begin{cases} i & \text{for } j \leq r \\ 1 & \text{for } j > r \end{cases}$$

Then, for $n = r+s = 2k$, we define the mapping $\gamma_{2k} : \mathbb{R}^{r,s} \to \mathrm{End}\left(\mathbb{C}^{2^n}\right) \cong \bigotimes^n \mathbb{C}(2)$ by

$$\begin{aligned} \gamma_{2k}(\mathbf{e}_{2j-1}) &:= \tau(2j-1)\, W \otimes \ldots \otimes W \otimes U \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}\,, \\ \gamma_{2k}(\mathbf{e}_{2j}) &:= \tau(2j)\, W \otimes \ldots \otimes W \otimes V \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}\,, \end{aligned}$$

where the matrices U and V are at position j, respectively. It is easy to check (Exercise 5.1.6) that the matrices $\gamma(\mathbf{e}_j)$ fulfil the relations (5.1.13). Thus, by universality, γ

extends to the algebra isomorphism under consideration. Analogously, for $n = 2k+1$, we set

$$\gamma_{2k+1}(\mathbf{e}_j) := \big(\gamma_{2k}(\mathbf{e}_j), \gamma_{2k}(\mathbf{e}_j)\big) \ , \quad 1 \leq j \leq 2k \, ,$$
$$\gamma_{2k+1}(\mathbf{e}_n) := (iW \otimes \ldots \otimes W, -iW \otimes \ldots \otimes W) \ .$$

We stress that the above explicit presentation of the isomorphisms (5.1.25) is by no means unique. ♦

Example 5.1.21 (*Clifford algebra of Minkowski space*) Recall the Minkowski space (M, η) from Example I/4.5.9. In the above notation, $M = \mathbb{R}^{1,3}$ and $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $\mu, \nu = 0, 1, 2, 3$, in the standard basis $\{\mathbf{e}_\mu\}$ of $\mathbb{R}^4$. Thus, the Clifford algebra of the Minkowski space coincides with $Cl_{1,3}$. By Proposition 5.1.19, we have $Cl_4^c = Cl_{1,3} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(4)$. Thus, passing to the complexification, we can represent the generators of $Cl_{1,3}$ explicitly in terms of complex 4×4-matrices. One of the most convenient choices for the isomorphism $\gamma : Cl_4^c \to \mathbb{C}(4) = \mathrm{End}\left(\mathbb{C}^4\right)$ is as follows:

$$\gamma : M \to \mathrm{End}\left(\mathbb{C}^4\right) , \quad \gamma(\mathbf{e}_\mu) := \begin{bmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{bmatrix} \equiv \gamma_\mu \, , \tag{5.1.26}$$

where $\tilde{\sigma}_0 = \sigma_0$ and $\tilde{\sigma}_i = -\sigma_i$, $i = 1, 2, 3$. Here, σ_0 is the identity matrix and σ_i denote the Pauli matrices. It is easy to check (Exercise 5.1.6) that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbb{1} \, . \tag{5.1.27}$$

Thus, γ extends to the unique algebra isomorphism $Cl_4^c \to \mathrm{End}\left(\Delta_4\right)$ given by Proposition 5.1.19.[6] We note that, associated with (5.1.26), we have the following presentation of the generators of $Cl_{4,0}$:

$$\mathbf{e}_1 := \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix} , \quad \mathbf{e}_k := \begin{bmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{bmatrix} , \quad k = 2, 3, 4 \, . \tag{5.1.28}$$

♦

Exercises

5.1.1 Prove the formulae (5.1.11) and (5.1.10).

5.1.2 Prove the isomorphisms (5.1.14).

5.1.3 Prove the formulae (5.1.18) and (5.1.19).

5.1.4 Prove the statements of Remark 5.1.17.

5.1.5 Prove the isomorphisms in (5.1.24).

5.1.6 Check the relations for the γ-matrices in Remark 5.1.20 and Example 5.1.21.

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[6] For the explicit formula, see point 2 of Example 5.2.10.

5.2 Spinor Groups

In this section, we exhibit natural group structures within a given Clifford algebra $Cl(V, \mathsf{q})$. As before, we assume that $\dim V < \infty$ and that the field $\mathbb{K}$ has characteristic zero. For simplicity, we will often omit the Clifford algebra product symbol.

In the sequel, elements $v \in V$ fulfilling $\mathsf{q}(v) = 0$ will be called isotropic and elements fulfilling $\mathsf{q}(v) \neq 0$ will be referred to as anisotropic. Note that every anisotropic element is invertible, with the inverse given by $v^{-1} = v/\mathsf{q}(v)$. Thus, endowed with the multiplication from $Cl(V, \mathsf{q})$, the set $Cl(V, \mathsf{q})^*$ of invertible elements of $Cl(V, \mathsf{q})$ acquires a group structure. $Cl(V, \mathsf{q})^*$ will be referred to as the group of units of $Cl(V, \mathsf{q})$. Using the parity automorphism p, we define the following Lie subgroup, called the Clifford group of (V, q):

$$\Gamma(V, \mathsf{q}) := \left\{ a \in Cl(V, \mathsf{q})^* : \mathsf{p}(a)va^{-1} \in V \text{ for all } v \in V \right\} . \tag{5.2.1}$$

By this definition, the Clifford group comes with a natural representation

$$\widetilde{\mathrm{Ad}} : \Gamma(V, \mathsf{q}) \to \mathrm{Aut}(V) , \quad \widetilde{\mathrm{Ad}}(a)v := \mathsf{p}(a)va^{-1} , \tag{5.2.2}$$

called the twisted adjoint representation.

Lemma 5.2.1 *The twisted adjoint representation has the following properties.*

1. *For any $a \in \Gamma(V, \mathsf{q})$, we have $\widetilde{\mathrm{Ad}}(\mathsf{p}(a)) = \widetilde{\mathrm{Ad}}(a)$.*
2. *For every anisotropic element $v \in V$, the mapping $\widetilde{\mathrm{Ad}}(v) : V \to V$ is the reflection about the hyperplane in V orthogonal to v, that is, for all $w \in V$,*

$$\widetilde{\mathrm{Ad}}(v)w = w - 2\frac{\eta(v, w)}{\mathsf{q}(v)} v . \tag{5.2.3}$$

3. *If q is non-degenerate, then the kernel of $\widetilde{\mathrm{Ad}}$ coincides with the multiplicative group $\mathbb{K}^* \cdot 1$ of non-zero multiples of the identity in $Cl(V, \mathsf{q})$.*

Proof To prove the first assertion, we apply $-\mathsf{p}$ to $\widetilde{\mathrm{Ad}}(a)v = \mathsf{p}(a)va^{-1}$. This yields

$$\widetilde{\mathrm{Ad}}(a)v = -\mathsf{p}(\widetilde{\mathrm{Ad}}(a)v) = av\mathsf{p}(a^{-1}) = \widetilde{\mathrm{Ad}}(\mathsf{p}(a))v .$$

Next, since $v^2 = \mathsf{q}(v) \cdot 1$, we have $v^{-1} = v(\mathsf{q}(v))^{-1}$ and, thus,

$$\mathsf{q}(v)\widetilde{\mathrm{Ad}}(v)w = -\mathsf{q}(v)vwv^{-1} = -vwv = v^2w - 2\eta(v, w)v = \mathsf{q}(v)w - 2\eta(v, w)v .$$

Thus, (5.2.3) holds. It remains to prove the third assertion. Since q is non-degenerate, we can choose a q-orthogonal basis $\mathbf{e}_1, \ldots \mathbf{e}_n$ in V such that $\mathsf{q}(\mathbf{e}_i) \neq 0$ for all $i = 1, \ldots, n$. Let $a \in \ker\left(\widetilde{\mathrm{Ad}}\right)$. Then, for any $v \in V$, $\mathsf{p}(a)v = va$ and, thus,

$$va_0 = a_0v , \quad -va_1 = a_1v , \tag{5.2.4}$$

where a_0 and a_1 denote the even and odd parts of a, respectively. Using (5.1.6), we may write $a_0 = p_0 + \mathbf{e}_1 p_1$ where p_0 and p_1 are polynomials in the generators $\mathbf{e}_2, \dots \mathbf{e}_n$. Clearly, p_0 is even and p_1 is odd. Using (5.2.4) with $v = \mathbf{e}_1$, we calculate

$$\mathbf{e}_1 p_0 + \mathbf{e}_1^2 p_1 = \mathbf{e}_1(p_0 + \mathbf{e}_1 p_1) = (p_0 + \mathbf{e}_1 p_1)\mathbf{e}_1 = p_0 \mathbf{e}_1 + \mathbf{e}_1 p_1 \mathbf{e}_1 = \mathbf{e}_1 p_0 - \mathbf{e}_1^2 p_1 .$$

Thus, $\mathbf{e}_1^2 p_1 = \mathsf{q}(\mathbf{e}_1) p_1 = 0$ and, hence, $p_1 = 0$. This shows that a_0 does not contain $\mathbf{e}_1$. Proceeding inductively, we obtain that a_0 does not contain any of the generators $\mathbf{e}_i$, that is, $a_0 = k \cdot 1$ where $k \in \mathbb{K}$. In the same way, one shows that a_1 does not contain any of the generators $\mathbf{e}_i$. Thus, being odd it must vanish. We conclude that $a = k \cdot 1$. Since, by assumption $a \neq 0$, we have $a \in \mathbb{K}^* \cdot 1$. ∎

Remark 5.2.2 By point 1 of Lemma 5.2.1, for every $v \in V$, we have

$$\widetilde{\mathrm{Ad}}(a^{-1}\mathsf{p}(a))v = v ,$$

and, by point 3 of that lemma, we conclude $\mathsf{p}(a) = ka$ with $k \in \mathbb{K}^*$. Moreover, since p is involutive, we obtain $k^2 = 1$. Now, by assumption, $\mathbb{K}$ has characteristic zero and, thus, the only solutions of this equation are $k = \pm 1$. We conclude that any element of $\Gamma(V, \mathsf{q})$ has a definite parity, that is, it is either even or odd. ♦

Let us denote by $\mathrm{O}(V, \mathsf{q})$ the orthogonal group of the quadratic space (V, q), that is, the subgroup of $\mathrm{Aut}(V, \mathsf{q})$ leaving q invariant. Correspondingly, let $\mathrm{SO}(V, \mathsf{q}) \subset \mathrm{O}(V, \mathsf{q})$ be the subgroup of transformations of determinant 1.

Theorem 5.2.3 *Let (V, q) be a quadratic space with q non-degenerate. Then, the twisted adjoint representation defines the short exact sequence*

$$1 \to \mathbb{K}^* \cdot 1 \to \Gamma(V, \mathsf{q}) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V, \mathsf{q}) \to 1 . \tag{5.2.5}$$

Proof By point 3 of Lemma 5.2.1, the kernel of $\widetilde{\mathrm{Ad}}$ coincides with $\mathbb{K}^* \cdot 1$. We show $\widetilde{\mathrm{Ad}}(\Gamma(V, \mathsf{q})) \subset \mathrm{O}(V, \mathsf{q})$. Using point 1 of Lemma 5.2.1, for any $v, w \in V$ and any $a \in \Gamma(v, \mathsf{q})$, we calculate

$$\begin{aligned} 2\eta(\widetilde{\mathrm{Ad}}(a)v, \widetilde{\mathrm{Ad}}(a)w) &= \widetilde{\mathrm{Ad}}(a)v \cdot \widetilde{\mathrm{Ad}}(a)w + \widetilde{\mathrm{Ad}}(a)w \cdot \widetilde{\mathrm{Ad}}(a)v \\ &= \widetilde{\mathrm{Ad}}(a)v \cdot \widetilde{\mathrm{Ad}}(\mathsf{p}(a))w + \widetilde{\mathrm{Ad}}(a)w \cdot \widetilde{\mathrm{Ad}}(\mathsf{p}(a))v \\ &= \mathsf{p}(a)(v \cdot w + w \cdot v)\mathsf{p}(a^{-1}) \\ &= 2\eta(v, w) . \end{aligned}$$

Finally, by the Cartan–Dieudonné Theorem, for a non-degenerate quadratic vector space (V, q), any element $R \in \mathrm{O}(V, \mathsf{q})$ can be written as a product of k reflections, $R = R_1 \dots R_k$ with $k \leq \dim V$.[7] But, by point 2 of Lemma 5.2.1, every reflection

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[7] See e.g. [23] or [439]. A reflection is, by definition, an orthogonal transformation $R \in \mathrm{O}(V, \mathsf{q})$ whose fixed point set $\ker(R - \mathbb{1})$ has codimension 1. It can be shown that any reflection is of the form given by the right hand side of (5.2.3) with v unique up to a non-zero scalar.

in (V, q) through an anisotropic vector belongs to $\widetilde{\mathrm{Ad}}(\Gamma(V, \mathsf{q}))$. Thus, since $\widetilde{\mathrm{Ad}} : \Gamma(V, \mathsf{q}) \to \mathrm{O}(V, \mathsf{q})$ is a homomorphism, there exist anisotropic elements $v_1, \dots, v_k$ in V such that $R_i = \widetilde{\mathrm{Ad}}(v_i)$ and, thus, $R = \widetilde{\mathrm{Ad}}(a)$ with $a = v_1 \dots v_k$. This implies that $\widetilde{\mathrm{Ad}}$ is surjective. ■

Remark 5.2.4

1. Since $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{K}^* \cdot 1$ and $\mathrm{im}(\widetilde{\mathrm{Ad}}) = \mathrm{O}(V, \mathsf{q})$, any element $a \in \Gamma(V, \mathsf{q})$ must be a product of anisotropic elements v_i of V, that is, $a = v_1 \dots v_k$ with $k \leq \dim V$. Clearly, $\mathsf{p}(a) = (-1)^k a$.
2. We denote
$$\Gamma^0(V, \mathsf{q}) := \Gamma(V, \mathsf{q}) \cap Cl^0(V, \mathsf{q})^*$$
and call it the special Clifford group. It clearly consists of products $v_1 \dots v_k$ with k even and we have the following short exact sequence induced from (5.2.5):
$$1 \to \mathbb{K}^* \cdot 1 \to \Gamma^0(V, \mathsf{q}) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V, \mathsf{q}) \to 1\,. \tag{5.2.6}$$
♦

Next, recall the canonical anti-automorphism t of $Cl(V, \mathsf{q})$ constructed in the proof of Proposition 5.1.5. By point 3 of Remark 5.1.6, it commutes with the parity automorphism p. The following is a direct consequence of the definition of $\Gamma(V, \mathsf{q})$ and is, therefore, left to the reader (Exercise 5.2.1).

Lemma 5.2.5 *The mappings* p *and* t *induce an automorphism and an anti-automorphism of* $\Gamma(V, \mathsf{q})$*, respectively.* ■

For any $a \in Cl(V, \mathsf{q})$, we define the anti-automorphism
$$a \mapsto \tilde{a} := \mathsf{t} \circ \mathsf{p}(a)\,. \tag{5.2.7}$$

Clearly, $\widetilde{ab} = \tilde{b}\tilde{a}$ and $\tilde{\tilde{a}} = a$. Correspondingly, we have a natural norm mapping
$$N : Cl(V, \mathsf{q}) \to Cl(V, \mathsf{q})\,, \quad N(a) := a\tilde{a}\,. \tag{5.2.8}$$

Note that, for any $v \in V$,
$$N(v) = -\mathsf{q}(v)\,. \tag{5.2.9}$$

Lemma 5.2.6 *Let* (V, q) *be a quadratic space with* q *non-degenerate. Then, the restriction of* N *to* $\Gamma(V, \mathsf{q})$ *is a group homomorphism* $\Gamma(V, \mathsf{q}) \to \mathbb{K}^* \cdot 1$. *Moreover,* $N(\mathsf{p}(a)) = N(a)$ *for any* $a \in \Gamma(V, \mathsf{q})$.

Proof Let $a \in \Gamma(V, \mathsf{q})$. Then, $\mathsf{p}(a) v a^{-1} \in V$ for any $v \in V$. Applying t, we obtain $\mathsf{t}(a)^{-1} v \tilde{a} = \mathsf{p}(a) v a^{-1}$, because t is the identity on V. Using this, we have

$$v = \mathsf{t}(a)\mathsf{p}(a)v(\tilde{a}a)^{-1} = \mathsf{p}(\tilde{a}a)v(\tilde{a}a)^{-1} = \widetilde{\mathrm{Ad}}(\tilde{a}a)v\,,$$

that is, $\tilde{a}a \in \ker\left(\widetilde{\mathrm{Ad}}\right)$. By Lemma 5.2.5, we also have $N(a) = a\tilde{a} \in \ker\left(\widetilde{\mathrm{Ad}}\right)$ and Theorem 5.2.3 implies $N(\Gamma(V,\mathsf{q})) \subset \mathbb{K}^* \cdot 1$.

We prove that the restriction of N is a homomorphism. Using $N(\Gamma(V,\mathsf{q})) \subset \mathbb{K}^* \cdot 1$, for any $a, b \in \Gamma(V,\mathsf{q})$, we calculate

$$N(ab) = ab\tilde{b}\tilde{a} = aN(b)\tilde{a} = N(a)N(b)\,.$$

Finally, again using $N(\Gamma(V,\mathsf{q})) \subset \mathbb{K}^* \cdot 1$, for any $a \in \Gamma(V,\mathsf{q})$,

$$N(\mathsf{p}(a)) = \mathsf{p}(a)\mathsf{p}(\tilde{a}) = \mathsf{p}(a\tilde{a}) = N(a)\,.$$

∎

We define[8]

$$\mathrm{Pin}(V,\mathsf{q}) := \{a \in \Gamma(V,\mathsf{q}) : N(a) = 1\} \tag{5.2.10}$$

and

$$\mathrm{Spin}(V,\mathsf{q}) := \mathrm{Pin}(V,\mathsf{q}) \cap \Gamma^0(V,\mathsf{q})\,. \tag{5.2.11}$$

By Lemma 5.2.6, the restriction of N to $\Gamma(V,\mathsf{q})$ is a group homomorphism $\Gamma(V,\mathsf{q}) \to \mathbb{K}^* \cdot 1$. Thus, $\mathrm{Pin}(V,\mathsf{q})$ and $\mathrm{Spin}(V,\mathsf{q})$ are normal subgroups of $\Gamma(V,\mathsf{q})$ and $\Gamma^0(V,\mathsf{q})$, respectively.

Definition 5.2.7 The groups $\mathrm{Pin}(V,\mathsf{q})$ and $\mathrm{Spin}(V,\mathsf{q})$ will be referred to as the pin group and the spin group of (V,q), respectively.

In general, the restrictions of $\widetilde{\mathrm{Ad}}$ to $\mathrm{Pin}(V,\mathsf{q})$ and $\mathrm{Spin}(V,\mathsf{q})$ are not surjective onto $\mathrm{O}(V,\mathsf{q})$ and $\mathrm{SO}(V,\mathsf{q})$, respectively. However, for a special class of base fields, called spin fields, surjectivity holds. In particular, $\mathbb{R}$ and $\mathbb{C}$ are spin fields. We refer to [407] for a detailed discussion.

Let us consider the Clifford algebra $Cl_{r,s}$ of the real vector space $V = \mathbb{R}^{r,s}$ endowed with the pseudo-Euclidean quadratic form given by (5.1.12) in some detail. In this case, we denote the group of units, the Clifford group, the pin group and the spin group by, respectively, $Cl^*_{r,s}$, $\Gamma_{r,s}$, $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$. Correspondingly, the orthogonal and the special orthogonal groups are denoted by $\mathrm{O}_{r,s}$ and $\mathrm{SO}_{r,s}$, respectively. We also write $\mathrm{Pin}(n) = \mathrm{Pin}_{n,0}$ and $\mathrm{Spin}(n) = \mathrm{Spin}_{n,0}$. Since $Cl_{r,s}$ is a finite-dimensional associative $\mathbb{R}$-algebra, $Cl^*_{r,s}$ is a Lie group with a global chart given by the natural inclusion mapping. By construction, $\Gamma_{r,s}$, $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$ are Lie subgroups of $Cl^*_{r,s}$. By Remark 5.1.17, $Cl^0_{r,s}$ and $Cl^0_{s,r}$ are isomorphic. This implies that $\mathrm{Spin}_{r,s}$ and $\mathrm{Spin}_{s,r}$ are isomorphic, too.[9] Theorem 5.2.3 implies the following.

[8]For $\mathbb{K} = \mathbb{R}$, one often defines $\mathrm{Pin}(V,\mathsf{q})$ by the condition $N(a) = \pm 1$. Then, Theorem 5.2.3 implies surjective mappings onto $\mathrm{O}(V,\mathsf{q})$ and $\mathrm{SO}(V,\mathsf{q})$, respectively. This leads to an obvious modification of Corollary 5.2.8.

[9]Note, however, that $\mathrm{Pin}_{r,s}$ and $\mathrm{Pin}_{s,r}$ are in general not isomorphic.

Corollary 5.2.8 *For every pair* (r, s)*,* $\mathrm{Spin}_{r,s}$ *is a double covering of the identity component* $\mathrm{SO}^0_{r,s}$*, that is, there is an exact sequence*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{r,s} \to \mathrm{SO}^0_{r,s} \to 1\,. \tag{5.2.12}$$

For $r \geq 2$ *or* $s \geq 2$*, the group* $\mathrm{Spin}_{r,s}$ *is connected.*

Proof Since the condition $N(a) = 1$ yields a normalization of generators only, the existence of the exact sequences is a direct consequence of Theorem 5.2.3. In particular, the intersection of $\ker(\widetilde{\mathrm{Ad}})$ with $\mathrm{Pin}_{r,s}$ is clearly $\mathbb{Z}_2$. It remains to prove the second assertion. The cases $(r, s) = (0, 1)$ and $(r, s) = (1, 0)$ are clearly trivial. For $(r, s) = (1, 1)$ one obtains $\mathrm{SO}^0_{1,1} = \mathbb{R}_+$ and $\mathrm{Spin}_{1,1} = \mathbb{Z}_2 \times \mathbb{R}_+$ which is disconnected. Now, assume $r \geq 2$ or $s \geq 2$. By (5.2.12), the kernel of $\mathrm{Spin}_{r,s} \to \mathrm{SO}^0_{r,s}$ is $\{1, -1\}$. Thus, it is enough to construct a path joining 1 and -1 in $\mathrm{Spin}_{r,s}$. By the above assumption, $\mathbb{R}^{r,s}$ contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or to $\mathbb{R}^{0,2}$. Thus, there exist two anisotropic orthogonal vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^{r,s}$ fulfilling $\mathsf{q}(\mathbf{e}_1) = \mathsf{q}(\mathbf{e}_2) = \pm 1$. Now,

$$t \mapsto \gamma(t) = (\mathbf{e}_1 \cos(t) + \mathbf{e}_2 \sin(t))(\mathbf{e}_2 \sin(t) - \mathbf{e}_1 \cos(t))\,, \quad t \in [0, \frac{\pi}{2}]\,,$$

is a continuous path with the required property. ■

Remark 5.2.9 The spin groups are, in general, not simply connected. Using the fact that $\mathrm{SO}^0_{r,s}$ is homotopic to the maximal compact subgroup $\mathrm{SO}(r) \times \mathrm{SO}(s)$, one obtains $\pi_1(\mathrm{SO}^0_{r,s}) = \pi_1(\mathrm{SO}(r)) \oplus \pi_1(\mathrm{SO}(s))$. Then, using

$$\pi_1(\mathrm{SO}(r)) = \begin{cases} 0 & \text{for} \quad r = 1 \\ \mathbb{Z} & \text{for} \quad r = 2 \\ \mathbb{Z}_2 & \text{for} \quad r > 2 \end{cases}$$

one can calculate $\pi_1(\mathrm{SO}^0_{r,s})$ for any pair (r, s). Next, using the natural embeddings $\mathrm{SO}(r) \times \mathrm{SO}(s) \to \mathrm{SO}^0_{r,s}$, together with the corresponding embeddings on the level of the spin group, one can calculate $\pi_1(\mathrm{Spin}_{r,s})$ as well, see [59] for a complete list. If both $r > 2$ and $s > 2$, then the fundamental group of $\mathrm{SO}_{r,s}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and, thus $\pi_1(\mathrm{Spin}_{r,s}) = \mathbb{Z}_2$ in that case. We conclude that the spin group is simply connected and, thus, that the covering $\lambda : \mathrm{Spin}_{r,s} \to \mathrm{SO}^0_{r,s}$ is universal in the cases $r > 2,\ s = 0, 1$, and $r = 0, 1\,,\ s > 2$ only. ♦

By Proposition 5.1.16, we have $\mathrm{Spin}_{r,s} \subset Cl^0_{r,s} \cong Cl_{s,r-1}$ for any $r \geq 1$. Thus, there are two possibilities for explicit matrix realizations of $\mathrm{Spin}_{r,s}$. We illustrate this for the spin group of the Minkowski space. Details are left to the reader (Exercise 5.2.3).

Example 5.2.10 (*Spin group of the Minkowski space*) We take up Example 5.1.21. Here, we consider the spin group $\mathrm{Spin}_{1,3} \subset Cl^0_{1,3} \cong Cl_{3,0}$ of (M, η).

1. We construct $\mathrm{Spin}_{1,3} \subset Cl_{3,0}$. By (5.1.18) and (5.1.14), $Cl_{3,0} \cong \mathbb{C}(2) = \mathrm{End}(\mathbb{C}^2)$. In terms of generators, this isomorphism reads:

$$\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Representing the elements $\{\mathbb{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$ of the vector space basis of $Cl_{3,0}$ in this way, one easily calculates:

$$Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \mathsf{p}(Z) = \begin{bmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{bmatrix}, \quad Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \tilde{Z} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $a, b, c, d \in \mathbb{C}$. Thus,

$$Z \cdot \mathsf{p}(Z)^\dagger = \det Z \cdot \mathbb{1}, \quad N(Z) = \det Z \cdot \mathbb{1}.$$

The second of these equations reduces $\mathbb{C}(2)$ to $\mathrm{SL}(2, \mathbb{C})$. We identify

$$M \to H(2, \mathbb{C}), \quad \mathbf{x} \mapsto \mathbf{x}_* := x^\mu \sigma_\mu,$$

where $H(2, \mathbb{C})$ is the space of Hermitean (2×2)-matrices, cf. Example I/5.1.13. For $g \in \mathrm{SL}(2, \mathbb{C})$, we have $g \cdot \mathsf{p}(g)^\dagger = \mathbb{1}$. Thus, via the automorphism $g \to (g^{-1})^\dagger$ of $\mathrm{SL}(2, \mathbb{C})$, the twisted adjoint representation may be identified with

$$\widetilde{\mathrm{Ad}}(g)\mathbf{x}_* = g\,\mathbf{x}_*\,g^\dagger, \quad g \in \mathrm{SL}(2, \mathbb{C}).$$

Finally, note that the hermiticity of $\mathbf{x}_*$ implies the hermiticity of $\widetilde{\mathrm{Ad}}(g)\mathbf{x}_*$ for any $g \in \mathrm{SL}(2, \mathbb{C})$. Thus, we obtain

$$\mathrm{Spin}_{1,3} \cong \mathrm{SL}(2, \mathbb{C}) \tag{5.2.13}$$

realized in $\mathrm{End}(\mathbb{C}^2)$. This is one of the special isomorphisms for low-dimensional spin groups which will be further discussed below. In this presentation, the universal covering $\lambda : \mathrm{Spin}_{1,3} \to \mathrm{SO}^0_{1,3}$ is given by $(\lambda(g)\mathbf{x})_* = g\,\mathbf{x}_*\,g^\dagger$, cf. Example I/5.1.13. Restricting λ to the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$, one obtains the universal covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, see Example I/5.1.11. This, together with (2.8.2), proves

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2), \quad \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{5.2.14}$$

2. We construct $\mathrm{Spin}_{1,3} \subset Cl^0_{1,3}$. By (5.1.19) and (5.1.14), $Cl_{1,3} \cong Cl_{0,2} \otimes Cl_{1,1} = \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2)$. The latter may be identified with a subalgebra of $\mathbb{C}(4)$,

$$Cl_{1,3} \cong \left\{ Z = \begin{bmatrix} z & w \\ w' & z' \end{bmatrix} : \; z, w \in \mathbb{C}(2) \right\}$$

via the mapping

$$\gamma : \mathbb{R}^{1,3} \to \mathbb{C}(4)\,, \quad \mathbf{x} \mapsto \gamma(\mathbf{x}) = \begin{bmatrix} 0 & \mathbf{x}_* \\ \mathbf{x}^* & 0 \end{bmatrix},$$

cf. (5.1.26). Here,[10] $\mathbf{x}^* = x^\mu \tilde{\sigma}_\mu$, $z = z^\mu \sigma_\mu$ and $z' = \bar{z}^\mu \tilde{\sigma}_\mu$ (and the same for w). One easily calculates

$$\mathsf{p}(Z) = \begin{bmatrix} z & -w \\ -w' & z' \end{bmatrix}.$$

By (5.2.11), we must require $N(Z) = 1$ and $Z \in \Gamma^0(\mathbb{R}^{1,3})$. The latter implies $w = 0$ and then, by point 1,

$$N(Z) = \begin{bmatrix} \det(z) & 0 \\ 0 & \overline{\det(z)} \end{bmatrix}.$$

Thus, we obtain $\det(z) = 1$. Now, applying the twisted adjoint representation for Z fulfilling $w = 0$ and $\det(z) = 1$, we obtain

$$\mathbf{x}_* \mapsto \widetilde{\mathrm{Ad}}(Z)\mathbf{x}_* = z\,\mathbf{x}_*\,z^\dagger\,.$$

Clearly, the hermiticity of $\mathbf{x}$ implies the hermiticity of $z\,\mathbf{x}\,z^\dagger$ and, thus,

$$\mathrm{Spin}_{1,3} = \left\{ Z = \begin{bmatrix} g & 0 \\ 0 & \dot{g} \end{bmatrix} : \; g \in \mathrm{SL}(2, \mathbb{C}) \right\}, \tag{5.2.15}$$

where $\dot{g} = (g^\dagger)^{-1}$. ♦

Example 5.2.11 (*Low-dimensional spin groups*) The following isomorphisms between low-dimensional spin groups and classical Lie groups can be confirmed by analogous arguments as in Example 5.2.10. For the compact spin groups, we have[11]

$$\begin{aligned} \mathrm{Spin}(2) &\cong \mathrm{U}(1)\,, \\ \mathrm{Spin}(3) &\cong \mathrm{SU}(2)\,, \\ \mathrm{Spin}(4) &\cong \mathrm{SU}(2) \times \mathrm{SU}(2)\,, \\ \mathrm{Spin}(5) &\cong \mathrm{Sp}(2)\,, \\ \mathrm{Spin}(6) &\cong \mathrm{SU}(4)\,. \end{aligned}$$

[10] See Example 5.1.21 for the notation.

[11] The first identity is trivial, the second and the third one were shown in Example 5.2.10 and the remaining two will be shown in Example 5.3.22.

For a discussion of Spin(7), Spin(8) and relations between spin groups and exceptional groups we refer to [8, 9, 286, 407, 439]. For the non-compact spin groups, we have[12]

$$
\begin{aligned}
&\mathrm{Spin}_{2,1} \cong \mathrm{SL}(2,\mathbb{R})\,,\\
&\mathrm{Spin}_{1,3} \cong \mathrm{SL}(2,\mathbb{C})\,,\ \ \mathrm{Spin}_{2,2} \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\\
&\mathrm{Spin}_{1,4} \cong \mathrm{Sp}_{1,1}(\mathbb{H})\,,\ \ \mathrm{Spin}_{2,3} \cong \mathrm{Sp}(4,\mathbb{R})\,,\\
&\mathrm{Spin}_{1,5} \cong \mathrm{SL}(2,\mathbb{H})\,,\ \ \mathrm{Spin}_{2,4} \cong \mathrm{SU}(2,2)\,,\ \ \mathrm{Spin}_{3,3} \cong \mathrm{SL}(4,\mathbb{R})\,,
\end{aligned}
$$

where $\mathrm{Sp}_{1,1}(\mathbb{H}) = \left\{g \in \mathbb{H}(2) : \bar{g}^{\mathrm{T}}\sigma_3 g = \sigma_3\right\}$. See [517] for detailed proofs. ♦

Next, let us consider the case $V = (\mathbb{C}^n, \mathsf{q})$ where q is the quadratic form given by the standard Hermitean form on $\mathbb{C}^n$. We denote $\mathrm{Pin}(n,\mathbb{C}) = \mathrm{Pin}(\mathbb{C}^n, \mathsf{q})$ and $\mathrm{Spin}(n,\mathbb{C}) = \mathrm{Spin}(\mathbb{C}^n, \mathsf{q})$. The following statements are left to the reader (Exercise 5.2.4).

Proposition 5.2.12 *The groups* $\mathrm{Pin}(n,\mathbb{C})$ *and* $\mathrm{Spin}(n,\mathbb{C})$ *are double covers of* $\mathrm{O}(n,\mathbb{C})$ *and* $\mathrm{SO}(n,\mathbb{C})$*, respectively. Moreover,* $\mathrm{Spin}(n,\mathbb{C})$ *is the universal covering of* $\mathrm{SO}(n,\mathbb{C})$ *and* $\mathrm{Spin}(n)$ *is its maximal compact subgroup.*

Note that $Cl_n^c = Cl_n \otimes \mathbb{C}$ contains both $\mathrm{Spin}(n) \subset Cl_n \otimes 1$ and $S^1 \cong \mathrm{U}(1) \subset 1 \otimes \mathbb{C}$.

Definition 5.2.13 (*Complex spin group*) The complex spin group[13] $\mathrm{Spin}^c(n)$ is the subgroup of $Cl_n \otimes \mathbb{C}$ generated by $\mathrm{Spin}(n)$ and by $\mathrm{U}(1)$.

Since obviously $\mathrm{Spin}(n) \cap \mathrm{U}(1) = \{1, -1\}$, we have an isomorphism

$$
\mathrm{Spin}^c(n) \cong (\mathrm{Spin}(n) \times \mathrm{U}(1))\,/\{\pm 1\} \equiv \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)\,, \tag{5.2.16}
$$

that is, elements of $\mathrm{Spin}^c(n)$ are equivalence classes $[(g,z)]$ of pairs $(g,z) \in \mathrm{Spin}(n) \times \mathrm{U}(1)$ under the equivalence relation $(g,z) \sim (-g,-z)$. Note that Corollary 5.2.8 immediately implies the following exact sequence

$$
1 \to \mathbb{Z}_2 \to \mathrm{Spin}^c(n) \xrightarrow{p} \mathrm{SO}(n) \times \mathrm{U}(1) \to 1\,, \tag{5.2.17}
$$

where

$$
p : \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)\,, \quad (g,z) \mapsto (\rho(g), z^2)\,, \tag{5.2.18}
$$

[12] Recall that, by Corollary 5.2.8, for $r \geq 2$ or $s \geq 2$, the group $\mathrm{Spin}_{r,s}$ is connected. Also recall that $\mathrm{Spin}_{r,s} = \mathrm{Spin}_{s,r}$.

[13] Also called the Spin^c-group.

and $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double covering given by (5.2.12). As an immediate consequence of this sequence, we obtain

$$\pi_1(\mathrm{Spin}^c(n)) \cong \mathbb{Z}\,. \tag{5.2.19}$$

Now, let $n = 2k$. Recall from Example 2.2.19 that we can view $\mathrm{U}(k)$ as a subgroup of $\mathrm{SO}(2n)$. Let

$$f : \mathrm{U}(k) \to \mathrm{SO}(2k) \times \mathrm{U}(1)\,, \quad f(a) := (a, \det(a)), \tag{5.2.20}$$

be the group homomorphism induced by this embedding. The following proposition shows that this homomorphism admits a natural lift to the Spin^c-group.

Proposition 5.2.14 *There exists a homomorphism* $F : \mathrm{U}(k) \to \mathrm{Spin}^c(2k)$ *such that*

$$\begin{tikzcd} & \mathrm{Spin}^c(2k) \arrow[d, "p"] \\ \mathrm{U}(k) \arrow[ur, "F"] \arrow[r, "f"'] & \mathrm{SO}(2k) \times \mathrm{U}(1)\,. \end{tikzcd}$$

Proof Given an element $a \in \mathrm{U}(k)$, choose a unitary basis $(\mathbf{e}_1, \dots, \mathbf{e}_k)$ in $\mathbb{C}^k$ such that

$$a = \mathrm{diag}\{e^{i\vartheta_1}, \dots, e^{i\vartheta_k}\}\,.$$

Let $\mathsf{J} : \mathbb{C}^k \to \mathbb{C}^k$ be the complex structure of $\mathbb{C}^k$. Then, $\mathbf{e}_j$ and $\mathsf{J}(\mathbf{e}_j)$ belong to Cl_n^c. We define

$$F(a) := \prod_{j=1}^{k} \big(\cos(\vartheta_j/2) + \sin(\vartheta_j/2)\mathbf{e}_j\mathsf{J}(\mathbf{e}_j)\big) \exp\left(\frac{i}{2}\sum_{j=1}^{k}\vartheta_j\right)\,. \tag{5.2.21}$$

It is easy to check that this is a group homomorphism (Exercise 5.2.6). By direct inspection, under this mapping, the above diagram becomes commutative. ■

Identify $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and consider the natural embeddings $\mathrm{SO}(n) \to \mathrm{SO}(n+2)$ and $\mathrm{SO}(2) \to \mathrm{SO}(n+2)$ induced from the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$. Correspondingly, $\mathrm{Spin}(n)$ and $\mathrm{U}(1) \cong \mathrm{SO}(2)$ may by viewed as subgroups of $\mathrm{Spin}(n+2)$. The intersection of these subgroups is $\{\pm 1\}$. This implies the existence of an injective homomorphism $f : \mathrm{Spin}^c(n) \to \mathrm{Spin}(n+2)$ such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Spin}^c(n) \arrow[r, "f"] \arrow[d, "p"'] & \mathrm{Spin}(n+2) \arrow[d, "\lambda"] \\ \mathrm{SO}(n) \times \mathrm{SO}(2) \arrow[r, "\iota"] & \mathrm{SO}(n+2) \end{tikzcd} \tag{5.2.22}$$

Finally, we discuss Lie algebra structures in $Cl(V, \mathsf{q})$. We assume that $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and that q be non-degenerate. Since $Cl(V, \mathsf{q})$ is an associative $\mathbb{K}$-algebra, it carries a natural Lie algebra structure. We denote its Lie bracket by $[\cdot, \cdot]$. Moreover, the group of units, the Clifford group, the pin group and the spin group are Lie groups. The Lie algebra $cl(V, \mathsf{q})^*$ of the group of units $Cl(V, \mathsf{q})^*$ clearly coincides with $Cl(V, \mathsf{q})$ viewed as a Lie algebra and there is an exponential mapping given by the usual exponential series (Exercise 5.2.5),

$$\exp : cl(V, \mathsf{q})^* \to Cl(V, \mathsf{q})^* , \quad \exp(A) = \frac{1}{n!} \sum_n A^n . \tag{5.2.23}$$

Using this, we can calculate the Lie algebra of the Clifford group. Limiting our attention to the special Clifford group $\Gamma^0(V, \mathsf{q})$, we obtain

$$Lie(\Gamma^0(V, \mathsf{q})) = \left\{A \in Cl^0(V, \mathsf{q}) : Av - vA \in V \text{ for all } v \in V\right\} .$$

Since, under the above assumptions, the restriction of $\widetilde{\mathrm{Ad}}$ to both the pin and the spin group[14] are covering homomorphisms onto subgroups of full dimension of the corresponding orthogonal groups, their Lie algebras clearly coincide with the Lie algebra of the orthogonal group. Let us denote the Lie algebras of the spin group and of the orthogonal group by $\mathrm{spin}(V, \mathsf{q})$ and $\mathfrak{o}(V, \mathsf{q})$, respectively. Consider the subspace

$$Cl_2(V, \mathsf{q}) := \operatorname{span}\left\{\mathbf{e}_i\mathbf{e}_j : 1 \le i < j \le \dim V\right\} \subset Cl(V, \mathsf{q}) ,$$

where $\{\mathbf{e}_i\}$ is a q-orthogonal basis of V. By (5.1.11) and by the defining relations (5.1.2), $Cl_2(V, \mathsf{q})$ is a Lie subalgebra of $Cl(V, \mathsf{q})$ with Lie bracket

$$[\mathbf{e}_i\mathbf{e}_j, \mathbf{e}_k\mathbf{e}_l] = 2\mathbf{e}_i(\eta_{kj}\mathbf{e}_l - \eta_{lj}\mathbf{e}_k) + 2(\eta_{ki}\mathbf{e}_l - \eta_{li}\mathbf{e}_k)\mathbf{e}_j , \tag{5.2.24}$$

where $\eta_{rs} = \eta(\mathbf{e}_r, \mathbf{e}_s)$. By (5.1.11), $\mathsf{c}(\bigwedge^2 V) \cong Cl_2(V, \mathsf{q})$ as vector spaces. Thus, we may endow $\bigwedge^2 V$ with the structure of a Lie algebra by setting

$$[\alpha, \beta]_{\wedge^2 V} := \mathsf{c}^{-1} \circ [\mathsf{c}(\alpha), \mathsf{c}(\beta)] . \tag{5.2.25}$$

Proposition 5.2.15 *The image of the mapping*[15]

$$\psi : \bigwedge^2 V \to \mathrm{End}(V) , \quad \psi(\alpha)v := -2\eta(v) \lrcorner \alpha \tag{5.2.26}$$

coincides with $\mathfrak{o}(V, \mathsf{q})$. *Moreover,* ψ *is a Lie algebra isomorphism.*

---

[14] Viewed as real Lie groups.

[15] In the formula below, $\psi(\alpha)v$ may be viewed as the supercommutator $[\mathsf{c}(\alpha), v]$, cf. [72] or [439].

The defining equation of ψ immediately implies

$$\psi(u \wedge v)w = 2\eta(w, v)u - 2\eta(w, u)v\,, \quad u, v, w \in V\,. \tag{5.2.27}$$

Proof For any $v, w \in V$ and any $\alpha \in \bigwedge^2 V$, we calculate

$$\begin{aligned}\eta(\psi(\alpha)v, w) &= \eta(w)\lrcorner(\psi(\alpha)v)\\ &= -2\eta(w)\lrcorner\,\eta(v)\lrcorner\,\alpha\\ &= 2\eta(v)\lrcorner\,\eta(w)\lrcorner\,\alpha\\ &= -\eta(v, \psi(\alpha)w)\,,\end{aligned}$$

showing that $\mathrm{im}(\psi) \subset \mathfrak{o}(V, \mathsf{q})$. Next, using (5.2.27) and (5.2.24), one shows that ψ is a homomorphism (Exercise 5.2.7). Finally, ψ is obviously injective and thus, by dimension counting, it is also surjective. ∎

Remark 5.2.16 Let $\{\mathbf{e}_i\}$ be a q-orthogonal basis in V and let $\{\vartheta^i\}$ be its dual, that is, $\eta^{-1}(\vartheta^i) = \eta^{ij}\mathbf{e}_j$. Then, using (5.2.27), one calculates

$$\frac{1}{4}\psi(\mathbf{e}_i \wedge \mathbf{e}_j)(\mathbf{e}_k) \wedge \eta^{-1}(\vartheta^k) = \mathbf{e}_i \wedge \mathbf{e}_j\,.$$

Thus, ψ is the inverse of the isomorphism $\kappa : \mathfrak{o}(V, \mathsf{q}) \to \bigwedge^2 V$ given by (2.2.38). This way, κ becomes a Lie algebra isomorphism. Combining it with $\mathsf{c} : \bigwedge^2 V \to Cl_2(V, \mathsf{q})$, we obtain the Lie algebra isomorphism

$$\varphi = \mathsf{c} \circ \kappa : \mathfrak{o}(V, \mathsf{q}) \to Cl_2(V, \mathsf{q})\,, \quad \varphi(A) = \frac{1}{4}\mathsf{c}\left(A(\mathbf{e}_k) \wedge \eta^{-1}(\vartheta^k)\right)\,, \tag{5.2.28}$$

which can be easily shown to be equal to (Exercise 5.2.8)

$$\varphi(A) = \frac{1}{4}\eta^{lm}\eta^{kn}\eta(A\,\mathbf{e}_k, \mathbf{e}_l)\,\mathbf{e}_m \cdot \mathbf{e}_n = \frac{1}{4}A^{lk}\mathbf{e}_l \cdot \mathbf{e}_k\,. \tag{5.2.29}$$

Under the isomorphism φ, the action of A on an element $v \in V$ is given by (Exercise 5.2.8):

$$A(v) = [\varphi(A), v]\,. \tag{5.2.30}$$

Via φ, $\mathrm{spin}(V, \mathsf{q})$ is naturally identified with $Cl_2(V, \mathsf{q})$. Thus, $\{\mathbf{e}_i\mathbf{e}_j : i < j\}$ form a natural basis in $\mathrm{spin}(V, \mathsf{q})$ corresponding to the basis $\{\psi(\mathbf{e}_i \wedge \mathbf{e}_j) : i < j\}$ in $\mathfrak{o}(V, \mathsf{q})$. By (5.2.27), the matrix of $\psi(\mathbf{e}_i \wedge \mathbf{e}_j)$ in the basis $\{\mathbf{e}_i\}$ coincides with the matrix $2E_{ij}$, where

$$(E_{ij})_{kl} = \eta_{lj}\eta_{ki} - \eta_{li}\eta_{kj}\,. \tag{5.2.31}$$

♦

The following proposition shows that the spin group is obtained by exponentiating $Cl_2(V,\mathsf{q})$ via the exponential mapping $\exp : Cl^0(V,\mathsf{q}) \to Cl^0(V,\mathsf{q})^*$.

Proposition 5.2.17 *The following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{o}(V,\mathsf{q}) \arrow[r, "\varphi"] \arrow[d, "\exp"] & Cl^0(V,\mathsf{q}) \arrow[d, "\exp"] \\
\mathrm{Spin}(V,\mathsf{q}) \arrow[r] & Cl^0(V,\mathsf{q})^*
\end{tikzcd}$$

Proof By (5.2.30), for any $A \in \mathfrak{o}(V,\mathsf{q})$ and $v \in V$, $A(v) = \mathrm{ad}(\varphi(A))(v)$ and, thus,

$$\exp(A)(v) = \exp(\mathrm{ad}(\varphi(A)))(v) = e^{\varphi(A)} v e^{-\varphi(A)} . \tag{5.2.32}$$

Since $\exp(A)(v) \in V$ and $\varphi(A) \in Cl^0(V,\mathsf{q})$, we have $e^{\varphi(A)} \in \Gamma^0(V,\mathsf{q})$. Since $\varphi(A)^{\mathrm{T}} = -\varphi(A)$, we obtain $N\big(e^{\varphi(A)}\big) = e^{\varphi(A)} e^{\varphi(A)^{\mathrm{T}}} = 1$. Thus, $e^{\varphi(A)} \in \mathrm{Spin}(V,\mathsf{q})$. ∎

Remark 5.2.18 On the right hand side of (5.2.32) we recognize the twisted adjoint action of $\mathrm{Spin}(V,\mathsf{q})$. Thus,

$$\widetilde{\mathrm{Ad}}\big(e^{\varphi(A)}\big) = \exp(A) , \quad \widetilde{\mathrm{Ad}}'\big(\varphi(A)\big) = \mathrm{ad}(\varphi(A)) = A . \tag{5.2.33}$$

♦

Exercises

5.2.1 Prove Lemma 5.2.5.

5.2.2 Prove the statements of Remark 5.2.9.

5.2.3 Work out the details of Example 5.2.10.

5.2.4 Prove Proposition 5.2.12.

5.2.5 Prove that the series in (5.2.5) converges.

5.2.6 Check that the mapping F defined by (5.2.21) is a group homomorphism.

5.2.7 Prove that the mapping ψ given by (5.2.26) is a homomorphism.

5.2.8 Prove the formulae (5.2.29) and (5.2.30).

5.3 Representations

In this section, we discuss representations of the Clifford algebra and of the spin group.

Definition 5.3.1 Let (V, q) be a quadratic vector space over a commutative field $\mathbf{k}$, let $\mathbb{K} \supset \mathbf{k}$ be a field containing $\mathbf{k}$ and let W be a finite-dimensional vector space over $\mathbb{K}$. A $\mathbb{K}$-representation of the Clifford algebra $Cl(V, \mathsf{q})$ is a $\mathbf{k}$-algebra homomorphism

$$\rho : Cl(V, \mathsf{q}) \to \mathrm{End}_{\mathbb{K}}(W) .$$

The representation space W is called a $Cl(V, \mathsf{q})$-module over $\mathbb{K}$.

Example 5.3.2 The Clifford algebra $Cl(V, \mathsf{q})$ itself, endowed with the module structure given by multiplication from the left, is a Clifford module. The exterior algebra $\bigwedge V$ is a Clifford module with the action given by the algebra homomorphism $\hat{F} : Cl(V, \mathsf{q}) \to \mathrm{End}(\bigwedge V)$ constructed in the proof of Proposition 5.1.10. Recall that on generators this action is given by the mapping

$$F : V \to \mathrm{End}(\textstyle\bigwedge V) , \quad F(v)\alpha := v \wedge \alpha + \eta(v) \lrcorner \alpha ,$$

cf. (5.1.7). The symbol mapping $\sigma : Cl(V, \mathsf{q}) \to \bigwedge V$ is the unique isomorphism of Clifford modules taking $1 \in Cl(V, \mathsf{q})$ to $1 \in \bigwedge V$. ♦

Let us discuss the $\mathbb{K}$-representations of $Cl_{r,s}$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Since we know the classification of these Clifford algebras in terms of matrix algebras, their representation theory is provided by the classical theory of simple associative algebras. By Theorem XVII.5.5 in [399], $\mathbb{K}(n) = \mathrm{End}(\mathbb{K}^n)$ is a simple ring and $\mathbb{K}^n$ is a simple $\mathbb{K}(n)$-module. By Corollary XVII.4.5. in [399], this simple module provides the unique irreducible representation of $\mathbb{K}(n)$. Correspondingly, the ring $\mathbb{K}(n) \oplus \mathbb{K}(n)$ has exactly two equivalence classes of irreducible representations given by projection onto the first and the second factor, respectively. Thus, by inspection of Table II in Sect. 1.4 of [407], one reads off the irreducible $\mathbb{K}$-representations of $Cl_{r,s}$. According to this table, the number of inequivalent irreducible representations is

$$\nu_{r,s} = \begin{cases} 2 & \text{if } \; r + 1 - s = 0 \pmod 4 \\ 1 & \text{otherwise} \end{cases}$$

Next, let us consider Cl^c_n. By Proposition 5.1.19,

$$Cl^c_{2k} \cong \mathbb{C}(2^k) , \quad Cl^c_{2k+1} \cong \mathbb{C}(2^k) \oplus \mathbb{C}(2^k) . \tag{5.3.1}$$

Thus, by the above cited theorem, Cl^c_{2k} has a unique faithful irreducible representation

$$\gamma_{2k} : Cl^c_{2k} \to \mathrm{End}\left(\Delta_{2k}\right) , \quad \Delta_{2k} = \mathbb{C}^{2^k} , \tag{5.3.2}$$

and Cl^c_{2k+1} has a faithful representation

$$\gamma_{2k+1} : Cl^c_{2k+1} \to \mathrm{End}\left(\Delta_{2k+1}\right) \oplus \mathrm{End}\left(\Delta_{2k+1}\right), \quad \Delta_{2k+1} = \mathbb{C}^{2^k}. \tag{5.3.3}$$

Thus, Cl^c_{2k+1} has two irreducible representations obtained by projecting onto the first and onto the second summand of $\Delta_{2k+1} \oplus \Delta_{2k+1}$, respectively. Explicit formulae for γ_n are given in Remark 5.1.20. By (5.1.25), the following diagram commutes:

$$\begin{tikzcd} Cl^c_{2k} \arrow[r] \arrow[d] & \mathrm{End}(\Delta_{2k}) \arrow[d, "\iota"] \\ Cl^c_{2k+1} \arrow[r] & \mathrm{End}(\Delta_{2k+1}) \oplus \mathrm{End}(\Delta_{2k+1}) \end{tikzcd} \tag{5.3.4}$$

Here, ι denotes the diagonal embedding. In the sequel, Δ_n will be called the space of complex n-spinors, or, the n-spinor module and the corresponding representation γ_n will be referred to as a spin representation of Cl^c_n. Frequently, we will omit the index and simply write γ.

For further reference, we include the following.

Remark 5.3.3 Let E be a complex $Cl(V, \mathsf{q})$-module and let $\dim V$ be even. Then, by Proposition 5.1.19, $Cl(V, \mathsf{q})^c \cong \mathrm{End}(\Delta_n)$ is simple and Δ_n is the unique irreducible representation. We have[16]

$$E \cong \Delta_n \otimes W, \tag{5.3.5}$$

where $W = \mathrm{Hom}_{Cl(V,\mathsf{q})^c}(\Delta_n, E)$ is the vector space of homomorphisms $\Delta_n \to E$ commuting with the $Cl(V, \mathsf{q})^c$-action. By Schur's Lemma, $\mathrm{End}(W) \cong \mathrm{End}_{Cl(V,\mathsf{q})^c}(E)$. Since $\mathrm{End}(E) \cong \mathrm{End}(\Delta_n) \otimes \mathrm{End}(W)$, we conclude

$$\mathrm{End}(E) \cong Cl(V, \mathsf{q})^c \otimes \mathrm{End}_{Cl(V,\mathsf{q})^c}(E). \tag{5.3.6}$$

Note that in the second factor $Cl(V, \mathsf{q})^c$ may be replaced by $Cl(V, \mathsf{q})$. ♦

Let us study the spin representations of Cl^c_n in more detail. For that purpose, we consider the chirality element

$$\Gamma_n := \mathrm{i}^n \mathrm{i}^{[\frac{n+1}{2}]} \mathsf{c}(\mathsf{v}), \tag{5.3.7}$$

where v is the natural volume element of $\mathbb{R}^n$ corresponding to a given orientation. For a chosen oriented orthonormal basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$ we have $\mathsf{v} = \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$ and

$$\Gamma_n = \mathrm{i}^n \mathrm{i}^{[\frac{n+1}{2}]} \mathbf{e}_1 \cdot \ldots \cdot \mathbf{e}_n. \tag{5.3.8}$$

[16]See Proposition 3.1.6 in [254].

Note that for $n = 2k$, we obtain

$$\Gamma_{2k} = i^k \mathbf{e}_1 \cdot \ldots \cdot \mathbf{e}_{2k} \,. \tag{5.3.9}$$

Clearly, Γ_n does not depend on the choice of the oriented orthonormal basis.

Lemma 5.3.4 *The chirality element has the following properties.*

1. $\Gamma_n^2 = 1$ *for all n.*
2. $\mathbf{x} \cdot \Gamma_n = (-1)^{n-1} \Gamma_n \cdot \mathbf{x}$ *for all $\mathbf{x} \in \mathbb{R}^n$.*

In particular, if n is odd, then Γ_n belongs to the centre of Cl_n^c. If n is even, then

$$a \cdot \Gamma_n = \Gamma_n \cdot \mathsf{p}(a)\,, \quad a \in Cl_n^c\,. \tag{5.3.10}$$

Proof The first statement is trivial. Next, by (5.1.13), for any $l = 1, \ldots, n$ we have $\mathbf{e}_l \cdot \mathsf{c}(\mathsf{v}) = (-1)^{n-1} \mathsf{c}(\mathsf{v}) \cdot \mathbf{e}_l$. This implies $\mathbf{x} \cdot \mathsf{c}(\mathsf{v}) = (-1)^{n-1} \mathsf{c}(\mathsf{v}) \cdot \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. This proves the second assertion. The latter immediately implies the remaining statements. ■

Since Γ_n is an involution, we have projectors

$$P^+ := \frac{1}{2}(1 + \Gamma_n)\,, \quad P^- := \frac{1}{2}(1 - \Gamma_n)\,, \tag{5.3.11}$$

fulfilling

$$P^+ + P^- = 1\,, \quad P^+P^- = P^-P^+ = 0\,. \tag{5.3.12}$$

Lemma 5.3.5 *If n is odd, then Γ_n induces a decomposition*

$$Cl_n^c = Cl_n^+ \oplus Cl_n^-\,, \quad Cl_n^\pm := P^\pm \cdot Cl_n^c = Cl_n^c \cdot P^\pm\,, \tag{5.3.13}$$

into isomorphic subalgebras fulfilling $\mathsf{p}(Cl_n^\pm) = Cl_n^\mp$.

Proof By Lemma 5.3.4, Γ_n is central. Thus, P^+ and P^- are central, too, and $Cl_n^\pm$ are ideals. Since Γ_n is an odd element, we have $\mathsf{p}(P^\pm) = P^\mp$. This implies $\mathsf{p}(Cl_n^\pm) = Cl_n^\mp$ showing, in particular, that the two subalgebras are isomorphic. ■

Clearly, the two summands in (5.3.3) correspond to Cl_n^+ and Cl_n^-, respectively. This can be checked explicitly by viewing the second isomorphism in (5.3.1) as

$$Cl_{2k+1}^c \cong Cl_{2k}^c \otimes (\mathbb{C} \oplus \mathbb{C}) \cong Cl_{2k}^c \otimes Cl_1^c$$

and using that the parity automorphism on Cl_1^c is given by $\mathsf{p}(u, v) = (v, u)$. Since $\mathsf{p}(Cl_n^\pm) = Cl_n^\mp$, we also conclude that the algebra $(Cl_n^c)^0$ is diagonally embedded in the decomposition (5.3.13),

$$(Cl_n^c)^0 = \left\{(a, \mathsf{p}(a)) \in Cl_n^+ \oplus Cl_n^- : a \in (Cl_n^c)^+\right\} . \tag{5.3.14}$$

Next, using $\gamma(\Gamma_n)^2 = \gamma\left(\Gamma_n^2\right) = 1$, we decompose the spinor module Δ_n for n even into eigenspaces of $\gamma(\Gamma_n)$ corresponding to the eigenvalues ± 1:

$$\Delta_n = \Delta_n^+ \oplus \Delta_n^- , \quad \Delta_n^\pm := \{\psi \in \Delta_n : \gamma(\Gamma_n)(\psi) = \pm\psi\} . \tag{5.3.15}$$

The projectors onto $\Delta_n^\pm$ are given by $\gamma(P^\pm)$.

Proposition 5.3.6 *For the complexified Clifford algebra Cl_n^c, the following hold.*

1. *If n is odd, then the two isomorphism classes of irreducible spinor modules are given by Δ_{n+1}^+ and Δ_{n+1}^-, respectively. In this case, there is a unique isomorphism class of irreducible $(Cl_n^c)^0$-modules of dimension $2^{\frac{n-1}{2}}$.*
2. *If n is even, then there are two isomorphism classes of irreducible $(Cl_n^c)^0$-modules, both of dimension $2^{\frac{n}{2}-1}$.*

Proof 1. Let n be odd. By (5.1.20), (5.3.1) and (5.3.15), we have

$$Cl_n^c \cong (Cl_{n+1}^c)^0 \cong \mathrm{End}^0(\Delta_{n+1}) = \mathrm{End}^0(\Delta_{n+1}^+ \oplus \Delta_{n+1}^-) . \tag{5.3.16}$$

If $F \in \mathrm{Hom}(\Delta_{n+1}^-, \Delta_{n+1}^+)$, then $\gamma(\Gamma) \circ F = -F \circ \gamma(\Gamma)$. Let $F = \gamma(a)$ with a even. Then, $\gamma(\mathsf{p}(a)) = \gamma(a)$ and, since Γ is central,

$$\gamma(\Gamma) \circ \gamma(a) = -\gamma(a) \circ \gamma(\Gamma) = -\gamma(a \cdot \Gamma) = -\gamma(\Gamma \cdot a) = -\gamma(\Gamma) \circ \gamma(a) ,$$

that is, $\gamma(a) = 0$ and thus $\mathrm{Hom}^0(\Delta_{n+1}^-, \Delta_{n+1}^+) = 0$. Also $\mathrm{Hom}^0(\Delta_{n+1}^+, \Delta_{n+1}^-) = 0$ by the same argument. Moreover, for $a \in Cl_{n+1}^c$ and $\psi_\pm \in \Delta_{n+1}^\pm$, we have

$$a\psi_\pm = \pm a\Gamma\psi_\pm = \pm\Gamma\mathsf{p}(a)\psi_\pm = \mathsf{p}(a)\psi_\pm .$$

Thus, if $a\Delta_{n+1}^\pm \subset \Delta_{n+1}^\pm$, then $a \in (Cl_{n+1}^c)^0$. As a consequence, we obtain the following decomposition of Cl_n^c into simple algebras:

$$Cl_n^c \cong \mathrm{End}(\Delta_{n+1}^+) \oplus \mathrm{End}(\Delta_{n+1}^-) . \tag{5.3.17}$$

Since Γ is central, the subspaces $\Delta_{n+1}^\pm$ are Cl_n^c-invariant. This shows that $\Delta_{n+1}^\pm$ are irreducible Cl_n^c-modules. Since they are distinguished by the action of Γ, they are inequivalent. Thus, $Cl_n^\pm \cong \mathrm{End}(\Delta_{n+1}^\pm)$. By (5.3.14), the restrictions to $(Cl_n^c)^0$ of the two irreducible representations of Cl_n^c coincide yielding a unique isomorphism class of $(Cl_n^c)^0$-modules.

2. Let n be even. Then, (5.3.16) and (5.3.17) imply

$$(Cl_n^c)^0 \cong \mathrm{End}(\Delta_n^+) \oplus \mathrm{End}(\Delta_n^-) .$$

By Lemma 5.3.4, the subspaces $\Delta_n^{\pm}$ are invariant under $(Cl_n^c)^0$. Thus, (5.3.15) is a decomposition of Δ_n into two inequivalent irreducible $(Cl_n^c)^0$-modules. ■

Clearly, the irreducible representations in Proposition 5.3.6 are all faithful. Since $\mathrm{Pin}_{r,s} \subset Cl_n^c$ and $\mathrm{Spin}_{r,s} \subset (Cl_n^c)^0$, with $r + s = n$, the faithful irreducible representations of Cl_n^c and $(Cl_n^c)^0$ constructed in Proposition 5.3.6 restrict to faithful representations of $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$ called pin and spin representations, respectively. For $\mathrm{Spin}_{r,s}$, we have[17]

$$\gamma_{r,s}^{\pm} : \mathrm{Spin}_{r,s} \to \mathrm{Aut}\left(\Delta_{r+s}^{\pm}\right), \quad r + s = 2k, \tag{5.3.18}$$

$$\gamma_{r,s} : \mathrm{Spin}_{r,s} \to \mathrm{Aut}(\Delta_{r+s}), \quad r + s = 2k + 1. \tag{5.3.19}$$

Proposition 5.3.7 *The pin representations of* $\mathrm{Pin}_{r,s}$ *and the spin representations of* $\mathrm{Spin}_{r,s}$ *are irreducible.*

Proof If a subspace of a pin representation is invariant under $\mathrm{Pin}_{r,s}$, then it is also invariant under the subalgebra of $Cl_{r,s}$ generated by $\mathrm{Pin}_{r,s}$. We show that this subalgebra coincides with all of $Cl_{r,s}$. For that purpose, it suffices to prove that V is spanned by linear combinations of elements of $\mathrm{Pin}_{r,s} \cap V$. Obviously, the span of $\mathrm{Pin}_{r,s} \cap V$ contains the open subset consisting of all elements $v \in V$ fulfilling $\mathsf{q}(v) > 0$ and, therefore, the whole of V. The assertion for $\mathrm{Spin}_{r,s}$ follows from the fact that

$$\mathrm{Spin}_{r,s} = \mathrm{Pin}_{r,s} \cap Cl_{r,s}^0,$$

because this implies that the subalgebra generated by $\mathrm{Spin}_{r,s}$ coincides with the intersection of the subalgebra generated by $\mathrm{Pin}_{r,s}$ with $Cl_{r,s}^0$. ■

Remark 5.3.8 (Spin^c*-representations*) Since the complex spin group $\mathrm{Spin}^c(n)$ is contained in the complexified Clifford algebra Cl_n^c, the spin representation Δ_n of $\mathrm{Spin}(n)$ extends to a representation of $\mathrm{Spin}^c(n)$ via

$$\gamma([(g, z)])(\psi) = z \cdot \gamma(g)(\psi), \tag{5.3.20}$$

for any $g \in \mathrm{Spin}(n)$, $z \in S^1$ and $\psi \in \Delta_n$. If n is even, then the splitting $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ is Spin^c-invariant and, thus, we have the two irreducible modules $\Delta_n^{\pm}$ as in the spin case. ♦

Example 5.3.9 (*Spin representations of* $\mathrm{Spin}_{1,3}$) By point 2 of Example 5.2.10,

$$\mathrm{Spin}_{1,3} = \left\{ Z = \begin{bmatrix} g & 0 \\ 0 & \dot{g} \end{bmatrix} : \ g \in \mathrm{SL}(2, \mathbb{C}) \right\}, \tag{5.3.21}$$

[17] Often, the representations $\gamma_{r,s}^{\pm}$ are called the half-spin representations.

where $\dot{g} = (g^\dagger)^{-1}$. This yields the two inequivalent irreducible spinor modules $S \cong \mathbb{C}^2$ and $\overline{S}^* \cong \mathbb{C}^2$ of $\mathrm{Spin}_{1,3}$, with S and $\overline{S}^*$ carrying the basic representation and the dual of the conjugate representation of $\mathrm{SL}(2,\mathbb{C})$, respectively, that is,

$$S \ni \phi \mapsto g\phi \in S\,, \quad \overline{S}^* \ni \tilde{\varphi} \to \dot{g}\tilde{\varphi} \in \overline{S}^*\,, \quad g \in \mathrm{SL}(2,\mathbb{C})\,.$$

In physics, S and $\overline{S}^*$ are called the space of left-handed and right-handed Weyl spinors, respectively. Their direct sum $S \oplus \overline{S}^* \cong \mathbb{C}^4$ is called the bispinor space. Choosing bases in these spaces, one obtains a frequently used calculus of dotted and undotted spinors, $\phi = (\phi^K)$ and $\tilde{\varphi} = (\tilde{\varphi}_{\dot{K}})$. ♦

Denote $n = r + s$. Since $\mathbb{R}^n \subset Cl_n \subset Cl_n^c$, via the spin representation, any vector $\mathbf{x} \in \mathbb{R}^n$ may be regarded as an endomorphism of Δ_n. We define

$$\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \to \Delta_n\,, \quad \mu(\mathbf{x} \otimes \psi) := \gamma(\mathbf{x})\psi\,. \tag{5.3.22}$$

Definition 5.3.10 The mapping μ defined by (5.3.22) will be referred to as the Clifford multiplication.

Usually, we will simply write $\mu(\mathbf{x} \otimes \psi) \equiv \mathbf{x} \cdot \psi$. Using the quantization isomorphism c, the Clifford multiplication may be extended to a mapping $\mu : \bigwedge \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \to \Delta_n$ as follows:

$$\alpha \cdot \psi \equiv \mu(\alpha \otimes \psi) := \gamma(\mathsf{c}(\alpha))\psi\,. \tag{5.3.23}$$

For any $\mathbf{x} \in \mathbb{R}^n$, $\alpha \in \bigwedge \mathbb{R}^n$ and $\psi \in \Delta_n$, the following holds (Exercise 5.3.1):

$$(\mathbf{x} \wedge \alpha) \cdot \psi = \mathbf{x} \cdot (\alpha \cdot \psi) + (\mathbf{x} \lrcorner \alpha) \cdot \psi\,. \tag{5.3.24}$$

As in Remark 5.2.9, we denote the covering homomorphism induced from $\widetilde{\mathrm{Ad}}$ and the induced Lie algebra homomorphism by

$$\lambda : \mathrm{Spin}_{r,s} \to \mathrm{SO}^0_{r,s}\,, \quad \mathrm{d}\lambda : \mathrm{spin}_{r,s} \to \mathfrak{so}_{r,s}\,.$$

These mappings are given by (5.2.33).

Proposition 5.3.11 *The Clifford multiplication has the following properties.*

1. *It is equivariant with respect to the* $\mathrm{Spin}_{r,s}$*-action,*[18] *that is, for any* $a \in \mathrm{Spin}_{r,s}$, $\alpha \in \bigwedge \mathbb{R}^n$ *and* $\psi \in \Delta_n$,

$$\gamma(a)(\alpha \cdot \psi) = (\lambda(a)\alpha) \cdot (\gamma(a)\psi)\,. \tag{5.3.25}$$

2. *Let* $n = 2k$. *Then, the Clifford multiplication with a non-zero* $\mathbf{x} \in \mathbb{R}^n$ *yields a vector space isomorphism* $\Delta^\pm \to \Delta^\mp$.

[18] In other words, it is a homomorphism of $\mathrm{Spin}_{(r,s)}$-representations.

Proof The proof of the first assertion is by induction with respect to the degree k of α. For $k = 1$, α is a vector $\mathbf{x} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\gamma(a)(\mathbf{x} \cdot \psi) &= \gamma(a)\gamma(\mathbf{x})(\psi) \\
&= \gamma(a)\gamma(\mathbf{x})\gamma(a^{-1})\gamma(a)(\psi) \\
&= \gamma(a\mathbf{x}a^{-1})\gamma(a)(\psi) \\
&= \gamma(\lambda(a)\mathbf{x})(\gamma(a)(\psi)) \\
&= (\lambda(a)\mathbf{x}) \cdot (\gamma(a)(\psi)) \,.
\end{aligned}$$

Now, assume that (5.3.25) holds for all elements $\alpha \in \bigwedge \mathbb{R}^n$ of degree $\leq k$. Then, using (5.3.24), for $\beta := \mathbf{x} \wedge \alpha$ we obtain

$$\begin{aligned}
\gamma(a)((\mathbf{x} \wedge \alpha) \cdot \psi) &= \gamma(a)\,(\mathbf{x} \cdot (\alpha \cdot \psi)) + \gamma(a)\,((\mathbf{x} \lrcorner \alpha) \cdot \psi) \\
&= (\lambda(a)\mathbf{x}) \cdot (\gamma(a)(\alpha \cdot \psi)) + (\lambda(a)(\mathbf{x} \lrcorner \alpha)) \cdot (\gamma(a)\psi) \\
&= (\lambda(a)\mathbf{x}) \cdot (\lambda(a)\alpha) \cdot (\gamma(a)\psi) + ((\lambda(a)\mathbf{x}) \lrcorner (\lambda(a)\alpha)) \cdot (\gamma(a)\psi) \\
&= ((\lambda(a)\mathbf{x}) \wedge (\lambda(a)\alpha)) \cdot (\gamma(a)\psi) \\
&= (\lambda(a)\beta) \cdot (\gamma(a)(\psi)) \,.
\end{aligned}$$

The second assertion is an immediate consequence of the fact that Γ anticommutes with any non-vanishing vector $\mathbf{x} \in \mathbb{R}^n$. ■

Now, let us focus on the case $n = 2k$. Then, there is a useful equivalent description of the spinor modules.[19] Consider the spinor module Δ_n together with its decomposition (5.3.15). As before, for $n = r + s$, we write $V = \mathbb{R}^{r,s}$, q for the pseudo-Euclidean quadratic form of V given by (5.1.12) and η for the corresponding bilinear form. The extensions of q and η to $V_{\mathbb{C}} = V \otimes \mathbb{C}$ are denoted by the same symbols. Recall that a subspace $W \subset V_{\mathbb{C}}$ is called isotropic if $\mathsf{q}(w) = 0$ for all $w \in W$. Given an isotropic subspace W, one can find a complementary isotropic subspace $W' \cong W^*$. For an oriented orthonormal basis $\{\mathbf{e}_i\}$ of $V_{\mathbb{C}}$ we define

$$W := \operatorname{span}\left\{\mathbf{e}_{2k-1} - i\,\mathbf{e}_{2k} : k = 1, \ldots, \frac{n}{2}\right\}, \quad W' := \operatorname{span}\left\{\mathbf{e}_{2k-1} + i\,\mathbf{e}_{2k} : k = 1, \ldots, \frac{n}{2}\right\},$$

and the isomorphism $\varphi : W \to (W')^*$ by $\varphi(w)(w') := \eta(w, w')$. It is now easy to check (Exercise 5.3.2) that for $v = w' + w$, $w \in W$ and $w' \in W'$,

$$\eta(w, w') = \frac{1}{2}\mathsf{q}(v)\,, \quad \eta(w, w) = 0 = \eta(w', w')\,. \tag{5.3.26}$$

[19] The following construction is at the heart of the general theory of spinor modules, see e.g. [439].

The corresponding decomposition $V_{\mathbb{C}} = W' \oplus W \cong W^* \oplus W$ is referred to as a complex polarization of V. We define $S_W := \bigwedge W^*$ and endow S_W with the structure of a Clifford module by defining the action of $V_{\mathbb{C}} \cong W^* \oplus W$ on S_W by[20]

$$\rho_W : W^* \oplus W \to \mathrm{End}(S_W)\,, \quad \rho_W(\zeta, w) := \sqrt{2}(\varepsilon(\zeta) + \iota(w))\,, \tag{5.3.27}$$

where ε and ι denote exterior multiplication and contraction, respectively. Using the anti-commutation relations for ε and ι, one can check that $\rho_W(\zeta, w)^2 = \mathsf{q}(\zeta, w)1$. Thus, by universality, ρ_W extends to a representation of the Clifford algebra Cl_n^c on S_W. By construction, ρ_W is faithful and, by dimension counting and by the uniqueness of the spinor module, we obtain the following.

Proposition 5.3.12 *For n even, the spinor module Δ_n is isomorphic to the Cl_n^c-module S_W.* ■

We decompose

$$S_W = \bigwedge W^* = \bigwedge{}^+ W^* \oplus \bigwedge{}^- W^* \tag{5.3.28}$$

with respect to the $\mathbb{Z}_2$-grading of the exterior algebra and denote $S_W^+ = \bigwedge^+ W^*$ and $S_W^- = \bigwedge^- W^*$.

Proposition 5.3.13 *The natural $\mathbb{Z}_2$-grading of S_W is compatible with the $\mathbb{Z}_2$-grading defined by the chirality element, that is, Γ_n acts as $+1$ on $\bigwedge^+ W^*$ and as -1 on $\bigwedge^- W^*$. As a consequence,*

$$\Delta_n^+ \cong S_W^+\,, \quad \Delta_n^- \cong S_W^-\,. \tag{5.3.29}$$

Proof We choose an oriented orthonormal basis $\{\mathbf{e}_i\}$ and denote $E_l := \frac{1}{\sqrt{2}}(\mathbf{e}_{2l-1} - i\,\mathbf{e}_{2l})$. It is easy to calculate Γ_n in this basis (Exercise 5.3.2):

$$\Gamma_n = (E_1\overline{E}_1 - 1) \cdot \ldots \cdot (E_k\overline{E}_k - 1)\,. \tag{5.3.30}$$

Now, denoting the basis elements of $S_W \cong \bigwedge W^*$ by $\overline{E}_{I_l} = \overline{E}_{i_1} \wedge \ldots \wedge \overline{E}_{i_l}$, where $I_l = \{i_1, \ldots, i_l\}$, the action of $\rho_W(E_i\overline{E}_i - 1)$ on $\overline{E}_{I_l}$ yields obviously $\overline{E}_{I_l}$ if $i \notin I_l$ and $-\overline{E}_{I_l}$ if $i \in I_l$. This implies

$$\rho_W(\Gamma_n)(\overline{E}_{I_l}) = (-1)^l \overline{E}_{I_l}\,,$$

which proves the assertion. ■

Correspondingly, we may consider $S^W := \bigwedge W$. Here, the action of $V_{\mathbb{C}} \cong W^* \oplus W$ is given by

$$\rho^W : W^* \oplus W \to \mathrm{End}(S^W)\,, \quad \rho^W(\zeta, w) := \sqrt{2}(\iota(\zeta) + \varepsilon(w))\,, \tag{5.3.31}$$

[20]The factor $\sqrt{2}$ is necessary in order to respect the Clifford algebra relations, because $\varepsilon(\zeta)\iota(w) + \iota(w)\varepsilon(\zeta) = \zeta(w) = \eta(\eta^{-1}(\zeta), w)$, cf. formula (2.7.33).

which provides a representation ρ^W of Cl_n^c on S^W. There is a natural non-degenerate pairing between S_W and S^W, given by

$$(\cdot,\cdot): \; S_W \otimes S^W \to \mathbb{C}, \quad (\phi,\psi) := \big(\iota(\phi^{\mathrm{T}})\psi\big)_{[0]}\,, \tag{5.3.32}$$

where the subscript [0] means taking the zero-order component in the exterior algebra and the superscript T is defined by

$$(\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k)^{\mathrm{T}} := \alpha_k \wedge \ldots \wedge \alpha_2 \wedge \alpha_1\,, \quad \alpha_i \in W^*\,.$$

Using (5.3.27) and (5.3.31), one proves (Exercise 5.3.4)

$$(\phi, \rho^W(\zeta, w)\psi) = (\rho_W(\zeta, w)\phi, \psi) \tag{5.3.33}$$

for any $\zeta \in W^*$, $w \in W$, $\phi \in S_W$ and $\psi \in S^W$. This, together with the non-degeneracy of the pairing, implies the following isomorphism of Clifford modules:

$$S^W \cong S_W^*. \tag{5.3.34}$$

Thus, we may call S^W the dual spinor module. Correspondingly, there is a natural non-degenerate pairing on S_W, given by

$$(\cdot,\cdot)_{S_W}: \; S_W \otimes S_W \to \bigwedge\nolimits^k W^*\,, \quad (\phi_1,\phi_2)_{S_W} := \big(\phi_1^{\mathrm{T}} \wedge \phi_2\big)_{[\mathrm{top}]}\,, \tag{5.3.35}$$

where the subscript [top] means taking the top-order component in the exterior algebra.[21] This pairing will be referred to as the canonical bilinear form on the spinor module. One shows (Exercise 5.3.4) that, for any $a \in Cl_n^c$,

$$(\rho_W(a)\phi, \psi)_{S_W} = (\phi, \rho_W(a^{\mathrm{T}})\psi)_{S_W}\,. \tag{5.3.36}$$

Thus, choosing a trivialization $\bigwedge^k W^* \cong \mathbb{C}$, via $(\cdot,\cdot)_{S_W}$ we may identify $S_W \cong S_W^*$ as Clifford modules. Combined with (5.3.34), this yields an isomorphism $S_W \cong S^W$.

Proposition 5.3.14 *Let* $\dim V = 2k$. *Then, the pairing* $(\cdot,\cdot)_{S_W}$ *is*

1. *symmetric if* $k = 0, 1 \bmod 4$,
2. *anti-symmetric if* $k = 2, 3 \bmod 4$.

Moreover, if $k = 0 \bmod 4$ *(respectively,* $k = 2 \bmod 4$*) it restricts to a non-degenerate symmetric (respectively, anti-symmetric) form on both* S_W^+ *and* S_W^-. *If* k *is odd,* $(\cdot,\cdot)_{S_W}$ *vanishes both on* S_W^+ *and* S_W^-, *thus, yielding a non-degenerate pairing between them.*

[21] In complete analogy, there is a pairing on S^W.

Proof For $\phi \in \bigwedge^l W^*$ and $\psi \in \bigwedge^{k-l} W^*$, we calculate

$$\begin{aligned}
(\psi, \phi)_{S_W} &= \psi^{\mathrm{T}} \wedge \phi \\
&= (-1)^{\frac{1}{2}(k-l)(k-l-1)} \psi \wedge \phi \\
&= (-1)^{\frac{1}{2}(k-l)(k-l-1)+l(k-l)} \phi \wedge \psi \\
&= (-1)^{\frac{1}{2}(k-l)(k-l-1)+l(k-l)+\frac{1}{2}l(l-1)} \phi^{\mathrm{T}} \wedge \psi \\
&= (-1)^{\frac{1}{2}k(k-1)} (\phi, \psi)_{S_W} .
\end{aligned}$$

This proves the first assertion. The remaining statements are left to the reader (Exercise 5.3.3). ■

In the following example, details are left to the reader (Exercise 5.3.5).

Example 5.3.15 Here, we take up Examples 5.1.21 and 5.2.10 where we discussed the Clifford algebra $Cl_{1,3}$ of the Minkowski space (M, η) and its spin group. Consider the complexification $M_{\mathbb{C}} = M \otimes \mathbb{C} \cong \mathbb{C}^4$ together with $Cl_4^c \cong Cl_{1,3}^c$. If $\{\mathbf{e}_i\}$ is the standard basis in $\mathbb{C}^4$, then $\tilde{\mathbf{e}}_0 = \mathbf{e}_0, \tilde{\mathbf{e}}_j = i\,\mathbf{e}_j$, with $j = 1, 2, 3$, is an orthonormal basis. As above, we pass to the basis defined by $E_l := \frac{1}{\sqrt{2}}(\tilde{\mathbf{e}}_{2l-1} - i\,\tilde{\mathbf{e}}_{2l})$ and $E'_l := \frac{1}{\sqrt{2}}(\tilde{\mathbf{e}}_{2l-1} + i\,\tilde{\mathbf{e}}_{2l})$, with $l = 1, 2$, and interchange the role of $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_3$ for convenience. In this basis, $\mathbf{z} \in M_{\mathbb{C}}$ reads

$$\mathbf{z} = \frac{1}{\sqrt{2}}(z^0 + z^3)E_1 + \frac{1}{\sqrt{2}}(z^0 - z^3)E'_1 + \frac{1}{\sqrt{2}}(z^1 - iz^2)E_2 - \frac{1}{\sqrt{2}}(z^1 + iz^2)E'_2 ,$$

where z^μ are complex coordinates in the standard basis. This yields the complex polarization $M_{\mathbb{C}} = W_+ \oplus W_-$, where

$$W_\pm = \left\{\mathbf{z} \in M_{\mathbb{C}} : \;\; z^0 \mp z^3 = 0 ,\; z^1 \pm iz^2 = 0\right\} . \tag{5.3.37}$$

Clearly, $W_+ \cong \mathbb{C}^2 \cong W_-$. We consider the spinor module $S = \bigwedge W_+$ and its decomposition into its irreducible components

$$S = S^+ \oplus S^- , \quad S^+ = \textstyle\bigwedge^+ W_+ = \bigwedge^0 W_+ \oplus \bigwedge^2 W_+ , \quad S^- = \bigwedge^- W_+ = \bigwedge^1 W_+ .$$

Clearly, $\{1, E := E_1 \wedge E_2\}$ and $\{E_1, E_2\}$ constitute bases in S^+ and S^-, respectively. Since $\rho(w) = \sqrt{2}\varepsilon(w)$ and $\rho(\eta(\overline{w})) = \sqrt{2}\iota(\eta(\overline{w}))$, we have

$$\begin{aligned}
\rho(E_i)1 = \sqrt{2}E_i , \quad \rho(E_i)E_j = \sqrt{2}\varepsilon_{ij}E , \quad \rho(E_i)E = 0 , \\
\rho(\overline{E}_i)1 = 0 , \quad \rho(\overline{E}_i)E_j = \sqrt{2}\delta_{ij}1 , \quad \rho(\overline{E}_i)E = \sqrt{2}\varepsilon_{ij}E_j ,
\end{aligned}$$

where ε_{ij} denotes the symplectic form on $\mathbb{C}^2$.[22] To describe the spin representation, it is enough to specify the action of M on S. Thus, for $v = w + \overline{w} \in M$, where $w \in W_+$,

[22]This is the natural bilinear pairing here, according to Proposition 5.3.14.

we must apply $\rho(v) = \sqrt{2}(\iota(\eta(\overline{w})) + \varepsilon(w))$ to elements of the basis of S. Taking $v_i^{\pm} = \overline{E_i} \pm E_i$, we obtain

$$\rho(v_i^{\pm})1 = \pm\sqrt{2}E_i\,, \quad \rho(v_i^{\pm})E_j = \sqrt{2}(\delta_{ij}1 \pm \varepsilon_{ij}E)\,, \quad \rho(v_i^{\pm})E = \sqrt{2}\varepsilon_{ij}E_j\,.$$

We know that the elements $\{\tilde{\mathbf{e}}_i\tilde{\mathbf{e}}_j\}$, with $i < j$, form a basis of the Lie algebra $\mathrm{spin}(4,\mathbb{C})$. Rewriting this basis in terms of the elements E_i and $\overline{E}_j$, one finds an explicit matrix representation of $\mathrm{spin}(4,\mathbb{C})$ with respect to the bases $\{1, E\}$ and $\{E_1, E_2\}$ in S^+ and S^-, respectively. From this representation one reads off that $\mathrm{spin}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. Then, by Proposition 5.2.17

$$\mathrm{Spin}(4,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})\,. \tag{5.3.38}$$

Finally, by Proposition 5.3.14, the bilinear form (5.3.36) should be anti-symmetric and should induce anti-symmetric bilinear forms on both S^+ and S^-. This can be checked by direct inspection. In the above bases, the bilinear forms on $S^{\pm}$ are given by the standard anti-symmetric form ε_{ij} with $i, j = 1, 2$. ♦

In the remainder of this section, we endow every spinor module with a natural Hermitean bilinear form, discuss its relation to the canonical bilinear form $(\cdot,\cdot)_{S_W}$ and draw important conclusions. We limit our attention to the Euclidean case and comment on the pseudo-Euclidean case at the end.

Thus, let (V, q) be a positive-definite quadratic space with bilinear form η. Extend η to $V_{\mathbb{C}}$ and consider the natural Hermitean bilinear form h on $V_{\mathbb{C}}$ associated with the complex bilinear form η via

$$\mathsf{h}(u, v) := \eta(\overline{u}, v)\,, \quad u, v \in V_{\mathbb{C}}\,. \tag{5.3.39}$$

Here, $v \mapsto \overline{v}$ denotes the complex conjugation mapping. Clearly, this mapping extends to a conjugate linear algebra automorphism $a \mapsto \overline{a}$ of $Cl(V, \mathsf{q})^c$. Combining this with the canonical anti-automorphism t, we obtain a conjugate linear anti-automorphism

$$a^* := \mathrm{t}(\overline{\mathrm{a}})\,, \tag{5.3.40}$$

that is, $(ab)^* = b^*a^*$ and $(\lambda a)^* = \overline{\lambda}a^*$ for $\lambda \in \mathbb{C}$. Let (E, ρ) be a complex representation of $Cl(V, \mathsf{q})^c$ endowed with a Hermitean structure. It is called unitary if

$$\rho(a^*) = \rho(a)^* \tag{5.3.41}$$

for all $a \in Cl(V, \mathsf{q})^c$. Thus, in particular, the generators $v \in V \subset Cl(V, \mathsf{q})^c$ act as self-adjoint operators on E. Clearly, for a unitary Clifford module, the representations of $\mathrm{Spin}(V)$ and $\mathrm{Pin}(V)$ preserve the Hermitean structure on E. Thus, they are unitary as well.

We extend h to $\bigwedge V_{\mathbb{C}}$ by setting $\mathsf{h}(\phi, \psi) = 0$ for $\phi, \psi \in \bigwedge V_{\mathbb{C}}$ having a different form degree and

$$\mathsf{h}(1,1) = 1\,, \quad \mathsf{h}(u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k) = \det\big(\mathsf{h}(u_i, v_j)\big)\,, \tag{5.3.42}$$

where $u_i, v_i \in V_{\mathbb{C}}$. Note that, with respect to the standard basis $\{\mathbf{e}_I\}$ of $\bigwedge V_{\mathbb{C}}$ induced from an η-orthonormal basis $\{\mathbf{e}_i\}$ of $V_{\mathbb{C}}$, h coincides with the standard Hermitean form on $\mathbb{C}^n$.

Example 5.3.16 We take up Example 5.3.2. Clearly, $\langle a, b\rangle := \mathrm{tr}(a^*b)$ defines a Hermitean inner product on $Cl(V,\mathsf{q})^c$. Then, $\langle a, vb\rangle = \langle va, b\rangle$, for any $v \in V \subset Cl(V,\mathsf{q})^c$ and $a, b \in Cl(V,\mathsf{q})^c$. Thus, endowed with the action by left multiplication, $Cl(V,\mathsf{q})^c$ is a unitary Clifford module. Next, it is easy to see that the quantization mapping intertwines the inner product on $\bigwedge V_{\mathbb{C}}$ given by (5.3.42) with the above inner product on $Cl(V,\mathsf{q})^c$ (Exercise 5.3.6). Thus, $\bigwedge V$ is a unitary Clifford module as well. ♦

Now, let $\dim V = 2k$ and let $V_{\mathbb{C}} = W' \oplus W \cong W^* \oplus W$ be a complex polarization. Then, by (5.3.26), W and W' are orthogonal with respect to h. Moreover, $\overline{W} = W'$. Thus, we can restrict h to W and to W' and then extend these restrictions via (5.3.42) to $\bigwedge W$ and $\bigwedge W'$, respectively. This way, we obtain a scalar product h^W on the spinor module S^W and, via $W' \cong W^*$, also a scalar product h_W on S_W.

Proposition 5.3.17 *The spinor modules S_W and S^W are unitary. In particular, the Hermitean forms h_W and h^W are* $\mathrm{Spin}(V)$*-invariant.*

Proof We write down the proof for S^W. It is enough to show that any $v \in V$ acts via ρ^W as a selfadjoint operator on $\bigwedge W$. Since v is real and $\overline{W} = W'$, with respect to the chosen complex polarization it decomposes as $v = \overline{w} + w$, where $w \in W$. We prove

$$\iota(\eta(\overline{w}))^* = \varepsilon(w)\,, \quad \varepsilon(w)^* = \iota(\eta(\overline{w}))\,.$$

On the one hand, for any $\phi = u_1\wedge\ldots\wedge u_k \in \bigwedge^k W$ and $\psi = v_1\wedge\ldots\wedge v_{k+1} \in \bigwedge^{k+1} W$,

$$\begin{aligned}\mathsf{h}^W(\iota(\eta(\overline{w}))^*\phi, \psi) &= \mathsf{h}(u_1 \wedge \ldots \wedge u_k, \iota(\eta(\overline{w}))(v_1 \wedge \ldots \wedge v_{k+1}))\\ &= \sum_{i=1}^k (-1)^{i-1}\mathsf{h}^W(u_1 \wedge \ldots \wedge u_k, \eta(\overline{w}, v_i)\, v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_{k+1})\\ &= \sum_{i=1}^k (-1)^{i-1}\mathsf{h}(w, v_i)\, \mathsf{h}^W(u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_{k+1})\,.\end{aligned}$$

On the other hand, $\mathsf{h}^W(\varepsilon(w)\phi, \psi) = \mathsf{h}^W(w \wedge u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_{k+1})$. Using (5.3.42) and expanding the determinant with respect to the first line, we obtain the assertion. Now, $\rho^W(v)^* = \sqrt{2}(\iota(\eta(\overline{w}))^* + \varepsilon(w)^*) = \sqrt{2}(\varepsilon(w) + \iota(\eta(\overline{w}))) = \rho^W(v)\,.$ ■

Let $\{\mathbf{e}_i\}$ be an h-orthonormal basis in W and let $\{\mathbf{e}_I\}$ be the induced basis in $S^W = \bigwedge W$. Let I^c denote the complement of I in $\{1,\ldots,k\}$. We choose $\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_k$ as the volume form and view the bilinear form $(\cdot,\cdot)_{S^W}$ as a $\mathbb{C}$-valued mapping.

Proposition 5.3.18 *Let* $\dim V = 2k$ *and let* $V_{\mathbb{C}} = W^* \oplus W$ *be a complex polarization. Then, the scalar product* h^W *on* S^W *and the canonical bilinear form* $(\cdot,\cdot)_{S^W}$ *are compatible, that is, there exists an anti-linear Spin*(V)*-equivariant mapping* $C : S^W \to S^W$ *such that*

$$\mathsf{h}^W(C(\mathbf{e}_I), \mathbf{e}_J) = (\mathbf{e}_I, \mathbf{e}_J)_{S^W} \,. \tag{5.3.43}$$

If the canonical bilinear form is symmetric, then $C^2 = \mathrm{id}$. *If it is anti-symmetric, then* $C^2 = -\,\mathrm{id}$. *The corresponding statements are true for* S_W.

Proof We have

$$(\mathbf{e}_I, \mathbf{e}_{I^c})_{S^W} = \varepsilon_I \,, \tag{5.3.44}$$

where $\varepsilon_I = \pm 1$. If $(\cdot,\cdot)_{S^W}$ is symmetric, then $\varepsilon_I = \varepsilon_{I^c}$. If it is anti-symmetric, then $\varepsilon_I = -\varepsilon_{I^c}$. Now, since both h^W and $(\cdot,\cdot)_{S^W}$ are non-degenerate, (5.3.43) defines an anti-linear isomorphism $C : S^W \to S^W$. Moreover, since S^W is unitary and since $(\cdot,\cdot)_{S^W}$ is $\mathrm{Spin}(n)$-invariant, C is $\mathrm{Spin}(V)$-equivariant. Comparing (5.3.43) with (5.3.44), we read off

$$C(\mathbf{e}_I) = \varepsilon_I \, \mathbf{e}_{I^c} \,. \tag{5.3.45}$$

This implies $C^2 = \mathrm{id}$ in the symmetric case and $C^2 = -\,\mathrm{id}$ in the anti-symmetric case. ■

Now, recall some basic terminology from representation theory. Let S be a Hermitean vector space carrying a unitary representation of a compact Lie group G. If there exists an anti-linear G-equivariant mapping $C : S \to S$ fulfilling $C^2 = \mathrm{id}$ or $C^2 = -\,\mathrm{id}$, then S is said to be of real or of quaternionic type, respectively. C is called the structure mapping. In the first case, S is the complexification of the real G-representation $S_{\mathbb{R}}$ given as the fixed point set of C. In the second case, C induces on S the structure of a quaternionic G-representation with scalar multiplication by the quaternions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ given by $\mathbf{i} = i$, $\mathbf{j} = C$ and $\mathbf{k} = \mathbf{ij}$. In both cases, C clearly provides an isomorphism of S and the dual module S^*. Consequently, such representations are referred to as self-dual. If a unitary G-representation is not self-dual, then it is said to be of complex type.

Combining Proposition 5.3.18 with Proposition 5.3.14, we obtain the following.

Theorem 5.3.19 *We have the following types of the spin representations of* $\mathrm{Spin}(n)$*:*

$$\begin{aligned}
n = 0 \quad mod\, 8 &: \ \Delta_n^{\pm} \ \textit{ of real type}\,,\\
n = 2, 6 \quad mod\, 8 &: \ \Delta_n^{\pm} \ \textit{ of complex type}\,,\\
n = 4 \quad mod\, 8 &: \ \Delta_n^{\pm} \ \textit{ of quaternionic type}\,,\\
n = 1, 7 \quad mod\, 8 &: \ \Delta_n \ \textit{ of real type}\,,\\
n = 3, 5 \quad mod\, 8 &: \ \Delta_n \ \textit{ of quaternionic type}\,.
\end{aligned}$$

Proof The first three assertions are immediate from Propositions 5.3.14 and 5.3.18. Consider the case $n = 2k - 1$ with k even. Then, by (5.3.17), $\Delta_n \cong \Delta_{n+1}^+$ and the restriction of the bilinear form yields a non-degenerate symmetric bilinear form for

$k = 0 \pmod 4$ and an anti-symmetric bilinear form for $k = 2 \pmod 4$, respectively. Finally, let $n = 2k - 1$ with k odd. Then, according to Proposition 5.3.14, the restriction of the canonical bilinear form to Δ^+_{n+1} vanishes. But instead one can take the $\mathrm{Spin}(n)$-invariant bilinear form

$$(\phi, \psi) := (\phi, \rho(\mathbf{e}_{n+1})\psi)_{S^W}, \quad \phi, \psi \in \Delta_{2k-1} = \Delta_{2(k-1)},$$

which is easily seen to be symmetric for $k = 1 \pmod 4$ and anti-symmetric for $k = 3 \pmod 4$, respectively. ■

Remark 5.3.20 (*Structure mapping*) Recall the explicit k-fold tensor product representation

$$\Delta_{2k} = \mathbb{C}^{2^k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

given by (5.3.2), together with the explicit presentation of the generators $\mathbf{e}_1$ and $\mathbf{e}_2$ of the spinor representation on $\mathbb{C}^2$,

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

provided in Example 5.1.13. Using this, from (5.3.45) one can read off the structure mapping C explicitly. Consider the case $n = 8k + 4$. Then, by Theorem 5.3.19, both $\Delta^\pm_n$ are of quaternionic type. For $n = 4$, we have $\Delta^+_4 = \mathbb{C}^2 = \Delta^-_4$ and Δ^+_4 and Δ^-_4 are spanned by $(\mathbf{e}_\varnothing, \mathbf{e}_{\{1,2\}})$ and $(\mathbf{e}_{\{1\}}, \mathbf{e}_{\{2\}})$, respectively. Taking into account that C must be anti-linear, for $\Delta_4 = \mathbb{C}^2 \otimes \mathbb{C}^2$, formula (5.3.45) yields:

$$C : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \quad C\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \otimes \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}\right) = \begin{bmatrix} -\overline{z_2} \\ \overline{z_1} \end{bmatrix} \otimes \begin{bmatrix} \overline{z_3} \\ \overline{z_4} \end{bmatrix}. \tag{5.3.46}$$

Then, $C^2 = -\,\mathrm{id}$, indeed. Moreover, using the explicit presentation of the Clifford multiplication on Δ_4 found in the proof of Proposition 5.1.15,

$$\mathbf{f}_1 = \mathbb{1} \otimes \mathbf{e}_1, \quad \mathbf{f}_2 = \mathbb{1} \otimes \mathbf{e}_2, \quad \mathbf{f}_3 = \mathbf{e}_1 \otimes i\,\mathbf{e}_1\mathbf{e}_2, \quad \mathbf{f}_4 = \mathbf{e}_2 \otimes i\,\mathbf{e}_1\mathbf{e}_2,$$

one checks by direct inspection (Exercise 5.3.7) that C commutes with the Clifford action,

$$C \circ \mathbf{f}_i = \mathbf{f}_i \circ C, \quad i = 1, 2, 3, 4. \tag{5.3.47}$$

This implies that C is equivariant with respect to the spin representation.[23] Using the above tensor product decomposition of Δ_{2k}, this construction may be easily extended to $n = 8k + 4$ yielding a quaternionic structure mapping C commuting with the Clifford multiplication. In a completely analogous way, the structure mappings of

[23] We know this already from Proposition 5.3.18.

the remaining cases provided by Theorem 5.3.19 may be constructed. For a complete list, we refer e.g. to [219].[24] ♦

Remark 5.3.21 (*Majorana spinors*) Let S be a complex spin representation. Then, S is called Majorana (resp. symplectic Majorana) if it admits a real (resp. quaternionic) structure mapping C. A spinor $\phi \in S$ is called Majorana if $C(\phi) = \phi$. We refer to [633] for more details. ♦

Example 5.3.22 (*Low-dimensional spin groups*) Recall Example 5.2.11. Here, we show that Theorem 5.3.19 yields elegant proofs of the isomorphisms between low-dimensional spin groups and classical Lie groups. We illustrate this by proving

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2)\,, \quad \mathrm{Spin}(6) \cong \mathrm{SU}(4)\,.$$

Since Δ_5 is a faithful 4-dimensional representation of quaternionic type, after identifying $\mathbb{C}^4 \cong \mathbb{H}^2$, we obtain an injective homomorphism $\varphi : \mathrm{Spin}(5) \to \mathrm{Sp}(2)$. By dimension counting, this must be an isomorphism. Next, Δ_6 is of complex type and decomposes into irreducible 4-dimensional representations, $\Delta_6 = \Delta_6^+ \oplus \Delta_6^-$. Thus, since the spin representation is unitary, we obtain injective homomorphisms $\varphi_\pm : \mathrm{Spin}(6) \to \mathrm{U}(4)$. Since $\mathrm{Spin}(6)$ is the covering group of a simple Lie group, it must be semisimple. Thus, its image under $\varphi_\pm$ must lie in $\mathrm{SU}(4)$. Again, by dimension counting, we conclude that $\varphi_\pm$ are isomorphisms. ♦

An analysis similar to that in Theorem 5.3.19 has also been carried out for the pseudo-Euclidean case, see [15, 286] for a detailed presentation. Here, we focus on the construction of a $\mathrm{Spin}(V)$-invariant Hermitean form on the spinor module.[25] Given this form, one can then proceed as in the positive-definite case. Recall from Proposition 5.3.17 that, for a positive-definite η, the Hermitean forms on the spinor modules are $\mathrm{Spin}(V)$-invariant. In the pseudo-Euclidean case, the situation is more complicated. It can be shown that, here, there does not exist a positive definite $\mathrm{Spin}(V)$-invariant Hermitean form at all. In particular, the canonical Hermitean form on the spinor module is only invariant with respect to the maximal compact subgroup of the spin group, see [59] for details. There exists, however, an indefinite invariant Hermitean form, defined as follows.

Take the canonical (positive-definite) Hermitean form

$$\mathsf{h}(\phi, \psi) := \phi^\dagger \psi\,, \quad \phi, \psi \in \Delta_{r,s}\,, \tag{5.3.48}$$

on $\Delta_{r+s} \cong \mathbb{C}^{2^k}$, where $r + s = 2k$. Let $\{\mathbf{e}_i\}$ be an orthonormal basis in $\mathbb{R}^{r,s}$. Any vector $\mathbf{x} \in \mathbb{R}^{r,s}$ may be decomposed as $\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_-$, where $\mathbf{x}_+ \in \mathrm{span}\,\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ and $\mathbf{x}_- \in \mathrm{span}\,\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_{r+s}\}$. By the explicit presentation of the Clifford action provided in Remark 5.1.20, we have $\gamma(\mathbf{e}_j)^\dagger = \eta_{jj}\gamma(\mathbf{e}_j)$ for all $j = 1, \ldots, n$. Thus,

[24] Note, however, the different conventions there.

[25] The $\mathrm{Spin}(V)$-invariance is relevant for applications in geometry and physics. In particular, one wants to construct $\mathrm{Spin}(V)$-invariant Lagrangians for field theoretical models.

$$\mathsf{h}(\mathbf{x}_+ \cdot \phi, \psi) = \mathsf{h}(\phi, \mathbf{x}_+ \cdot \psi), \quad \mathsf{h}(\mathbf{x}_- \cdot \phi, \psi) = -\mathsf{h}(\phi, \mathbf{x}_- \cdot \psi). \tag{5.3.49}$$

We define

$$\Gamma_r = \begin{cases} \mathbf{e}_1 \cdot \mathbf{e}_2 \cdot \ldots \cdot \mathbf{e}_r & \text{if } r = 0, 1 \pmod 4 \\ i\,\mathbf{e}_1 \cdot \mathbf{e}_2 \cdot \ldots \cdot \mathbf{e}_r & \text{if } r = 2, 3 \pmod 4 \end{cases} \tag{5.3.50}$$

Then, $\Gamma_r^2 = 1$ and $\mathbf{x}_+ \cdot \Gamma_r = (-1)^{r-1} \Gamma_r \cdot \mathbf{x}_+$ and $\mathbf{x}_- \cdot \Gamma_r = (-1)^r \Gamma_r \cdot \mathbf{x}_-$. Thus,

$$\mathsf{h}(\Gamma_r \cdot \phi, \psi) = \mathsf{h}(\phi, \Gamma_r \cdot \psi), \tag{5.3.51}$$

for any $\phi, \psi \in \Delta_{r+s}$. Now, we can define a modified Hermitean form:

$$\mathsf{h}_\Delta(\phi, \psi) := \mathsf{h}(\Gamma_r \cdot \phi, \psi), \quad \phi, \psi \in \Delta_{r,s}. \tag{5.3.52}$$

Proposition 5.3.23 *The bilinear form* h_Δ *has the following properties.*

1. *It defines an indefinite Hermitean form of index* 2^{k-1}.
2. *It is* $\mathrm{Spin}_{r,s}$*-invariant.*
3. *For any* $\mathbf{x} \in \mathbb{R}^n$ *and any* $\phi, \psi \in \Delta_{r,s}$,

$$\mathsf{h}_\Delta(\mathbf{x} \cdot \phi, \psi) + (-1)^r \mathsf{h}_\Delta(\phi, \mathbf{x} \cdot \psi) = 0. \tag{5.3.53}$$

Proof The matrix $\gamma(\Gamma_r)$ is non-singular and has 2^{k-1} positive and 2^{k-1} negative eigenvalues. Moreover, by (5.3.51),

$$\overline{\mathsf{h}_\Delta(\phi, \psi)} = \overline{\mathsf{h}(\Gamma_r \cdot \phi, \psi)} = \mathsf{h}(\psi, \Gamma_r \cdot \phi) = \mathsf{h}(\Gamma_r \cdot \psi, \phi) = \mathsf{h}_\Delta(\psi, \phi).$$

This proves the first assertion. Next, take any $\mathbf{x} \in \mathbb{R}^n$ and decompose $\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_-$. Then, using (5.3.49), we calculate

$$\begin{aligned} \mathsf{h}_\Delta(\mathbf{x} \cdot \phi, \psi) &= \mathsf{h}(\Gamma_r \cdot \mathbf{x} \cdot \phi, \psi) \\ &= \mathsf{h}(\Gamma_r \cdot \mathbf{x}_+ \cdot \phi, \psi) + \mathsf{h}(\Gamma_r \cdot \mathbf{x}_- \cdot \phi, \psi) \\ &= (-1)^{r-1} \mathsf{h}(\mathbf{x}_+ \cdot \Gamma_r \cdot \phi, \psi) + (-1)^r \mathsf{h}(\mathbf{x}_- \cdot \Gamma_r \cdot \phi, \psi) \\ &= (-1)^{r-1} \mathsf{h}(\Gamma_r \cdot \phi, \mathbf{x} \cdot \psi) \\ &= (-1)^{r-1} \mathsf{h}_\Delta(\phi, \mathbf{x} \cdot \psi). \end{aligned}$$

This proves the third assertion. Finally, let $g = \mathbf{x}_1 \cdot \ldots \cdot \mathbf{x}_{2m} \in \mathrm{Spin}_{r,s}$. Then, using (5.3.53) together with $N(g) = 1$, we obtain

$$\mathsf{h}_\Delta(g \cdot \phi, g \cdot \psi) = (-1)^{2rm} \mathsf{h}_\Delta(\phi, \psi) = \mathsf{h}_\Delta(\phi, \psi).$$

■

In applications, we will usually denote $\mathsf{h}_\Delta = \langle \cdot, \cdot \rangle$.

Remark 5.3.24 One can take

$$\Gamma_s = \begin{cases} (-1)^{[\frac{s+1}{2}]} \mathbf{e}_{r+1} \cdot \mathbf{e}_{r+2} \cdot \ldots \cdot \mathbf{e}_{r+s} & \text{if } s = 0, 1 \pmod 4 \\ i(-1)^{[\frac{s+1}{2}]} \mathbf{e}_{r+1} \cdot \mathbf{e}_{r+2} \cdot \ldots \cdot \mathbf{e}_{r+s} & \text{if } s = 2, 3 \pmod 4 \end{cases} \tag{5.3.54}$$

and define an (equivalent) modified Hermitean form replacing Γ_r by Γ_s in (5.3.52). ♦

Example 5.3.25 We take up Example 5.3.9. Using the presentation of the Clifford multiplication given by (5.1.26), we obtain the following Hermitean form on the bispinor space $S \oplus \overline{S}^*$ over the Minkowski space:

$$\mathsf{h}_\Delta(\Psi_1, \Psi_2) = \Psi_1^\dagger \gamma^0 \Psi_2 = \phi_1^\dagger \tilde{\varphi}_2 + \tilde{\varphi}_1^\dagger \phi_2 \,, \tag{5.3.55}$$

for any $\Psi_1, \Psi_2 \in S \oplus \overline{S}^*$ decomposed as in Example 5.3.9. Comparing this with (5.3.21), the $\mathrm{Spin}_{1,3}$-invariance of h_Δ is obvious. ♦

Exercises

5.3.1 Prove formula (5.3.24).

5.3.2 Prove the formulae (5.3.26) and (5.3.30).

5.3.3 Complete the proof of Proposition 5.3.14.

5.3.4 Prove formulae (5.3.33) and (5.3.36).

5.3.5 Work out the details of Example 5.3.15.

5.3.6 Show that the quantization mapping intertwines the inner products in Cl_n^c and $\bigwedge V_{\mathbb{C}}$ as defined in Example 5.3.16.

5.3.7 Prove formula (5.3.47).

5.4 Spin Structures and Spin^c-Structures

Now, we consider a real orientable n-dimensional Riemannian vector bundle $\pi : E \to M$. Recall from Corollary 4.8.4 that E is orientable iff $\mathsf{w}_1(E) = 0$. Moreover, if $\mathsf{w}_1(E) = 0$, then the distinct orientations on E are in one-to-one correspondence with elements of $H^0_{\mathbb{Z}_2}(M)$. By Example 1.6.6, the associated frame bundle of E may be reduced to the bundle of orthonormal frames $O(E)$ and every choice of an orientation yields a further reduction to the bundle of oriented orthonormal frames $O_+(E)$. By Corollary 5.2.8, there is an exact sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{j} \mathrm{Spin}(n) \xrightarrow{\lambda} \mathrm{SO}(n) \to 1\,. \tag{5.4.1}$$

Thus, we have a covering homomorphism $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ with kernel $\mathbb{Z}_2$. By Remark 5.2.9, the latter is universal for $n > 2$.

Definition 5.4.1 (*Spin structure*) Let $\pi : E \to M$ be a real orientable n-dimensional Riemannian vector bundle with $n > 2$. Then, a spin structure on E is a pair $(S(E), \Lambda)$, where $S(E)$ is a principal $\mathrm{Spin}(n)$-bundle over M and $\Lambda : S(E) \to O_+(E)$, together with λ, is a vertical bundle morphism.

Two spin structures $(S_1(E), \Lambda_1)$ and $(S_2(E), \Lambda_2)$ are called equivalent if there exists a $\mathrm{Spin}(n)$-equivariant mapping $F : S_1(E) \to S_2(E)$ fulfilling $\Lambda_2 \circ F = \Lambda_1$.

Note that, by (5.4.1), Λ is a two-sheeted covering.

By Example 5.2.11, $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Thus, for $n = 2$, we take for $\lambda : \mathrm{U}(1) \to \mathrm{U}(1)$ the connected two-fold covering. For $n = 1$, a spin structure is defined as a two-fold covering of M.

Remark 5.4.2 In the terminology of Sect. 2.2, a spin structure is yet another example of an H-structure. In the terminology of Sect. 1.6, $O_+(E)$ is a λ-extension of $S(E)$, or, since λ is surjective, $S(E)$ is a $\mathrm{Spin}(n)$-extension of $O_+(E)$. We have

$$O_+(E) = S(E) \times_{\mathrm{Spin}(n)} \mathrm{SO}(n)\,, \tag{5.4.2}$$

or, on the level of vector bundles,

$$E \cong O_+(E) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong S(E) \times_{\mathrm{Spin}(n)} \mathbb{R}^n\,. \tag{5.4.3}$$

This yields an equivalent definition of a spin structure: a spin structure on E is a pair $(S(E), \varphi)$, where $S(E)$ is a principal $\mathrm{Spin}(n)$-bundle over M and

$$\varphi : E \to S(E) \times_{\mathrm{Spin}(n)} \mathbb{R}^n$$

is an isomorphism of oriented Riemannian vector bundles. ♦

Let us discuss the question of existence and uniqueness of spin structures.

Theorem 5.4.3 *Let $\pi : E \to M$ be a real oriented Riemannian vector bundle. Then, there exists a spin structure on E iff the second Stiefel–Whitney class $\mathsf{w}_2(E)$ vanishes. Moreover, if $\mathsf{w}_2(E) = 0$, then the isomorphism classes of spin structures on E are in one-to-one correspondence with the elements of $H^1_{\mathbb{Z}_2}(M)$.*

Proof By Proposition 3.7.5, the exact sequence (5.4.1) induces a fibration of classifying spaces,

$$B\mathbb{Z}_2 \xrightarrow{Bj} B\mathrm{Spin}(n) \xrightarrow{B\lambda} B\mathrm{SO}(n)\,. \tag{5.4.4}$$

By Appendix G, $B\mathbb{Z}_2$ coincides with the Eilenberg–MacLane space $K(\mathbb{Z}_2, 1)$ and thus, by the discussion in Sect. 4.8, the fibration (5.4.4) implies the sequence

$$K(\mathbb{Z}_2, 1) \xrightarrow{Bj} B\mathrm{Spin}(n) \xrightarrow{B\lambda} B\mathrm{SO}(n) \xrightarrow{\theta} K(\mathbb{Z}_2, 2)\,. \tag{5.4.5}$$

Using Corollary 3.6.9 and $[M, K(\mathbb{Z}_2, n)] = H^n_{\mathbb{Z}_2}(M)$, we derive from (5.4.5) the following exact sequence of pointed sets:

$$\begin{aligned} \ldots \longrightarrow [M, \mathrm{Spin}(n)] \xrightarrow{\lambda_*} [M, \mathrm{SO}(n)] \xrightarrow{\Omega\theta_*} H^1_{\mathbb{Z}_2}(M) \xrightarrow{Bj_*} [M, B\mathrm{Spin}(n)] \\ \xrightarrow{B\lambda_*} [M, B\mathrm{SO}(n)] \xrightarrow{\theta_*} H^2_{\mathbb{Z}_2}(M)\,. \end{aligned} \tag{5.4.6}$$

Now, a principal $\mathrm{SO}(n)$-bundle P admits a 2-fold covering by a principal $\mathrm{Spin}(n)$-bundle iff it is contained in the image of $B\lambda_*$, that is, according to the exactness of this sequence iff $\theta_*(P) = 0$. But, by definition of the Stiefel–Whitney classes, we have

$$\theta_*(P) = \mathsf{w}_2(P)\,.$$

The second statement also follows from the exactness of the sequence (5.4.6). ■

The most important special case is provided by the choice $E = \mathrm{T}M$.

Definition 5.4.4 (*Spin manifold*) A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle.

Since the Stiefel–Whitney classes of a manifold M are, by definition, the Stiefel–Whitney classes of $\mathrm{T}M$, Theorem 5.4.3 implies the following.

Corollary 5.4.5 *An oriented Riemannian manifold M admits a spin structure iff its second Stiefel–Whitney class $\mathsf{w}_2(M)$ vanishes. Moreover, if $\mathsf{w}_2(M) = 0$, then the isomorphism classes of spin structures on M are in one-to-one correspondence with the elements of $H^1_{\mathbb{Z}_2}(M)$.* ■

Remark 5.4.6 Let $L_+(M)$ be the bundle of oriented linear frames of M. We show that, for any Riemannian metric on M, the manifolds $O_+(M)$ and $L_+(M)$ are homotopy equivalent. For that purpose, let $j : O_+(M) \to L_+(M)$ be the natural inclusion mapping and let $p : L_+(M) \to O_+(M)$ be defined by the standard orthonormalization procedure of linear frames. Then, clearly $p \circ j = \mathrm{id}_{O_+(M)}$. Since, for any $u \in L_+(M)$, the image $j \circ p(u)$ is obtained from u by a transformation from $\mathrm{GL}_+(\mathbb{R}^n)$ and, since $\mathrm{GL}_+(\mathbb{R}^n)$ is connected, we conclude that $j \circ p$ is homotopic to the identity on $L_+(M)$. This implies that the choice of a spin structure for a given Riemannian metric uniquely determines a spin structure for any other Riemannian metric. In this sense, a spin structure does not depend on the choice of the Riemannian metric. ♦

Remark 5.4.7 (*Pseudo-Riemannian manifolds*) If (M, g) is a pseudo-Riemannian manifold with signature (r, s), then one has the following existence criterion, see [59] and further references therein: let $\mathrm{T}M = E^r \oplus E^s$ be a decomposition of $\mathrm{T}M$ into a time-like (positive definite) subbundle E^r and an orthogonal space-like (negative definite) subbundle E^s. The manifold M admits a spin structure iff $\mathsf{w}_2(M) = \mathsf{w}_1(E^r) \cup \mathsf{w}_1(E^s)$. In particular, a time- or a space-orientable pseudo-Riemannian manifold admits a spin structure if its second Stiefel–Whitney class vanishes. The spin structures are classified by the group $H^1_{\mathbb{Z}_2}(M)$. We refer the reader to [59] for a discussion of special classes of examples important in geometry and physics. ♦

We continue with a number of examples.

Example 5.4.8 (*2-connected manifolds*) If M is 2-connected, then the Hurewicz Theorem, together with the Universal Coefficient Theorem, implies that $H^1_{\mathbb{Z}_2}(M)$ and $H^2_{\mathbb{Z}_2}(M)$ vanish. Thus, M carries a unique spin structure. Examples of this type are spheres of dimension $n > 2$, simply connected Lie groups and the Stiefel manifolds $\mathrm{S}_{\mathbb{K}}(k, l)$ of k-frames in $\mathbb{K}^l$ fulfilling $d(l - k + 1) \geq 4$, see Theorem 3.4.10. Here, d is the dimension of $\mathbb{K}$ over $\mathbb{R}$. ♦

Example 5.4.9 (*Spin Structure of* S^4) Consider $M = \mathrm{S}^4$ which fits into the class of manifolds described by Example 5.4.8. Let us calculate the spin structure $S(\mathrm{S}^4)$ explicitly. By Example 5.2.11, we have $\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $\mathrm{Spin}(5) = \mathrm{Sp}(2)$. Thus, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[r, "i"] \arrow[d, "\lambda"'] & \mathrm{Sp}(2) \arrow[d, "\lambda'"] \\
\mathrm{SO}(4) \arrow[r, "i'"] & \mathrm{SO}(5)
\end{tikzcd}$$

Here, λ and λ' are the covering homomorphisms and i and i' denote the natural inclusion homomorphisms. Next, recall from Example 1.1.24 and Remark 1.1.25 that

$$\mathrm{S}^4 \cong S_{\mathbb{R}}(1, 5) \cong \mathrm{SO}(5)/\mathrm{SO}(4) \tag{5.4.7}$$

and

$$\mathbb{H}\mathrm{P}^1 \cong G_{\mathbb{H}}(1, 2) \cong \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \,. \tag{5.4.8}$$

By Example 1.1.18, the bundle of oriented orthonormal frames $O_+(\mathrm{S}^4)$ coincides with the principal $\mathrm{SO}(4)$-bundle $\mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(4)$ and, by (5.4.8), $\mathrm{Sp}(2)$ carries the structure of a principal $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-bundle over S^4. Thus, the pair (λ', λ) of Lie group homomorphisms defines a morphism of principal bundles:

$$\begin{tikzcd}
\mathrm{Sp}(2) \arrow[r, "\lambda'"] \arrow[d] & O(\mathrm{S}^4) \arrow[d] \\
\mathrm{S}^4 \arrow[r, "\mathrm{id}"] & \mathrm{S}^4
\end{tikzcd}$$

Since λ' is a 2-fold covering, we conclude that $\mathrm{Sp}(2)$, viewed as a principal $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-bundle over S^4, coincides with the (unique) spin structure $S(\mathrm{S}^4)$. ♦

Example 5.4.10 (*Projective spaces*) Consider the n-dimensional projective space $\mathbb{K}\mathrm{P}^n$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then,

1. $\mathbb{R}\mathrm{P}^n$ is spin iff $n = 3 \pmod 4$.
2. $\mathbb{C}\mathrm{P}^n$ is spin iff n is odd.
3. $\mathbb{H}\mathrm{P}^n$ is spin for all n.

In case 1 we have two spin structures and in the remaining cases the spin structures are unique. By Example 4.5.3, the total Stiefel–Whitney class of $\mathbb{K}\mathrm{P}^n$ is

$$\mathsf{w} = 1 + \mathsf{w}_1 + \mathsf{w}_2 + \ldots = (1 + \xi)^{n+1} ,$$

where ξ is the generator of the $\mathbb{Z}_2$-cohomology ring. This generator has degree 1, 2 and 4 for, respectively, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Now, one has to analyze the conditions $\mathsf{w}_1 = 0$ and $\mathsf{w}_2 = 0$ for each case. For $\mathbb{K} = \mathbb{R}$, they are equivalent to $\mathsf{w}_1 = (n + 1)\xi$ and $\mathsf{w}_2 = \binom{n+1}{2}\xi^2 = 0$, that is,

$$(n + 1) = 0 \pmod 2 , \quad \frac{n(n + 1)}{2} = 0 \pmod 2 .$$

Moreover, $H^1_{\mathbb{Z}_2}(\mathbb{R}\mathrm{P}^n)$ is generated by ξ. This yields the assertion. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, the proof is obvious.

In special cases, one can give a proof by simple geometric arguments, see e.g. Proposition 3.3 in [554] where it is shown that $\mathbb{C}\mathrm{P}^2$ cannot carry a spin structure. ♦

Example 5.4.11 Let (M, g) be an oriented 4-dimensional Riemannian manifold. Consider the Hodge decomposition

$$\bigwedge\nolimits^2 \mathrm{T}^*M = \bigwedge\nolimits^2{}_+ \mathrm{T}^*M \oplus \bigwedge\nolimits^2{}_- \mathrm{T}^*M ,$$

see (2.8.17). Since $\bigwedge^2_+ \mathbb{R}^4 \cong \mathfrak{so}(3) = \mathrm{spin}(3)$, the subbundles $E_\pm = \bigwedge^2_\pm \mathrm{T}M$ are Riemannian with the fibre metric induced from the Killing form on $\mathfrak{so}(3)$. Let $O(E_\pm)$ be the principal SO(3)-bundles of (positive or negative) orthonormal frames of $E_\pm$ and let $S(E_\pm)$ be the corresponding spin bundles. It is easy to show (Exercise 5.4.1) that

$$E_\pm \cong \mathrm{Ad}(S(E_\pm)) . \tag{5.4.9}$$

♦

Example 5.4.12 (*Compact Riemann surfaces*) For a compact Riemann surface of genus g, there are exactly 2^{2g} distinct spin structures. We refer to [407] for their explicit construction. ♦

Example 5.4.13 Any complex manifold M is orientable, because the realification of a complex vector bundle is orientable, see Sect. 4.2. Moreover, by Corollary 4.4.7/1, $\mathsf{w}_2(M)$ is the mod 2-reduction of the first Chern class $\mathrm{c}_1(M)$. Thus, a complex manifold is spin iff $\mathrm{c}_1(M) = 0 \pmod 2$. ♦

Since a spin structure is a principal bundle, the ordinary theory of connections as developed in Chap. 1 applies. Since $\Lambda : S(M) \to O_+(M)$ is a covering, any connection form ω in $O_+(M)$ lifts uniquely to a connection form $\hat{\omega}$ in $S(M)$. The latter is defined via the commutative diagram

$$\begin{array}{ccc} \mathrm{T}S(M) & \xrightarrow{\hat{\omega}} & \mathrm{spin}(n) \\ {\scriptstyle \Lambda'}\downarrow & & \downarrow{\scriptstyle d\lambda} \\ \mathrm{T}O_+(M) & \xrightarrow{\omega} & \mathfrak{so}(n) \end{array}$$

Uniqueness follows from the fact that $d\lambda$ is an isomorphism of Lie algebras. Explicitly,

$$\hat{\omega} = (\mathrm{d}\lambda)^{-1}\Lambda^*\omega\,. \tag{5.4.10}$$

On the other hand, according to Corollary 1.3.14, any connection in $S(M)$ induces a unique connection in $O_+(M)$.

Definition 5.4.14 Let (M, g) be an oriented Riemannian spin manifold and let $(S(M), \Lambda)$ be a chosen spin structure on M. Let ω be the Levi-Civita connection of g viewed as a principal connection on $O_+(M)$. Then, the unique lift $\hat{\omega}$ defined by (5.4.10) will be referred to as the spin connection on $S(M)$.

Finally, we show that the notion of a spin structure extends to the notion of a Spin^c-structure in an obvious way. Let

$$\lambda := \mathrm{pr}_1 \circ p : \ \mathrm{Spin}^c(n) \to \mathrm{SO}(n)\,, \quad \sigma := \mathrm{pr}_2 \circ p : \ \mathrm{Spin}^c(n) \to \mathrm{U}(1)\,,$$

be the natural homomorphisms defined by the sequence (5.2.17).

Definition 5.4.15 (Spin^c-*structure*) Let $\pi : E \to M$ be a real orientable n-dimensional Riemannian vector bundle with $n > 2$. Then, a *Spin^c*-structure on E is a pair $(S^c(E), \Lambda)$, where $S^c(E)$ is a principal Spin$^c(n)$-bundle over M and $\Lambda : S^c(E) \to O_+(E)$, together with λ, is a vertical principal bundle morphism.

Clearly, by the above definition, since $\mathrm{U}(1)$ and $\mathrm{Spin}(n)$ are Lie subgroups of $\mathrm{Spin}^c(n)$, we have

(a) $S^c(E)$ factorised with respect to the natural right $\mathrm{U}(1)$-action is isomorphic to $O_+(E)$.
(b) $S^c(E)$ factorised with respect to the natural right $\mathrm{Spin}(n)$-action is a principal $\mathrm{U}(1)$-fibre bundle over M which we denote by P.
(c) We have a two-fold covering $S^c(E) \to O_+(E) \times_M P$, where $O_+(E) \times_M P$ is the fibre product[26] of principal bundles over M with structure group $\mathrm{SO}(n) \times \mathrm{U}(1)$.

Associated with P, we have a complex line bundle

$$L := P \times_\sigma \mathbb{C} \tag{5.4.11}$$

[26]Cf. Remark 1.1.9/2.

which is referred to as the fundamental (or determinant) line bundle of the Spin^c-structure. As in Remark 5.4.2, on the level of vector bundles, we have

$$E \cong O_+(E) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong S^c(E) \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n .$$

Thus, as before, identifying $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and considering the natural embedding $i : \mathrm{SO}(n) \times \mathrm{SO}(2) \to \mathrm{SO}(n+2)$ induced from the decomposition $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$, we obtain

$$E \oplus L \cong S^c(E) \times_{i \circ p} (\mathbb{R}^n \oplus \mathbb{R}^2) . \tag{5.4.12}$$

Now, by the commutative diagram (5.2.22), we have the following.

Proposition 5.4.16 *An oriented Riemannian vector bundle E over M admits a* Spin^c*-structure iff there exists a complex line bundle L over M such that $E \oplus L$ admits a spin structure.*

Using this criterion, it is easy to discuss the obstruction against the existence of a Spin^c-structure.

Proposition 5.4.17 *An oriented Riemannian vector bundle E admits a* Spin^c*-structure iff its second Stiefel–Whitney class $\mathsf{w}_2(E)$ is the mod 2 reduction of a cohomology class from $H^2_{\mathbb{Z}}(M)$.*

Proof By Proposition 5.4.16 and Theorem 5.4.3, E admits a Spin^c-structure iff there exist a line bundle L such that $\mathsf{w}_2(E \oplus L) = 0$, that is, by the Whitney Sum Formula, iff

$$\mathsf{w}_2(E \oplus L) = \mathsf{w}_2(E) + \mathsf{w}_2(L) + \mathsf{w}_1(E)\mathsf{w}_1(L) = 0 .$$

Since E and L are oriented, we conclude $\mathsf{w}_2(E) + \mathsf{w}_2(L) = 0$. Since these are mod 2 classes, this implies

$$\mathsf{w}_2(E) = \mathsf{w}_2(L) .$$

But, $\mathsf{w}_2(L)$ is the mod 2 reduction of $\mathrm{c}_1(L)$. This proves the assertion in one direction. Conversely, if $\mathsf{w}_2(E)$ is the mod 2 reduction of an integral cohomology class α, then we can find a complex line bundle L with first Chern class α. ∎

Let $\mathrm{Spin}^c(E)$ be the set of Spin^c-structures on E. By assigning to a Spin^c-structure the first Chern class of P, we obtain a mapping

$$\mathrm{Spin}^c(E) \to H^2_{\mathbb{Z}}(M) .$$

It can be shown [219] that the Spin^c-structures of E are classified by $H^2_{\mathbb{Z}}(M)$.

Example 5.4.18 Note that via the natural inclusion mapping $\iota : \mathrm{Spin}(n) \to \mathrm{Spin}^c(n)$, every spin structure $S(E)$ induces a Spin^c-structure of E. The latter is obtained by taking the fibre product with the trivial principal $\mathrm{U}(1)$-bundle P_0,

$$S^c(E) = S(E) \times_M P_0 \,.$$

♦

Example 5.4.19 Assume that E admits a complex structure, that is, $O_+(E)$ admits a $\mathrm{U}(k)$-reduction Q. Then, by Proposition 5.2.14, there exists a homomorphism $F : \mathrm{U}(k) \to \mathrm{Spin}^c(2k)$ projecting onto $\mathrm{SO}(2k) \times \mathrm{U}(1)$. Thus,

$$S^c(E) := Q \times_{\mathrm{U}(k)} \mathrm{Spin}^c(2k)$$

is a Spin^c-structure of E. ♦

Clearly, the most important example is $E = \mathrm{T}M$.

Definition 5.4.20 (Spin^c-*manifold*) Let M be an oriented Riemannian manifold. If $\mathrm{T}M$ admits a Spin^c-structure, then M is called a Spin^c-manifold.

By Example 5.4.18, every spin manifold has a canonical Spin^c-structure and, by Example 5.4.19, every almost complex manifold has a canonical Spin^c-structure, too. The following deep theorem holds [305, 681].

Theorem 5.4.21 (Wu–Hirzebruch–Hopf) *Every compact orientable* 4-*dimensional manifold is* Spin^c.

As in the case of a spin structure, the ordinary theory of connections applies here. For a given $\mathrm{Spin}^c(n)$-structure $S^c(E)$ on E, a connection form ω on $S^c(E)$ takes values in the Lie algebra

$$\mathrm{spin}^c(n) = \mathrm{spin}(n) \oplus i\mathbb{R} \cong \mathfrak{so}(n) \oplus i\mathbb{R} \,.$$

Let there be given connection forms on $O_+(E)$ and P, respectively. Then, by Remark 1.3.17, they induce a connection form on the fibre product $O_+(E) \times_M P$ and, since $S^c(E) \to O_+(E) \times_M P$ is a covering, the latter lifts to a unique connection form on $S^c(E)$. Conversely, given a connection form on $S^c(E)$, it induces connection forms on $O_+(E)$ and P, respectively.

Exercise

5.4.1 Prove formula (5.4.9).

5.5 Clifford Modules and Dirac Operators

We introduce a variety of vector bundle structures associated with Clifford modules and, in particular, with spinor modules. These structures can be defined for arbitrary pseudo-Riemannian vector bundles $E \to M$, but in applications in geometry and physics the special case $E = \mathrm{T}M$ with M being a pseudo-Riemannian manifold is the most important one. We rather focus on the Riemannian case.

First, observe that the basic representation of $\mathrm{SO}(n)$ on the Euclidean space $(\mathbb{R}^n, \mathsf{q})$ induces an action of $\mathrm{SO}(n)$ by algebra homomorphisms on the tensor algebra over $\mathbb{R}^n$ which leaves the ideal $\mathscr{I}_{\mathsf{q}}(\mathbb{R}^n)$ defined in Sect. 5.1 invariant. Thus, we obtain a representation of $\mathrm{SO}(n)$ on the Clifford algebra Cl_n by algebra homomorphisms:

$$\rho_n : \mathrm{SO}(n) \to \mathrm{Aut}(Cl_n)\,. \tag{5.5.1}$$

Definition 5.5.1 (*Clifford bundle*) Let E be an oriented Riemannian vector bundle of rank n and let $O_+(E)$ be the bundle of oriented orthonormal frames. Then, the associated algebra bundle

$$Cl(E) := O_+(E) \times_{\rho_n} Cl_n \tag{5.5.2}$$

will be referred to as the Clifford bundle of E. For an oriented Riemannian manifold (M, g), the bundle $Cl(\mathrm{T}M)$ will be called the Clifford bundle of M. It will be denoted by $Cl(M)$.

By analogy, one defines the Clifford bundle of a Hermitean vector bundle using the extension of ρ_n to Cl_n^c. For example, we can take the complexification

$$Cl^c(E) = Cl(E) \otimes \mathbb{C} = O_+(E) \times_{\rho_n} Cl_n^c\,. \tag{5.5.3}$$

Below, we will often not distinguish in notation between $Cl(E)$ and $Cl^c(E)$ and just write $Cl(E)$ for both.

Note that $Cl(E)$ is a bundle of Clifford algebras over M. In particular, the fibrewise multiplication in $Cl(E)$ provides the space of sections of $Cl(E)$ with a natural algebra structure. It follows that all Clifford algebra operations carry over to Clifford bundles. In particular, the parity automorphism induces a vertical bundle automorphism of $Cl(E)$ and, thus, we obtain a decomposition

$$Cl(E) = Cl^0(E) \oplus Cl^1(E) \tag{5.5.4}$$

corresponding to (5.1.5). Moreover, the vector space isomorphism given by Proposition 5.1.10 induces an vector bundle isomorphism

$$\bigwedge E \cong Cl(E)\,. \tag{5.5.5}$$

Second, we consider bundles of modules over the Clifford bundle, that is, for a given Riemannian (or Hermitean) vector bundle $E \to M$, the fibre at $m \in M$ of such a bundle is a left module over $Cl(E_m)$. In particular, for $E = \mathrm{T}M$, the fibre at m is a left module over $Cl(\mathrm{T}_m M)$. Such bundles will be referred to as Clifford module bundles. We give the definition for the case $E = \mathrm{T}M$. The generalization to an arbitrary Riemannian (or Hermitean) vector bundle will then be obvious.

Definition 5.5.2 (*Clifford module bundle over $Cl(M)$*) Let (M, g) be a Riemannian manifold and let $\mathscr{E} \to M$ be a real (or complex) vector bundle. If there exists a mapping $c : \mathrm{T}M \to \mathrm{End}(\mathscr{E})$ fulfilling

$$c(X)^2 = \mathsf{g}(X, X)\,\mathrm{id}_{\mathscr{E}_m} \tag{5.5.6}$$

for every $X \in \mathrm{T}_m M$, then c is referred to as a Clifford mapping and $\mathscr{E}$ as a Clifford module bundle over $Cl(M)$.

By the universal property, since $\mathrm{T}M \subset Cl(M)$ generates $Cl(M)$ fibrewise, c induces a unique homomorphism

$$\hat{c} : Cl(M) \to \mathrm{End}(\mathscr{E}) \tag{5.5.7}$$

of algebra bundles fulfilling $\hat{c}(X) = c(X)$ for any $X \in \mathrm{T}_m M$. This justifies the terminology.

The special case when the typical fibre of a complex Clifford module bundle $\mathscr{E}$ coincides with a spinor module Δ_n is of particular importance. Such a bundle will be referred to as a spinor bundle over $Cl(M)$. Let us assume that the Riemannian manifold (M, g) admits a spin structure $(S(M), \Lambda)$. Then, we have a canonically associated bundle,

$$\mathscr{S}(M) := S(M) \times_\gamma \Delta_n\,, \tag{5.5.8}$$

where γ denotes the spinor representation.

Definition 5.5.3 (*Spinor bundle*) The vector bundle $\mathscr{S}(M)$ will be referred to as the spinor bundle of (M, g) relative to the fixed spin structure $S(M)$.

In the sequel, $\mathscr{S}(M)$ will also be called the canonical spinor bundle. The generalization to a Hermitean vector bundle carrying a spin structure is obvious. Clearly, $\mathscr{S}(M)$ is a Clifford module bundle with the Clifford mapping given by the spinor representation γ. This follows from the fact that (Exercise 5.5.1)

$$Cl(M) \cong S(M) \times_{\mathrm{Spin}(n)} Cl_n \tag{5.5.9}$$

with the action of $\mathrm{Spin}(n)$ on Cl_n given by conjugation,

$$\mathrm{Ad} : \mathrm{Spin}(n) \times Cl_n \to Cl_n\,, \quad \mathrm{Ad}(g)a := gag^{-1}\,.$$

Now, the spinor representation of Cl_n on Δ_n induces a fibrewise action of the associated bundle $Cl(M) \cong S(M) \times_{\mathrm{Spin}(n)} Cl_n$ on $\mathscr{S}(M)$. Note that Remark 5.3.3 implies the following.

Remark 5.5.4 Let $\mathscr{E}$ be a complex Clifford module bundle. Then, the isomorphism (5.3.6) implies

$$\mathrm{End}(\mathscr{E}) \cong Cl^c(M) \otimes \mathrm{End}_{Cl(M)}(\mathscr{E})\,. \tag{5.5.10}$$

Moreover, since locally every Riemannian manifold admits a spin structure, (5.3.5) implies the following local structure for any Clifford module bundle $\mathscr{E}$:

$$\mathscr{E}_{\restriction U} \cong \mathscr{S}(U) \otimes \mathscr{W}\,, \tag{5.5.11}$$

where $U \subset M$ is an open subset, $\mathscr{S}(U)$ is the spinor bundle with respect to a chosen spin structure on U and $\mathscr{W} = \mathrm{Hom}_{Cl(M)}(\mathscr{S}(U), \mathscr{E})$. If (M, g) admits a spin structure, then (5.5.11) holds globally. ♦

According to (5.3.15), for $n = 2k$, $\mathscr{S}(M)$ splits into a direct sum of subbundles,

$$\mathscr{S}(M) = \mathscr{S}^+(M) \oplus \mathscr{S}^-(M)\,, \quad \mathscr{S}^\pm(M) = S(M) \times_\gamma \Delta_n^\pm\,.$$

Remark 5.5.5 Let $n = 2k$. By point 2 of Proposition 5.3.11, the Clifford multiplication with any non-vanishing vector $\mathbf{x} \in \mathbb{R}^n$ yields vector space isomorphisms $\Delta_n^\pm \to \Delta_n^\mp$. This implies that the Clifford mapping c is odd, that is, for any $X \in \mathrm{T}_m M$, we have a bundle isomorphism $c(X) : \mathscr{S}^\pm(M) \to \mathscr{S}^\mp(M)$. ♦

Remark 5.5.6 By Remark 5.3.8, in a completely analogous way, we may consider the $\mathrm{Spin}^c(n)$-representation on Δ_n given by (5.3.20). Thus, we can build the spinor bundle

$$\mathscr{S}^c(M) := S^c(M) \times_{\mathrm{Spin}^c(n)} \Delta_n \tag{5.5.12}$$

with respect to a fixed Spin^c-structure. Moreover, if n is even, then we have a natural splitting

$$\mathscr{S}^c(M) = \mathscr{S}^c_+(M) \oplus \mathscr{S}^c_-(M) \tag{5.5.13}$$

corresponding to the spinor module splitting $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. Many considerations in the sequel, spelled out for $\mathscr{S}(M)$, hold true for that case as well. ♦

Remark 5.5.7 (*Projective spinor bundle*) In 4-dimensional Riemannian geometry, the projectivization of spinor bundles plays an important role. Let M be an oriented 4-dimensional spin manifold. Consider the irreducible spinor modules $\Delta_4^\pm \cong \mathbb{C}^2$ of $\mathrm{Spin}(4)$. Then, for the corresponding projective spaces $\mathrm{P}(\Delta_4^\pm)$ we have

$$\mathrm{P}(\Delta_4^\pm) \cong \mathbb{C}\mathrm{P}^1 \cong \mathrm{Sp}(1)/\mathrm{U}(1)\,. \tag{5.5.14}$$

Thus, $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts naturally on $\mathrm{P}(\Delta_4^\pm)$. Indeed, denoting by $\lambda_\mp : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1)$ the Lie group homomorphisms given by projection onto the first and second component, respectively, we define the left actions

$$\sigma_\mp : (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times (\mathrm{Sp}(1)/\mathrm{U}(1)) \to \mathrm{Sp}(1)/\mathrm{U}(1)\,, \quad \sigma_\mp(h)([g]) := [\lambda_\mp(h)g]\,.$$

Consequently, we can build the associated projective spinor bundles

$$\mathrm{P}^\pm(M) := S(M) \times_{\sigma_\pm} \mathrm{P}(\Delta_4^\pm)\,. \tag{5.5.15}$$

In particular, by Example 5.4.9, $S(\mathrm{S}^4) = \mathrm{Sp}(2)$, where $\mathrm{Sp}(2)$ is viewed as a principal $(\mathrm{Sp}(1)\times\mathrm{Sp}(1))$-bundle over S^4. Then, using (5.5.14), we obtain $\mathrm{P}^{\pm}(\mathrm{S}^4) = \mathrm{Sp}(2)\times_{\sigma_\pm} \mathrm{Sp}(1)/\mathrm{U}(1)$ and, thus,

$$\mathrm{P}^{+}(\mathrm{S}^4) \cong \mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))\,, \quad \mathrm{P}^{-}(\mathrm{S}^4) = \mathrm{Sp}(2)/(\mathrm{U}(1)\times\mathrm{Sp}(1))\,. \tag{5.5.16}$$

♦

Remark 5.5.8 Let M be an oriented 4-dimensional spin manifold endowed with a conformal structure.[27] We show that $\mathrm{P}^{\pm}(M)$ carry natural almost complex structures. Since the Clifford multiplication with any non-vanishing vector of $\mathbb{R}^n$ yields vector space isomorphisms $\Delta_4^{\pm} \to \Delta_4^{\mp}$, for any non-zero spinor $\phi \in \mathscr{S}^{-}(M)_m$ at a point $m \in M$, the Clifford multiplication $X \mapsto X\cdot\phi$ with $X \in \mathrm{T}_m M$ yields a real vector space isomorphism $\mathrm{T}_m M \cong \mathscr{S}^{+}(M)_m$ which endows $\mathrm{T}_m M$ with a complex structure. It can be easily seen that the latter is compatible with any metric from the conformal class and that it induces an orientation on $\mathrm{T}_m M$ which is opposite to the chosen orientation of M, see [218] for a detailed proof. Clearly, by multiplying ϕ with a nonvanishing complex number, we obtain the same complex structure, that is, the complex structures constructed this way are parameterized by the projective spaces $\mathrm{P}^{-}(M)_m$. Since the stabilizer of $[\phi] \in \mathrm{P}^{-}(M)_m$ is clearly $\mathrm{U}(1)\times\mathrm{Sp}(1)$, we get

$$\mathrm{P}^{-}(M)_m \cong (\mathrm{Sp}(1)\times\mathrm{Sp}(1))/(\mathrm{U}(1)\times\mathrm{Sp}(1)) \cong \mathrm{SO}(4)/\mathrm{U}(2)\,.$$

Let us fix a Riemannian metric in the conformal class. Then, the spin connection of this metric yields a splitting of $\mathrm{TP}^{-}(M)$ into the vertical distribution V and a horizontal complement Γ,

$$\mathrm{TP}^{-}(M) = V \oplus \Gamma\,.$$

Now, we can endow $\mathrm{P}^{-}(M)$ with an almost complex structure as follows. On V we take the natural complex structures of the fibres which are complex projective lines. On the horizontal part Γ at the point $[\phi] \in \mathrm{P}^{-}(M)_m$ we take the complex structure of $\mathrm{T}_m M$ constructed above.

It can be shown that the almost complex structure on $\mathrm{P}^{-}(M)$ constructed in this way is integrable iff M is self-dual, see Theorem 4.1 in [37]. Moreover, one can show that conformally equivalent metrics yield the same complex structure. For our purposes, the most important example is $M = \mathrm{S}^4$ which is clearly self-dual, cf. Example 2.8.10.

Obviously, $\mathrm{P}^{+}(M)$ may be discussed in a similar manner. ♦

In applications, Clifford module bundles are usually endowed with additional structures. These will be explained next.

[27] The assumption that M be spin can be dropped, see [37].

Definition 5.5.9 Let $\mathscr{E} \to M$ be a real (or complex) Clifford module bundle endowed with a Riemannian (or Hermitean) fibre metric h. If the Clifford mapping $c : \mathrm{T}M \to \mathrm{End}(\mathscr{E})$ maps every $X \in \mathrm{T}M$ to a self-adjoint endomorphism,[28]

$$\mathsf{h}(\Phi, c(X)\Psi) = \mathsf{h}(c(X)\Phi, \Psi) \tag{5.5.17}$$

for any $\Phi, \Psi \in \mathscr{E}_m$ and any $X \in \mathrm{T}_m M$, then $\mathscr{E}$ will be referred to as a Riemannian (or Hermitean) Clifford module bundle. It will be denoted by $(\mathscr{E}, \mathsf{h})$.

In the sequel, it will be often convenient to write $\langle \cdot, \cdot \rangle$ instead of h.

Consider the case $\mathscr{E} = \mathscr{S}(M)$. If we take the Hermitean fibre metric induced from the canonical (positive-definite) Hermitean form[29]

$$\mathsf{h}(\phi, \psi) := \phi^\dagger \psi\,, \quad \phi, \psi \in \Delta_n\,, \tag{5.5.18}$$

then, by (5.3.49), we have $\mathsf{h}(\mathbf{x} \cdot \phi, \psi) = \mathsf{h}(\phi, \mathbf{x} \cdot \psi)$ and, thus, the condition (5.5.17) is fulfilled.

Finally, we consider Riemannian (or Hermitean) Clifford module bundles endowed with a connection compatible with the fibre metric and with the module structure in a sense to be explained. From now on, if there will be no danger of confusion, Clifford mappings will be often denoted by the dot operation,

$$c(X)\Phi = X \cdot \Phi\,.$$

Moreover, we will always assume that the Riemannian manifold under consideration be oriented without further mentioning it.

Thus, let (M, g) be an n-dimensional Riemannian manifold. Note that the Levi-Civita connection of g induces a connection in the Clifford bundle

$$Cl(M) = O_+(M) \times_{\rho_n} Cl_n\,,$$

as well as in its complexification. We denote this connection by ∇^{g}. The Lie algebra homomorphism induced by (5.5.1) is $\rho_n' : \mathfrak{so}(n) \to \mathrm{Der}(Cl_n)$, where $\mathrm{Der}(Cl_n)$ is the Lie algebra of derivations of Cl_n. Consequently, by (1.4.2), ∇^{g} acts as a derivation in the algebra of sections of $Cl(M)$,

$$\nabla^{\mathsf{g}}(\zeta \cdot \chi) = (\nabla^{\mathsf{g}}\zeta) \cdot \chi + \zeta \cdot (\nabla^{\mathsf{g}}\chi)\,, \tag{5.5.19}$$

for any $\zeta, \chi \in \Gamma^\infty(Cl(M))$. Thereby, $\Gamma^\infty(Cl^0(M))$ and $\Gamma^\infty(Cl^1(M))$ are left invariant. Moreover, under the canonical identification $Cl(M) \cong \bigwedge \mathrm{T}M$, ∇^{g} leaves

[28] In textbooks using the convention $j(v)^2 = -\mathsf{q}(v)1$ instead of (5.1.2), $c(X)$ is assumed to be skew-adjoint. Both (5.5.17) and its skew-adjoint counterpart are equivalent to the requirement that the Hermitean form be invariant under the Clifford action by unit vectors, that is, $\langle c(\mathbf{e})\Phi, c(\mathbf{e})\Psi \rangle = \langle \Phi, \Psi \rangle$ for any $\mathbf{e} \in \mathrm{T}_m M$ fulfilling $\mathsf{g}(\mathbf{e}, \mathbf{e}) = 1$.

[29] See (5.3.48).

$\Gamma^\infty\big(\bigwedge^k \mathrm{T}M\big)$ invariant and coincides there with the covariant derivatives defined by the representations $\bigwedge^k \rho_n$.

Definition 5.5.10 (*Dirac bundle*) Let $(\mathscr{E}, \mathsf{h})$ be a Riemannian (or Hermitean) Clifford module bundle over a Riemannian manifold (M, g) endowed with an h-compatible connection ∇. Then, ∇ is called a Clifford connection if it is a module derivation, that is,

$$\nabla(\zeta \cdot \Phi) = \nabla^{\mathsf{g}}(\zeta) \cdot \Phi + \zeta \cdot \nabla\Phi\,, \tag{5.5.20}$$

for any $\zeta \in \Gamma^\infty(Cl(M))$ and $\Phi \in \Gamma^\infty(\mathscr{E})$. A Clifford module bundle $(\mathscr{E}, \mathsf{h})$ over (M, g) endowed with a Clifford connection ∇ will be referred to as a Dirac bundle over (M, g). It will be denoted by $(\mathscr{E}, \mathsf{h}, \nabla)$.

Since $C^\infty(M) \subset \Gamma^\infty(Cl(M))$, formula (5.5.20) implies

$$\nabla_X(f \cdot \Phi) = X(f) \cdot \Phi + f \cdot \nabla_X\Phi\,, \tag{5.5.21}$$

for any $X \in \mathfrak{X}(M), f \in C^\infty(M)$ and $\Phi \in \Gamma^\infty(\mathscr{E})$. Since $\mathrm{T}M \subset Cl(M)$, it implies

$$\nabla_X(Y \cdot \Phi) = \nabla^{\mathsf{g}}_X(Y) \cdot \Phi + Y \cdot \nabla_X\Phi\,, \tag{5.5.22}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\Phi \in \Gamma^\infty(\mathscr{E})$. Clearly, by (5.5.19), ∇^{g} itself is Clifford.

Example 5.5.11 Let (M, g) be a Riemannian manifold endowed with a spin structure $(S(M), \Lambda)$ and let $\mathscr{S}(M) = S(M) \times_\gamma \Delta_n$ be the canonically associated spinor bundle, endowed with the fibre metric induced from the scalar product on Δ_n. Then, the unique spin connection in $S(M)$ induces a canonical connection in $\mathscr{S}(M)$ which is Clifford. Indeed, the representations $\gamma : \mathrm{Spin}(n) \to \mathrm{Aut}(\Delta_n)$ and $\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{Aut}(Cl_n)$ preserve the module multiplication, that is,

$$\gamma(g)(a \cdot \psi) = (\mathrm{Ad}(g)a) \cdot (\gamma(g)\psi)\,,$$

for any $g \in \mathrm{Spin}(n)$, $a \in Cl_n$ and $\psi \in \Delta_n$. Differentiating this equation at the identity of $\mathrm{Spin}(n)$ yields the assertion. ♦

Now we are prepared to introduce the following basic notion.[30]

Definition 5.5.12 (*Dirac operator*) Let $(\mathscr{E}, \mathsf{h}, \nabla)$ be a Dirac bundle over a Riemannian manifold (M, g). Then, the first order differential operator $\mathrm{D} : \Gamma^\infty(\mathscr{E}) \to \Gamma^\infty(\mathscr{E})$ defined by

$$\mathrm{D} := \mathrm{i}\, c \circ \mathsf{g}^{-1} \circ \nabla :\ \Gamma^\infty(\mathscr{E}) \xrightarrow{\nabla} \Gamma^\infty(\mathrm{T}^*M \otimes \mathscr{E}) \xrightarrow{\mathsf{g}^{-1}} \Gamma^\infty(\mathrm{T}M \otimes \mathscr{E}) \xrightarrow{c} \Gamma^\infty(\mathscr{E})$$

[30] The imaginary unit is added to make the Dirac operator self-adjoint. This is the standard convention in physics. In most mathematical textbooks, the Clifford multiplication is chosen to be skew-adjoint, cf. formula (5.5.17) and the associated comment. Then, there is no place for adding an i.

will be referred to as the Dirac operator of $(\mathscr{E}, \mathsf{h}, \nabla)$. The operator D^2 will be called the Dirac Laplacian.

Remark 5.5.13

1. From (5.5.21) we obtain
$$[\mathrm{D}, f] = \mathrm{i}\, c(\mathrm{d}f)\,. \tag{5.5.23}$$

2. Let $\{e_j\}$, $j = 1, \dots, n$, be a local oriented orthonormal frame on M and let $\{\vartheta^j\}$ be its dual coframe. In the sequel, we will often write $c_j := c(e_j)$. Then, by (2.1.30), locally we have $\nabla\Phi = \sum_{j=1}^n \vartheta^j \otimes \nabla_{e_j}\Phi$ and, thus,
$$\mathrm{D}(\Phi) = i\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\Phi \equiv i\sum_{j=1}^{n} c_j \nabla_{e_j}\Phi\,, \tag{5.5.24}$$
for any $\Phi \in \Gamma^\infty(\mathscr{E})$.
3. The notions of Dirac bundle and Dirac operator naturally extend to the pseudo-Riemannian case. ♦

Using the natural volume form $\mathsf{v_g}$ on M and the fibre metric $\mathsf{h} = \langle\cdot,\cdot\rangle$, we endow the space $\Gamma^\infty(\mathscr{E})$ with a natural L^2-inner product,
$$\langle\Phi_1, \Phi_2\rangle_{L^2} := \int_M \langle\Phi_1, \Phi_2\rangle \mathsf{v_g}\,, \quad \Phi_1, \Phi_2 \in \Gamma^\infty(\mathscr{E})\,. \tag{5.5.25}$$

In the sequel, we will limit our attention to sections having a finite L^2-norm. This requirement is always fulfilled for M compact or for sections with compact support.

Proposition 5.5.14 *With respect to the natural L^2-inner product on $\Gamma^\infty(\mathscr{E})$, the Dirac operator is formally self-adjoint,*
$$\langle \mathrm{D}\Phi_1, \Phi_2\rangle_{L^2} = \langle\Phi_1, \mathrm{D}\Phi_2\rangle_{L^2}\,.$$

Proof To calculate $\langle\mathrm{D}\Phi_1, \Phi_2\rangle$ at any point $m \in M$, we can use the local formula (5.5.24). Then, using (5.5.17) and (5.5.22), together with the compatibility condition for ∇ in the form given by (2.6.2), we calculate
$$\begin{aligned}
\langle\mathrm{D}\Phi_1, \Phi_2\rangle &= -\mathrm{i}\sum_j \langle e_j \cdot \nabla_{e_j}\Phi_1, \Phi_2\rangle \\
&= -\mathrm{i}\sum_j \langle\nabla_{e_j}\Phi_1, e_j \cdot \Phi_2\rangle \\
&= -\mathrm{i}\sum_j \left\{e_j\left(\langle\Phi_1, e_j\cdot\Phi_2\rangle\right) - \langle\Phi_1, \left(\nabla^{\mathsf{g}}_{e_j} e_j\right)\cdot\Phi_2\rangle\right\} + \mathrm{i}\sum_j \langle\Phi_1, e_j\cdot\nabla_{e_j}\Phi_2\rangle\,.
\end{aligned}$$

Let $\{\vartheta^j\}$ be the coframe dual to $\{e_j\}$. Defining $\alpha := \sum_j \langle \Phi_1, e_j \cdot \Phi_2 \rangle \vartheta^j$ and using (2.1.50), together with Remark 2.7.5, we obtain

$$\langle \mathrm{D}\Phi_1, \Phi_2 \rangle = i\, \mathrm{d}^* \alpha + \langle \Phi_1, \mathrm{D}\Phi_2 \rangle \,.$$

This implies the assertion. ■

Remark 5.5.15 Under the assumption that the Riemannian manifold (M, g) be complete, one can show that the Dirac operator is an (unbounded) essentially selfadjoint operator on $L^2(\mathscr{E})$, see Sect. 1I.5 in [407] or Sect. 4.1 in [219]. Moreover, we will see that the Dirac operator has a pure point spectrum, see Proposition 5.7.11. ♦

Let us discuss two basic examples.

Example 5.5.16 (*The Clifford bundle*) Consider the Clifford bundle $Cl(M)$ over (M, g) endowed with its canonical Riemannian connection induced from the Levi-Civita connection of g. Recall that $Cl(M)$ is a bundle of left modules over itself by left Clifford multiplication. By Proposition 5.1.10, the symbol mapping provides a vector bundle isomorphism $\sigma : Cl(M) \to \bigwedge \mathrm{T}^*M$. The latter allows us to transport the Clifford module bundle structure from $Cl(M)$ to $\bigwedge \mathrm{T}^*M$. Then,

$$c : \mathrm{T}M \to \mathrm{End}(\textstyle\bigwedge \mathrm{T}^*M) \,, \quad c(X)\alpha = \mathsf{g}(X) \wedge \alpha + X \lrcorner \alpha \,,$$

cf. formula (5.1.8). Thus, the Dirac operator of $Cl(M)$ takes the form

$$\mathrm{D}\alpha = i \sum_j c_j \nabla_{e_j} \alpha = i \sum_j \left(\mathsf{g}(e_j) \wedge \nabla_{e_j} \alpha + e_j \lrcorner \nabla_{e_j} \alpha \right) \,.$$

Using (2.2.47) and (2.7.23), we obtain

$$\mathrm{D}\alpha = \mathrm{i}(\mathrm{d} - \mathrm{d}^*)\alpha \,. \tag{5.5.26}$$

♦

Example 5.5.17 (*The canonical spinor bundle*) Consider the canonical spinor bundle $\mathscr{S}(M) = S(M) \times_\gamma \Delta_n$ of (M, g) relative to a fixed spin structure. As we have seen, $\mathscr{S}(M)$ is a Clifford module bundle with the Clifford mapping given by the spinor representation γ. For historical reasons, the Dirac operator of $\mathscr{S}(M)$ will be denoted by $\not{D}$. We have

$$\not{D}\Phi = \mathrm{i} \sum_{j=1}^{n} c_j \nabla_{e_j} \Phi \,, \quad \Phi \in \Gamma^\infty(\mathscr{S}(M)) \,, \tag{5.5.27}$$

where ∇ is the spin connection of g, that is, using (5.2.29), we obtain

$$\nabla \Phi = \mathrm{d}\Phi + \sum_{i<j} \omega_{ij} c_i c_j \Phi \,.$$

Here, ω_{ij} are the coefficients of the spin connection form. In particular, for the Minkowski space, the Clifford bundle is trivial. Thus, the Clifford action is given by the spinor representation

$$\gamma : M \to \mathrm{End}\left(\mathbb{C}^4\right), \quad \gamma(\mathbf{e}_\mu) := \begin{bmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{bmatrix} \equiv \gamma_\mu ,$$

cf. (5.1.26). This yields the Dirac operator of relativistic quantum mechanics in a convenient representation. ♦

Given a Dirac bundle, one may construct a whole family of associated Dirac bundles as follows.

Remark 5.5.18 Let $(\mathscr{E}, \mathsf{h}, \nabla)$ be a Dirac bundle over a Riemannian manifold (M, g) and let $(E, \mathsf{h}^E, \nabla^E)$ be any Riemannian (or Hermitean) vector bundle over M endowed with a compatible connection ∇^E. Then, we can endow the tensor product bundle[31] $\mathscr{E} \otimes E$ with the tensor product metric and with the structure of a bundle of left modules over $Cl(M)$ by setting

$$\zeta \cdot (\Phi \otimes s) := \zeta \cdot \Phi \otimes s ,$$

where $\zeta \in Cl(\mathrm{T}_m M)$, $\Phi \in \mathscr{E}_m$ and $s \in E_m$. Clearly, this formula defines a Clifford mapping for $\mathscr{E} \otimes E$. Moreover, we equip $\mathscr{E} \otimes E$ with the canonical tensor product connection $\nabla \otimes \nabla^E$, defined by

$$\left(\nabla \otimes \nabla^E\right)(\Phi \otimes s) := (\nabla \Phi) \otimes s + \Phi \otimes \left(\nabla^E s\right) , \qquad (5.5.28)$$

cf. Remark 1.5.9/3. It is easy to prove (Exercise 5.5.2) that $\nabla \otimes \nabla^E$ is formally self-adjoint and fulfils (5.5.20). Correspondingly, we have a naturally associated Dirac operator D_E. The tensor product bundle $\mathscr{E} \otimes E$ endowed with the product metric and with the canonical connection is usually referred to as a twisted Clifford module bundle and D_E is called the twisted Dirac operator.

In particular, assume that E is associated with a principal bundle P and ∇^E corresponds to a connection form ω. Consider the following special cases:
(a) Let $\mathscr{E} = Cl(M)$. This bundle is associated with $O_+(M)$ and carries a natural connection induced from the Levi-Civita connection ω^0 of g.
(b) Assume that M is spin and consider $\mathscr{S}(M)$. The latter is associated with $S(M)$ and carries the spin connection ω^s of g.

By Remark 1.5.9/3, in both cases the tensor product connection $\nabla \otimes \nabla^E$ corresponds to the natural connection on the fibre product $O_+(M) \times_M P$ or $S(M) \times_M P$, respectively, given by (1.3.16). ♦

[31] If E is Riemannian, then this is a tensor product over $\mathbb{R}$.

Exercises

5.5.1 Prove that (5.5.9) defines a vector bundle isomorphism.

5.5.2 Prove that the tensor product connection $\nabla \otimes \nabla^E$ defined in Remark 5.5.18 is compatible with the fibre metric, that is, it is formally self-adjoint, and satisfies the derivation property (5.5.20).

5.6 Weitzenboeck Formulae

Here, we take up the discussion of second order differential operators from Sect. 2.7. We derive the counterpart of the Weitzenboeck Theorem 2.7.11 for any Dirac bundle $(\mathscr{E}, \mathsf{h}, \nabla)$ over a Riemannian manifold (M, g). This will be of fundamental importance in the sequel. Here, the Weitzenboeck curvature operator $\mathfrak{R}^{\mathscr{E}} : \Gamma^\infty(\mathscr{E}) \to \Gamma^\infty(\mathscr{E})$ is defined by

$$\mathfrak{R}^{\mathscr{E}}(\Phi) := -\frac{1}{2} \sum_{j,k} c_j c_k \mathsf{R}^{\mathscr{E}}(e_j, e_k)\Phi \,, \tag{5.6.1}$$

where $\mathsf{R}^{\mathscr{E}} \in \Omega^2(M, \mathrm{End}(\mathscr{E}))$ is the curvature endomorphism form of ∇ and $\{e_i\}$ is an oriented local orthonormal frame. We also recall the Bochner-Laplace operator $\nabla^*\nabla : \Gamma^\infty(\mathscr{E}) \to \Gamma^\infty(\mathscr{E})$, cf. Definition 2.7.8. The latter is formally self-adjoint and, by (2.7.31),

$$\nabla^*\nabla\Phi = -\sum_i \left(\nabla_{e_i}\nabla_{e_i}\Phi - \nabla_{\nabla_{e_i}e_i}\Phi\right) . \tag{5.6.2}$$

By expanding $\nabla_{e_i}e_i$, this formula may be rewritten as

$$\nabla^*\nabla\Phi = -\sum_i \nabla_{e_i}\nabla_{e_i}\Phi + \sum_{i,j} \mathsf{g}(e_j, \nabla_{e_i}e_i)\nabla_{e_j}\Phi \,. \tag{5.6.3}$$

Theorem 5.6.1 (Weitzenboeck Formula for the Dirac operator) *Let* $(\mathscr{E}, \mathsf{h}, \nabla)$ *be a Dirac bundle over a Riemannian manifold* (M, g) *and let* D *be its Dirac operator. Then, for any* $\Phi \in \Gamma^\infty(\mathscr{E})$,

$$\mathrm{D}^2\Phi = \nabla^*\nabla\Phi + \mathfrak{R}^{\mathscr{E}}(\Phi) \,. \tag{5.6.4}$$

Proof Let $\{e_j\}$ be a local orthonormal frame. Using (5.5.22), we calculate

$$
\begin{aligned}
\mathrm{D}^2\Phi &= -\sum_{i,j} c_i \nabla_{e_i}\left(c_j \nabla_{e_j}\Phi\right)\\
&= -\sum_{i,j} c_i\left(\nabla_{e_i}e_j \cdot \nabla_{e_j}\Phi\right) - \sum_{i,j} c_i c_j \nabla_{e_i}\nabla_{e_j}\Phi\\
&= -\sum_{i,j,k} \mathsf{g}(\nabla_{e_i}e_j, e_k)\, c_i c_k \nabla_{e_j}\Phi - \sum_{i,j} c_i c_j \nabla_{e_i}\nabla_{e_j}\Phi\\
&= -\sum_{i,j} \mathsf{g}(\nabla_{e_i}e_j, e_i)\nabla_{e_j}\Phi - \sum_{i} \nabla_{e_i}\nabla_{e_i}\Phi\\
&\quad - \sum_{j,i\neq k} \mathsf{g}(\nabla_{e_i}e_j, e_k)\, c_i c_k \nabla_{e_j}\Phi - \sum_{i\neq j} c_i c_j \nabla_{e_i}\nabla_{e_j}\Phi\,.
\end{aligned}
$$

By (5.6.3), the sum of the first two terms coincides with $\nabla^*\nabla\Phi$. Using (2.1.46), together with the fact that ∇ is torsionless, we find

$$
-\sum_{j,i\neq k} \mathsf{g}(\nabla_{e_i}e_j, e_k)\, c_i c_k \nabla_{e_j}\Phi = \frac{1}{2}\sum_{i,j} c_i c_j \nabla_{[e_i,e_j]}\Phi\,.
$$

Thus, by (2.1.32) and (5.6.1), the sum of the third and the fourth term in the above calculation is equal to

$$
-\frac{1}{2}\sum_{i,j} c_i c_j \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}\right)\Phi = \mathfrak{R}^{\mathscr{E}}(\Phi)\,.
$$

∎

Next, we will find a refinement of the Weitzenboeck Formula which corresponds to the natural algebra bundle isomorphism (5.5.10),

$$
\mathrm{End}(\mathscr{E}) \cong Cl^c(M) \otimes \mathrm{End}_{Cl(M)}(\mathscr{E})\,.
$$

As before, let $\mathsf{R}^{\mathscr{E}}$ be the curvature endomorphism form of ∇, let ∇^{g} be the Levi-Civita connection of g and let R be the Riemann curvature of g. Moreover, let $\mathsf{R}^{\nabla^{\mathsf{g}}} \in \Omega^2(M, \mathrm{End}(\mathscr{E}))$ be the curvature endomorphism form of ∇^{g} viewed as a connection in the Clifford bundle $Cl(M)$. By (5.2.29), for every $X, Y \in \mathfrak{X}(M)$,

$$
\mathsf{R}^{\nabla^{\mathsf{g}}}(X, Y) = \frac{1}{4}\sum_{l,k} \mathsf{g}(\mathsf{R}(X, Y)e_k, e_l)\, c_l c_k\,, \tag{5.6.5}
$$

where $\{e_j\}$ is a g-orthonormal frame.

Lemma 5.6.2 *Let* $(\mathscr{E}, \mathsf{h}, \nabla)$ *be a Dirac bundle over the Riemannian manifold* (M, g). *Then, for any* $X, Y, Z \in \mathfrak{X}(M)$*, we have*

$$
[\mathsf{R}^{\mathscr{E}}(X, Y), c(Z)] = c(\mathsf{R}(X, Y)Z)\,, \tag{5.6.6}
$$

$$
[\mathsf{R}^{\nabla^{\mathsf{g}}}(X, Y), c(Z)] = c(\mathsf{R}(X, Y)Z)\,. \tag{5.6.7}
$$

Moreover, the curvature endomorphism form of ∇ *uniquely decomposes as*

$$\mathsf{R}^{\mathscr{E}} = \mathsf{R}^{\nabla^{\mathsf{g}}} + \mathrm{F}^{\mathscr{E}}\,, \tag{5.6.8}$$

where $\mathrm{F}^{\mathscr{E}} \in \Omega^2(M, \mathrm{End}_{Cl(M)}(\mathscr{E}))$.

Proof To show (5.6.6), we work in a local holonomic frame $\{e_l = \partial_l\}$. For $X = e_i$, $Y = e_j$ and $Z = e_k$, the compatibility condition (5.5.22) reads

$$\nabla_j(c_k\phi) = (\nabla^{\mathsf{g}}_j e_k)\cdot\phi + c_k\nabla_j\phi\,,$$

for any local section ϕ in $\mathscr{E}$. Thus,

$$\nabla_i\nabla_j(c_k\phi) = (\nabla^{\mathsf{g}}_i\nabla^{\mathsf{g}}_j e_k)\cdot\phi + (\nabla^{\mathsf{g}}_j e_k)\cdot\nabla_i\phi + (\nabla^{\mathsf{g}}_i e_k)\cdot\nabla_j\phi + c_k\nabla_i\nabla_j\phi\,.$$

Writing down this equation with i and j exchanged and subtracting it from the first equation, we obtain the assertion. To prove (5.6.7), we chose an orthonormal local frame $\{e_l\}$. Then, for $X = e_i$, $Y = e_j$ and $Z = e_a$, we calculate

$$[\mathsf{R}^{\nabla^{\mathsf{g}}}(e_i, e_j), c_a] = \frac{1}{4}\sum_{l,k}\mathsf{R}_{ijkl}[c_l c_k, c_a] = \sum_l \mathsf{R}_{ijal}c_l = \mathsf{R}(e_i, e_j)c_a\,.$$

Here, we have used (2.3.15) and $[e_l e_k, e_a] = 0$ if $k = l$ or if k, l and a are all distinct. By (5.6.6) and (5.6.7), $[\mathsf{R}^{\mathscr{E}}(X, Y) - \mathsf{R}^{\nabla^{\mathsf{g}}}(X, Y), c(Z)] = 0$. This yields (5.6.8). ■

Definition 5.6.3 The element $\mathrm{F}^{\mathscr{E}} \in \Omega^2(M, \mathrm{End}_{Cl(M)}(\mathscr{E}))$ will be referred to as the twisting curvature of the Dirac bundle $\mathscr{E}$.

Theorem 5.6.4 (Lichnerowicz) *Let* $\mathscr{E}$ *be a Dirac bundle over the Riemannian manifold* (M, g) *and let* D *be its Dirac operator. Then,*

$$\mathrm{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathsf{Sc} + \mathfrak{F}^{\mathscr{E}}\,, \tag{5.6.9}$$

where Sc *denotes the scalar curvature of* (M, g) *and*

$$\mathfrak{F}^{\mathscr{E}} = -\frac{1}{2}\sum_{j,k} c_j c_k \mathrm{F}^{\mathscr{E}}(e_j, e_k) \tag{5.6.10}$$

is the Weitzenboeck curvature operator of $\mathrm{F}^{\mathscr{E}}$ *written in an orthonormal frame* $\{e_i\}$.

Proof Let

$$\mathfrak{R}^{\mathsf{g}} = -\frac{1}{2}\sum_{j,k} c_j c_k \mathsf{R}^{\nabla^{\mathsf{g}}}(e_j, e_k)$$

be the Weitzenboeck curvature operator of $\mathsf{R}^{\nabla^{\mathsf{g}}}$. Then, by (5.6.8), $\mathfrak{R}^{\mathscr{E}} = \mathfrak{R}^{\mathsf{g}} + \mathfrak{F}^{\mathscr{E}}$ and the Weitzenboeck Formula (5.6.4) yields

$$\mathrm{D}^2 \;=\; \nabla^*\nabla + \mathfrak{R}^{\mathsf{g}} + \mathfrak{F}^{\mathscr{E}}\,.$$

Thus, it remains to show that $\mathfrak{R}^{\mathsf{g}} = \frac{1}{4}\mathsf{Sc}$. Using (5.6.5), together with (2.3.15) and (2.3.16), for any local g-orthonormal frame $\{e_k\}$ on M we calculate

$$\begin{aligned}
\mathfrak{R}^{\mathsf{g}} &= \frac{1}{8}\sum_{i,j,k,l} \mathsf{g}(\mathsf{R}(e_i,e_j)e_k,e_l)c_ic_jc_kc_l\\
&= \frac{1}{24}\sum_{l,i\neq j\neq k\neq i}\left\{\mathsf{g}\left(\mathsf{R}(e_i,e_j)e_k+\mathsf{R}(e_k,e_i)e_j+\mathsf{R}(e_j,e_k)e_i,e_l\right)\right\}c_ic_jc_kc_l\\
&\quad+\frac{1}{8}\sum_{i,j,l}\left\{\mathsf{g}(\mathsf{R}(e_i,e_j)e_i,e_l)c_ic_jc_i+\mathsf{g}(\mathsf{R}(e_i,e_j)e_j,e_l)c_ic_jc_j\right\}c_l\\
&= -\frac{1}{4}\sum_{i,j,l}\mathsf{g}(\mathsf{R}(e_i,e_j)e_i,e_l)e_je_l\\
&= \frac{1}{4}\sum_{j,l}\mathsf{Ric}(e_j,e_l)c_jc_l\,,
\end{aligned}$$

and thus, by (2.7.40), $\mathfrak{R}^{\mathsf{g}} = \frac{1}{4}\mathsf{Sc}$. ∎

Let us analyze Theorem 5.6.1 for the Dirac operators of Examples 5.5.16 and 5.5.17. For the canonical spinor bundle, we immediately obtain the following.

Corollary 5.6.5 (Lichnerowicz) *For the Dirac operator $\not{D}$ of the canonical spinor bundle $\mathscr{S}(M)$, the Lichnerowicz Formula reads*

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathsf{Sc}\,. \tag{5.6.11}$$

∎

Next, let (M,g) be an oriented Riemannian manifold carrying a Spin^c-structure $S^c(M)$ and let P be the corresponding principal U(1)-bundle. Let ω be the Levi-Civita connection on $O_+(M)$ and let τ be a connection on P. Then, via the two-fold covering $S^c(M) \to O_+(E)\times_M P$, these connections define a unique connection ω^τ on $S^c(M)$. Let $\mathscr{S}^c(M)$ be the corresponding canonical spinor bundle[32] endowed with the Dirac operator D_τ defined by ω^τ,

$$\mathrm{D}_\tau\Phi = \mathrm{i}\sum_j e_j\cdot\nabla_{e_j}\Phi\,,\quad \nabla\Phi = \mathrm{d}\Phi + \frac{1}{2}\sum_{i<j}\omega_{ij}c_ic_j\Phi + \frac{1}{2}\tau\cdot\Phi\,. \tag{5.6.12}$$

[32] We leave it to the reader to check in detail that $\mathscr{S}^c(M)$ is a Dirac bundle.

The decomposition (5.5.10) reads

$$\mathrm{End}(\mathscr{S}^c(M)) \cong Cl^c(M) \otimes \mathrm{End}(L)\,,$$

where L is the associated fundamental line bundle defined by (5.4.11). Using this, together with (5.2.18), we see that in this case the twisting curvature endomorphism is given by the curvature endomorphism form $\mathrm{F}^\tau \in \mathrm{End}(L)$ of the curvature $\Omega_\tau = \mathrm{d}\tau$. The latter is given by $\frac{1}{2}\Omega_\tau$. Thus, by (5.6.1), its Weitzenboeck curvature operator is given by

$$\mathfrak{F}^\tau = -\frac{1}{4}\sum_{j,k} c_j c_k\, \Omega_\tau(e_j, e_k) = -\frac{1}{2}\mathsf{c}(\Omega_\tau)\,. \tag{5.6.13}$$

Thus, Theorem 5.6.4 implies the following.

Corollary 5.6.6 *For the Dirac operator* D_τ *of the spinor bundle* $\mathscr{S}^c(M)$*, the Lichnerowicz Formula reads*

$$\mathrm{D}_\tau^2\Phi = \nabla^*\nabla\Phi + \frac{1}{4}\mathsf{Sc}\Phi - \frac{1}{2}\mathsf{c}(\Omega_\tau)\Phi\,. \tag{5.6.14}$$

■

Next, let us turn to the exterior bundle.

Example 5.6.7 (*Twisted exterior bundle*) Consider the left $Cl(M)$-module bundle

$$\mathscr{E} = \bigwedge \mathrm{T}^*M$$

with its Dirac operator $\mathrm{D} = \mathrm{i}(\mathrm{d} - \mathrm{d}^*)$, see (5.5.26). Then,

$$\mathrm{D}^2\alpha = -(\mathrm{d} - \mathrm{d}^*)(\mathrm{d} - \mathrm{d}^*)\alpha = (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\alpha$$

and, thus, by (2.7.14), D^2 coincides with the Hodge-Laplace operator,

$$\mathrm{D}^2 = \Box\,. \tag{5.6.15}$$

Thus, in the case under consideration, the Weitzenboeck Formula (5.6.4) reproduces Theorem 2.7.11:

$$\Box \;=\; \nabla^{\omega^0 *}\nabla^{\omega^0}\alpha + \mathfrak{R}^\Lambda(\alpha)\,,$$

where ∇^{ω^0} is the covariant derivative of the Levi-Civita connection and

$$\mathfrak{R}^\Lambda = \mathsf{R}_{ijkl}\varepsilon^i\iota^j\varepsilon^k\iota^l\,,$$

cf. formulae (2.7.39) and (2.7.38). Now, let us consider the twisted Dirac bundle

$$\mathscr{E} = \bigwedge \mathrm{T}^*M \otimes E\,,$$

where $(E, \mathsf{h}^E, \nabla^E)$ is some Riemannian (or Hermitean) vector bundle over M endowed with a compatible connection ∇^E and where $\mathscr{E}$ is endowed with the canonical tensor product connection $\nabla = \nabla^{\omega^0} \otimes \nabla^E$, cf. Remark 5.5.18. Let D_E be the Dirac operator of $\mathscr{E}$. Clearly,

$$\mathsf{R}^{\mathscr{E}} = \mathsf{R}^{\Lambda} + \mathsf{R}^{E} ,$$

where R^{Λ} and R^{E} are the curvature endomorphism forms of $\bigwedge \mathrm{T}^*M$ and E, respectively. Now, Theorem 5.6.1 implies the following Weitzenboeck Formula for this case:

$$\mathrm{D}_E^2 = \nabla^*\nabla + \mathfrak{R}^{\Lambda} + \mathfrak{R}^{E} , \tag{5.6.16}$$

where $\mathfrak{R}^{\Lambda}$ and $\mathfrak{R}^{E}$ are the Weitzenboeck curvature endomorphisms of $\bigwedge \mathrm{T}^*M$ and E, respectively. As a direct consequence of Lemma 2.7.19, we obtain

$$\mathrm{D}_E = \mathrm{i}(\mathrm{d}_\omega - \mathrm{d}_\omega^*) ,$$

where ω is the connection form of ∇^E. Thus, by (2.7.52),

$$\mathrm{D}_E^2 = \mathrm{d}_\omega \circ \mathrm{d}_\omega^* + \mathrm{d}_\omega^* \circ \mathrm{d}_\omega = \Box_\omega . \tag{5.6.17}$$

This yields an alternative proof of the Generalized Weitzenboeck Formula 2.7.20. ♦

Recall from Sect. 2.7 that the Weitzenboeck Formula may be used to get insight into the relation between curvature and topology, cf. Proposition 2.7.14 and Corollary 2.7.15. Here, in particular, we obtain information about harmonic spinors, that is, sections of $\mathscr{S}(M)$ fulfilling $\not{D}\Phi = 0$.

Corollary 5.6.8 *Let (M, g) be a compact spin manifold. Then,*

1. *if the scalar curvature of g is positive, then (M, g) admits no harmonic spinors,*[33]
2. *if the scalar curvature of g vanishes identically, then every harmonic spinor on (M, g) is globally parallel.*

Proof Assume $\not{D}\Phi = 0$ for some $\Phi \in \Gamma^\infty(\mathscr{S}(M))$. Then, integrating the identity (5.6.11) applied to Φ with respect to the canonical volume form v_g yields

$$\frac{1}{4}\int_M \mathsf{Sc} \parallel \Phi \parallel^2 \mathsf{v}_\mathsf{g} = -\langle \nabla^*\nabla\Phi, \Phi\rangle_{L^2} = -\langle \nabla\Phi, \nabla\Phi\rangle_{L^2} .$$

This implies both statements. ♦

Exercises

5.6.1 Prove formula (5.6.1).

5.6.2 Prove formula (5.6.17).

[33]This statement also holds under the weaker assumption that the scalar curvature be non-negative and strictly positive at some point.

5.7 Elliptic Complexes. The Hodge Theorem

In this section, we assume that (M, g) is an oriented compact n-dimensional Riemannian manifold.

Let E and F be vector bundles over M. Recall that a differential operator

$$P : \Gamma^\infty(E) \to \Gamma^\infty(F)$$

of order k is a local linear mapping. While the notion of linearity is obvious, the notion of locality needs some explanation. In abstract terms, it means that P factors through the k-jet bundle $J^k(E)$.[34] However, here, we prefer a more direct working definition. In local coordinates on $U \subset M$, the operator P can be represented as

$$P = \sum_{|\alpha|\leq k} P_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}\,, \tag{5.7.1}$$

where, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, P_α is a vector bundle morphism from E to F over U symmetric in the indices of α.

Given vector bundles E and F and a differential operator $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$, one defines the formal adjoint $P^* : \Gamma^\infty(F^*) \to \Gamma^\infty(E^*)$ acting between the spaces of sections of the dual bundles F^* and E^* by setting

$$\int_M \chi(P\phi)\, \mathrm{v_g} = \int_M (P^*\chi)\phi\, \mathrm{v}_g\,, \tag{5.7.2}$$

for any $\phi \in \Gamma^\infty(E)$ and $\chi \in \Gamma^\infty(F^*)$. It is easy to show that the formal adjoint exists and that it is unique.

It is easy to check that the k-th order coefficients of P given by (5.7.1) transform as a tensor field $M \to S^k(\mathrm{T}M) \otimes \mathrm{Hom}(E, F)$ over U. Here, $S^k(\mathrm{T}M)$ denotes the k-fold symmetric tensor product of $\mathrm{T}M$. This suggests the following definition.

Definition 5.7.1 (*Principal symbol*) Let $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ be a differential operator of order k and let $\pi : \mathrm{T}M \to M$ be the canonical bundle projection. The principal symbol of P is a mapping which assigns to each point $\xi \in \mathrm{T}^*M$ a mapping $\sigma_\xi(P) : E_{\pi(\xi)} \to F_{\pi(\xi)}$ defined by

$$\sigma_\xi(P) := \mathrm{i}^k \sum_{|\alpha|=k} P_\alpha(\pi(\xi))\xi^\alpha\,, \tag{5.7.3}$$

where $\xi = \sum_j \xi_j \mathrm{d}x^j$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

---

[34] That is, there is a vector bundle morphism $\varphi_P : J^k(E) \to F$ such that $P = \varphi_P \circ j_k$, where $j_k : \Gamma^\infty(E) \to \Gamma^\infty(J^k(E))$ is the k-th jet prolongation. This means that $P(s)(m)$ is determined by the germ of the section s at the point m. Conversely, by a theorem of Peetre, any linear local operator is differential.

Formula (5.7.3) defines local sections over T^*U of the bundle

$$\mathrm{Hom}(\pi^*(E), \pi^*(F)) \to \mathrm{T}^*M$$

which glue together to a global section. By definition, this section is homogeneously polynomial of degree k along the fibres of T^*M. Thus, the principal symbol is a bundle morphism

$$\sigma(P): \ \pi^*(E) \to \pi^*(F)\,.$$

Remark 5.7.2 By multilinearization, we may identify the space of those sections of $\mathrm{Hom}(\pi^*(E), \pi^*(F))$ which are homogeneously polynomial of degree k along the fibres of T^*M with the space of sections of the bundle $S^k(\mathrm{T}M) \otimes \mathrm{Hom}(E, F) \to M$. That is, the symbol may be also viewed as a section

$$\sigma(P) : M \to S^k(\mathrm{T}M) \otimes \mathrm{Hom}(E, F)\,.$$

♦

If $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ and $Q : \Gamma^\infty(F) \to \Gamma^\infty(L)$ are differential operators over M, then their principal symbols fulfil the following (Exercise 5.7.1):

$$\sigma_\xi(Q + P) = \sigma_\xi(Q) + \sigma_\xi(P)\,, \tag{5.7.4}$$

$$\sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P)\,. \tag{5.7.5}$$

If E and F are Riemannian or Hermitean, then

$$\sigma_\xi(P^*) = (\sigma_\xi(P))^\dagger\,, \tag{5.7.6}$$

where P^* is the formal adjoint with respect to the L^2-inner products.

Definition 5.7.3 (*Elliptic differential operator*) A differential operator P is called elliptic if its principal symbol $\sigma_\xi(P)$ is a vector space isomorphism for all $\xi \neq 0$.

By (5.7.6), P is elliptic iff P^* is elliptic.

Proposition 5.7.4 *Let* D *be the Dirac operator of a Dirac bundle* $(\mathscr{E}, \mathsf{h}, \nabla)$ *over a Riemannian manifold* (M, g)*. Then, for any* $\xi \in \mathrm{T}^*M$*,*

$$\sigma_\xi(\mathrm{D}) = -\mathsf{g}^{-1}(\xi)\,, \quad \sigma_\xi(D^2) = \parallel \xi \parallel^2\,, \tag{5.7.7}$$

where the symbols on the right denote Clifford multiplication with the vector $-\mathsf{g}^{-1}(\xi)$ *and with the scalar* $\parallel \xi \parallel^2 = \mathsf{g}^{-1}(\xi, \xi)$*, respectively. In particular, both* D *and* D^2 *are elliptic.*

Using the identification $\mathrm{T}^*M \cong \mathrm{T}M$, it is common to write $\sigma_\xi(\mathrm{D}) = -\xi$.

Proof For any $m \in M$, choose a local chart with coordinates $\{x^j\}$ in a neighbourhood of m such that m corresponds to 0 and $e_j = \partial_j = \mathsf{g}^{-1}\left(\mathrm{d}x^j\right)$. Then, using (5.5.24), up to zero-order terms we obtain $\mathrm{D} = \mathrm{i}\sum_j c_j\partial_j$ and, thus,

$$\sigma_\xi(\mathrm{D})\Phi = \mathrm{i}^2\sum_j \xi^j c_j\Phi = -c(\mathsf{g}^{-1}(\xi))\Phi\,, \quad \Phi \in \Gamma^\infty(\mathscr{E})\,.$$

For D^2, using (5.5.6), we obtain

$$\sigma_\xi(\mathrm{D}^2)\Phi = \sum_{j,k}\xi^j\xi^k c_j c_k\Phi = \mathsf{g}^{-1}(\xi,\xi)\,\mathrm{id}_{\mathscr{E}_m}\Phi\,, \quad \Phi \in \Gamma^\infty(\mathscr{E})\,.$$

■

Now, recall Remark 5.5.15. For a Dirac bundle $\mathscr{E}$ over a complete Riemannian manifold (M,g), the Dirac operator viewed as an operator on $L^2(\mathscr{E})$ is unbounded and self-adjoint. Thus, we have the full theory of self-adjoint operators on Hilbert spaces at our disposal. However, for many purposes, in particular, for purposes of index theory one needs a functional analytic setting in which the operators are bounded and which in a sense accounts for the degree of differentiability. This setting is provided by the theory of Sobolev spaces. This is an established part of modern analysis and there is a number of textbook presentations, see e.g. [501]. So, here we only make some elementary remarks for further reference.[35]

Given a vector bundle E over (M,g) endowed with a fibre metric $\langle\cdot,\cdot\rangle$ and a compatible connection ∇, using the Riemannian metric g, one defines the inner product

$$\langle\phi,\psi\rangle_{W^k} := \int_M \left\{\langle\phi,\psi\rangle + \langle\nabla\phi,\nabla\psi\rangle + \ldots + \langle\nabla^k\phi,\nabla^k\psi\rangle\right\}\mathsf{v}_\mathsf{g}\,. \tag{5.7.8}$$

Then, by definition, the Sobolev space $W^k(E)$ is the completion

$$W^k(E) := \overline{\{\phi \in C^\infty(E) : \|\,\phi\,\|_{W^k} < \infty\}}\,. \tag{5.7.9}$$

Note that $W^k(E)$ is a Hilbert space for any k.[36] In particular, $W^0(E) = L^2(E)$ and we obviously have $\|\,\phi\,\|_{W^k} \le \|\,\phi\,\|_{W^{k'}}$ for $k' < k$. The Sobolev norm induced from (5.7.8) depends on g, $\langle\cdot,\cdot\rangle$ and ∇. However, it is easy to see that different choices of these data lead to equivalent norms, that is, as a topological vector space, $W^k(E)$ depends only on the underlying vector bundle. Moreover, one can check that, for compact M, the Sobolev norm $\|\,\phi\,\|_{W^k}$ is equivalent to the norm defined by the scalar product

[35] Actually, for purposes of this chapter, the short presentations of Sobolev theory in [246, 407] or [535] are sufficient.

[36] There is a Banach space version based on L^p-norms which, however, we do not need here.

$$\langle \phi, \psi \rangle_{W^k} := \sum_i \sum_{|\alpha| \le k} \int_{U_i} \left\langle \frac{\partial^{|\alpha|}}{\partial x^\alpha} \phi, \frac{\partial^{|\alpha|}}{\partial x^\alpha} \psi \right\rangle \mathrm{d}^n x \,, \tag{5.7.10}$$

where $\{U_i\}$ is some finite covering of M by local coordinates $\{x^j\}$ (Exercise 5.7.2).

We can extend the above definition to negative k by duality, that is, W^{-k} is the dual of W^k with respect to the L^2-pairing.[37] As a consequence, one obtains the following sequence of embeddings

$$\mathscr{S} \subset W^\infty \subset \ldots \subset W^1 \subset W^0 = L^2 \subset W^{-1} \subset \ldots W^{-\infty} \subset \mathscr{S}' \,.$$

Here, $W^\infty = \bigcap_k W^k$, $W^{-\infty} = \bigcup_k W^k$ and $\mathscr{S}'$ denotes the space of tempered distributions. The statements of the following proposition are immediate consequences of the definition of W^k (Exercise 5.7.4).

Proposition 5.7.5

1. *For any $k' > k$, there is a bounded inclusion $W^{k'} \to W^k$.*
2. *Every covariant derivative is a bounded mapping $\nabla : W^k(E) \to W^{k-1}(E)$.*
3. *Any vector bundle morphism $\varphi : E \to F$ covering a diffeomorphism extends to a bounded mapping $W^k(E) \to W^k(F)$ for every k.*
4. *Any differential operator $P : C^\infty(E) \to C^\infty(F)$ of order p extends to a bounded mapping $W^k(E) \to W^{k-p}(E)$ for all k.*
5. *If $V \subset W^k(E)$ is a finite-dimensional subspace, then we have the L^2-orthogonal direct sum decomposition*
$$W^k(E) = V \oplus V^\perp \,.$$

The following two lemmas are of basic importance.

Lemma 5.7.6 (Rellich) *The inclusion $W^{k'} \to W^k$ is compact for $k' > k \ge 0$.*

Lemma 5.7.7 (Sobolev) *If $k > \frac{1}{2} \dim M + p$, then $W^k \subset C^p(E)$ and the embedding is continuous.*

Finally, the formal adjoint of a differential operator P defined by (5.7.2) extends to a bounded operator between Sobolev spaces. In detail, if $P : W^k(E) \to W^l(F)$, then $P^* : W^{-l}(F) \to W^{-k}(E)$ is given by

$$(P\phi, \chi) = (\phi, P^*\chi) \,. \tag{5.7.11}$$

Now, let us study the Dirac operator D of a Dirac bundle $\mathscr{E}$ (or of a twisted version $\mathscr{E} \otimes E$) in the context of Sobolev spaces. Our presentation is along the lines of [219] and [212]. By Proposition 5.7.5, we obtain bounded Sobolev extensions

[37] In fact, W^{-k} can be endowed with a Hilbert space structure via the Fourier transform of (5.7.8). This way, W^k can be defined for any real number k. This is of importance in the theory of pseudo-differential operators.

$$\mathrm{D} : W^k \to W^{k-1} , \quad \mathrm{D}^2 : W^k \to W^{k-2} . \tag{5.7.12}$$

In particular, if we view the Dirac operator as a mapping $\mathrm{D} : W^1(\mathscr{E}) \to L^2(\mathscr{E})$, we can calculate

$$\begin{aligned} \| \mathrm{D}\psi \|_{L^2}^2 &= \sum_{i,j} \int_M \langle c_i \nabla_{e_i} \psi, c_j \nabla_{e_j} \psi \rangle \mathsf{v_g} \\ &= \sum_{i,j} \int_M \langle \nabla_{e_i} \psi, \frac{1}{2}(c_i c_j + c_j c_i) \nabla_{e_j} \psi \rangle \mathsf{v_g} \\ &= \sum_{i} \int_M \| \nabla_{e_i} \psi \|^2 \, \mathsf{v_g} \\ &= n \| \nabla \psi \|_{L^2}^2 , \end{aligned}$$

where $n = \dim M$. Thus,

$$\| \mathrm{D}\psi \|_{L^2}^2 = n \| \nabla \psi \|_{L^2}^2 \leq n \| \psi \|_{W^1}^2 \ . \tag{5.7.13}$$

In the sequel, one of our main objectives will be to prove that D is Fredholm. This notion is at the heart of index theory.

Definition 5.7.8 (*Fredholm operator*) Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Then, T is called Fredholm if its kernel and cokernel are both finite-dimensional. The integer

$$\mathrm{ind}(T) := \dim(\ker T) - \dim(\mathrm{coker}\, T)$$

is referred to as the index of T.

Often, $\mathrm{ind}(T)$ is also called the analytic index of T.

Using the Closed Graph Theorem, one can show that every Fredholm operator has a closed range, see Lemma 2.1 in [29]. This implies (Exercise 5.7.3)

$$H_2 = \mathrm{im}\, T \oplus \ker T^* , \tag{5.7.14}$$

and, hence,

$$\mathrm{coker}\ T = H_2/T(H_1) \cong \ker\ T^* . \tag{5.7.15}$$

We conclude

$$\mathrm{ind}(T) = \dim(\ker\ T) - \dim(\ker\ T^*) . \tag{5.7.16}$$

A key role in the analysis below is played by the Weitzenboeck Formula 5.6.1. By point 3 of Proposition 5.7.5, the Weitzenboeck curvature operator is bounded, that is, there exists $c > 0$ such that

$$-c \parallel \psi \parallel_{L^2}^2 \leq \langle \psi, \mathfrak{R}^\nabla \psi \rangle \leq c \parallel \psi \parallel_{L^2}^2 \ . \tag{5.7.17}$$

Lemma 5.7.9 *Let $\mathscr{E}$ be a Dirac bundle over a compact Riemannian manifold* (M, g) *and let* D *be its Dirac operator. Then, for all $\psi \in W^1(\mathscr{E})$,*

$$\parallel \psi \parallel_{W^1}^2 -(c+1) \parallel \psi \parallel_{L^2}^2 \leq \parallel \mathrm{D}\psi \parallel_{L^2}^2 \leq \parallel \psi \parallel_{W^1}^2 +(c-1) \parallel \psi \parallel_{L^2}^2 \ . \tag{5.7.18}$$

Moreover, the mapping

$$\psi \to \parallel \psi \parallel_*^2 := \parallel \psi \parallel_{L^2}^2 + \parallel \mathrm{D}\psi \parallel_{L^2}^2 \tag{5.7.19}$$

defines a norm $\parallel \cdot \parallel_*$ which is equivalent to the W^1-norm.

Proof It suffices to prove the assertions for $\psi \in \Gamma^\infty(\mathscr{E})$. Rewrite (5.6.4) as

$$\psi + \nabla^* \nabla \psi = \mathrm{D}^2 \psi + (1 - \mathfrak{R}^\nabla)\psi \,, \tag{5.7.20}$$

take the L^2-scalar product of this equation with ψ and use (5.7.17). This immediately yields (5.7.18). Next, using (5.7.13) and (5.7.18), we derive

$$\frac{1}{n}\left(\parallel \psi \parallel_{L^2}^2 + \parallel \mathrm{D}\psi \parallel_{L^2}^2\right) \leq \parallel \psi \parallel_{W^1}^2 \leq \parallel \mathrm{D}\psi \parallel_{L^2}^2 +(c+1) \parallel \psi \parallel_{L^2}^2 \ . \tag{5.7.21}$$

This inequality yields the proof of the second assertion. ∎

Remark 5.7.10 (*Gårding Inequality*) By (5.7.18), we have

$$\parallel \psi \parallel_{W^1}^2 \leq C(\parallel \psi \parallel_{L^2}^2 + \parallel \mathrm{D}\psi \parallel_{L^2}^2) \,, \tag{5.7.22}$$

which is usually referred to as the Gårding Inequality. By a simple local argument, we have $\parallel \psi \parallel_{W^{k+1}} \leq C_1 \sum_i \parallel \partial_i \psi \parallel_{W^k}$. Using this, together with the fact that both ∂_i and $[\mathrm{D}, \partial_i]$ are first order operators, by induction, one easily shows (Exercise 5.7.5)

$$\parallel \psi \parallel_{W^{k+1}}^2 \leq C_k(\parallel \psi \parallel_{W^k}^2 + \parallel \mathrm{D}\psi \parallel_{W^k}^2) \,, \tag{5.7.23}$$

which is usually referred to as the basic elliptic estimate. ♦

Let us denote the spectrum of the self-adjoint operator D on $L^2(\mathscr{E})$ by $\sigma(\mathrm{D})$.

Proposition 5.7.11 *Let $\mathscr{E}$ be a Dirac bundle over the compact Riemannian manifold* (M, g) *with Dirac operator* D*. Then, the following hold.*

1. *The closure $\overline{\mathrm{D}} = \mathrm{D}^*$ of* D *is defined on $W^1(\mathscr{E}) \subset L^2(\mathscr{E})$.*
2. *If $\lambda \notin \sigma(\overline{\mathrm{D}})$, then $(\mathrm{D} - \lambda)^{-1} : L^2(\mathscr{E}) \to L^2(\mathscr{E})$ is a compact operator.*
3. *There is a complete orthonormal basis $\psi_1, \psi_2, \ldots$ of $L^2(\mathscr{E})$ consisting of eigenvectors of* D*, $\mathrm{D}\psi_n = \lambda_n \psi_n$. Moreover, the eigenspaces are all finite-dimensional and $\lim_{n\to\infty} |\lambda_n| = \infty$.*

Proof 1. Let $\psi \in \mathscr{D}(\overline{\mathrm{D}})$ belong to the domain of definition of $\overline{\mathrm{D}}$. Then, there exists a sequence $\{\psi_n\}$ of elements of $\Gamma^\infty(\mathscr{E})$ such that $\psi_n \to \psi$ in L^2 and $\mathrm{D}(\psi_n)$ converges in L^2. Thus, by (5.7.18), $\{\psi_n\}$ is a Cauchy sequence in $W^1(\mathscr{E})$ and, thus, ψ_n converges to some element $\tilde{\psi} \in W^1(\mathscr{E})$. Since the embedding $W^1(\mathscr{E}) \to L^2(\mathscr{E})$ is continuous, ψ and $\tilde{\psi}$ must coincide, that is, $\psi \in W^1(\mathscr{E})$. Conversely, if $\psi \in W^1(\mathscr{E})$, then it clearly belongs to $\mathscr{D}(\overline{\mathrm{D}})$.

2. We rewrite the inequality (5.7.18) as

$$\| (\mathrm{D}-\lambda)^{-1}(\mathrm{D}-\lambda)\psi \|^2_{W^1} \leq 2 \| (\mathrm{D}-\lambda)\psi \|^2_{L^2} + (1+2\lambda^2+c) \| \psi \|^2_{L^2} \ .$$

Denoting $\phi = (\mathrm{D}-\lambda)\psi$, we obtain

$$\| (\mathrm{D}-\lambda)^{-1}\phi \|^2_{W^1} \leq 2 \| \phi \|^2_{L^2} + (1+2\lambda^2+c) \| (\mathrm{D}-\lambda)^{-1}\phi \|^2_{L^2} \ .$$

Since $(\mathrm{D}-\lambda)^{-1}$ is bounded in $L^2(\mathscr{E})$, there exists a number $C > 0$ such that

$$\| (\mathrm{D}-\lambda)^{-1}\phi \|^2_{W^1} \leq C \| \phi \|^2_{L^2} \ .$$

Thus, the image of $(\mathrm{D}-\lambda)^{-1}$ is contained in $W^1(\mathscr{E})$ and the assertion follows from the compactness of the embedding $W^1(\mathscr{E}) \to L^2(\mathscr{E})$.

3. The third assertion follows from the standard spectral theory of compact self-adjoint operators. If we choose $\lambda \notin \sigma(\overline{\mathrm{D}})$ real, then $(\mathrm{D}-\lambda)^{-1}$ is of this type. Thus, there exists a complete orthonormal basis $\{\psi_n\}$ in $L^2(\mathscr{E})$, such that

$$(\mathrm{D}-\lambda)^{-1}\psi_n = \mu_n\psi_n \,, \quad \mu_n \neq 0 \,, \quad \lim_{n\to\infty} \mu_n = 0 \,.$$

This implies $\mathrm{D}\psi_n = \lambda_n\psi_n$ with eigenvalues given by $\lambda_n = (\mu_n^{-1} + \lambda)$ and fulfilling $\lim_{n\to\infty} |\lambda_n| = \infty$. Moreover, every eigenspace is finite-dimensional. ■

Corollary 5.7.12 *There exists a real number $C > 0$ such that*

$$|\langle \mathrm{D}\phi, \phi\rangle_{L^2}| \geq C \| \phi \|^2_{L^2}$$

for all $\phi \in W^1(\mathscr{E})$ which are orthogonal to $\ker(\mathrm{D})$.

Proof By point 2 of Proposition 5.7.11, we can decompose $\phi = \sum_n' c_n\psi_n$, where the sum is taken over all eigenvectors corresponding to non-vanishing eigenvalues. Then, using the orthonormality of the set $\{\psi_n\}$, we obtain

$$|\langle \mathrm{D}\phi, \phi\rangle_{L^2}| = \sum_n{}' |c_n|^2|\lambda_n| \geq |\lambda_1| \sum_n{}' |c_n|^2 = |\lambda_1| \| \phi \|^2_{L^2} \,,$$

where λ_1 is the lowest non-vanishing eigenvalue which exists according to $\lim_{n\to\infty} |\lambda_n| = \infty$. ■

Remark 5.7.13 (*Elliptic regularity*) It turns out that the eigenfunctions of a Dirac operator are smooth. This is a basic principle in the theory of elliptic operators which, in the context of Dirac operators, may be proved by elementary means. We outline the idea of this proof and refer to [212] for details. First, by point 1 of Proposition 5.7.11, every eigenfunction of a Dirac operator D belongs to $W^1(\mathscr{E})$. Next, starting from the Gårding inequality, by simple iteration type arguments, one proves that

$$\| \psi \|_{W^{k+2}} \leq C(\| \mathrm{D}^2\psi \|_{W^k} + \| \psi \|_{W^k}) , \tag{5.7.24}$$

for any $\psi \in W^{k+2}(\mathscr{E})$, $k \geq 0$. Using some analytic tools,[38] from this estimate one may conclude the following: if $\psi \in W^k(\mathscr{E})$ and $\mathrm{D}^2\psi \in W^k(\mathscr{E})$, then $\psi \in W^{k+2}(\mathscr{E})$. Iterating this argument one concludes that the eigenfunctions ψ_n belong to $W^k(\mathscr{E})$ for all k and, thus, by the Sobolev Lemma, they are smooth. ♦

Remark 5.7.14 (*The spectrum of the Dirac operator*) Let us summarize what we have learnt about the spectrum of D. We have an orthogonal direct sum decomposition

$$L^2(\mathscr{E}) = \bigoplus_\lambda H_\lambda \tag{5.7.25}$$

into a sum of countably many finite-dimensional subspaces H_λ. Each H_λ is an eigenspace of D with eigenvalue λ consisting of smooth sections. The eigenvalues λ form a discrete subset of $\mathbb{R}$ and fulfil $\lim_{n\to\infty} |\lambda_n| = \infty$. ♦

Theorem 5.7.15 *Let $\mathscr{E}$ be a Dirac bundle over a compact Riemannian manifold (M, g). Then, its Dirac operator $\mathrm{D} : W^{k+1}(\mathscr{E}) \to W^k(\mathscr{E})$ with $k \geq 0$ is a Fredholm operator with index zero. Moreover,*

$$W^k(\mathscr{E}) = \ker \mathrm{D} \oplus \mathrm{im}(\mathrm{D}) . \tag{5.7.26}$$

Proof We prove that $\ker \mathrm{D}$ and $L^2(\mathscr{E})/\,\mathrm{im}(\mathrm{D})$ are finite-dimensional vector spaces of the same dimension.

(a) The basic elliptic estimate (5.7.23) implies

$$\| \psi \|^2_{W^{k+1}} \leq C_k \| \psi \|^2_{W^k} , \tag{5.7.27}$$

for any $\psi \in \ker(\mathrm{D})$. Now, choose a sequence $\{\psi_n\}$ fulfilling $\| \psi \|^2_{W^{k+1}} \leq 1$ and $\mathrm{D}\psi_n = 0$. Then, by the Rellich Lemma, there exists a subsequence which is W^k-convergent and, by (5.7.27), this subsequence is Cauchy in the W^{k+1}-norm. Thus, by completeness of $W^{k+1}(\mathscr{E})$, there exists a W^{k+1}-convergent subsequence. This proves compactness of the unit ball and, thus, $\ker(\mathrm{D})$ is finite-dimensional.

(b) We prove that $\mathrm{im}(\mathrm{D})$ is closed in $W^k(\mathscr{E})$. For that purpose, we decompose[39]

[38] Either difference quotients or Friedrich mollifiers.

[39] Cf. point 5 of Proposition 5.7.5.

$$W^{k+1}(\mathscr{E}) = \ker(\mathrm{D}) \oplus (\ker(\mathrm{D}))^{\perp}$$

and restrict D to $(\ker(\mathrm{D}))^{\perp}$. Then, it is injective. Let $\psi = \lim_{n\to\infty} \mathrm{D}\psi_n$ belong to the closure of $\mathrm{im}(\mathrm{D})$. Then, $\{\psi_n\}$ is W^{k+1}-bounded: assume that this is not the case. Then, there exists a subsequence $\{\psi_m\}$ such that $\| \psi_m \|_{W^{k+1}} \to \infty$ and the sequence

$$\varphi_m := \frac{\psi_m}{\| \psi_m \|_{W^{k+1}}}$$

consists of elements whose W^{k+1}-norm is equal to 1. Moreover, $\lim_{m\to\infty} \mathrm{D}\varphi_m = 0$ in the W^k-norm. By the Rellich Lemma, there exists a W^k-convergent subsequence $\{\varphi_l\}$ and, by the Gårding inequality, $\{\varphi_l\}$ converges to some $\hat{\varphi}$ in the W^{k+1}-norm. By continuity, $\mathrm{D}\hat{\varphi} = 0$. But, on the other hand, $\| \hat{\varphi} \|_{W^k} = 1$. By the injectivity of D, this is a contradiction. This shows that $\{\psi_n\}$ is W^{k+1}-bounded, indeed. Thus, again applying the Rellich Lemma and the Gårding inequality, we obtain a W^{k+1}-convergent subsequence whose limit $\hat{\psi}$ satisfies $\mathrm{D}\hat{\psi} = \psi$. Thus, the image is closed.

(c) We decompose

$$W^{k}(\mathscr{E}) = \ker(\mathrm{D}) \oplus (\ker(\mathrm{D}))^{\perp}$$

and prove $\mathrm{im}(\mathrm{D}) = (\ker(\mathrm{D}))^{\perp}$. By point (b), it is enough to show that $\mathrm{im}(\mathrm{D})$ is dense in $(\ker(\mathrm{D}))^{\perp}$: let $\eta \in W^{-k}(\mathscr{E})$ such that

$$\eta(\mathrm{D}\psi) = 0$$

for all $\psi \in W^{k+1}(\mathscr{E})$. By the Hahn–Banach Theorem, it is enough to show that the restriction of η to $(\ker(\mathrm{D}))^{\perp}$ vanishes. By assumption, $\mathrm{D}^*\eta = 0$, where

$$\mathrm{D}^* : W^{-k}(\mathscr{E}) \to W^{-(k+1)}(\mathscr{E})$$

is the Sobolev extension of the formal adjoint defined by (5.7.11). By elliptic regularity, η is smooth and, therefore, D^* coincides with the formal adjoint of D when applied to η. Thus, by the self-adjointness of D,

$$\mathrm{D}^*\eta = \mathrm{D}\eta\,.$$

Thus, $\eta \in \ker(\mathrm{D})$, that is, the restriction of η to $(\ker(\mathrm{D}))^{\perp}$ vanishes, indeed. ■

Remark 5.7.16 Theorem 5.7.15 and elliptic regularity imply the following.

1. The quotient space $\mathrm{coker}(\mathrm{D})$ may be represented by a subspace consisting of smooth sections. Thus, the index of D does not depend on the Sobolev extension used.

2. Since $\ker(\mathrm{D}) \subset \Gamma^\infty(\mathscr{E})$, formula (5.7.26) implies

$$\Gamma^\infty(\mathscr{E}) = \ker(\mathrm{D}) \oplus \mathrm{im}(\mathrm{D}) . \tag{5.7.28}$$

♦

Using the elliptic estimate (5.7.24) for D^2, by the same arguments as in the above proof, we obtain the following.

Theorem 5.7.17 *Let $\mathscr{E}$ be a Dirac bundle over a compact Riemannian manifold (M, g) with Dirac operator $\mathrm{D} : W^{k+2}(\mathscr{E}) \to W^{k+1}(\mathscr{E})$, where $k \geq 0$. Then, its square $\mathrm{D}^2 : W^{k+2}(\mathscr{E}) \to W^k(\mathscr{E})$ is a Fredholm operator with index zero. Moreover,*

$$W^k(\mathscr{E}) = \ker(\mathrm{D}^2) \oplus \mathrm{im}(\mathrm{D}^2) . \tag{5.7.29}$$

■

Let us apply Theorem 5.7.17 to the important special case of the twisted Dirac bundle $\mathscr{E} = \bigwedge \mathrm{T}^*M \otimes E$ with its Dirac operator D_E. By Example 5.6.7,

$$\mathrm{D}_E^2 = \mathrm{d}_\omega \circ \mathrm{d}_\omega^* + \mathrm{d}_\omega^* \circ \mathrm{d}_\omega = \Box_\omega . \tag{5.7.30}$$

We extend d_ω and d_ω^* to operators

$$\begin{aligned} \mathrm{d}_\omega &: W^{k+1}(\textstyle\bigwedge^p \mathrm{T}^*M \otimes E) \to W^k(\textstyle\bigwedge^{p+1} \mathrm{T}^*M \otimes E) , \\ \mathrm{d}_\omega^* &: W^k(\textstyle\bigwedge^{p+1} \mathrm{T}^*M \otimes E) \to W^{k-1}(\textstyle\bigwedge^p \mathrm{T}^*M \otimes E) . \end{aligned}$$

Then,

$$\Box_\omega : \; W^{k+1}(\textstyle\bigwedge \mathrm{T}^*M \otimes E) \to W^{k-1}(\textstyle\bigwedge \mathrm{T}^*M \otimes E) . \tag{5.7.31}$$

Thus, Theorem 5.7.17 implies the following.

Theorem 5.7.18 (Hodge Decomposition Theorem) *The following L^2-orthogonal direct sum decomposition holds:*

$$W^{k-1}(\textstyle\bigwedge \mathrm{T}^*M \otimes E) = \ker(\Box_\omega) \oplus \mathrm{im}(\Box_\omega) . \tag{5.7.32}$$

■

Again, by elliptic regularity, we have $\ker(\Box_\omega) \subset \Gamma^\infty(\bigwedge \mathrm{T}^*M \otimes E)$. Thus, we obtain the Hodge Decomposition Theorem 2.7.2 as a special case.

As a consequence of Theorem 5.7.18, the bounded linear mapping

$$\Box_\omega : \ker(\Box^\omega)^\perp \to \mathrm{im}(\Box_\omega) \tag{5.7.33}$$

is bijective, where $\mathrm{im}(\Box_\omega)$ is a closed subspace and thus a Hilbert space itself. Hence, by the Open Mapping Theorem, (5.7.33) is an isomorphism. Taking the inverse and extending it by 0 to $\ker(\Box_\omega)$, we obtain a bounded linear operator

$$\mathrm{G}_\omega : W^{k-1}(\bigwedge \mathrm{T}^*M \otimes E) \to W^{k+1}(\bigwedge \mathrm{T}^*M \otimes E)\,, \tag{5.7.34}$$

called the Green's operator of $\Box_\omega$.

Remark 5.7.19

1. Clearly, if $\xi \in \ker(\Box_\omega)$, then $\mathrm{G}_\omega \Box_\omega \xi = 0$. Moreover, by definition of G_ω, if $\xi \in \ker(\Box_\omega)^\perp$, then $\mathrm{G}_\omega \Box_\omega \xi = \xi$. Thus, the bounded linear operator $\mathrm{G}_\omega \Box_\omega$ on $W^{k+1}(\bigwedge \mathrm{T}^*M \otimes E)$ is the L^2-orthogonal projector onto the subspace $\ker(\Box_\omega)^\perp$.
2. By definition of G_ω, if $\chi \in \mathrm{im}(\Box_\omega)$, then $\Box_\omega \mathrm{G}_\omega \chi = \chi$ and if $\chi \in \mathrm{im}(\Box_\omega)^\perp$, then $\Box_\omega \mathrm{G}_\omega \chi = 0$. Thus, the bounded linear operator $\Box_\omega \mathrm{G}_\omega$ on $W^{k-1}(\bigwedge \mathrm{T}^*M \otimes E)$ is the L^2-orthogonal projector onto the subspace $\mathrm{im}(\Box_\omega)$. ♦

The above results are special cases of general results holding true in the theory of elliptic operators. This general theory heavily rests on the calculus of pseudo-differential operators. In more detail, for an elliptic operator $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ of order p over a compact manifold M, the following hold true, see [407]:

(a) For any open subset $U \subset M$ and any $\phi \in W^k(E)$, the smoothness of $(P\phi)_{\restriction U}$ implies the smoothness of $\phi_{\restriction U}$.
(b) For every k, P extends to a Fredholm operator $P : W^k(E) \to W^{k-p}(F)$ with $\dim(\ker P)$, $\dim(\mathrm{coker}\, P)$ and $\mathrm{ind}(P)$ being independent of k.
(c) For every k, the norms $\|\cdot\|_{W^k}$ and $\|\cdot\|_{W^{k-p}} + \|P\cdot\|_{W^{k-p}}$ are equivalent.

As a direct consequence of these facts, for every elliptic self-adjoint differential operator $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$, one obtains

(d) The operator P shares the spectral properties listed in Remark 5.7.14.
(e) There is an L^2-orthogonal direct sum decomposition

$$\Gamma^\infty(E) = \ker P \oplus \mathrm{im}\, P\,. \tag{5.7.35}$$

In the remainder of this section, we will consider the following natural generalization of an elliptic operator.

Definition 5.7.20 (*Elliptic complex*) Let $\mathfrak{E} = (E_0, \ldots, E_n)$ be a finite collection of Riemannian (or Hermitean) vector bundles over a manifold M and let $P = (P_0, \ldots, P_{n-1})$ be a collection of differential operators $P_k : \Gamma^\infty(E_k) \to \Gamma^\infty(E_{k+1})$ of order p. The pair $(\mathfrak{E}, P)$ is called a complex if $P_{k+1} \circ P_k = 0$. It is called elliptic if

$$\ker(\sigma_\xi(P_k)) = \mathrm{im}(\sigma_\xi(P_{k-1}))\,, \tag{5.7.36}$$

for every $0 \neq \xi \in \mathrm{T}^*M$.

We will be mainly interested in the case $p = 1$.[40] Let us define

$$E^e := \bigoplus_k E_{2k}\,, \quad E^o := \bigoplus_k E_{2k+1}\,, \tag{5.7.37}$$

and associated mappings $P^e : \Gamma^\infty(E^e) \to \Gamma^\infty(E^o)$ and $P^o : \Gamma^\infty(E^o) \to \Gamma^\infty(E^e)$ by

$$P^e := \sum_k (P_{2k} + P_{2k-1}^*)\,, \quad P^o := \sum_k (P_{2k+1} + P_{2k}^*)\,. \tag{5.7.38}$$

Note that $(P^e)^* = P^o$. Moreover, let us consider the associated Laplace operators,

$$\Box_k := P_{k-1}P_{k-1}^* + P_k^* P_k : \;\; \Gamma^\infty(E_k) \to \Gamma^\infty(E_k)\,. \tag{5.7.39}$$

Then, the Laplace operator of $(\mathfrak{E}, P)$ is defined by

$$\Box := \sum_k \Box_k = P^o P^e + P^e P^o = \Box_e + \Box_o\,, \tag{5.7.40}$$

where $\Box_e$ and $\Box_o$ are the restrictions of $\Box$ to E^e and E^o, respectively. It is easy to show the following (Exercise 5.7.6).

Proposition 5.7.21 *The following statements are equivalent:*

1. $(\mathfrak{E}, P)$ *is an elliptic complex.*
2. $\Box_k$ *is elliptic for all* k.
3. P^e *is elliptic.* ■

Now, let us limit our attention to compact Riemannian manifolds (M, g) again. Then, by the above discussion, every element P_k of an elliptic complex $(\mathfrak{E}, P)$ extends to a Fredholm operator and, thus, we can define the cohomology groups of $(\mathfrak{E}, P)$ by

$$H^k(\mathfrak{E}, P) := \ker(P_k)/\operatorname{im}(P_{k-1}) \tag{5.7.41}$$

and its index by

$$\operatorname{ind}(\mathfrak{E}, P) := \sum_k (-1)^k \dim(H^k(\mathfrak{E}, P))\,. \tag{5.7.42}$$

Associated with the above family of Laplace operators, one has a generalized Hodge Theorem.[41] The latter implies

$$H^k(\mathfrak{E}, P) = \ker(\Box_k)\,. \tag{5.7.43}$$

[40] One can also consider the more general case when the P_i are of different order [32].

[41] See e.g. Theorem 1.5.2 in [246].

Then,

$$\begin{aligned}\operatorname{ind}(\mathfrak{E}, P) &= \sum_k (-1)^k \dim(\ker \Box_k) \\ &= \dim(\ker \Box_e) - \dim(\ker \Box_o) \\ &= \dim(\ker(P^{e*}P^e)) - \dim(\ker(P^e P^{e*})) \\ &= \dim(\ker P^e) - \dim(\ker P^{e*})\,.\end{aligned}$$

Thus,

$$\operatorname{ind}(\mathfrak{E}, P) = \operatorname{ind}(P^e)\,. \tag{5.7.44}$$

This reduces the computation of the index to the computation of the index of a two-term complex, that is, of a single elliptic operator. In this context, one often says that one can use the operators P^e or P^o to roll up the elliptic complex.

We close this section by considering the classical examples of elliptic complexes. They will be taken up again in Sect. 5.9.

Example 5.7.22 (*De Rham complex*) Consider $E_k := \bigwedge^k \mathrm{T}^*M$ and take for P_k the exterior differential

$$\mathrm{d}_k : \Gamma^\infty(\bigwedge^k \mathrm{T}^*M) \to \Gamma^\infty(\bigwedge^{k+1}\mathrm{T}^*M)\,.$$

As before, we denote the operations of exterior multiplication and contraction by ε and ι, respectively. Since $\mathrm{d}^2 = 0$, we must only check the ellipticity condition (5.7.36). Let $\xi \neq 0$. Clearly,

$$\sigma_\xi(\mathrm{d}_k)(\alpha) = i\xi \wedge \alpha\,, \tag{5.7.45}$$

for any $\alpha \in \bigwedge^k \mathrm{T}^*M$. Thus, $\operatorname{im}(\sigma_\xi(\mathrm{d}_{k-1})) \subset \ker(\sigma_\xi(\mathrm{d}_k))$. To prove the converse inclusion, let $\alpha \in \ker(\sigma_\xi(\mathrm{d}_k))$, that is, $\xi \wedge \alpha = 0$. Choose a local coordinate system $\{x^j\}$ such that $\xi = \mathrm{d}x^1$. Then, $\alpha = \mathrm{d}x^1 \wedge \beta$ with $\beta \in \bigwedge^{k-1}\mathrm{T}^*M$. This shows $\alpha \in \operatorname{im}(\sigma_\xi(\mathrm{d}_{k-1}))$. Thus, the de Rham complex is elliptic with the principal symbol given by $\sigma(\mathrm{d}_k) = i\varepsilon$. We denote it by $\mathfrak{E}_{\mathrm{dR}}(M)$.

Next, consider the formal adjoint $\mathrm{d}_k^* : \Gamma^\infty(\bigwedge^{k+1}\mathrm{T}^*M) \to \Gamma^\infty(\bigwedge^k \mathrm{T}^*M)$. Then, (2.7.23) immediately implies

$$\sigma_\xi(\mathrm{d}_k^*)(\alpha) = -i\mathsf{g}^{-1}(\xi) \lrcorner \alpha\,,$$

that is, $\sigma(\mathrm{d}_k^*) = -i\iota \circ \mathsf{g}^{-1}$. Next, since $\Box = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, (5.7.5) and (2.7.33) imply

$$\sigma_\xi(\Box) = \varepsilon(\xi)\iota(\mathsf{g}^{-1}(\xi)) + \iota(\mathsf{g}^{-1}(\xi))\varepsilon(\xi) = \parallel \xi \parallel^2 \cdot 1\,.$$

This shows that $\Box$ is elliptic.[42] Finally, by (5.7.41) and (5.7.42), the cohomology groups of the de Rham complex coincide with the de Rham cohomology groups of M and, thus, its index coincides with the Euler characteristic $\chi(M)$. ♦

Example 5.7.23 (*Signature complex*) Let (M, g) be an even-dimensional oriented compact Riemannian manifold. Denote $\dim M = 2n$. Consider the Clifford bundle $Cl(M)$ of (M, g). By Example 5.5.16, $Cl(M)$ is isomorphic to $\bigwedge \mathrm{T}^*M$ as a Clifford module bundle. Under this identification, the Clifford mapping of $Cl(M)$ is given by

$$c : \mathrm{T}M \to \mathrm{End}(\bigwedge \mathrm{T}^*M)\,, \quad c(X)\alpha = \mathsf{g}(X) \wedge \alpha + X \lrcorner \alpha\,,$$

and the Dirac operator reads $\mathrm{D}\alpha = \mathrm{i}(\mathrm{d} - \mathrm{d}^*)\alpha$. Now, recall that the chirality element $\Gamma_{2n} := \mathrm{i}^n \mathsf{c}(\mathsf{v})$ implies a natural decomposition $Cl_n^c = Cl_n^+ \oplus Cl_n^-$ of the complexified Clifford algebra, cf. (5.3.7) and (5.3.13). Clearly, Γ_{2n} induces an involutive automorphism of $Cl(M) \otimes \mathbb{C}$ yielding a splitting of that bundle. It is easy to check (Exercise 5.7.7) that, under the identification with $\bigwedge \mathrm{T}^*M \otimes \mathbb{C}$, this involutive automorphism is given by

$$\tau : \bigwedge^k \mathrm{T}^*M \otimes \mathbb{C} \to \bigwedge^{2n-k} \mathrm{T}^*M \otimes \mathbb{C}\,, \quad \tau(\alpha) := \mathrm{i}^{n+k(k+1)} * \alpha\,. \tag{5.7.46}$$

Since $\tau^2 = \mathrm{id}$, we can decompose

$$\bigwedge \mathrm{T}^*M \otimes \mathbb{C} = \bigwedge^+ \mathrm{T}^*M \oplus \bigwedge^- \mathrm{T}^*M \tag{5.7.47}$$

into subbundles of elements corresponding to eigenvalues ± 1 of τ. Next, it is easy to show (Exercise 5.7.9) that

$$c(X) \circ \tau + \tau \circ c(X) = 0\,, \quad X \in \mathfrak{X}(M)\,, \tag{5.7.48}$$

and, correspondingly,

$$\mathrm{D} \circ \tau + \tau \circ \mathrm{D} = 0\,. \tag{5.7.49}$$

This is in accordance with point 2 of Lemma 5.3.4. By (5.7.49), the restrictions of D to the subbundles $\bigwedge^+ \mathrm{T}^*M$ and $\bigwedge^- \mathrm{T}^*M$ yield mappings

$$\mathrm{d}_\pm : \Gamma^\infty(\bigwedge^\pm \mathrm{T}^*M) \to \Gamma^\infty(\bigwedge^\mp \mathrm{T}^*M)\,, \tag{5.7.50}$$

and, thus, a complex

$$0 \longrightarrow \Gamma^\infty(\bigwedge^+ \mathrm{T}^*M) \xrightarrow{\mathrm{d}_+} \Gamma^\infty(\bigwedge^- \mathrm{T}^*M) \longrightarrow 0\,,$$

which will be referred to as the signature complex of M and will be denoted by $\mathfrak{E}_{\mathrm{sgn}}(M)$. It may be viewed as obtained by rolling up the de Rham complex using

[42] Clearly, this also follows from Example 5.6.7.

the Z_2-grading defined by (5.7.47). Clearly, d_- is the adjoint of d_+. By Proposition 5.7.4, D is elliptic and, thus, d_+ and d_- are elliptic, too. We define

$$\sigma(M) := \mathrm{ind}(\mathrm{d}_+) \tag{5.7.51}$$

and call it the signature of M. By (5.7.16), we have $\sigma(M) = \dim(\ker(\mathrm{d}_+)) - \dim(\ker(\mathrm{d}_-))$ and, using $\ker(\mathrm{d}_+^*) \subset \mathrm{im}(\mathrm{d}_+)^\perp$, we obtain

$$\sigma(M) = \dim(\ker(\Box^+)) - \dim(\ker(\Box^-))\,, \tag{5.7.52}$$

where $\Box^+ = \mathrm{d}_-\mathrm{d}_+$ and $\Box^- = \mathrm{d}_+\mathrm{d}_-$. Clearly, if we change the orientation of M, then d_+ and d_- are interchanged and, thus, the signature changes its sign. Moreover, we have

$$\sigma(M) = 0\,, \quad \text{for } \dim M = 2 \pmod 4\,. \tag{5.7.53}$$

Indeed, in this case, one can check that complex conjugation yields an isomorphism $\bigwedge^+ \mathrm{T}^*M \cong \bigwedge^- \mathrm{T}^*M$ which clearly implies the assertion (Exercise 5.7.8). This shows that only the case $\dim M = 4k$ is interesting. Here, we have

$$\sigma(M) = \dim(\ker(\Box^+_{2k})) - \dim(\ker(\Box^-_{2k}))\,, \quad \text{for } \dim M = 4k\,, \tag{5.7.54}$$

where $\Box^\pm_{2k}$ denote the restrictions of $\Box^\pm$ to the subspaces of form degree $2k$. To prove this statement, observe that the mappings

$$\varphi_\pm : \textstyle\bigwedge^p \mathrm{T}^*M \to (\bigwedge^p \mathrm{T}^*M \oplus \bigwedge^{4k-p} \mathrm{T}^*M)^\pm\,, \quad \varphi_\pm(\alpha) := \dfrac{1}{2}(\alpha \pm \tau\alpha)\,, \tag{5.7.55}$$

are isomorphisms of vector bundles intertwining $\Box^+_k$ with $\Box^-_k$ (Exercise 5.7.10). This implies

$$(\textstyle\bigwedge^p \mathrm{T}^*M \oplus \bigwedge^{4k-p} \mathrm{T}^*M)^+ \cong \bigwedge^p \mathrm{T}^*M \cong (\bigwedge^p \mathrm{T}^*M \oplus \bigwedge^{4k-p} \mathrm{T}^*M)^-\,,$$

for every $p \neq 4k - p$. Thus, all contributions in (5.7.52) cancel except for those corresponding to form degree $p = 2k$.

Finally, the Hodge Theorem implies via $\ker(\Box_{2k}) \cong H^{2k}_{\mathrm{dR}}(M)$ a purely topological formula for the signature as follows. For a closed, connected, oriented manifold of dimension $2n$, one defines a pairing

$$\mathsf{s}_M : H^n_{\mathrm{dR}}(M) \times H^n_{\mathrm{dR}}(M) \to \mathbb{R}\,, \quad \mathsf{s}_M([\alpha],[\beta]) := \int_M \alpha \wedge \beta\,. \tag{5.7.56}$$

If $H^n_{\mathrm{dR}}(M) = 0$, we put $\mathsf{s}_M = 0$. This is a symmetric, non-degenerate bilinear form on $H^n_{\mathrm{dR}}(M)$ called the intersection form of M. Let (b^+, b^-) be the signature of the quadratic form corresponding to s_M. Now, for $\dim M = 4k$ we have $\tau = *$ and, thus, for a (real) $2k$-form α representing an element of $(H^{2k}_{\mathrm{dR}}(M))_\pm$ we have

$$\int_M \alpha \wedge \alpha = \pm \parallel \alpha \parallel_{L^2}^2 \ .$$

Thus,

$$\sigma(M) = b^+ - b^- \,, \tag{5.7.57}$$

that is, the signature of M coincides with the index of the intersection form. ♦

Example 5.7.24 (*Spin complex*) As before, let (M, g) be a $2n$-dimensional oriented compact Riemannian manifold. Consider the canonical spinor bundle

$$\mathscr{S}(M) = S(M) \times_\gamma \Delta_n$$

relative to a chosen spin structure on M, cf. formula (5.5.8). Since $\dim M = 2n$, it splits into a direct sum of subbundles,

$$\mathscr{S}(M) = \mathscr{S}^+(M) \oplus \mathscr{S}^-(M) \,, \quad \mathscr{S}^\pm(M) = S(M) \times_\gamma \Delta_n^\pm \,.$$

By Example 5.5.17, the Dirac operator of $\mathscr{S}(M)$ is given by

$$\not{D}\Phi = i \sum_{j=1}^n c_j \nabla_{e_j} \Phi \,, \quad \Phi \in \Gamma^\infty(\mathscr{S}(M)) \,,$$

where ∇ is the spin connection. By Remark 5.5.5, for $\dim M = 2n$, the Clifford mapping c implies a bundle isomorphism $c(X) : \mathscr{S}^\pm(M) \to \mathscr{S}^\mp(M)$ for any nowhere vanishing vector field X on M. This induces a splitting of the Dirac operator,

$$\not{D}^\pm : \Gamma^\infty\big(\mathscr{S}^\pm(M)\big) \to \Gamma^\infty\big(\mathscr{S}^\mp(M)\big) \,. \tag{5.7.58}$$

By Proposition 5.7.4, $\not{D}$ is elliptic. Thus, $\not{D}^\pm$ are elliptic, too, and we obtain an elliptic complex

$$0 \longrightarrow \Gamma^\infty(\mathscr{S}^+(M)) \xrightarrow{\not{D}^+} \Gamma^\infty(\mathscr{S}^-(M)) \longrightarrow 0 \,,$$

which will be referred to as the spin complex of (M, g) with respect to the chosen spin structure. Clearly, $\not{D}^-$ is the adjoint of $\not{D}^+$. The index of this complex, that is, the index of $\not{D}$ will be shown to coincide with the Â-genus[43] $\hat{\mathrm{A}}(M)$ of the manifold M, see Corollary 5.9.1.

Let E be a Riemannian (or Hermitean) vector bundle over M endowed with a compatible connection. Consider the tensor product $\mathscr{S}(M) \otimes E$. By Remark 5.5.18, there is a natural associated twisted Dirac operator $\not{D}_E$ with the Clifford action given by $\gamma \otimes \mathrm{id}$. Thus, $\not{D}_E$ is elliptic and the same construction as above yields the twisted spin complex

[43] See Sect. 4.7.

$$0 \longrightarrow \Gamma^\infty(\mathscr{S}^+(M) \otimes E) \xrightarrow{\not{D}_E^+} \Gamma^\infty(\mathscr{S}^-(M) \otimes E) \longrightarrow 0\,. \tag{5.7.59}$$

♦

Example 5.7.25 (*Dolbeault complex*) Let M be a compact complex manifold of complex dimension n. Recall from Example 2.2.10 that its canonically associated almost complex structure J induces a splitting

$$\bigwedge^k \mathrm{T}^*{}_{\mathbb{C}} M = \bigoplus_{p+q=k} \bigwedge^{p,q} M\,, \quad \bigwedge^{p,q} M = \bigwedge^p \mathrm{T}^{*1,0} M \otimes \bigwedge^q \mathrm{T}^{*0,1} M\,. \tag{5.7.60}$$

The canonical projections $\Pi^{p,q} : \bigwedge^k \mathrm{T}^*{}_{\mathbb{C}} M \to \bigwedge^{p,q} M$ induce mappings

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)\,, \quad \bar{\partial} : \Omega^{p,q} M \to \Omega^{p,q+1}(M)$$

defined by

$$\partial := \Pi^{p+1,q} \circ \mathrm{d}\,, \quad \bar{\partial} := \Pi^{p,q+1} \circ \mathrm{d}\,. \tag{5.7.61}$$

Since, by assumption, J is integrable, Corollary 2.2.15 implies

$$\partial^2 = 0\,, \quad \bar{\partial}^2 = 0\,, \quad \bar{\partial} \circ \partial + \partial \circ \bar{\partial} = 0\,. \tag{5.7.62}$$

Thus, for any p,

$$\ldots \longrightarrow \Omega^{p,q-1}(M) \xrightarrow{\bar{\partial}} \Omega^{p,q}(M) \xrightarrow{\bar{\partial}} \Omega^{p,q+1}(M) \longrightarrow \ldots\,, \tag{5.7.63}$$

is a complex of differential operators, called the Dolbeault complex. Usually, one restricts attention to $p = 0$. By (5.7.45), the symbol of $\bar{\partial}$ is given by

$$\sigma_\xi(\bar{\partial})(\alpha) = i\xi^{0,1} \wedge \alpha\,, \tag{5.7.64}$$

where $\xi = \xi^{1,0} + \xi^{0,1}$ is the decomposition implied from (5.7.60). We conclude that the Dolbeault complex is elliptic. The index of the Dolbeault complex is referred to as the arithmetic genus of the manifold M. It is denoted by $\mathrm{Ag}(M)$.

Now, let M be additionally endowed with a Riemannian metric g compatible with J, that is, $\mathsf{g}(X, Y) = \mathsf{g}(\mathsf{J}X, \mathsf{J}Y)$ for any $X, Y \in \mathfrak{X}(M)$. Then, the Dolbeault complex fits into the general framework of this section. Indeed, by (2.2.10), for any local g-orthonormal frame $\{e_k\}$ on M, the $(1, 0)$- and $(0, 1)$-components of $\mathrm{T}M$ are locally spanned by $\{e_k - i\mathsf{J}e_k\}$ and $\{e_k + i\mathsf{J}e_k\}$, respectively. By the compatibility of g and J, both components are g-isotropic. Thus, the corresponding decomposition of the exterior bundle is, pointwise, a special case of the construction of the Clifford modules S_W and S^W in Sect. 5.3 with the Clifford action induced from (5.3.27).

Finally, Proposition 2.6.6 ensures that the Dolbeault complex may be twisted with a vector bundle E endowed with a fibre metric and a compatible connection. ♦

Exercises

5.7.1 Prove formulae (5.7.4)–(5.7.6).

5.7.2 Prove that the scalar products (5.7.8) and (5.7.10) on $\Gamma^\infty(E)$ define equivalent norms. Use this to show that the topology so defined does not depend on the choice of g, $\langle\cdot,\cdot\rangle$, ∇ or a covering of M by local charts.

5.7.3 Prove the isomorphism (5.7.14).

5.7.4 Prove the statements of Proposition 5.7.5.

5.7.5 Prove the elliptic estimate (5.7.23).

5.7.6 Prove Proposition 5.7.21.

5.7.7 Prove that, under the isomorphism $Cl(M)\otimes\mathbb{C}\cong\bigwedge\mathrm{T}^*M\otimes\mathbb{C}$, the involution induced from the chirality element coincides with the involution τ defined by (5.7.46).

5.7.8 Consider Example 5.7.23. Show that in case $n=2k+1$ complex conjugation yields an isomorphism $\bigwedge^+\mathrm{T}^*M\cong\bigwedge^-\mathrm{T}^*M$.

5.7.9 Prove the formulae (5.7.48) and (5.7.49).

5.7.10 Prove that the formula (5.7.55) defines isomorphisms of vector bundles.

5.7.11 Prove that $\mathrm{sign}(\mathbb{C}\mathrm{P}^{2k})=1$.

5.8 The Atiyah–Singer Index Theorem

In this section, some of the analytic details will be omitted. This applies, in particular, to standard Sobolev-type arguments. For a full treatment of the subject we refer to the classical papers by Atiyah, Bott, Getzler, Gilkey, McKean, Patody, Segal and Singer [32, 34, 39, 40, 242, 243, 245, 435], as well as to the monographs [72, 246, 407, 533].

The discussion in the previous section suggests to consider the following general setting.

Definition 5.8.1 (*Graded Dirac bundle*) A graded Dirac bundle is a Dirac bundle $\mathscr{E}$ endowed with an involutive self-adjoint vertical bundle automorphism $\tau:\mathscr{E}\to\mathscr{E}$ anticommuting with the Clifford action and with the Dirac operator D of $\mathscr{E}$.

The operator τ will be called the grading operator. Note that anticommuting with D is equivalent to commuting with the underlying Clifford connection. Also note that the Examples 5.7.23, 5.7.24 and 5.7.25 are of that type.

Let there be given a graded Dirac bundle $\mathscr{E}$ over a compact Riemannian manifold (M, g). By involutivity, τ has (fibrewise) the eigenvalues ± 1 and, thus, we may decompose

$$\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^- \,. \tag{5.8.1}$$

This way, $\mathscr{E}$ becomes a $\mathbb{Z}_2$-graded Clifford module bundle. In the sequel, we will be concerned with even-dimensional oriented manifolds M.[44] In that case, there is always a canonical grading induced from the chirality element Γ, cf. (5.3.7) and Lemma 5.3.4.

In the present context, it is quite common and convenient to use the terminology of superspaces, see e.g. [72, 535]. In this language, $\mathscr{E}$ is a superbundle, its fibres are superspaces and the algebra bundle $\mathrm{End}(\mathscr{E})$ is a superalgebra bundle, that is, τ acting by conjugation induces a decomposition $\mathrm{End}(\mathscr{E}) = \mathrm{End}(\mathscr{E})_0 \oplus \mathrm{End}(\mathscr{E})_1$ into an even and an odd part fulfilling

$$\mathrm{End}(\mathscr{E}_m)_i \cdot \mathrm{End}(\mathscr{E}_m)_j \subset \mathrm{End}(\mathscr{E}_m)_{(i+j \text{ mod } 2)} \,,$$

for every $m \in M$. For any $A_0 \in \mathrm{End}(\mathscr{E}_m)_0$ and $A_1 \in \mathrm{End}(\mathscr{E}_m)_1$, we have

$$\tau(A_0 + A_1)\tau = A_0 - A_1 \,.$$

Associated with the above decomposition, we have a natural notion of parity. We say that an even element $A_0 \in \mathrm{End}(\mathscr{E})_0$ has parity $|A_0| = 0$ and an odd element $A_1 \in \mathrm{End}(\mathscr{E})_1$ has parity $|A_1| = 1$. Using this, one can endow $\mathrm{End}(\mathscr{E})$ with the structure of a Lie superalgebra bundle by defining the super-commutator fibrewise as the bilinear extension of

$$[A, B]_\tau := A \cdot B - (-1)^{|A||B|} B \cdot A \,.$$

Moreover, the following notion of supertrace relative to the grading τ is useful. For an even element A, we define

$$\mathrm{str}_{\mathscr{E}}(A) := \mathrm{Tr}(\tau A) \,. \tag{5.8.2}$$

Then,

$$\mathrm{str}_{\mathscr{E}}(A) = \mathrm{Tr}(A_{++}) - \mathrm{Tr}(A_{--}) \,, \tag{5.8.3}$$

where A_{++} and A_{--} are the diagonal blocks of A with respect to the decomposition (5.8.1). In particular, for an odd element A, we have $\mathrm{Str}_{\mathscr{E}}(A) = 0$. One easily shows the following (Exercise 5.8.1):

$$\mathrm{str}_{\mathscr{E}}([A, B]_\tau) = 0 \,. \tag{5.8.4}$$

---

[44] We will see soon that the index vanishes if M is odd-dimensional.

Below, we also need the superalgebra $\mathfrak{A}$ of bounded operators on $L^2(\mathscr{E})$ and their supertrace. Clearly, the decomposition (5.8.1) induces the decomposition

$$L^2(\mathscr{E}) = L^2(\mathscr{E}^+) \oplus L^2(\mathscr{E}^-)\,. \tag{5.8.5}$$

Next, viewing τ as an operator acting on $L^2(\mathscr{E})$, we obtain a corresponding decomposition $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ into an even and an odd part fulfilling

$$\mathfrak{A}_i \cdot \mathfrak{A}_j \subset \mathfrak{A}_{i+j \bmod 2}\,.$$

Note that $L^2(\mathscr{E}^\pm)$ are the eigenspaces of τ corresponding to the eigenvalues ± 1. For any $a_0 \in \mathfrak{A}_0$ and $a_1 \in \mathfrak{A}_1$, we have $\tau(a_0 + a_1)\tau = a_0 - a_1$. As above, associated with the decomposition of $\mathfrak{A}$, we have a natural notion of parity. We say that an even element $a_0 \in \mathfrak{A}_0$ has parity $|a_0| = 0$ and an odd element $a_1 \in \mathfrak{A}_1$ has parity $|a_1| = 1$. Using this, one can endow $\mathfrak{A}$ with the structure of a Lie superalgebra by defining the super-commutator as

$$[a, b]_\tau := a \cdot b - (-1)^{|a||b|} b \cdot a\,.$$

For any trace-class operator $a \in \mathfrak{A}$, the supertrace is defined by

$$\mathrm{Str}_{\mathscr{E}}(a) := \mathrm{Tr}(\tau a)\,. \tag{5.8.6}$$

As above, we have

$$\mathrm{Str}_{\mathscr{E}}(a) = \mathrm{Tr}(a_{++}) - \mathrm{Tr}(a_{--})\,, \tag{5.8.7}$$

where a_{++} and a_{--} are the diagonal blocks of a with respect to the decomposition (5.8.5). Moreover, for any odd element a, we have $\mathrm{Str}_{\mathscr{E}}(a) = 0$. Finally,

$$\mathrm{Str}_{\mathscr{E}}([a, b]_\tau) = 0\,, \tag{5.8.8}$$

provided either a or b are of trace class (Exercise 5.8.1).

Now, recall from Remark 5.3.3 that any complex $Cl(V, \mathsf{q})$-module E is of the form $E \cong \Delta_n \otimes W$, where $W = \mathrm{Hom}_{Cl(V,\mathsf{q})^c}(\Delta_n, E)$, and

$$\mathrm{End}(E) \cong Cl(V, \mathsf{q})^c \otimes \mathrm{End}_{Cl(V,\mathsf{q})}(E)\,. \tag{5.8.9}$$

Here, $\mathrm{End}_{Cl(V,\mathsf{q})}(E)$ may be identified with $\mathrm{End}(W)$. Correspondingly, by Remark 5.5.4, locally we have $\mathscr{E}_{\restriction U} \cong \mathscr{S}(U) \otimes \mathscr{W}$ with $\mathscr{W} = \mathrm{Hom}_{Cl(U)}(\mathscr{S}(U), \mathscr{E})$ and

$$\mathrm{End}(\mathscr{E}_{\restriction U}) \cong Cl^c(U) \otimes \mathrm{End}_{Cl(U)}(\mathscr{E}_{\restriction U})\,. \tag{5.8.10}$$

Thus, the supertrace $\mathrm{str}_{\mathscr{E}}$ boils down to the product of supertraces over the factors on the right hand side of this equation. We write down the relevant notions on the algebraic level of equation (5.8.9) and then extend them to $\mathscr{E}$ fibrewise. To start with, recall that the chirality element Γ_n of Cl_n^c, given by formula (5.3.8), endows Δ_n with a $\mathbb{Z}_2$-grading which is called the canonical grading of the spinor module. Let $\{\mathbf{e}_i\}$ be an orthonormal basis of V and, for each subset $I \subset I_n = \{1, \ldots, n\}$, let $\mathbf{e}_I = 0$ if $I = \varnothing$ and $\mathbf{e}_I = \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k}$ if $I = \{i_1, \ldots, i_k\}$ and $i_1 < \ldots < i_k$. Then, with respect to the canonical grading, we have

$$\operatorname{str}_{\Delta_n}(\mathbf{e}_I) = \begin{cases} (-2i)^{\frac{n}{2}} & \text{if } I = I_n \\ 0 & \text{otherwise} \end{cases} \tag{5.8.11}$$

see Exercise 5.8.3. Then, for any $a \in Cl_n^c$,

$$\operatorname{str}_{\Delta_n}(a) = (-2i)^{\frac{n}{2}} \sigma(a)_{\restriction [n]} ,$$

where σ is the symbol mapping given by (5.1.10) and $[n]$ means taking the n-form part.[45] Then, for any $L = a \otimes F \in \operatorname{End}(E)$, we have

$$\operatorname{str}_E(L) = (-2i)^{\frac{n}{2}} \sigma(a)_{\restriction [n]} \operatorname{str}_W(F) . \tag{5.8.12}$$

This formula extends fibrewise to $\mathscr{E}$. Now, recall that on the bundle level a decomposition $\mathscr{E} = \mathscr{S}(M) \otimes \mathscr{W}$ holds in general only locally. To avoid such a decomposition one introduces the following notion of relative supertrace. Since $\operatorname{str}_{\Delta_n}(\Gamma_n) = 2^{\frac{n}{2}}$, for $L = \Gamma_n \otimes F \in Cl(V, \mathsf{q})^c \otimes \operatorname{End}(W)$, we obtain

$$\operatorname{str}_W(F) = 2^{-\frac{n}{2}} \operatorname{str}_E(L) .$$

Motivated by this formula, we define the relative supertrace of $F \in \operatorname{End}_{Cl(V,\mathsf{q})}(E)$ by

$$\operatorname{str}_{E|\Delta_n}(F) := 2^{-\frac{n}{2}} \operatorname{str}_E(\Gamma_n F) . \tag{5.8.13}$$

If W is ungraded, then

$$\operatorname{str}_{E|\Delta_n}(F) = \operatorname{tr}_W(F) = 2^{-\frac{n}{2}} \operatorname{tr}_E(F) .$$

By analogy with (5.8.13), we define the relative supertrace on $\mathscr{E}$ fibrewise by

$$\operatorname{str}_{\mathscr{E}|\mathscr{S}}(A_m) := 2^{-\frac{n}{2}} \operatorname{str}_{\mathscr{E}}(\Gamma_n(m) A_m) , \tag{5.8.14}$$

where $A_m \in \operatorname{End}(\mathscr{E}_m)$ and $\Gamma_n(m)$ is the chirality element corresponding to $\mathsf{v}_{\mathsf{g}_m}$.

Now, let us consider the Dirac operator D of $\mathscr{E}$. Since it anticommutes with τ, we get a Fredholm complex

$$0 \longrightarrow \Gamma^\infty(\mathscr{E}^+) \xrightarrow{\mathrm{D}^+} \Gamma^\infty(\mathscr{E}^-) \longrightarrow 0 . \tag{5.8.15}$$

[45] We identify $\bigwedge^n V^* \cong \mathbb{R}$ via the canonical volume form of q.

In this setting, D is referred to as a graded Dirac operator. As in the examples of the previous section, the adjoint of D^+ is $\mathrm{D}^- : \Gamma^\infty(\mathscr{E}^-) \to \Gamma^\infty(\mathscr{E}^+)$. Thus, according to (5.7.44) and (5.7.16), the index of this complex is given by

$$\mathrm{ind}(\mathrm{D}) := \dim(\ker\ \mathrm{D}^+) - \dim(\ker\ \mathrm{D}^-)\,. \tag{5.8.16}$$

We are going to study ind(D) within the setting described above. For that purpose, heat kernels are of basic importance.

Remark 5.8.2 (*Heat kernels*) Note that $\psi(t) = \mathrm{e}^{-t\mathrm{D}^2}\psi_0$ is a solution to the heat equation

$$\frac{\partial \psi}{\partial t} + \mathrm{D}^2\psi = 0 \tag{5.8.17}$$

for any $\psi_0 \in L^2(\mathscr{E})$. Therefore, $\mathrm{e}^{-t\mathrm{D}^2}$ will be called the heat operator. By standard arguments, for $t > 0$, $\psi(t)$ is the unique smooth solution to (5.8.17) fulfilling $\lim_{t\to 0}\psi(t) = \psi_0$. Moreover,

$$\lim_{t\to\infty} \psi(t) = \mathrm{P}_{\ker \mathrm{D}}(\psi_0)\,, \quad \| \psi(t) \| \leq \| \psi_0 \|,$$

where $\mathrm{P}_{\ker \mathrm{D}}$ is the orthogonal projection onto $\ker \mathrm{D} \subset L^2(\mathscr{E})$, see Proposition 4.2.2 in [212]. It is easy to show that these statements also hold true for any $\psi_0 \in W^k(\mathscr{E})$ with $k > 0$. This implies that $\mathrm{e}^{-t\mathrm{D}^2} : L^2(\mathscr{E}) \to W^k(\mathscr{E})$ is bounded for any $t > 0$ and $k \geq 0$. Thus, the Sobolev Lemma implies that

$$\mathrm{e}^{-t\mathrm{D}^2} : L^2(\mathscr{E}) \to \Gamma^\infty(\mathscr{E})\,, \quad t > 0\,,$$

is bounded. Such an operator is referred to as a smoothing operator. Moreover, using the natural L^2-pairing (5.7.11), one extends the heat operator to a bounded mapping $\mathrm{e}^{-t\mathrm{D}^2} : W^{-k}(\mathscr{E}) \to L^2(\mathscr{E})$, for any $t > 0$ and $k \geq 0$, and one shows that $\mathrm{e}^{-t\mathrm{D}^2} : W^{-k}(\mathscr{E}) \to \Gamma^\infty(\mathscr{E})$ is smoothing, too, for any $t > 0$. Now, by the Schwartz Kernel Theorem, $\mathrm{e}^{-t\mathrm{D}^2}$ admits a smooth kernel k, called the heat kernel of D^2,

$$\left(\mathrm{e}^{-t\mathrm{D}^2}\phi\right)(p) = \int_M \mathrm{k}_t(p,q)\phi(q)\mathrm{v_g}(q)\,, \quad \phi \in \Gamma^\infty(\mathscr{E})\,. \tag{5.8.18}$$

More precisely, denote by $p_i : M \times M \to M$ the projections onto the first and the second factor, respectively. Then,

$$\mathscr{E} \boxtimes \mathscr{E}^* := p_1^*\mathscr{E} \otimes p_2^*\mathscr{E}^*$$

is a vector bundle over $M \times M$ and k_t is a smooth family of sections in $\mathscr{E} \boxtimes \mathscr{E}^*$. For an orthonormal basis $\{\psi_n\}$ of $L^2(\mathscr{E})$ consisting of eigensections of D^2 with (non-negative) eigenvalues λ_k, we have

$$\mathrm{k}_t(p,q) = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k} \psi_k(p) \otimes \overline{\psi_k(q)}\,. \tag{5.8.19}$$

One shows that

(a) the heat kernel satisfies the heat equation with respect to both variables,
(b) for each smooth section ϕ,

$$\int_M \mathrm{k}_t(p,q)\phi(q)\mathsf{v}_{\mathsf{g}}(q) \to \phi(p) \tag{5.8.20}$$

uniformly in p as $t \to 0$.

Moreover, the heat kernel is the unique time-dependent section of $\mathscr{E} \boxtimes \mathscr{E}^*$ which is of class C^2 in p and q and of class C^1 in t and which has the properties (a) and (b), see [72, 533].

Finally, since D^2 is smoothing and has a smooth kernel, $\mathrm{e}^{-t\mathrm{D}^2}$ is trace class for all $t > 0$, see Theorem 8.12 in [533]. ♦

In the first step, we prove the following important formula [435].

Proposition 5.8.3 (McKean–Singer Formula) *Let $\mathscr{E}$ be a graded Dirac bundle with grading τ and let* D *be its Dirac operator. Then, for any $t > 0$,*

$$\mathrm{ind(D)} = \mathrm{Str}_{\mathscr{E}}\left(e^{-t\mathrm{D}^2}\right). \tag{5.8.21}$$

Proof The assertion follows from the spectral theorem for the positive self-adjoint operator D^2. Clearly, the decomposition of D^2 with respect to (5.8.1) is given by

$$\mathrm{D}^2 = \begin{bmatrix} \mathrm{D}^-\mathrm{D}^+ & 0 \\ 0 & \mathrm{D}^+\mathrm{D}^- \end{bmatrix}.$$

Let $n_\lambda^\pm$ be the dimensions of the λ-eigenspaces $H_\lambda^\pm$ of the restrictions $\mathrm{D}^-\mathrm{D}^+$ and $\mathrm{D}^+\mathrm{D}^-$ of D^2 to $L^2(\mathscr{E}^\pm)$, respectively. Then,

$$\mathrm{Str}_{\mathscr{E}}\left(\mathrm{e}^{-t\mathrm{D}^2}\right) = \sum_{\lambda \geq 0} (n_\lambda^+ - n_\lambda^-)\,\mathrm{e}^{-t\lambda}\,.$$

Let $\psi \in H_\lambda^+$. Then, $\mathrm{D}^+\mathrm{D}^-\mathrm{D}^+\psi = \lambda\mathrm{D}^+\psi$, that is, $\mathrm{D}^+\psi \in H_\lambda^-$ is an eigenspinor field for $\mathrm{D}^+\mathrm{D}^-$ with eigenvalue λ. Thus, for every $\lambda \neq 0$, D^+ maps H_λ^+ isomorphically onto H_λ^-. This implies $n_\lambda^+ = n_\lambda^-$ for any $\lambda > 0$. Consequently, only $n_0^+ - n_0^- = \dim(\ker\,\mathrm{D}^+) - \dim(\ker\,\mathrm{D}^-)$ remains in the above sum. ■

By the McKean–Singer Formula and (5.8.18) (Exercise 5.8.2),

$$\mathrm{ind(D)} = \mathrm{Str}_{\mathscr{E}}\left(e^{-t\mathrm{D}^2}\right) = \int_M \mathrm{str}_{\mathscr{E}_q}\left(\mathrm{k}_t(q,q)\right)\mathsf{v}_{\mathsf{g}}(q)\,. \tag{5.8.22}$$

Here, the integrand is the fibrewise supertrace of the endomorphism $\mathrm{k}_t(q, q) \in \mathrm{End}(\mathscr{E}_q)$.

Example 5.8.4 (*Heat kernel of the Laplacian on* $\mathbb{R}^n$) Consider the Laplace operator Δ on $\mathbb{R}^n$. Its heat kernel is easily calculated (Exercise 5.8.4):

$$\mathrm{k}_t(\mathbf{x}, \mathbf{y}) = (4\pi t)^{-\frac{n}{2}} \, \mathrm{e}^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}} . \tag{5.8.23}$$

♦

Example 5.8.5 (*Heat kernel of the harmonic oscillator*) Consider the Hamilton operator of the harmonic oscillator on $\mathbb{R}$,

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2 x^2 .$$

Since this self-adjoint operator is quadratic both in differentiation and in multiplication, it is plausible to make the following ansatz:

$$\mathrm{k}_t(x, y) = \mathrm{e}^{a(t)\frac{x^2}{2} + a(t)\frac{y^2}{2} + b(t)xy + c(t)} .$$

Then, denoting the derivative with respect to t by a dot, we calculate

$$\begin{aligned}
&\dot{\mathrm{k}}_t(x, y) + H\, \mathrm{k}_t(x, y) \\
&\quad = \left(\dot{a}(t)\frac{x^2}{2} + \dot{a}(t)\frac{y^2}{2} + \dot{b}(t)xy + \dot{c}(t) - (a(t)x + b(t)y)^2 - a(t) + \omega^2 x^2 \right) \mathrm{k}_t(x, y) ,
\end{aligned}$$

for any $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Thus, the heat equation implies

$$\dot{c} = a , \quad \dot{a} = 2b^2 = 2(a^2 - \omega^2) . \tag{5.8.24}$$

Solving this system (Exercise 5.8.5) yields

$$\begin{aligned}
a(t) &= -\coth(2(t - t_0)), \\
b(t) &= -\frac{1}{\sinh(2(t - t_0))}, \\
c(t) &= -\frac{1}{2} \log(\sinh(2(t - t_0))) + c_0 .
\end{aligned}$$

Finally, using the initial condition (5.8.20), we obtain $t_0 = 0$ and $c_0 = -\frac{1}{2} \log(2\pi)$ and, thus,

$$\mathrm{k}_t(x, y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \left(\frac{2t}{\sinh(2t)}\right)^{\frac{1}{2}} \exp\left(-\frac{x^2 + y^2}{2\tanh(2t)} + \frac{xy}{\sinh(2t)}\right), \tag{5.8.25}$$

which is referred to as Mehler's Formula. ♦

The next important observation is that the index is homotopy invariant. To show this, let us consider a continuous family D_s, $s \in [0, 1]$, of graded Dirac operators on a complex vector bundle $\mathscr{E}$ which means that all data (the Riemannian metric g, the Clifford action c, the fibre metric on $\mathscr{E}$ and the connection ∇) entering the definition of D vary continuously with s preserving, of course, all compatibility conditions. Then, $s \to \mathrm{D}_s$ is a continuous mapping from $[0, 1]$ to the space of bounded mappings $B(W^{k+1}(\mathscr{E}), W^k(\mathscr{E}))$ for any k, cf. (5.7.12).

Proposition 5.8.6 *Let* $s \mapsto \mathrm{D}_s$ *be a continuous family of graded Dirac operators. Then,* $\mathrm{ind}(\mathrm{D}_0) = \mathrm{ind}(\mathrm{D}_1)$.

Proof Since the heat kernel is smooth, formula (5.8.21) implies that $\mathrm{ind}(\mathrm{D}_s)$ is a smooth function of s. Thus, using Duhamel's Formula,[46] we calculate

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{ind}(\mathrm{D}_s)\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Str}_{\mathscr{E}}\left(\mathrm{e}^{-t(\mathrm{D}_s)^2}\right)\right) = -t\,\mathrm{Str}_{\mathscr{E}}\left(\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{D}_s, \mathrm{D}_s\mathrm{e}^{-t(\mathrm{D}_s)^2}\right]_\tau\right).$$

This quantity vanishes by (5.8.8). ■

To summarize our discussion up until now, Proposition 5.8.6 shows that the index of a graded Dirac operator D is a topological invariant and the McKean–Singer Formula suggests that this invariant can possibly be calculated via the heat kernel of D^2. It turns out that this idea is fruitful indeed. It leads to one of the proofs of the index theorem.[47] Note that the left hand side of (5.8.22) does not depend on t whereas the right hand side makes sense for all $t > 0$. This suggest that the limit of the right hand side as $t \to 0$ may be meaningful and that it might be possible to use this limit for calculating the index. Theorem 5.8.10 below substantiates this idea. To prove it, we use the following approximation concept for heat kernels.

Definition 5.8.7 Let $\mathscr{E}$ be a Dirac bundle with Dirac operator D and let $\mathrm{k}_t(p, q)$ be the heat kernel of D^2. Let k be a positive integer. Then, an approximate heat kernel of order k is a smooth t-dependent section $\tilde{\mathrm{k}}_t(p, q)$ of $\mathscr{E} \boxtimes \mathscr{E}^*$ fulfilling the initial condition (5.8.20) and

$$\left(\frac{\partial}{\partial t} + \mathrm{D}_p^2\right)\tilde{\mathrm{k}}_t(p, q) = t^k \phi_t(p, q),$$

where ϕ_t is a C^k-section of $\mathscr{E} \boxtimes \mathscr{E}^*$ depending continuously on t for $t \geq 0$ and where D_p denotes the Dirac operator applied in the p-variable.

[46] See e.g. Sect. 2.7 in [72] for a proof.

[47] For a comparison of the different proofs available, see [87].

By standard Sobolev-type arguments, see [533, Chap. 7], one shows the following.

Lemma 5.8.8 *Let $\mathscr{E}$ be a Dirac bundle with Dirac operator* D *and let* $\mathrm{k}_t(p,q)$ *be the heat kernel of* D^2. *Then, for every k there exists a $k' \geq k$ such that for any approximate heat kernel $\tilde{\mathrm{k}}_t(p,q)$ of order k', we have*

$$\mathrm{k}_t(p,q) - \tilde{\mathrm{k}}_t(p,q) = t^k \phi_t(p,q)\,,$$

where ϕ_t is a C^k-section of $\mathscr{E} \boxtimes \mathscr{E}^*$ depending continuously on t for $t \geq 0$. ■

In the sequel, Taylor expansions of geometric objects in a geodesic chart will be used.

Remark 5.8.9 (*Taylor expansions*) We take up Remarks 1.7.19 and 2.1.30. Let (M, g) be a Riemannian manifold and let $x^1, \ldots, x^n$ be normal coordinates of a geodesic chart (U, κ) centered at $m \in M$ such that the local holonomic frame $\{\partial_i\}$ is orthonormal at m. Construct a local synchronous frame $\mathfrak{e} = (e_1, \ldots, e_n)$ on U for the Levi-Civita connection on $\mathrm{T}M$ by parallel transporting the tangent space basis $\{\partial_j\}$ at m along the geodesics through m, cf. Remark 1.7.19. By construction, $\mathfrak{e}$ is orthonormal and coincides with $\{\partial_i\}$ at m. Thus, (1.7.17) implies

$$\Gamma^k_{il}(\mathbf{x}) \sim -\frac{1}{2}\mathsf{R}^k_{ijl}(0)x^j + 0(\|\mathbf{x}\|^2)\,, \tag{5.8.26}$$

where Γ^k_{il} are the Christoffel symbols of the Levi-Civita connection and R^k_{ijl} are the components of the Riemann curvature in normal coordinates, respectively. In particular, we have $\Gamma^k_{il}(0) = 0$. A similar Taylor-type expansion holds for the metric:

$$\mathsf{g}_{ij}(\mathbf{x}) = \delta_{ij} - \frac{1}{3}\sum_{k,l}\mathsf{R}_{iklj}(0)x^k x^l + 0(\|\mathbf{x}\|^3)\,. \tag{5.8.27}$$

The proof of this formula is in complete analogy to the proof of (1.7.17). Let $\{\theta^j\}$ be the coframe dual to $\mathfrak{e}$, let $\omega^i{}_j$ be the components of the Levi-Civita connection in this frame and let $X^r = \sum_i x^i \partial_i$ be the radial vector field. Then,

$$X^r \lrcorner \theta^i = x^i\,, \quad X^r \lrcorner \omega^i{}_j = 0\,, \quad \mathsf{g}_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j = \delta_{ij}\theta^i \otimes \theta^j\,. \tag{5.8.28}$$

Clearly, the tautological form θ on M may be expressed with respect to both the holonomic frame $\{\partial_j\}$ and the synchronous frame $\{e_j\}$,

$$\theta = \sum_j \mathrm{d}x^j \partial_j = \theta^j e_j\,,$$

and we may decompose $\theta^j = \theta^j{}_k \mathrm{d}x^k$. Then, $\mathsf{g}_{ij} = \delta_{kl}\theta^k{}_i\theta^l{}_j$. Thus, it is enough to find the Taylor expansion for the coefficient functions $\theta^i{}_j$. Using the relations (5.8.28), by analogous arguments as in Remark 1.7.19, one obtains (Exercise 5.8.6)

$$(X^r \circ X^r + X^r)\theta^i{}_j = -\sum_{k,l} \mathsf{R}^i_{klj}(0)x^k x^l \,. \tag{5.8.29}$$

This implies (5.8.27). For a detailed presentation of the arguments, we also refer to [34]. ♦

The basic idea is now to take the following counterpart of (5.8.23) on the Riemannian manifold (M, g) as the first approximation to the true heat kernel:

$$h_t(p, q) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{\mathrm{d}(p, q)}{4t}\right), \tag{5.8.30}$$

where $\mathrm{d}(p, q)$ denotes the geodesic distance between p and q.

Theorem 5.8.10 (Heat kernel asymptotics) *Let $\mathscr{E}$ be a Dirac bundle over a compact Riemannian manifold (M, g) and let D be its Dirac operator. Let k_t be the heat kernel of D^2. Then,*

1. *as $t \to 0$, there is an asymptotic expansion*[48]

$$\mathrm{k}_t(p, q) \sim h_t(p, q) \sum_{j=0}^{\infty} t^j\, a_j(p, q)\,, \tag{5.8.31}$$

   where the a_j are smooth sections of $\mathscr{E} \boxtimes \mathscr{E}^*$. This expansion is valid in the Banach space $C^r(\mathscr{E} \boxtimes \mathscr{E}^*)$ for any integer $r \geq 0$.
2. *The values $a_j(p, p)$ along the diagonal are given in terms of algebraic expressions involving the metric and the connection coefficients, together with their derivatives. In particular, $a_0(p, p)$ is the identity endomorphism of $\mathscr{E}$.*

Our proof is along the lines of Theorem 7.15 in [533].

Proof By Lemma 5.8.8, it is enough to show that there exist smooth sections a_j of $\mathscr{E} \boxtimes \mathscr{E}^*$ such that for each k the partial sum

$$S_t(p, q) = h_t(p, q) \sum_{j=0}^{J} t^j a_j(p, q)$$

is an approximate heat kernel of order k for all sufficiently large J. Since h_t is of order t^∞ outside any neighbourhood of the diagonal in $M \times M$, it clearly suffices to determine the sections $a_j(p, q)$ for p near q. Thus, we may use a local geodesic coordinate system $x^1, \ldots, x^n$ centered at q. We denote the determinant of the metric g by g, the geodesic distance from q to p by r, that is, $r^2 = \mathsf{g}_{ij}x^i x^j$. Then, one calculates, see Exercise 5.8.7,

[48] Recall that $f(t) \sim \sum_{k=0}^{\infty} a_k(t)$ is called an asymptotic expansion for a function f on $\mathbb{R}_+$ if, for any n, almost all the partial sums of the series approximate f to within an error of order t^n. Clearly, the series need not converge.

$$h_t^{-1}\left(\frac{\partial}{\partial t}+\mathrm{D}_p^2\right)(h_t\phi)=\frac{\partial\phi}{\partial t}+\mathrm{D}_p^2\phi+\frac{r}{4gt}\frac{\partial g}{\partial r}\phi+\frac{1}{t}\nabla_{r\frac{\partial}{\partial r}}\phi\,, \tag{5.8.32}$$

for any local section ϕ of $\mathscr{E}\boxtimes\mathscr{E}^*$. Now, seeking a solution to the heat equation in the form $h_t\phi$, we expand $\phi\sim\phi_0+t\phi_1+t^2\phi_2+\ldots$, with the ϕ_j not depending on t, insert this expansion into (5.8.32) and put the coefficient functions of each power of t equal to zero. This yields the following system of equations:

$$\nabla_{r\frac{\partial}{\partial r}}\phi_j+\left(j+\frac{r}{4g}\frac{\partial g}{\partial r}\right)\phi_j=-\mathrm{D}_p^2\phi_{j-1}\,, \tag{5.8.33}$$

where $j=0,1,2,\ldots$ and $\phi_{-1}=0$. This is a system of ordinary differential equations along each ray starting from q which may be solved recursively. Note that the first of these equations ($j=0$) simply reads

$$\nabla_{\frac{\partial}{\partial r}}\left(g^{\frac{1}{4}}\phi_0\right)=0\,, \tag{5.8.34}$$

showing that ϕ_0 is uniquely determined by its initial value $\phi_0(0)$. We put $\phi_0(0)=1$, the identity endomorphism of S_q. This suggests to rewrite the remaining equations by incorporating the factor $g^{\frac{1}{4}}$ as well. This yields (Exercise 5.8.8):

$$\nabla_{\frac{\partial}{\partial r}}\left(r^jg^{\frac{1}{4}}\phi_j\right)=-r^{j-1}g^{\frac{1}{4}}\mathrm{D}_p^2\phi_{j-1}\,, \tag{5.8.35}$$

for any $j\geq 1$. Thus, every ϕ_j is determined by ϕ_{j-1} up to an additive term of order r^{-j} near $r=0$. If we require smoothness at $r=0$, this term must vanish and, thus, all ϕ_j are uniquely determined by the initial condition $\phi_0(0)=1$.

To summarize, we have constructed local representatives $\phi_j(x)$ of the heat kernel coefficients $a_j(p,q)$ for p near q. By standard Sobolev-type arguments, one shows that, for $J>\frac{1}{2}\dim M+k$,

$$\mathrm{k}_t^J(p,q)=h_t(p,q)\sum_{j=0}^{J}t^ja_j(p,q)$$

is an approximate heat kernel of order k.

To prove the second assertion, note that $a_j(p,p)$ is given locally by $\phi_j(0)$. Thus, it is enough to expand both sides of (5.8.33), or (5.8.35), in a Taylor series about the origin. Then, the coefficients $\phi_j(0)$ may be iteratively calculated in terms of algebraic expressions involving the metric and the connection coefficients, together with their derivatives, indeed. ■

Example 5.8.11 To illustrate the second assertion in Theorem 5.8.10, let us find the first two coefficients of the heat kernel expansion. First, from (5.8.34) and the initial condition, we read off $\phi_0=g^{-\frac{1}{4}}$. Substituting this into (5.8.35) and using the Weitzenboeck Formula 5.6.1 we obtain

$$\phi_1(0) = -(\mathrm{D}^2\phi_0)(0) = \sum_i \left(\frac{\partial}{\partial x^i}\right)^2 (g^{-\frac{1}{4}})(0) - \mathfrak{R}^{\mathscr{E}}(0)\,. \tag{5.8.36}$$

From (5.8.27), we conclude

$$g^{-\frac{1}{4}}(\mathbf{x}) = 1 + \frac{1}{12}\sum_{i,j,l} x^j x^l \mathsf{R}_{ijli}(0) + 0(\|\mathbf{x}\|^3)\,. \tag{5.8.37}$$

This entails

$$\sum_i \left(\frac{\partial}{\partial x^i}\right)^2 (g^{-\frac{1}{4}}) = \frac{1}{6}\sum_{j,l} \mathsf{R}_{jllj} = \frac{1}{6}\mathsf{Sc}$$

at the origin and, therefore,

$$a_0(q,q) = 1\,, \quad a_1(q,q) = \frac{1}{6}\mathsf{Sc}(q) - \mathfrak{R}^{\mathscr{E}}(q)\,. \tag{5.8.38}$$

Thus, the first non-trivial heat kernel coefficient is given by the scalar curvature of (M, g) and by the Weitzenboeck curvature operator of the Dirac bundle $\mathscr{E}$. ♦

Combining Theorem 5.8.10 with the McKean–Singer Formula in the form of (5.8.22), we obtain the following.

Corollary 5.8.12 *Let $\mathscr{E}$ be a graded Dirac bundle over a compact Riemannian manifold (M, g) and let D be its Dirac operator. Then, the index of D is zero if the dimension of M is odd. If n is even, then*

$$\operatorname{ind}\mathrm{D} = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_M \operatorname{str}_{\mathscr{E}_q}\big(a_{\frac{n}{2}}(q,q)\big)\mathsf{v}_{\mathsf{g}}(q)\,. \tag{5.8.39}$$

Proof By (5.8.31) and (5.8.22), we have

$$\operatorname{ind}\mathrm{D} \sim \frac{1}{(4\pi)^{\frac{n}{2}}}\sum_{j=0}^{\infty}\left(\int_M \operatorname{str}_{\mathscr{E}_q}\big(a_j(q,q)\big)\mathsf{v}_{\mathsf{g}}(q)\right) t^{j-\frac{n}{2}}\,.$$

Since the left hand side is constant, both assertions follow. ■

This corollary reduces the calculation of the index of D to the calculation of the integral over the heat kernel coefficient of D^2 of order $\frac{n}{2}$.

For the further analysis of formula (5.8.39), let us fix a point $q \in M$ and let $\exp_q : \mathrm{T}_q M \to M$ be the exponential mapping of (M, g). Then, for $p = \exp_q(X)$ in a neighbourhood of q, we denote

$$\mathrm{k}_t(X) := \mathrm{k}_t(\exp_q(X), q) \in \operatorname{Hom}\big(\mathscr{E}_q, \mathscr{E}_{\exp_q(X)}\big)\,.$$

We trivialize the bundle $\mathscr{E}$ over an open neighbourhood U centered at q by choosing a synchronous framing combined with a local geodesic chart, cf. Remarks 1.7.19 and 5.8.9. In more detail, we choose a local geodesic chart (U, κ) centered at q and identify the fibres $\mathscr{E}_q$ and $\mathscr{E}_{\exp_q(X)}$ via the parallel transport operator along the radial geodesic from q to $\exp_q(X)$. Clearly, the geodesic chart provides a local trivialization of $\mathrm{T}M$ and, thus, of $Cl(M)$ over U as well. Now, recall (5.8.10),

$$\mathrm{End}(\mathscr{E}_{\restriction U}) \cong Cl^c(U) \otimes \mathrm{End}_{Cl(U)}(\mathscr{E}_{\restriction U})\,. \tag{5.8.40}$$

In the above local trivializations, the Clifford action boils down to the action of $Cl(\mathrm{T}_qM)$ on the fibre $\mathscr{E}_q$ and, thus, $\mathrm{End}_{Cl(M)}(\mathscr{E})$ is locally trivial as well, with fibre $\mathrm{End}_{Cl(\mathrm{T}_qM)}(\mathscr{E}_q) = \mathrm{End}(W)$. Thus, for a chosen point $q \in M$, we may view the heat kernel as

$$\mathrm{k}_t(X) \in \mathrm{End}(\mathscr{E}_q) \cong Cl^c(\mathrm{T}_qM) \otimes \mathrm{End}(W)\,. \tag{5.8.41}$$

Let $x^1, \dots, x^n$ be the normal coordinates of the chosen geodesic chart and let $\{\partial_j\}$ and $\{e_j\}$ be the holonomic and the (orthonormal) synchronous frames, respectively. Recall that the latter coincide at the point q. In these normal coordinates, we will write $\mathrm{k}_t(\mathbf{x})$ for $\mathrm{k}_t(X)$. Let $\{\mathbf{e}_i\}$ be the orthonormal basis of T_qM given by $\mathbf{e}_i = e_i(q)$. Moreover, as before, for each subset $I \subset I_n = \{1, \dots, n\}$, let $\mathbf{e}_I = 0$ if $I = \varnothing$ and $\mathbf{e}_I = \mathbf{e}_{i_1} \dots \mathbf{e}_{i_k}$ if $I = \{i_1, \dots, i_k\}$ and $i_1 < \cdots < i_k$. In this basis, (5.8.31) takes the form

$$\mathrm{k}_t(\mathbf{x}) \sim h_t(\mathbf{x}) \sum_{j=0}^{\infty} \sum_I t^j\, \mathbf{e}_I \otimes a_{j,I}(\mathbf{x})\,, \tag{5.8.42}$$

where the coefficients $a_{j,I}(\mathbf{x})$ are $\mathrm{End}(W)$-valued. Now, by (5.8.30) and (5.8.12),

$$\mathrm{str}_{\mathscr{E}_q}\big(\mathrm{k}_t(0)\big) \sim \frac{(-2i)^{\frac{n}{2}}}{(4\pi t)^{\frac{n}{2}}} \sum_{j=0}^{\infty} t^j\, \mathrm{str}_{\mathscr{E}_q|\Delta_n}(a_{j,I_n}(0))\,. \tag{5.8.43}$$

To summarize, (5.8.39) takes the form

$$\mathrm{ind}\,\mathrm{D} = (2\pi i)^{-\frac{n}{2}} \int_M \mathrm{str}_{\mathscr{E}_q|\Delta_n}\big(a_{\frac{n}{2},I_n}(q,q)\big)\mathsf{v}_{\mathsf{g}}(q)\,. \tag{5.8.44}$$

Next, let us analyze the local representative $\mathbb{A} \in \Omega^1(U, \mathrm{End}(\mathscr{E}_q))$ of the Clifford connection ∇ in the chosen synchronous framing $\{e_i\}$. We denote $\mathbf{c}_i = c(\mathbf{e}_i)$.

Lemma 5.8.13 *In the local trivialization defined by a synchronous framing,*

$$\mathbb{A}_i(\mathbf{x}) = \frac{1}{8} \sum_{j,k,l} \mathsf{R}_{ijkl}(0) x^j \mathbf{c}_k \mathbf{c}_l + \sum_{k,l} \alpha_{ikl}(\mathbf{x}) \mathbf{c}_k \mathbf{c}_l + \beta_i(\mathbf{x})\,, \tag{5.8.45}$$

where $\mathsf{R}_{ijkl}(0)$ *are the Riemann curvature coefficients at the origin with respect to the holonomic frame* $\{\partial_j\}$, $\alpha_{ikl} \in C^\infty(U)$ *are functions of order* $0(\|\mathbf{x}\|^2)$ *and* $\beta_i \in C^\infty(U, \mathrm{End}(W))$ *are functions of order* $0(\|\mathbf{x}\|)$.

Proof By (1.7.17), we have $\mathbb{A}(0) = 0$ and

$$\mathbb{A}_i(\mathbf{x}) \sim -\frac{1}{2}\mathsf{R}^{\mathscr{E}}_{ij}(0)x^j + 0(\|\mathbf{x}\|^2)\,, \tag{5.8.46}$$

where $\mathsf{R}^{\mathscr{E}}$ is the curvature endomorphism of ∇. By (5.6.8), $\mathsf{R}^{\mathscr{E}} = \mathsf{R}^{\nabla^{\mathsf{g}}} + \mathrm{F}^{\mathscr{E}}$, where

$$\mathsf{R}^{\nabla^{\mathsf{g}}}(X, Y) = \frac{1}{4}\sum_{l,k} \mathsf{g}(\mathsf{R}(X, Y)(e_k), e_l)\, c_l c_k \tag{5.8.47}$$

is the curvature endomorphism of ∇^{g} viewed as a connection in the Clifford bundle $Cl(M)_{\restriction U}$ and $\mathrm{F}^{\mathscr{E}} \in \Omega^2(U, \mathrm{End}(W))$ is the twisting curvature of the Dirac bundle $\mathscr{E}$. Since, at the origin, e_i and ∂_i coincide, the contribution of $\mathsf{R}^{\nabla^{\mathsf{g}}}$ coincides with the first term in (5.8.45) up to a term of order $0(\|\mathbf{x}\|^2)$. Since the Clifford action on W is trivial, the contribution of $\mathrm{F}^{\mathscr{E}}$ is simply a function of order $0(\|\mathbf{x}\|)$. ■

Now, we are prepared to prove the Atiyah–Singer Index Theorem. The proof we give is based on a method developed by Getzler [242, 243], which is often referred to as Getzler rescaling.[49] By (5.5.5), $Cl(M)$ and $\bigwedge \mathrm{T}^*M$ may be identified as Clifford module bundles. In our trivialization, this boils down to the Clifford module isomorphism

$$\bigwedge \mathrm{T}^*_q M \cong Cl(\mathrm{T}^*_q M)\,, \tag{5.8.48}$$

with the left Clifford action on $\bigwedge \mathrm{T}^*_q M$ given by $c(\alpha) = \varepsilon(\alpha) + \iota(\alpha)$, cf. Example 5.3.2. Now, given a function ϕ on $\mathbb{R}_+ \times U$ with values in $\bigwedge \mathrm{T}^*_q M \otimes \mathrm{End}(W)$, we define the Getzler rescaling operator by

$$(\delta_\lambda \phi)(t, \mathbf{x}) := \sum_{j=0}^{n} \lambda^{-j} \phi\big(\lambda^2 t, \lambda \mathbf{x}\big)_{[j]}\,, \tag{5.8.49}$$

for $0 < \lambda \leq 1$. Here, the index $[j]$ means restriction to the form degree j. This implies the rescaling $\hat{\delta}_\lambda A := \delta_\lambda A \delta_\lambda^{-1}$ for any operator A acting on functions of the above type. In particular, we obtain

$$\hat{\delta}_\lambda \partial_t = \lambda^{-2}\partial_t\,, \quad \hat{\delta}_\lambda \partial_j = \lambda^{-1}\partial_j\,, \quad \hat{\delta}_\lambda \varepsilon(\alpha) = \lambda^{-1}\varepsilon(\alpha)\,, \quad \hat{\delta}_\lambda \iota(\alpha) = \lambda\iota(\alpha)\,, \tag{5.8.50}$$

for $\alpha \in \mathrm{T}^*_q M$. We will write $\varepsilon^i = \varepsilon(\mathrm{d}x^i)$ and $\iota^i = \mathsf{g}^{ij}\,\iota(\partial_j)$.

[49]We use the Getzler calculus in a purely operational manner. For a deeper discussion we refer to [72, 533].

As a last ingredient, we need the relative Chern character form of the bundle $\mathscr{E}$. It is defined as follows. The twisting curvature endomorphism $\mathrm{F}^{\mathscr{E}}$ of ∇ is a 2-form with values in the real vector bundle

$$\mathfrak{u}_{Cl^c(M)}(\mathscr{E}) := \mathfrak{u}(\mathscr{E}) \cap \mathrm{End}_{Cl^c(M)}(\mathscr{E})\,,$$

where $\mathfrak{u}(\mathscr{E}) \subset \mathrm{End}(\mathscr{E})$ is the subbundle of skew-adjoint endomorphisms, see Sect. 4.6. For $A_m \in \mathfrak{u}_{Cl^c(M)}(\mathscr{E})$, one has $\mathrm{e}^{\mathrm{i}A_m/4\pi} \in \mathrm{End}_{Cl^c(M)}(\mathscr{E}_m)$, so that one can define a section $q^{\mathscr{E}}$ in the bundle $\mathrm{FPS}\big(\mathfrak{u}_{Cl^c(M)}(\mathscr{E})\big)$ of formal power series, see Sect. 4.6, by

$$q^{\mathscr{E}}_m(A_m) := \mathrm{str}_{\mathscr{E}|\mathscr{S}}\left(\exp(-A_m/4\pi\mathrm{i})\right)\,, \quad m \in M\,. \tag{5.8.51}$$

By definition, the relative Chern character form of $\mathscr{E}$ is

$$\mathrm{ch}(\mathscr{E}|\mathscr{S}) := h_{\mathrm{F}^{\mathscr{E}}}(q^{\mathscr{E}})\,,$$

with $h_{\mathrm{F}^{\mathscr{E}}}$ given by (4.6.33). One easily shows that this form is closed and that its de Rham cohomology class, which we denote by the same symbol, does not depend on the choice of the connection, see Sect. 3 in [526]. According to Remark 4.6.10, we write

$$\mathrm{ch}(\mathscr{E}|\mathscr{S}) = \mathrm{str}_{\mathscr{E}|\mathscr{S}}\big(\exp(-\mathrm{F}^{\mathscr{E}}/2\pi i)\big)\,. \tag{5.8.52}$$

Theorem 5.8.14 (Atiyah–Singer) *Let $\mathscr{E}$ be a graded Dirac bundle over an even-dimensional oriented compact Riemannian manifold (M, g) and let D be its Dirac operator. Then,*

$$\mathrm{ind}\,\mathrm{D} = \int_M \hat{\mathrm{A}}(M) \wedge \mathrm{ch}(\mathscr{E}|\mathscr{S})\,, \tag{5.8.53}$$

where $\hat{\mathrm{A}}(M)$ is the $\hat{\mathrm{A}}$-genus form of M. In the integrand, the component of form degree $\dim M$ *is taken.*

Proof Let $\dim M = n$. We define the rescaled heat kernel by

$$\mathrm{k}^{\lambda}_t(\mathbf{x}) := \lambda^n(\delta_\lambda \mathrm{k})_t(\mathbf{x})\,. \tag{5.8.54}$$

Since the heat kernel satisfies the heat equation, we have

$$(\partial_t + \lambda^2\delta_\lambda \mathrm{D}^2\delta_\lambda^{-1})\mathrm{k}^{\lambda}_t = 0\,.$$

Thus, k^{λ}_t is the heat kernel of the rescaled operator $P_\lambda := \lambda^2\hat{\delta}_\lambda \mathrm{D}^2$. In the first step, we prove that the limit $P_0 = \lim_{\lambda\to 0} P_\lambda$ exists by explicitly calculating it. For that purpose, we work in the local trivialization of $\mathscr{E}$ over a neighbourhood U centered at q obtained by the above described synchronous framing. By the Weitzenboeck Formula (5.6.9),

$$\mathrm{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathsf{Sc} + \mathfrak{F}^{\mathscr{E}}\,, \tag{5.8.55}$$

and by (2.7.31), for any local frame $\{e_j\}$ we have

$$\nabla^*\nabla = -\mathsf{g}^{ij}\left(\nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i}e_j}\right)\,.$$

In the synchronous frame, $\nabla_{\partial_i} = \partial_i + \mathbb{A}_i$ with $\mathbb{A}_i$ given by (5.8.45). In order to be able to apply the rescaling mappings, we must consistently use the isomorphism (5.8.48). Then, using the fact that the Clifford action on W is trivial, we obtain

$$\begin{aligned}P_\lambda(\mathbf{x}) = &-\sum_{i,j}\mathsf{g}^{ij}(\lambda\mathbf{x})(\partial_i + \lambda(\hat{\delta}_\lambda\mathbb{A}_i)(\mathbf{x}))(\partial_j + \lambda(\hat{\delta}_\lambda\mathbb{A}_j)(\mathbf{x}))\\ &-\lambda\sum_{i,j,k}(\mathsf{g}^{ij}\Gamma^k_{ij})(\lambda\mathbf{x})(\partial_k + \lambda(\hat{\delta}_\lambda\mathbb{A}_k)(\mathbf{x}))\\ &+\frac{\lambda^2}{4}\mathsf{Sc}(\lambda\mathbf{x}) + \lambda^2\big(\hat{\delta}_\lambda\mathfrak{F}^{\mathscr{E}}\big)(\mathbf{x})\,.\end{aligned} \tag{5.8.56}$$

Here, Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection in normal coordinates. By Lemma 5.8.13, we have

$$\begin{aligned}\lambda(\hat{\delta}_\lambda\mathbb{A}_i)(\mathbf{x}) = &\frac{1}{8}\sum_{j,k,l}\mathsf{R}_{ijkl}(0)x^j(\varepsilon^k + \lambda^2\iota^k)(\varepsilon^l + \lambda^2\iota^l)\\ &+\lambda^{-1}\sum_{k,l}\alpha_{ikl}(\lambda\mathbf{x})(\varepsilon^k + \lambda^2\iota^k)(\varepsilon^l + \lambda^2\iota^l) + \lambda\beta_i(\lambda\mathbf{x})\,.\end{aligned}$$

Since the functions α_{ikl} and β_i are of order $0(\|\mathbf{x}\|^2)$ and $0(\|\mathbf{x}\|)$, respectively, we obtain

$$\lim_{\lambda\to 0}\lambda(\hat{\delta}_\lambda\mathbb{A}_i)(\mathbf{x}) = \frac{1}{4}\sum_j\Omega_{ij}x^j\,,$$

where

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l}\mathsf{R}_{ijkl}(0)\varepsilon^k\varepsilon^l$$

acts on $\bigwedge\mathrm{T}^*_qM$ by exterior multiplication. Thus, Ω_{ij} may be viewed as an antisymmetric $(n\times n)$-matrix with values in the even part $\mathfrak{A}$ of the exterior algebra of T^*_qM which is a finite-dimensional commutative algebra (over $\mathbb{C}$) with unit. By the above arguments, the limit $\lambda\to 0$ of the covariant derivative exists and is equal to $\partial_i + \frac{1}{4}\sum_j\Omega_{ij}x^j$. Next, using the Taylor expansions for g_{ij} and Γ^k_{ij} derived in Remark 5.8.9, we see that the second and the third term in (5.8.56) vanish in the limit $\lambda\to 0$. Finally, by (5.6.10), the limit of the last term is simply $\mathfrak{F}^{\mathscr{E}}(0)$. Using the Taylor expansion of g_{ij} once again, we obtain

$$P_0 = \lim_{\lambda \to 0} P_\lambda = -\sum_i \big(\partial_i + \frac{1}{4}\sum_j \Omega_{ij} x^j\big)^2 + \mathfrak{F}^{\mathscr{E}}(0)\,. \tag{5.8.57}$$

Under the identification (5.8.48), $\mathfrak{F}^{\mathscr{E}}(0)$ becomes an element of $\mathfrak{A} \otimes \mathrm{End}(W)$ acting on $\bigwedge \mathrm{T}_q^* M$ by exterior multiplication. This finishes the first step of the proof.

In the second step, we calculate the heat kernel k_t^0 of P_0. We denote $\mathrm{F} = \mathfrak{F}^{\mathscr{E}}(0)$ and

$$\mathrm{H} = -\sum_i \big(\partial_i + \frac{1}{4}\sum_j \Omega_{ij} x^j\big)^2 .$$

This is the Hamiltonian of a generalized harmonic oscillator. Then, by the above discussion, $P_0 = \mathrm{H} + \mathrm{F}$ is a differential operator acting on $\mathfrak{A} \otimes \mathrm{End}(W)$-valued functions on U. Since $\mathfrak{A}$ is commutative, the operators H and F commute. Thus, $\mathrm{e}^{-tP_0} = \mathrm{e}^{-t\mathrm{H}}\mathrm{e}^{-t\mathrm{F}}$. Since Ω is an antisymmetric $(n \times n)$-matrix with values in the 2-forms we can choose the orthonormal basis in $\mathrm{T}_q M$ so that Ω is represented by a block-diagonal matrix,

$$\Omega = \bigoplus_{p=1}^{\frac{n}{2}} \Omega_p\,, \quad \Omega_p = \begin{bmatrix} 0 & -\omega_p \\ \omega_p & 0 \end{bmatrix}.$$

Then, H decouples into a sum of operators of the form

$$h = -\big(\partial_x + \frac{1}{4}\omega y\big)^2 - \big(\partial_y - \frac{1}{4}\omega x\big)^2 = -\big(\partial_x^2 + \partial_y^2\big) - \frac{\omega^2}{16}\big(x^2 + y^2\big) + \frac{1}{2}\big(x\partial_y - y\partial_x\big)$$

and it remains to calculate the heat kernel of this operator. By the uniqueness of the heat kernel, we can seek a rotationally invariant solution. Then, the last term in h will not contribute and, apart from the fact that we must replace ω by $i\omega$, we have a sum of two harmonic oscillator Hamiltonians. Using Mehler's Formula (5.8.25), we obtain (Exercise 5.8.9)

$$\mathrm{k}_t^0(\mathbf{x}) = (4\pi t)^{-\frac{n}{2}} \det{}^{\frac{1}{2}} \left(\frac{t\Omega/2}{\sinh(t\Omega/2)}\right) \mathrm{e}^{-\frac{1}{4t}\langle \frac{t\Omega}{2}\coth(\frac{t\Omega}{2})\mathbf{x},\mathbf{x}\rangle} \mathrm{e}^{-t\mathrm{F}}\,, \tag{5.8.58}$$

and, thus,

$$\mathrm{k}_t^0(0) = (4\pi t)^{-\frac{n}{2}} \det{}^{\frac{1}{2}} \left(\frac{t\Omega/2}{\sinh(t\Omega/2)}\right) \mathrm{e}^{-t\mathrm{F}}\,. \tag{5.8.59}$$

This finishes the second step of the proof.

In the third step, we show that the index of D may be expressed in terms of the heat kernel coefficients of $\mathrm{k}_t^0(0)$. For that purpose, consider the asymptotic expansion of the rescaled heat kernel k_t^λ. Applying the rescaling mapping to k_t as given by (5.8.42) and using the isomorphism (5.8.48), we obtain

$$\mathrm{k}_t^\lambda(\mathbf{x}) \sim \lambda^n\, h_{\lambda^2 t}(\lambda\mathbf{x}) \sum_{j=0}^{\infty} \sum_I t^j\, \lambda^{2j-|I|} a_{j,I}(\lambda\mathbf{x})\, \mathrm{d}x^I\,, \tag{5.8.60}$$

where as usual $\mathrm{d}x^I := \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$ if $I = \{i_1, \ldots, i_k\}$ and $i_1 < \ldots < i_k$. Thus,

$$\mathrm{k}_t^\lambda(0) \sim (4\pi)^{-\frac{n}{2}} \sum_{j=0}^{\infty} \sum_I t^{j-\frac{n}{2}}\, \lambda^{2j-|I|} a_{j,I}(0)\, \mathrm{d}x^I\,. \tag{5.8.61}$$

Without giving a proof here, we use the fact that the coefficients of the asymptotic expansion (5.8.60) depend continuously on λ, see Theorem 2.48 in [72]. This implies that the asymptotic expansion of k_t^0 can be obtained as the limit $\lambda \to 0$ of the asymptotic expansion (5.8.60). Thus, let

$$\mathrm{k}_t^0(0) = (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} P_j\big(\Omega/2, -\mathrm{F}\big) t^j \tag{5.8.62}$$

be the Taylor series of (5.8.59). Since Ω and F are nilpotent elements of the exterior algebra, this series converges for all values of t. Then, comparing coefficients, we read off

$$P_j\big(\Omega/2, -\mathrm{F}\big) = \lim_{\lambda\to 0} \sum_I \lambda^{2j-|I|} a_{j,I}(0)\, \mathrm{d}x^I\,,$$

that is, $a_{j,I}(0) = 0$ for $j > \frac{|I|}{2}$ and

$$P_j\big(\Omega/2, -\mathrm{F}\big) = \sum_{|I|=2j} a_{j,I}(0)\, \mathrm{d}x^I\,. \tag{5.8.63}$$

But, $|I| \leq n$ and, thus, $P_j \neq 0$ for $j = 0, 1, \ldots, \frac{n}{2}$ only. This implies

$$\mathrm{k}_t^0(0) = (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}} P_j\big(\Omega/2, -\mathrm{F}\big) t^j\,. \tag{5.8.64}$$

Taking the supertrace of this equation and using (5.8.11), together with (5.8.63), we obtain

$$\mathrm{str}_{\mathscr{E}_q}\big(\mathrm{k}_t^0(0)\big) = (4\pi t)^{-\frac{n}{2}} \mathrm{str}_{\mathscr{E}_q}\Big(P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big)\Big) t^{\frac{n}{2}} = \frac{(-2i)^{\frac{n}{2}}}{(4\pi t)^{\frac{n}{2}}}\, \mathrm{str}_{\mathscr{E}_q|\Delta_n}(a_{\frac{n}{2},I_n}(0)) t^{\frac{n}{2}}\,.$$

Comparing with (5.8.44), we conclude

$$\mathrm{ind}\,\mathrm{D} = (4\pi)^{-\frac{n}{2}} \int_M \mathrm{str}_{\mathscr{E}_q}\Big(P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big)\Big) \mathsf{v}_{\mathsf{g}}(q)\,, \tag{5.8.65}$$

that is, the index of D is given by the coefficients of the heat kernel expansion of P_0, indeed. This finishes the third step of the proof.

Finally, it remains to calculate the integrand of (5.8.65). Comparing (5.8.64) with (5.8.59), we see that

$$P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big) = \det{}^{\frac{1}{2}}\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)\exp(-\mathrm{F})_{\restriction[n]}\,,$$

where $[n]$ means taking the n-form part of the right hand side. Since the summands in the Taylor expansion of the first factor on the right hand side are just differential forms on U, by (5.8.11), we have

$$\mathrm{str}_{\mathscr{E}_q}\big(P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big)\big)\mathsf{v}_{\mathsf{g}} = (-2i)^{\frac{n}{2}}\det{}^{\frac{1}{2}}\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)\mathrm{str}_{\mathscr{E}_q|\Delta_n}\left(\exp(-\mathrm{F})\right)_{\restriction[n]}\,.$$

Since P_j is a homogeneous polynomial of degree j,

$$P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big) = (2\pi i)^{\frac{n}{2}}P_{\frac{n}{2}}\big(\Omega/4\pi i, -\mathrm{F}/2\pi i\big)\,,$$

and, using (4.7.18), we have

$$\mathrm{str}_{\mathscr{E}_q}\big(P_{\frac{n}{2}}\big(\Omega/2, -\mathrm{F}\big)\big)\mathsf{v}_{\mathsf{g}} = (-2i)^{\frac{n}{2}}(2\pi i)^{\frac{n}{2}}\hat{\mathrm{A}}(M)\,\mathrm{ch}(\mathscr{E}|\mathscr{S})_{\restriction[n]}\,.$$

Inserting this into (5.8.65) yields the assertion. ∎

Often, the right hand side of (5.8.53) is referred to as the topological index of the Dirac bundle. In this language, the Atiyah–Singer Index Theorem states that the analytical index of a Dirac operator is equal to the topological index of its Dirac bundle.

Remark 5.8.15 (*Local Index Theorem*) Note that in the proof of Theorem 5.8.14 we have actually obtained a much stronger result which is usually referred to as the Local Index Theorem: for every $q \in M$, the limit $\lim_{t\to 0}\mathrm{str}_{\mathscr{E}_q}\mathrm{k}_t(q,q)\mathsf{v}_{\mathsf{g}}(q)$ exists and is given by

$$\lim_{t\to 0}\mathrm{str}_{\mathscr{E}_q}\mathrm{k}_t(q,q)\mathsf{v}_{\mathsf{g}}(q) = \Big(\hat{\mathrm{A}}(M)\wedge\mathrm{ch}(\mathscr{E}|\mathscr{S})\Big)_{\restriction[n]}(q)\,. \tag{5.8.66}$$

♦

Remark 5.8.16 (*Family Index Theorem*) Both the Index Theorem 5.8.14 and the Local Index Theorem generalize to the case of families of Dirac operators [40, Part IV]. It turns out that the heat kernel methods developed above may be extended to this situation in a quite straightforward way. This has been shown by Bismut [78], see also Chap. 10 in [72] for a detailed presentation. Here, we only explain the setting and formulate the result.[50]

Consider a smooth fibre bundle $\pi : M \to Y$, where M and Y are compact connected manifolds of dimensions $n+m$ and m, respectively, with n even. That is, π is a smooth mapping and for every open subset $U \subset Y$ the inverse image $\pi^{-1}(U)$ is diffeomorphic to $U \times X$, where X is an n-dimensional compact manifold.[51] Denote the bundle of vertical vectors on M with respect to the fibre structure by $\mathrm{V}M$. Assume that the fibration $\pi : M \to Y$ is endowed with the following additional structures:

(a) a fibre metric g^{V},
(b) a projection $P : \mathrm{T}M \to \mathrm{V}M$,
(c) a $\mathrm{Spin}(n)$-structure $S(\mathrm{V}M) \to M$ on the vertical bundle $\mathrm{V}M$.

By points (a) and (b), we have a canonical connection ∇^{V} on $\mathrm{V}M$ defined as follows. First note that the kernel of P defines a horizontal distribution on M and, thus, a splitting $\mathrm{T}M = \mathrm{V}M \oplus \ker P$, that is, P defines a connection in $\pi : M \to Y$. Now, take any metric g^Y on Y, lift it to $\ker P$ and combine this lift with g^{V} to a Riemannian metric g^M on M. Take its Levi-Civita connection ∇^M and project it to $\mathrm{V}M$,

$$\nabla^{\mathrm{V}} := P\,\nabla^M\,P\,.$$

It is easy to show that this is a connection on $\mathrm{V}M$ which does not depend on the choice of g^Y. Moreover, the restrictions of this connection to the fibres of π coincide with the Levi-Civita connections on the fibres. To summarize, $\mathrm{V}M$ carries the structure of a Hermitean vector bundle with a connection which is compatible with the metric. Next, by point (c), $\mathrm{V}M$ admits a spin structure $S(\mathrm{V}M)$ and, thus, the connection ∇^{V} naturally lifts to a connection on $S(\mathrm{V}M)$. By construction, for every $y \in Y$, the restriction of this connection to $S(\mathrm{V}M)_{\restriction M_y}$ coincides with the spin connection corresponding to the Levi-Civita connection of the fibre metric g^{V}.

Now, assume we are given a Clifford module bundle $\mathscr{E}$ over the Clifford bundle $Cl(\mathrm{V}M)$. Since $\mathrm{V}M$ carries a spin structure, $\mathscr{E}$ has the form

$$\mathscr{E} = \mathscr{S}_V \otimes E\,, \tag{5.8.67}$$

where $\mathscr{S}_V$ is a spinor bundle associated with $S(\mathrm{V}M)$ and E is a vector bundle given by $E = \mathrm{Hom}_{Cl(\mathrm{V}M)}(\mathscr{S}_{\mathrm{V}}, \mathscr{E})$. The spinor bundle may be viewed as a tensor product $\mathscr{S}_{\mathrm{V}} = \mathscr{S} \otimes V^\rho$, where $\mathscr{S}$ is the canonical spinor bundle, ρ is a complex representation of $\mathrm{Spin}(n)$ and V^ρ is the corresponding vector bundle associated with $S(\mathrm{V}M)$. Since n is even, the natural splitting of $\mathscr{S}$ induces a splitting of $\mathscr{S}_{\mathrm{V}}$ into the chirality components $\mathscr{S}_{\mathrm{V}}^{\pm}$. The natural spin connection on $S(\mathrm{V}M)$ induces connections on the bundles $\mathscr{S}_{\mathrm{V}}^{\pm}$. As usual, we assume that E carries a Hermitean fibre metric g^E and a

[50]For the very formulation of the result, a shorthand version of the detailed description below would be sufficient. However, we present the full structure which then may be taken as the starting point for reading the proof of Bismut.

[51]Then, $\pi : M \to Y$ may be viewed as associated with a principal $\mathrm{Diff}(X)$-bundle over Y.

compatible connection ∇^E. By (5.8.67), the twisting curvature of $\mathscr{E}$ simply coincides with the curvature of ∇^E.

Given the above described structures, we obtain a family of Dirac bundles

$$\mathscr{E}_y^{\pm} = \left(\mathscr{S}_{\mathrm{V}}^{\pm} \otimes E\right)_{\restriction M_y}$$

with an associated family of Dirac operators $\mathrm{D}_y : C^\infty\left(\mathscr{E}_y^+\right) \to C^\infty\left(\mathscr{E}_y^-\right)$. Using the bundle metric in $\mathscr{E}_y^{\pm}$ and the natural volume form on M_y, we obtain L^2-completions $H_y^{\pm}$ of the above C^∞-spaces. The latter fit together to continuous[52] Hilbert bundles $H^{\pm} \to Y$. Correspondingly, the Dirac operators combine to a bundle mapping $\mathrm{D} : H^+ \to H^-$. In this context, one can prove a Local Index Theorem for the Dirac family D and, as a corollary one obtains the Atiyah–Singer Index Theorem for D:

$$\mathrm{ch}(\mathrm{Ind}(\mathrm{D})) = \int_{M/Y} \hat{A}(\mathrm{V}M)\,\mathrm{ch}(E)\,. \tag{5.8.68}$$

Here, $\mathrm{ch}(\mathrm{Ind}(\mathrm{D}))$ is the Chern character of the index bundle, see Appendix E, $\hat{A}(\mathrm{V}M)$ is the $\hat{A}$-genus of the bundle $\mathrm{V}M$ for the connection ∇^{V} and $\mathrm{ch}(E)$ is the Chern character form of the bundle E. The symbol $\int_{M/Y}$ means integration over the fibres of $\pi : M \to Y$. ♦

In the literature, there can be found many generalized index theorems, e.g. algebraic index theorems for formal deformation quantizations, see [190, 483, 511] and references therein. It is interesting to note, see [484], that an application of the algebraic index theorem to the case of the cotangent bundle endowed with the canonical symplectic form and the deformation quantization given by the asymptotic pseudodifferential calculus reproduces the Atiyah–Singer Index Theorem.

Exercises

5.8.1 Prove formula (5.8.1).

5.8.2 Let $\mathrm{k}(t, p, q)^+$ be the heat kernel of $\mathrm{D}^-\mathrm{D}^+$. Show that

$$\mathrm{str}_{\mathscr{E}_q}\left(\mathrm{k}(t, q, q)^+\right) = \sum_k \mathrm{e}^{-t\lambda_k} |\psi_k^+(q)|^2\,,$$

where $\{\psi_k^+\}$ is an orthonormal basis of eigensections with $\mathrm{D}^-\mathrm{D}^+\psi_k^+ = \lambda_k \psi_k^+$. Conclude that

$$\mathrm{Tr}\,\mathrm{e}^{-t\mathrm{D}^-\mathrm{D}^+} = \int_M \mathrm{str}_{\mathscr{E}_q}\left(\mathrm{k}(t, q, q)^+\right)\mathsf{v_g}(q)\,.$$

5.8.3 Prove formula (5.8.11). *Hint.* Show that, for any $I \neq I_n$, there exists an index i such that $\mathbf{e}_I = -\frac{1}{2}[\mathbf{e}_i, \mathbf{e}_i\mathbf{e}_I]_\tau$. Then, by (5.8.4), $\mathrm{tr}_\Delta\,\mathbf{e}_I$ vanishes.

[52]These bundles are not smooth, because the composition $L^2 \times C^\infty \to L^2$ is not smooth.

5.8.4 Confirm formula (5.8.23).

5.8.5 Confirm the solutions to (5.8.24) given in Example 5.8.5.

5.8.6 Prove formula (5.8.28).

5.8.7 Prove formula (5.8.32). *Hint.* First, show that

$$\nabla h = -\frac{h}{2t} r \frac{\partial}{\partial r}\,, \quad \frac{\partial h}{\partial t} + \Delta h = \frac{rh}{4gt}\frac{\partial g}{\partial r}\,.$$

5.8.8 Confirm formula (5.8.35).

5.8.9 Confirm formula (5.8.58).

5.9 Applications

In this section, we discuss some consequences of the Atiyah–Singer Index Theorem. The reader can find a lot of further applications in Chap. IV of [407].

To start with, the following is an immediate consequence of Theorem 5.8.14.

Corollary 5.9.1 (Atiyah–Singer) *If M is a spin manifold and $\mathscr{E}$ is the canonical spinor bundle, then the index of the Dirac operator $\not{D}$ coincides with the $\hat{A}$-genus of M.*

This implies:

(a) the index of the Dirac operator does not depend on the spin structure.
(b) the $\hat{A}$-genus of a spin manifold is an integer.

Point (b) may be sharpened as follows.

Proposition 5.9.2 *Let M be a compact spin manifold such that* $\dim M = 4 \bmod 8$. *Then, $\hat{A}(M)$ is an even integer.*

Proof By Theorem 5.3.19, for $n = 4 \bmod 8$ the spinor representations Δ_n are of quaternionic type. By Remark 5.3.20, the corresponding structure mappings $C : \Delta_n \to \Delta_n$ commute with the Clifford multiplication and are $\mathrm{Spin}(n)$-equivariant. The same is true for the structure mappings $C_\pm : \Delta_n^\pm \to \Delta_n^\pm$ of the corresponding irreducible components of Δ_n. Now, the $\mathrm{Spin}(n)$-equivariance

$$C \circ \gamma(g) = \gamma(g) \circ C\,, \tag{5.9.1}$$

for any $g \in \mathrm{Spin}(n)$, implies that C may be extended to a fibre-preserving mapping of the spinor bundle which we denote by the same letter. Differentiating (5.9.1), we obtain that C commutes with the covariant derivative of the spin connection,

$$\nabla_X \circ C = C \circ \nabla_X\,,$$

for any $X \in \mathfrak{X}(M)$. This property, together with the fact that C commutes with the Clifford multiplication, implies that the Dirac operator commutes with C. Correspondingly, we have $\not{D}^+ \circ C_+ = C_- \circ \not{D}^+$. Thus, the kernel and the cokernel of $\not{D}^+$ are quaternionic vector spaces and, consequently, their complex dimension is even. ∎

Combining Corollary 5.9.1 with Bochner-type arguments, we obtain the following.

Proposition 5.9.3 (Lichnerowicz) *Let M be a compact spin manifold admitting a metric of strictly positive scalar curvature. Then, the $\hat{\mathrm{A}}$-genus of M must vanish.*

Proof By point 1 of Corollary 5.6.8, $\ker \not{D} = \ker \not{D}^2 = 0$. Since

$$\ker \not{D} = \ker \not{D}^+ \oplus \ker \not{D}^- ,$$

this implies $\operatorname{ind} \not{D} = \hat{A}(M) = 0$, ∎

Next, we turn to the analysis of the Atiyah–Singer Index Theorem for the Dirac bundle

$$\mathscr{E} = Cl^c(M) \cong \bigwedge \mathrm{T}^*M \otimes \mathbb{C} .$$

This is a left $Cl(M)$-module bundle with the Clifford mapping of $Cl(M)$ given by

$$c : \mathrm{T}M \to \mathrm{End}(\bigwedge \mathrm{T}^*M) , \quad c(X)\alpha = \mathsf{g}(X) \wedge \alpha + X \lrcorner \alpha ,$$

cf. formula (5.1.8). Its Dirac operator is induced from the de Rham complex $\mathfrak{C}_{\mathrm{dR}}(M)$ and is given by

$$\mathrm{D}\alpha = i(\mathrm{d} - \mathrm{d}^*)\alpha ,$$

see Examples 5.5.16 and 5.7.22. Recall that the index of the de Rham complex coincides with the Euler characteristic $\chi(M)$.

Now, let us analyze the right-hand side of (5.8.53) for that case. Since

$$\bigwedge V^* \otimes \mathbb{C} \cong Cl^c(V) = \mathrm{End}(\Delta_n) \cong \Delta_n^* \otimes \Delta_n ,$$

for any even-dimensional vector space V, the typical fibre of $\mathscr{E}$ is $E = \Delta_n \otimes \Delta_n^*$, that is, the twisting vector space is $W = \Delta_n^*$. Note that, in this situation, besides the canonical grading $\tau_0 = \Gamma_n \otimes \mathrm{id}$ we have a grading $\tau = \omega_n \otimes \omega_n$ given by the volume form $\omega_n = \mathbf{e}_{I_n}$ of Cl_n. Clearly, this is the natural grading induced from that on $\bigwedge V^*$. So, we are going to consider this grading here.

Recall that the Euler form of an oriented Riemannian manifold M is defined by $\mathsf{e}(M) := \mathsf{e}(\mathrm{T}M)$.

Lemma 5.9.4 *The following holds.*

$$\hat{A}(M) \wedge \mathrm{ch}(\mathscr{E}|\mathscr{S}) = \mathsf{e}(M) .$$

Proof Let R and R^{Λ} denote the curvature endomorphism forms of the Levi-Civita connections on $\mathrm{T}M$ and $\bigwedge \mathrm{T}^*M \otimes \mathbb{C}$, respectively, and let $\mathfrak{o}(\mathrm{T}M) \subset \mathrm{End}(\mathrm{T}M)$ denote the subbundle of skew-symmetric endomorphisms. By (4.7.17), $\hat{A}(M) = h_{\mathsf{R}}(r^M)$, where r^M is the section in $\mathrm{FPS}\big(\mathfrak{o}(\mathrm{T}M)\big)$ given by

$$r_m^M(A_m) := \det{}^{\frac{1}{2}} \left(\frac{\frac{\mathrm{i}A_m}{8\pi}}{\sinh\left(\frac{\mathrm{i}A_m}{8\pi}\right)} \right).$$

By Remark 4.6.21, $\mathsf{e}(M) = h_{\mathsf{R}}(\varepsilon^M)$, where ε^M is the section in $\mathrm{Pol}\big(\mathfrak{o}(\mathrm{T}M)\big)$ defined by

$$\varepsilon_m^M(A_m) := \mathrm{pf}\left(\frac{A_m}{4\pi}\right).$$

By formula (5.8.52), $\mathrm{ch}(\mathscr{E}|\mathscr{S}) = h_{\mathrm{F}^{\mathscr{E}}}(q^{\Lambda})$, where $\mathrm{F}^{\mathscr{E}}$ is the twisting curvature endomorphism form of the Levi-Civita connection on $\bigwedge \mathrm{T}^*M \otimes \mathbb{C}$ and q^{Λ} is the section in $\mathrm{FPS}\big(\mathfrak{u}_{Cl^c(M)}(\mathscr{E})\big)$ defined by (5.8.51). To prove the assertion, it suffices to show

$$h_{\mathsf{R}}(r^M) \wedge h_{\mathrm{F}^{\mathscr{E}}}(q^{\Lambda}) = h_{\mathsf{R}}(\varepsilon^M)\,. \tag{5.9.2}$$

For that purpose, we rewrite $h_{\mathrm{F}^{\mathscr{E}}}$ in terms of h_{R}. First, to calculate $\mathrm{F}^{\mathscr{E}}$, we choose a local orthonormal frame $\{e_i\}$ in $\mathrm{T}M$. According to (2.7.36), in this frame the curvature endomorphism form of the Levi-Civita connection on $\bigwedge \mathrm{T}^*M \otimes \mathbb{C}$ is given by

$$\mathsf{R}^{\Lambda}(e_i, e_j) = -\mathsf{g}(\mathsf{R}(e_i, e_j)e_k, e_l)\varepsilon^k \iota^l\,. \tag{5.9.3}$$

Let c_i and b_i denote the local sections in $\mathrm{End}(Cl^c(M))$ defined fibrewise by Clifford multiplication by e_i from the left and the right, respectively. Clearly, the b_i take values in $\mathrm{End}_{Cl^c(M)}(\mathscr{E})$. Under the isomorphism with $\bigwedge \mathrm{T}^*M \otimes \mathbb{C}$,

$$c_i = \varepsilon(e_i) + \iota(e_i)\,, \quad b_i = \varepsilon(e_i) - \iota(e_i)\,. \tag{5.9.4}$$

Using

$$c_i c_j + c_j c_i = 2\delta_{ij}\,, \quad b_i b_j + b_j b_i = -2\delta_{ij} \tag{5.9.5}$$

(Exercise 5.9.1) and the symmetry properties of the curvature, we obtain

$$\mathsf{R}^{\Lambda}(e_i, e_j) = -\frac{1}{4}\,\mathsf{g}\big(\mathsf{R}(e_i, e_j)e_k, e_l\big)(c^k c^l - b^k b^l)\,.$$

Since the first summand coincides with (5.8.47), the Weitzenboeck Formula yields

$$\mathrm{F}^{\mathscr{E}} = \frac{1}{4}\,\mathsf{g}(\mathsf{R}\,e_k, e_l) b^k b^l\,. \tag{5.9.6}$$

According to (5.2.29), then

$$\mathrm{F}^{\mathscr{E}} = \rho \circ \varphi \circ \mathsf{R}\,,$$

where $\varphi : \mathfrak{o}(\mathrm{T}M) \to Cl_2(M)$ is the vertical vector bundle isomorphism which is fibrewise defined by (5.2.28) and $\rho : Cl_2(M) \to \mathfrak{u}_{Cl^c(M)}(Cl^c(M))$ is the vertical vector bundle morphism assigning to $\xi \in Cl_2(\mathrm{T}_m M)$ the endomorphism of $Cl^c(\mathrm{T}_m M)$ defined by right multiplication by ξ. By Lemma 4.6.18/2, then

$$h_{\mathrm{F}^{\mathscr{E}}}(q^\Lambda) = h_{\mathrm{R}}(q^\Lambda \circ \rho \circ \varphi)\,.$$

Plugging this into (5.9.2) and using that h_{R} is an algebra homomorphism, we find that it suffices to show

$$r^M \cdot (q^\Lambda \circ \rho \circ \varphi) = \varepsilon^M\,.$$

Fibrewise, this boils down to the assertion that the identity[53]

$$\det{}^{\frac{1}{2}}\left(\frac{\mathrm{i}A/2}{\sinh(\mathrm{i}A/2)}\right)\,\mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)}\right) = \mathrm{pf}(A)\,,\quad A \in \mathfrak{o}(V)\,, \tag{5.9.7}$$

holds for every oriented Euclidean vector space V of dimension $2l$. Here, $\mathrm{str}_{\mathrm{rel}}$ denotes the relative supertrace on $\mathrm{End}_{Cl^c(V)}\left(Cl^c(V)\right)$ associated with the involution defined by simultaneous left and right multiplication by the natural volume form on V. In what follows, calculations are left to the reader (Exercise 5.9.2). In an appropriate oriented orthonormal basis, A has block diagonal form

$$A = \mathrm{diag}(A_1, \ldots, A_l)\,,\quad A_k = x_k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\,,\quad x_k \in \mathbb{R}\,,$$

and we have

$$\det{}^{\frac{1}{2}}\left(\frac{\mathrm{i}A/2}{\sinh(\mathrm{i}A/2)}\right) = \prod_{k=1}^{l} \det{}^{\frac{1}{2}}\left(\frac{\mathrm{i}A_k/2}{\sinh(\mathrm{i}A_k/2)}\right)\,, \tag{5.9.8}$$

$$\mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)}\right) = \prod_{k=1}^{l} \mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A_k)}\right)\,, \tag{5.9.9}$$

$$\mathrm{pf}(A) = \prod_{k=1}^{l} \mathrm{pf}(A_k)\,. \tag{5.9.10}$$

Thus, it suffices to prove (5.9.7) in two dimensions. Using an appropriate ordered orthonormal basis $\{e_1, e_2\}$ in V, we compute

$$\sinh(\mathrm{i}A/2) = \mathrm{i}\sinh(x/2)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and thus

[53] We rescale $A/4\pi \mapsto A$.

$$\det{}^{\frac{1}{2}}\left(\frac{\mathrm{i}A/2}{\sinh(\mathrm{i}A/2)}\right) = \frac{x/2}{\sinh(x/2)}\,. \tag{5.9.11}$$

Using (5.2.29), we furthermore find

$$\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)} = \cosh(x/2) - \mathrm{i}\sinh(x/2)b_1b_2\,.$$

Hence, by (5.8.13),

$$\mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)}\right) = \frac{1}{2}\mathrm{str}_{Cl^c(M)}\big(\Gamma_V(\cosh(x/2) - \mathrm{i}\sinh(x/2)b_1b_2)\big)\,.$$

As a left $Cl^c(M)$-module, $Cl^c(M)$ is isomorphic to $\Delta_V \otimes \Delta_V^*$. Via this isomorphism, left multiplication by Γ_V corresponds to $\gamma(\Gamma_V) \otimes \mathrm{id}_{\Delta_V^*}$, the endomorphism b_1b_2 corresponds to $\mathrm{id}_{\Delta_V} \otimes \gamma^{\mathrm{T}}(e_1e_2)$ and the involution corresponds to

$$\gamma(e_1e_2) \otimes \gamma^{\mathrm{T}}(e_1e_2) = -\gamma(\Gamma_V) \otimes \gamma^{\mathrm{T}}(\Gamma_V)\,.$$

Writing str_{Δ_V} and $\mathrm{str}_{\Delta_V^*}$ for the supertrace on Δ_V and Δ_V^* defined by the canonical involutions $\gamma(\Gamma_V)$ and $\gamma^{\mathrm{T}}(\Gamma_V)$, respectively, we thus obtain

$$\begin{aligned}\mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)}\right) = \frac{1}{2}\Big\{&\cosh(x/2)\mathrm{str}_{\Delta_V}\big(\gamma(\Gamma_V)\big)\mathrm{str}_{\Delta_V^*}\big(\mathrm{id}_{\Delta_V^*}\big)\\ &- \mathrm{i}\sinh(x/2)\mathrm{str}_{\Delta_V}\big(\gamma(\Gamma_V)\big)\mathrm{str}_{\Delta_V^*}\big(\gamma^{\mathrm{T}}(e_1e_2)\big)\Big\}\,.\end{aligned}$$

By (5.8.11), this yields

$$\mathrm{str}_{\mathrm{rel}}\left(\mathrm{e}^{\mathrm{i}\rho\circ\varphi(A)}\right) = 2\sinh(x/2)\,.$$

It follows that the left hand side of (5.9.7) equals x. This coincides with $\mathrm{pf}(A)$. ■

This Lemma implies the following classical theorem.

Theorem 5.9.5 (Gauß–Bonnet) *The Euler characteristic of an even-dimensional oriented manifold M is given by*

$$\chi(M) = \int_M \mathsf{e}(M)\,. \tag{5.9.12}$$

■

More generally, let us consider the de Rham complex twisted with a complex vector bundle E over M, denoted by $\mathfrak{E}_{\mathrm{d}R}(M,E)$. By the Atiyah–Singer Index Theorem and by Lemma 5.9.4, we have

$$\mathrm{ind}\left(\mathfrak{E}_{\mathrm{dR}}(M,E)\right) = \int_M \mathrm{ch}(E) \wedge \mathsf{e}(M)\,.$$

Since $\mathsf{e}(M)$ is of top degree, we conclude

$$\operatorname{ind}\left(\mathfrak{E}_{\mathrm{dR}}(M,E)\right) = \operatorname{rank}(E)\,\chi(M)\,. \tag{5.9.13}$$

By analogy with Theorem 5.9.5, one can derive the following classical theorems corresponding to the elliptic complexes discussed in Examples 5.7.23 and 5.7.25. For the signature complex one obtains the following.

Theorem 5.9.6 (Hirzebruch) *Let M be an oriented Riemannian manifold of dimension divisible by* 4*. Then, the signature of M is given by*

$$\sigma(M) = \int_M L(M)\,, \tag{5.9.14}$$

where L is the L-genus of the manifold.[54] ■

As an application, consider the case $\dim M = 4$. In view of (4.7.11), the Hirzebruch Signature Theorem implies

$$\sigma(M) = \frac{1}{3}\mathfrak{p}_1(M)\,.$$

Moreover, by (4.7.15),

$$\hat{A}(M) = -\frac{1}{24}\mathfrak{p}_1(M)\,.$$

Thus,

$$\sigma(M) = -8\hat{A}(M)\,. \tag{5.9.15}$$

Since, on a spin manifold, the $\hat{A}$-genus is an integer, this implies that on a compact 4-dimensional spin manifold, the signature is divisible by 8. Combining this with Proposition 5.9.2, we obtain the following classical theorem of Rohlin [535].

Theorem 5.9.7 (Rohlin) *The signature of a compact* 4*-dimensional spin manifold is divisible by* 16. ■

More generally, let us consider the signature complex twisted with a vector bundle E, denoted by $\mathfrak{E}_{\mathrm{sgn}}(M,E)$. As for the de Rham complex, it is easy to calculate its index. One obtains

$$\operatorname{ind}\left(\mathfrak{E}_{\mathrm{sgn}}(M,E)\right) = \sum_{2j+4k=n} \int_M 2^j \mathrm{ch}_j(E) \wedge L_k(M)\,, \tag{5.9.16}$$

where $n = \dim M$. Thus, for $n = 4$, we get[55]

---

[54] See Example 4.7.3.

[55] The factor $4 = 2^{\frac{4}{2}}$ in front of the second term comes from the supertrace formula (5.8.43).

$$\operatorname{ind}\left(\mathfrak{E}_{\mathrm{sgn}}(M, E)\right) = \operatorname{rank}(E)\,\chi(M) + 4\,\mathrm{ch}_2(E)\,. \tag{5.9.17}$$

Finally, we apply the Atiyah–Singer Index Theorem to the Dolbeault complex.

Theorem 5.9.8 (Riemann–Roch) *Let M be a compact complex Riemannian manifold. Then, its arithmetic genus* $\mathrm{Ag}(M)$ *is given by*

$$\mathrm{Ag}(M) = \int_M \mathrm{Td}(M)\,, \tag{5.9.18}$$

where Td *is the Todd genus of the manifold.*[56] ■

Exercises

5.9.1 Confirm the anticommutation relations (5.9.5).

5.9.2 Prove the formulae (5.9.8)–(5.9.11).

[56] See Example 4.7.3.

Chapter 6
The Yang–Mills Equation

In this chapter we study pure gauge theories. In Sect. 6.1, we present the geometric model of gauge theory including the basics concerning the structure of the classical configuration space. Next, in Sect. 6.2, we formulate the action functional and show that (anti-)self-dual solutions correspond to absolute minima of the Yang–Mills action. Sections 6.3, 6.4, 6.5, and 6.6 are devoted to a systematic study of instantons. First, we present the BPST-instanton family in detail, including the topological description and a detailed discussion of the role of the conformal invariance of the Yang–Mills equation. Next, we present the famous ADHM-construction providing solutions on S^4 with arbitrary instanton number. We limit our attention to the gauge group $G = \mathrm{Sp}(1)$ and only comment on solutions for the other classical groups. The proof that the ADHM-construction yields all instanton solutions is highly nontrivial. Roughly speaking, it goes as follows: first, one reinterprets the ADHM data in terms of complex geometry on the twistor space $\mathbb{C}\mathrm{P}^3$ and, using these complex data, one applies the Horrocks construction yielding algebraic vector bundles over $\mathbb{C}\mathrm{P}^3$ of a certain type. Second, by deep results of algebraic geometry, all algebraic vector bundles of this type arise from the Horrocks construction. Third, one uses the Atiyah–Ward correspondence to complete the proof. While we discuss points 1 and 3 in detail, point 2 is beyond the scope of this book. Finally, we study the instanton moduli space and we outline how it is used for the study of the topology of differentiable 4-manifolds. In Sect. 6.7, we present the classical stability analysis of the Yang–Mills equation as developed by Bourguignon and Lawson and, in Sect. 6.8, we discuss non-minimal solutions.

6.1 Gauge Fields. The Configuration Space

A classical pure Yang–Mills theory consists of the following structural elements.

(a) The theory is defined on a principal fibre bundle (P, M, G, Ψ, π) called the gauge principal bundle. Here, the base manifold M represents the spacetime and the

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structure group G plays the role of the gauge group. In the sequel, G will always be compact.

(b) A gauge potential mediating the fundamental interaction under consideration is given by a connection form ω on P and the field strength is given by the curvature form Ω of ω. These objects are related by the Structure Equation (1.4.9) and Ω fulfils the Bianchi identity (1.4.10),

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega], \quad D_\omega \Omega = 0 .$$

Any local section $s\colon U \to \pi^{-1}(U)$ provides a local representation of ω and Ω, respectively, in terms of objects on M,

$$\mathscr{A} = s^*\omega , \quad \mathscr{F} = s^*\Omega . \tag{6.1.1}$$

By Proposition 1.3.11, Corollary 1.3.12 and Remark 1.4.15/1, ω and Ω may be reconstructed from any system of local representatives $\mathscr{A}$ and $\mathscr{F}$ corresponding to a chosen bundle atlas of P.

(c) An active local gauge transformation is given by a vertical automorphism $\vartheta \in \mathrm{Aut}_M(P)$ with corresponding equivariant mapping $u \in \mathrm{Hom}_G(P, G)$,

$$\vartheta^*\omega = \mathrm{Ad}(u^{-1}) \circ \omega + u^*\theta , \quad \vartheta^*\Omega = \mathrm{Ad}(u^{-1}) \circ \Omega , \tag{6.1.2}$$

cf. Proposition 1.8.7 and Remark 1.8.8/1. Below, for simplicity, we will write

$$\vartheta^*\omega = \omega^{(u)} .$$

By Remark 1.8.8/2, for local representatives $\mathscr{A}$ and $\mathscr{F}$ of ω and Ω, respectively, one has

$$\mathscr{A}^{(\rho)} = \mathrm{Ad}(\rho^{-1}) \circ \mathscr{A} + \rho^*\theta , \quad \mathscr{F}^{(\rho)} = \mathrm{Ad}(\rho^{-1}) \circ \mathscr{F} , \tag{6.1.3}$$

where $\rho = u \circ s$. By (1.3.15) and (1.4.19), the latter formulae may also be interpreted passively, that is, as transformations corresponding to a change of a local trivialization of P.

Remark 6.1.1 Usually, in this book, local gauge potentials $\mathscr{A}$ are written down in 'geometrical units', that is, their components have the unit of inverse length. In physics, especially in quantum field theory, it is often relevant to make the coupling constant e of the gauge theory under consideration transparent. Moreover, physicists often choose a system of units where $c = 1 = \hbar$ and they prefer to work with Hermitean quantities. Then, the gauge potential $\mathscr{A}$ must be replaced by $ie\mathscr{A}$. We call the latter a physical representation and we will refer to it in some places. Note that, in this representation, not the physical representative $\mathscr{A}$ itself but $ie\mathscr{A}$ is the local representative of a connection form. Sometimes, the choice $c = 1 = \hbar$ is not convenient. Then, in the CGS system, $ie\mathscr{A}$ should be replaced by $\frac{ie}{\hbar c}\mathscr{A}$ and in the SI system it should be replaced by $\frac{ie}{\hbar}\mathscr{A}$, respectively. ♦

In the remainder of this section, we introduce the configuration space and we construct the action functional for Yang–Mills theory. For these purposes, we apply the notions and structures discussed in Sect. 2.7 to the case $E = \mathrm{Ad}(P)$, that is, we endow the adjoint bundle with the structure of a Riemannian vector bundle. To do so, from now on we assume:

1. the spacetime manifold M is endowed with a Riemannian or a pseudo-Riemannian metric g,
2. the Lie algebra $\mathfrak{g}$ of G carries an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.[1]

Then, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ induces via (2.6.4) a fibre metric on $\mathrm{Ad}(P)$ and, via formula (2.7.48), we have an L^2-inner product[2] on $\Omega^k(M, \mathrm{Ad}(P))$,

$$\langle\alpha, \beta\rangle_{L^2} = \int_M \alpha \dot{\wedge} * \beta\,. \tag{6.1.4}$$

Next, consider a connection form ω on P and its covariant exterior derivative $\mathrm{d}_\omega : \Omega^p(M, \mathrm{Ad}(P)) \to \Omega^{p+1}(M, \mathrm{Ad}(P))$, cf. Definition 1.5.1. Given the above L^2-structure, we may define the covariant exterior coderivative

$$\mathrm{d}_\omega^*\alpha : \Omega^{p+1}(M, \mathrm{Ad}(P)) \to \Omega^p(M, \mathrm{Ad}(P))$$

via (2.7.51),

$$\langle \mathrm{d}_\omega\alpha, \beta\rangle_{L^2} = \langle\alpha, \mathrm{d}_\omega^*\beta\rangle_{L^2}\,, \tag{6.1.5}$$

and the generalized Hodge-Laplacian, cf. (2.7.52),

$$\Box_\omega := \mathrm{d}_\omega \circ \mathrm{d}_\omega^* + \mathrm{d}_\omega^* \circ \mathrm{d}_\omega : \ \Omega^p(M, \mathrm{Ad}(P)) \to \Omega^p(M, \mathrm{Ad}(P))\,. \tag{6.1.6}$$

Now, let us discuss the configuration space of a Yang–Mills theory. By Remark 1.3.8, the set of connections $\mathscr{C}$ on a principal fibre bundle P carries the structure of an infinite-dimensional affine space with the corresponding translation vector space given by

$$\mathscr{T} = \Omega^1(M, \mathrm{Ad}(P)) \cong \Omega^1_{\mathrm{Ad,hor}}(P, \mathfrak{g})\,. \tag{6.1.7}$$

This space will be referred to as the classical configuration space of the gauge field theory under consideration. By point c) above, $\mathscr{C}$ is acted upon by the group of vertical

[1] Note that such an inner product always exists if G is compact. In that case, it may be obtained from any auxiliary inner product by averaging over the group with respect to the Haar measure. In many applications, the gauge group G is compact and semisimple. Then, for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ one can choose the negative of the Killing form k. Compactness implies that $-k$ is positive-definite, cf. Sect. 5.5 of Volume I.

[2] We must restrict ourselves to square integrable forms. In particular, we may consider forms with compact support.

automorphisms $\mathrm{Aut}_M(P)$. This group will be denoted by $\mathscr{G}$ and will be referred to as the group of local gauge transformations. Note that (6.1.2) defines a right action. If necessary, one can pass to a left action by viewing gauge transformations as mappings $\omega \mapsto (\vartheta^{-1})^*\omega$.

Remark 6.1.2 Depending on the context, $\mathscr{G}$ will be viewed as $\mathrm{Hom}_G(P, G)$ or, equivalenty, as the space of sections of the associated bundle $P \times_G G$, cf. Sect. 1.8. There is yet another useful point of view. Note that the adjoint action of G induces a bundle mapping

$$\Phi : P \times_G G \to \mathrm{End}(\mathrm{Ad}(P)), \quad \Phi([(p, g)])([(p, X)]) := [(p, \mathrm{Ad}(g)X)],$$

whose kernel coincides with the center of $\mathscr{G}$. Clearly, this definition does not depend on the choice of the representative of $[(p, X)] \in \mathrm{Ad}(P)$. This shows that local gauge transformations may be viewed as sections of the vector bundle $\mathrm{End}(\mathrm{Ad}(P))$. Then, (6.1.2) may be rewritten as follows:

$$\omega^{(u)} = \omega + u^{-1}\nabla_\omega u. \tag{6.1.8}$$

♦

In the sequel, for many purposes, it will be necessary to pass to a Sobolev completion of $\mathscr{C}$ and $\mathscr{G}$.[3] In this way, $\mathscr{C}$ will become an infinite-dimensional Hilbert manifold and $\mathscr{G}$ an infinite-dimensional Hilbert-Lie group. To be able to define such a completion, we assume that G be a compact connected linear Lie group. Moreover, in places where the Sobolev completion is essential, we will deal with the case of M being a compact connected orientable Riemannian manifold. So, we also make this assumption here. We stress, however, that Sobolev completions for noncompact manifolds exist as well, see the work of Eichhorn [178] and Eichhorn and Heber [179]. For any vector bundle E, let $W^k(E)$ denote the Hilbert space of cross sections of E of Sobolev class k, cf. (5.7.8). We denote

$$\Omega^p_k(M, \mathrm{Ad}(P)) := W^k\big(\textstyle\bigwedge^p \mathrm{T}^*M \otimes \mathrm{Ad}(P)\big).$$

These spaces are endowed with the natural L^2-inner product (6.1.4). In this Hilbert space setting, the translation vector space $\mathscr{T}$ is defined as

$$\mathscr{T} = \Omega^1_k(M, \mathrm{Ad}(P)) \tag{6.1.9}$$

and the configuration space $\mathscr{C}$ is defined as the completion with respect to the metric induced from the W^k-norm on $\mathscr{T}$. In this way, $\mathscr{C}$ becomes an affine Hilbert space with translation Hilbert space $\mathscr{T}$. In particular,

$$\mathrm{T}\mathscr{C} = \mathscr{C} \times \mathscr{T}. \tag{6.1.10}$$

[3]For basics of the theory of Sobolev spaces, we refer to Sect. 5.7.

Remark 6.1.3 In the sequel, as usual, the tangent space to $\mathscr{C}$ at a point $\omega \in \mathscr{C}$ will be identified with the translation vector space,

$$\mathrm{T}_\omega \mathscr{C} = \mathscr{T} . \tag{6.1.11}$$

However, one may also consider the affine tangent space $\omega + \mathscr{T}$. ♦

To turn $\mathscr{G}$ into a Hilbert Lie group, we choose an n such that $G \subset \mathfrak{gl}(n, \mathbb{C})$ and take the associated vector bundle

$$P \times_G \mathfrak{gl}(n, \mathbb{C}) ,$$

where G acts on $\mathfrak{gl}(n, \mathbb{C})$ by conjugation. Then, $P \times_G G$ is a vertical subbundle of $P \times_G \mathfrak{gl}(n, \mathbb{C})$ and, hence,

$$\Gamma^\infty(P \times_G G) \subset \Gamma^\infty(P \times_G \mathfrak{gl}(n, \mathbb{C})) .$$

By definition, $\mathscr{G}$ is the closure of $\Gamma^\infty(P \times_G G)$ in $W^{k+1}(P \times_G \mathfrak{gl}(n, \mathbb{C}))$.

We will assume $k > \dim(M)/2 + 1$. Then, the Sobolev Lemma 5.7.7 ensures that multiplication of a W^{k+1}-function by a W^l-function, where $\dim(M)/2 < l \leq k$, yields a W^l-function. It follows that $\mathscr{G}$ is a group, acting via (6.1.2) on $\mathscr{C}$. Note that the elements of $\mathscr{C}$ and $\mathscr{G}$ are of class C^1 and C^2, respectively. In particular, $\mathscr{G}$ may be viewed as consisting of vertical automorphisms of P of class C^2 or of sections of class C^2 of the associated bundles $P \times_G G$ or $\mathrm{End}(\mathrm{Ad}(P))$, respectively, cf. Remark 6.1.2. In fact, one can prove that $\mathscr{G}$ is a Hilbert-Lie group with Lie algebra

$$\mathrm{L}\mathscr{G} = W^{k+1}(\mathrm{Ad}(P)) \tag{6.1.12}$$

and exponential mapping

$$\exp_{\mathscr{G}}(\xi)(p) = \exp_G(\xi(p)) , \quad \xi \in \mathrm{L}\mathscr{G} ,\ p \in P , \tag{6.1.13}$$

and that the $\mathscr{G}$-action on $\mathscr{C}$ is smooth [455], [478], [591]. Many properties of finite-dimensional Lie groups carry over to infinite-dimensional Hilbert Lie groups, see [92].

Next, we extend the covariant exterior derivative d_ω to an operator

$$\mathrm{d}_\omega : \Omega^p_{k+1}(M, \mathrm{Ad}(P)) \to \Omega^{p+1}_k(M, \mathrm{Ad}(P)) ,$$

and its dual to

$$\mathrm{d}^*_\omega : \Omega^{p+1}_k(M, \mathrm{Ad}(P)) \to \Omega^p_{k-1}(M, \mathrm{Ad}(P)) .$$

Composition then yields bounded linear operators

$$\Delta_\omega := \mathrm{d}^*_\omega \circ \mathrm{d}_\omega : \quad \Omega^p_{k+1}(M, \mathrm{Ad}(P)) \to \Omega^p_{k-1}(M, \mathrm{Ad}(P)) , \tag{6.1.14}$$

and

$$\Box_\omega := \mathrm{d}_\omega \circ \mathrm{d}^*_\omega + \mathrm{d}^*_\omega \circ \mathrm{d}_\omega : \quad \Omega^p_{k+1}(M, \mathrm{Ad}(P)) \to \Omega^p_{k-1}(M, \mathrm{Ad}(P)) .$$

Note that the mapping

$$\mathscr{C} \to \mathrm{B}\big(\Omega^p_{k+1}(M, \mathrm{Ad}(P)), \Omega^{p+1}_k(M, \mathrm{Ad}(P)\big) , \quad \omega \mapsto \mathrm{d}_\omega ,$$

is continuous linear and, hence, smooth. The same is true for the mapping $\omega \mapsto \mathrm{d}^*_\omega$. Hence, the mappings

$$\omega \mapsto \Delta_\omega , \quad \omega \mapsto \Box_\omega ,$$

are continuous and smooth, because they factorize into continuous linear mappings. Moreover, we note the following equivariance properties:

$$O_{\omega^{(u)}} = \mathrm{Ad}(u^{-1}) \circ O_\omega \circ \mathrm{Ad}(u) , \quad \omega \in \mathscr{C}, u \in \mathscr{G} , \tag{6.1.15}$$

where O stands for, respectively, d, d^*, Δ and $\Box$.

In sharp contrast to Maxwell theory, in a Yang–Mills theory the configuration space $\mathscr{C}$ acquires a nontrivial stratified structure under the action of $\mathscr{G}$. This structure will be investigated in detail in Chap. 8. As we know from Part I, the orbit types constituting the stratification are labeled by conjugacy classes of stabilizers of the group action. Thus, let us find the stabilizer

$$\mathscr{G}_\omega := \{u \in \mathscr{G} : \omega^{(u)} = \omega\}$$

of $\omega \in \mathscr{C}$ with respect to the action of $\mathscr{G}$. It turns out that $\mathscr{G}_\omega$ is determined by the holonomy of ω. Thus, recall the definitions[4] of the holonomy group $\mathscr{H}_{p_0}(\omega)$ and of the holonomy bundle $P_{p_0}(\omega)$ of a connection ω based at $p_0 \in P$. Note that, in the Sobolev context, $P_{p_0}(\omega)$ is a vertical subbundle of class C^2, because ω is C^1.

Lemma 6.1.4 *Let $p_0 \in P$ and $\omega \in \mathscr{C}$. Then, for $u \in \mathscr{G}$, one has $u \in \mathscr{G}_\omega$ iff the restriction of u to $P_{p_0}(\omega)$ is constant.*

Proof Let $\gamma : [0, 1] \to P$ be an ω-horizontal curve starting at p_0. Then,

(a) for every $u \in \mathscr{G}$, the curve $\vartheta_u \circ \gamma$ is $\omega^{(u)}$-horizontal and starts at $\vartheta_u(p_0)$,
(b) for every $g \in G$, the curve $\Psi_g \circ \gamma$ is ω-horizontal and starts at $\Psi_g(p_0)$.

First, let $u \in \mathscr{G}_\omega$. Then $\omega^{(u)} = \omega$ and hence $\vartheta_u \circ \gamma$ is ω-horizontal. By uniqueness of the horizontal lift, it must then coincide with the curve $\Psi_{u(p_0)} \circ \gamma$, because the latter is also ω-horizontal, starts at $\vartheta_u(p_0) = \Psi_{u(p_0)}(p_0)$ and projects to the same curve in M. Thus, for all t,

$$\Psi_{u(\gamma(t))}\big(\gamma(t)\big) = \vartheta_u \circ \gamma(t) = \Psi_{u(p_0)}\big(\gamma(t)\big)$$

[4] Cf. Definitions 1.7.6 and 1.7.13.

and hence $u(\gamma(t)) = u(p_0)$. This shows that u is constant on $P_{p_0}(\omega)$.

Conversely, if u is constant on $P_{p_0}(\omega)$, it is constant along all ω-horizontal curves γ starting at p_0. Then, $\vartheta_u \circ \gamma = \Psi_{u(p_0)} \circ \gamma$. It follows that $\omega^{(u)}$-horizontal curves are also ω-horizontal and vice versa. This implies $\omega^{(u)} = \omega$. ■

Theorem 6.1.5 (Stabilizer Theorem) *$\mathscr{G}_\omega$ is a compact Lie subgroup of $\mathscr{G}$ with Lie algebra given by*

$$\mathrm{L}\mathscr{G}_\omega = \ker(\nabla^\omega) = \{\xi \in \mathrm{L}\mathscr{G} : \xi_{\restriction P_{p_0}(\omega)} = \mathrm{const}\}\,. \tag{6.1.16}$$

$\mathscr{G}_\omega$ is isomorphic to $\mathrm{C}_G(\mathscr{H}_{p_0}(\omega))$, the centralizer of the holonomy group in G.

Proof By Lemma 6.1.4,

$$\mathscr{G}_\omega = \{u \in \mathscr{G} : u_{\restriction P_{p_0}(\omega)} = \mathrm{const}\}\,. \tag{6.1.17}$$

Let $\xi \in \mathrm{L}\mathscr{G}$. Then, $\nabla^\omega \xi = 0$ iff $\xi_{\restriction P_{p_0}(\omega)} = \mathrm{const}$, that is, iff $\exp_{\mathscr{G}}(\xi)_{\restriction P_{p_0}(\omega)} = \mathrm{const}$. The second equivalence follows from (6.1.13). Thus,

$$\exp_{\mathscr{G}}(\mathrm{L}\mathscr{G}) \cap \mathscr{G}_\omega = \exp_{\mathscr{G}}(\ker(\nabla^\omega))\,.$$

Since $\ker(\nabla^\omega)$ is a closed subspace of the Hilbert space $\mathrm{L}\mathscr{G}$, the right hand side is a submanifold of $\mathscr{G}$. Since the left hand side is a neighbourhood of the unit element of $\mathscr{G}_\omega$, it follows that $\mathscr{G}_\omega$ is a Lie subgroup of $\mathscr{G}$ with Lie algebra given by (6.1.16). The argument is analogous to the finite-dimensional case, see [92, Sect. III.1.3].

Next, consider the natural group homomorphism

$$\Phi_{p_0} : \mathscr{G} \to G\,, \quad u \mapsto u(p_0)\,.$$

Since, by our choice of k, convergence in W^{k+1} implies pointwise convergence, Φ_{p_0} is a continuous Lie group homomorphism and, hence, smooth. Due to (6.1.17), the restriction of Φ_{p_0} to the subgroup $\mathscr{G}_\omega$ is injective, hence, a Lie group isomorphism onto its image. The image is

$$\Phi_{p_0}(\mathscr{G}_\omega) = \mathrm{C}_G(\mathscr{H}_{p_0}(\omega))\,.$$

To see this, recall that $\mathscr{H}_{p_0}(\omega)$ is the structure group of $P_{p_0}(\omega)$. Thus, inclusion from left to right is due to equivariance of the elements of $\mathscr{G}$. For the converse inclusion it suffices to note that, for any $a \in \mathrm{C}_G(\mathscr{H}_{p_0}(\omega))$, the function on $P_{p_0}(\omega)$ with constant value a is equivariant and, hence, can be equivariantly prolonged to P, thus becoming an element of $\mathscr{G}_\omega$. ■

Remark 6.1.6 As an immediate consequence of the fact that $\mathscr{G}_\omega$ is an (embedded) Lie subgroup, the projection $\mathscr{G} \to \mathscr{G}/\mathscr{G}_\omega$ defines a locally trivial principal $\mathscr{G}_\omega$-bundle [92, Sect. 6.2.4]. ♦

Finally, we introduce the gauge orbit space $\mathscr{M}$. It is obtained from $\mathscr{C}$ by factorizing with respect to the group action (6.1.2):

$$\mathscr{M} := \mathscr{C}/\mathscr{G}\,.$$

At this stage, this is just a topological quotient. It will be equipped with additional structure later. Note that $\mathscr{M}$ is the space of classes of gauge equivalent potentials, the 'true' configuration space. In [476] it was shown that the mapping

$$\mathscr{C} \times \mathscr{G} \to \mathscr{C} \times \mathscr{C}\,, \quad (\omega, u) \mapsto (\omega, \omega^{(u)})\,,$$

is closed. Together with the compactness of the stabilizers, this implies the following, see Corollary I/6.3.3/3 or [93, III, Sect. 4].

Theorem 6.1.7 *The action of $\mathscr{G}$ on $\mathscr{C}$ is proper.*

This, in turn, has the following immediate consequences[5]:

(a) The orbits of the action of $\mathscr{G}$ on $\mathscr{C}$ are closed.
(b) The orbit space $\mathscr{M}$ is Hausdorff.

In the sequel, an orthogonal splitting of the tangent bundle into the vertical distribution $\mathfrak{V}$ spanned by the tangent spaces to the orbits and a horizontal complement $\mathfrak{H}$ will be of fundamental importance:

$$\mathrm{T}\mathscr{C} = \mathfrak{V} \oplus \mathfrak{H}\,. \tag{6.1.18}$$

This decomposition formula will be proved below. First, to calculate $\mathfrak{V}$, consider a smooth element $\xi \in \mathrm{L}\mathscr{G}$, the corresponding curve $t \mapsto \exp_{\mathscr{G}}(t\xi)$ and the curve

$$t \mapsto \gamma(t) := \exp_{\mathscr{G}}(-t\xi)\,\omega\,\exp_{\mathscr{G}}(t\xi) + \exp_{\mathscr{G}}(-t\xi)\,\mathrm{d}\,\exp_{\mathscr{G}}(-t\xi)\,, \tag{6.1.19}$$

on the gauge orbit through $\omega \in \mathscr{C}$. The tangent vector to this curve at ω is

$$\mathrm{d}\xi + [\omega, \xi] = \nabla^{\omega}\xi \in \Omega^1(M, \mathrm{Ad}(P))\,.$$

Thus, the tangent space to the orbit at ω coincides with the image $\nabla^{\omega}(\Omega^0(\mathrm{Ad}(P)))$. Clearly, this characterization carries over to the Sobolev completion

$$\nabla^{\omega} : W^{k+1}(\mathrm{Ad}(P)) \to W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P))\,.$$

This provides the following presentation of infinitesimal gauge transformations.

Remark 6.1.8 (Infinitesimal gauge transformations) Let $\xi \in \mathrm{L}\mathscr{G}$, take $t \mapsto \rho(t) = \exp(t\xi)$, insert it into (6.1.8) and differentiate with respect to t at $t = 0$. This yields

$$\omega^{(\xi)} = \omega + \nabla^{\omega}\xi\,. \tag{6.1.20}$$

[5] This is proved by the same arguments as in the proof of Proposition I/6.3.4.

Let η be the Maurer–Cartan form on $\mathscr{G}$. As in the finite-dimensional case, this is the left-invariant 1-form on $\mathscr{G}$ generated by the identity endomorphism of the Lie algebra, that is,

$$\eta_{\mathbb{1}} = \mathrm{id}_{\mathrm{L}\mathscr{G}} \ .$$

Then, for a left invariant vector field ξ_* on $\mathscr{G}$ generated by $\xi \in \mathrm{L}\mathscr{G}$, we have $\eta(\xi_*) = \xi$. We denote the differential on $\mathscr{C}$ by δ and its restriction to the orbits of $\mathscr{G}$ by $\hat{\delta}$. Then, $\hat{\delta}\omega(\xi_*) = \omega^{(\xi)} - \omega$ and we obtain

$$\hat{\delta}\omega(\xi_*) = \nabla^{\omega}\xi = \nabla^{\omega}\eta(\xi_*) \,,$$

and, thus,

$$\hat{\delta}\omega = \nabla^{\omega} \circ \eta \,.$$

♦

To find $\mathfrak{H}_{\omega}$, consider the Laplace operator Δ_{ω} given by (6.1.14) acting on zero-forms,

$$\Delta_{\omega} = \nabla^{\omega *} \circ \nabla^{\omega} : \quad W^{k+1}(\mathrm{Ad}(P)) \to W^{k-1}(\mathrm{Ad}(P)) \,.$$

Recall that it is elliptic and that, by elliptic regularity,

$$\ker(\Delta_{\omega}) \subset \Gamma^{\infty}(\mathrm{Ad}(P)) \,.$$

Moreover, applying the Hodge Theorem 5.7.18 to the case of 0-forms, we obtain

$$W^{k-1}(\mathrm{Ad}(P)) = \ker(\Delta_{\omega}) \oplus \mathrm{im}(\Delta_{\omega}) \,. \tag{6.1.21}$$

By Remark 5.7.19, the orthogonal projectors onto $\mathrm{im}(\Delta_{\omega})$ and $\ker(\Delta_{\omega})$ are given by

$$\Delta_{\omega}\mathrm{G}_{\omega} \,, \quad \mathbb{1} - \Delta_{\omega}\mathrm{G}_{\omega} \,, \tag{6.1.22}$$

respectively. Here, G_{ω} is the Green's operator (5.7.34) of Δ_{ω}. In addition, since $\mathrm{im}(\nabla^{\omega}) \perp \ker(\nabla^{\omega *})$,

$$\ker(\Delta_{\omega}) = \ker(\nabla^{\omega}) \,. \tag{6.1.23}$$

Moreover, since $\mathrm{im}(\Delta_{\omega}) \subset \mathrm{im}(\nabla^{\omega *})$ and $\mathrm{im}(\nabla^{\omega *}) \perp \ker(\nabla^{\omega})$, the decomposition (6.1.21) implies

$$\mathrm{im}(\Delta_{\omega}) = \mathrm{im}(\nabla^{\omega *}) \,. \tag{6.1.24}$$

Finally, (6.1.23) and (6.1.24) imply

$$\nabla^{\omega}\mathrm{G}_{\omega}\Delta_{\omega} = \nabla^{\omega} \,, \quad \Delta_{\omega}\mathrm{G}_{\omega}\nabla^{\omega *} = \nabla^{\omega *} \,. \tag{6.1.25}$$

Theorem 6.1.9 *For every $\omega \in \mathscr{C}$, one has the L^2-orthogonal decomposition*

$$W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P)) = \mathrm{im}(\nabla^\omega) \oplus \ker(\nabla^{\omega *}) .$$

The orthogonal projectors onto $\mathrm{im}(\nabla^\omega)$ *and* $\ker(\nabla^{\omega *})$ *are given by*

$$\mathbf{v}_\omega = \nabla^\omega \mathrm{G}_\omega \nabla^{\omega *} , \quad \mathbf{h}_\omega = \mathrm{id} - \mathbf{v}_\omega , \tag{6.1.26}$$

respectively.

Proof We show that the bounded linear operator

$$\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *} : W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P)) \to W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P))$$

is the L^2-orthogonal projector onto the subspace $\mathrm{im}(\nabla^\omega)$ and

$$\ker(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) = \ker(\nabla^{\omega *}) .$$

Using (6.1.25), we obtain

$$\left(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}\right)^2 = \nabla^\omega \mathrm{G}_\omega \Delta_\omega \mathrm{G}_\omega \nabla^{\omega *} = \nabla^\omega \mathrm{G}_\omega \nabla^{\omega *} ,$$

that is, $\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}$ is a projector. As a consequence,

$$\mathrm{im}(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) = \ker(\mathbb{1} - \nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) ,$$

hence $\mathrm{im}(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *})$ is closed, and one has

$$W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P)) = \mathrm{im}(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) \oplus \ker(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) . \tag{6.1.27}$$

Since $\mathrm{G}_\omega = 0$ on $\ker(\Delta_\omega)$, the Hodge decomposition and (6.1.24) imply

$$\mathrm{im}(\mathrm{G}_\omega) = \mathrm{im}(\mathrm{G}_\omega \Delta_\omega) = \mathrm{im}(\mathrm{G}_\omega \nabla^{\omega *}) .$$

Since, in addition, $\mathrm{im}(\mathrm{G}_\omega) = \ker(\nabla^\omega)^\perp$, we conclude

$$\mathrm{im}(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) = \mathrm{im}(\nabla^\omega \mathrm{G}_\omega) = \mathrm{im}(\nabla^\omega) .$$

Since G_ω is injective on $\mathrm{im}(\nabla^{\omega *}) = \mathrm{im}(\Delta_\omega)$,

$$\ker(\nabla^\omega \mathrm{G}_\omega \nabla^{\omega *}) = \ker(\nabla^{\omega *}) .$$

Since $\mathrm{im}(\nabla^\omega)$ and $\ker(\nabla^{\omega *})$ are L^2-orthogonal, the assertion follows. ■

Remark 6.1.10 From Theorem 6.1.9, we conclude

$$\mathfrak{V}_\omega = \mathrm{im}(\nabla^\omega) , \quad \mathfrak{H}_\omega = \ker(\nabla^{\omega *}) . \tag{6.1.28}$$

Thus, by (6.1.15), the distributions $\mathfrak{V}$ and $\mathfrak{H}$ are equivariant,

$$\mathfrak{V}_{\omega^{(u)}} = (\mathfrak{V}_\omega)^{(u)}, \quad \mathfrak{H}_{\omega^{(u)}} = (\mathfrak{H}_\omega)^{(u)}.$$

Correspondingly, for any $u \in \mathscr{G}$,

$$\mathrm{G}_{\omega^{(u)}} = \mathrm{Ad}(u^{-1}) \circ \mathrm{G}_\omega \circ \mathrm{Ad}(u), \tag{6.1.29}$$

and, thus,

$$\mathbf{v}_{\omega^{(u)}} = \mathrm{Ad}(u^{-1}) \circ \mathbf{v}_\omega \circ \mathrm{Ad}(u), \quad \mathbf{h}_{\omega^{(u)}} = \mathrm{Ad}(u^{-1}) \circ \mathbf{h}_\omega \circ \mathrm{Ad}(u). \tag{6.1.30}$$

♦

6.2 The Yang–Mills Equation. Self-dual Connections

Now, we come to the dynamics of the Yang–Mills system. Typically, the dynamical equations for a model of classical field theory are obtained as the Euler–Lagrange equations of a variational principle for the physical action built from the fields. In a gauge theory, the action functional should be gauge invariant. At this point, the reader may wish to consult Chap. 4 of Volume I. In Sect. I/4.6 we have discussed the Maxwell equations in some detail. There, we have used the L^2-scalar product on the space of (square-integrable) 2-forms on Minkowski space M to construct an invariant 4-form (the Lagrangian) from the electromagnetic 2-form f,

$$L(A) = -\frac{1}{2} f \wedge * f$$

and to build the physical action $S(A) = \int_M L(A)$. Here, A is a gauge potential for f, that is, $f = \mathrm{d}A$.[6] The variational principle for this action yields the second group[7] of the (source-free) Maxwell equations in the vacuum,

$$\mathrm{d}^* f = 0.$$

We extend this to the Yang–Mills case. Using the L^2-scalar product on $\Omega^2(M, \mathrm{Ad}(P))$ given by (6.1.4), we define the following gauge invariant functional on the configuration space[8]:

$$S : \mathscr{C} \to \mathbb{R}, \quad S(\omega) = \frac{1}{2} \int_M \Omega \dot{\wedge} * \Omega. \tag{6.2.1}$$

[6] We have used the notation of Volume I here.

[7] The first group of Maxwell equations is of purely geometric character. It says that the 2-form f is closed. In terms of connection theory, this equation clearly coincides with the Bianchi identity.

[8] The factor $\frac{1}{2}$ is chosen according to the conventions used in physics.

This quantity will be referred to as the Yang–Mills action. Accordingly, the n-form $L(\omega) = \frac{1}{2}\Omega \dot{\wedge} * \Omega$ will be called the Lagrange density or, simply, the Lagrangian of the Yang–Mills theory.

Remark 6.2.1 Depending on the context, alternatively, we will write

$$S(\omega) = \frac{1}{2}\langle \Omega, \Omega \rangle_{L^2} = \frac{1}{2}\|\Omega\|^2 = \frac{1}{2}\int_M |\Omega|^2 \mathsf{v}_{\mathsf{g}}\,, \tag{6.2.2}$$

where $|\Omega|^2$ is defined by

$$\Omega \dot{\wedge} * \Omega = |\Omega|^2 \mathsf{v}_{\mathsf{g}}\,.$$

By formula (4.5.12) of Volume I, the sign of $|\Omega|^2$ depends on the signature of g. If g is Riemannian it is positive, on Minkowski space it is negative. ♦

Next, we derive the field equations of a pure Yang–Mills theory. First, recall that any connection fulfils the Bianchi identity $D_\omega \Omega = 0$, cf. Proposition 1.4.11. In the sequel, if not otherwise stated, we will view the curvature form Ω as an element of $\Omega^2(M, \mathrm{Ad}P)$. Then, the Bianchi identity takes the form

$$\mathrm{d}_\omega \Omega = 0\,. \tag{6.2.3}$$

This identity yields the first group of field equations of a Yang–Mills theory. For the Abelian case, $G = \mathrm{U}(1)$, it coincides with the first group of Maxwell's equations. We derive the second group of field equations by postulating a variational principle for the Yang–Mills action (6.2.1),

$$\delta S(\omega) = 0\,. \tag{6.2.4}$$

Let us derive the Euler–Lagrange equations corresponding to this variational principle. Since the configuration space $\mathscr{C}$ is an affine space with translation vector space $\mathscr{T}$, we have

$$\mathrm{T}_\omega \mathscr{C} = \mathscr{T}$$

and the derivative of S at ω in the direction of $\alpha \in \mathrm{T}_\omega \mathscr{C}$ is given by

$$\delta S_\omega(\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\restriction_0 S(\omega + t\alpha)\,.$$

By the Structure Equation, the curvature of the connection form $\omega + t\alpha$ is given by

$$\Omega_t = \Omega + t\mathrm{d}_\omega \alpha + \frac{t^2}{2}[\alpha, \alpha]\,. \tag{6.2.5}$$

Using this and (2.7.51), we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} S(\omega + t\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0} \left(\frac{1}{2}\langle \Omega_t, \Omega_t\rangle_{L^2}\right)$$
$$= \langle \Omega, \mathrm{d}_\omega \alpha\rangle_{L^2}$$
$$= \langle \mathrm{d}_\omega^* \Omega, \alpha\rangle_{L^2} .$$

Since the L^2-inner product is non-degenerate, we conclude that $\delta_\omega S = 0$ iff

$$\mathrm{d}_\omega^* \Omega = 0 . \tag{6.2.6}$$

This is the Euler–Lagrange equation of the above variational principle. It will be referred to as the (pure) Yang–Mills equation. Keeping in mind the analogy with Maxwell electrodynamics mentioned above, one may call (6.2.3) the first group and (6.2.6) the second group of Yang–Mills equations.

Definition 6.2.2 A solution to the Yang–Mills equation will be called a Yang–Mills connection.

Remark 6.2.3

1. In terms of local representatives $\mathscr{A}_\mu$ and $\mathscr{F}_{\mu\nu}$, Eq. (6.2.6) takes the form (Exercise 6.2.1)
$$\partial_\mu \mathscr{F}^{\mu\nu} + [\mathscr{A}_\mu, \mathscr{F}^{\mu\nu}] = 0 . \tag{6.2.7}$$
2. For the Abelian group $G = \mathrm{U}(1)$, we have $\mathfrak{g} = i\mathbb{R}$. In this case, all commutators vanish and we obtain $\mathrm{d}_\omega = \mathrm{d}$. Thus, (6.2.3) and (6.2.6) take the form
$$\mathrm{d}\Omega = 0 , \quad \mathrm{d}^* \Omega = 0 .$$
Writing f for the local representative of Ω, we obtain the (sourcefree) Maxwell equations
$$\mathrm{d}f = 0 , \quad \mathrm{d}^* f = 0$$
as a special case of the Yang–Mills equation.
3. One easily shows (Exercise 6.2.2) that a connection ω fulfils the Yang–Mills equation iff $\Box_\omega \Omega = 0$.

For the remainder of this section, we assume that M is a 4-dimensional oriented Riemannian manifold and that G is a compact connected Lie group. These assumptions have the following immediate consequences:

(a) Since G is compact, we may choose the $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ to be positive definite. Then, (6.1.4) defines a norm $\| \cdot \|$ on $\Omega^2(M, \mathrm{Ad}(P))$.
(b) Since M is 4-dimensional, by (2.8.17), we have the decomposition

$$\textstyle\bigwedge^2 \mathrm{T}^* M = \bigwedge^2{}_+ \mathrm{T}^* M \oplus \bigwedge^2{}_- \mathrm{T}^* M , \tag{6.2.8}$$

into the fibrewise eigenspaces of the Hodge star operator and a corresponding decomposition of the space of 2-forms $\Omega^2(M)$.

Clearly, the decomposition (6.2.8) extends to $\Omega^2(M, E)$ for any associated bundle E and persists for any Sobolev completion (under the assumptions made on G and M). Now, recall the notion of (anti-)self-duality from Sect. 2.8.

Definition 6.2.4 A connection form ω on a principal bundle $P(M, G)$ is called self-dual or anti-self-dual, if its curvature form $\Omega \in \Omega^2(M, \mathrm{Ad}(P))$ is self-dual or anti-self-dual, respectively.

Proposition 6.2.5 *Every self-dual or anti-self-dual connection is a Yang–Mills connection.*

Proof This is an immediate consequence of the Bianchi identity. ■

We show that the property of (anti-)self-duality is a conformal invariant.

Lemma 6.2.6 *The Hodge star operator on a Riemannian manifold* (M, g) *restricted to* 2*-forms is conformally invariant iff* $\dim M = 4$.

Proof Let $\dim M = k$ and let $\varphi : M \to M$ be a conformal transformation, that is, there exists a nowhere vanishing $f \in C^\infty(M)$ such that $\varphi^*\mathsf{g} = f^2\mathsf{g}$. Then, $\det(\varphi^*\mathsf{g}) = f^{2k} \det(\mathsf{g})$ and, thus, the volume forms are related by

$$\mathsf{v}_{\varphi^*\mathsf{g}} = f^k\, \mathsf{v}_{\mathsf{g}}\,.$$

On the other hand, for $\alpha \in \Omega^2(M)$, we have

$$(\varphi^*\mathsf{g})^{-1}(\alpha) = \frac{1}{f^4}\, \mathsf{g}^{-1}(\alpha)\,.$$

This implies

$$(\varphi^*\mathsf{g})^{-1}(\alpha) \lrcorner\, \mathsf{v}_{\varphi^*\mathsf{g}} = f^{k-4}\mathsf{g}^{-1}(\alpha) \lrcorner\, \mathsf{v}_{\mathsf{g}}\,,$$

that is, the star operators defined by g and by $\varphi^*\mathsf{g}$ coincide iff $k = 4$. ■

Note that this proof may be extended to conformal mappings between Riemannian manifolds (of dimension 4).

Proposition 6.2.7 *Let* (N, h) *and* (M, g) *be oriented* 4*-dimensional Riemannian manifolds and let* $\varphi : N \to M$ *be a conformal orientation preserving diffeomorphism. Let* (P, M, G, π) *be a principal fibre bundle. If* ω *is a self-dual (or anti-self-dual) connection on* P*, then the pullback of* ω *under* φ *is a self-dual (or anti-self-dual) connection on the pullback bundle* φ^*P.

Proof For clearness of presentation, in this proof, we denote the curvature form of ω, viewed as an element of $\Omega^2_{\mathrm{Ad,hor}}(P, \mathfrak{g})$ by $\tilde{\Omega}$ and, viewed as an element of $\Omega^2(M, \mathrm{Ad}(P))$, by $\overline{\Omega}$. Let us denote the Hodge star operators corresponding to h and g by $*_{\mathsf{h}}$ and $*_{\mathsf{g}}$, respectively, and let $\vartheta : \varphi^* P \to P$ be the natural principal bundle morphism projecting onto φ. By assumption, $*_{\mathsf{g}}\overline{\Omega} = \pm\overline{\Omega}$. We have to show that $\vartheta^*\tilde{\Omega}$ is (anti-)self-dual with respect to the metric h. For that purpose, for $y \in N$, $(y, p) \in \varphi^* P \subset N \times P$, $Y_1, Y_2 \in \mathrm{T}_y N$ and $Z_1, Z_2 \in \mathrm{T}_p P$ such that $\pi'(Z_i) = \varphi'(Y_i)$, we calculate

$$
\begin{aligned}
\overline{(\vartheta^*\tilde{\Omega})}_y(Y_1, Y_2) &= \iota_{(y,p)} \circ (\vartheta^*\tilde{\Omega})_{(y,p)}((Y_1, Z_1), (Y_2, Z_2)) \\
&= \iota_{(y,p)} \circ \tilde{\Omega}_p(\vartheta'(Y_1, Z_1), \vartheta'(Y_2, Z_2)) \\
&= \iota_{(y,p)} \circ \iota_p^{-1} \circ \overline{\Omega}_{\pi(p)}(\pi'(Z_1), \pi'(Z_2)) \\
&= \iota_{(y,p)} \circ \iota_p^{-1} \circ \overline{\Omega}_{\pi(p)}(\varphi'(Y_1), \varphi'(Y_2)) \\
&= \iota_{(y,p)} \circ \iota_p^{-1} \circ (\varphi^*\overline{\Omega})_y(Y_1, Y_2)\,,
\end{aligned}
$$

that is,

$$
\overline{(\vartheta^*\tilde{\Omega})}_y = \iota_{(y,p)} \circ \iota_p^{-1} \circ (\varphi^*\overline{\Omega})_y\,. \tag{6.2.9}
$$

Here, $\iota_p^{-1} : \mathrm{Ad}(P) \to \mathfrak{g}$ and $\iota_{(y,p)} : \mathfrak{g} \to \mathrm{Ad}(\varphi^* P)$ and the composition $\iota_{(y,p)} \circ \iota_p^{-1}$ is the fibre mapping of the bundle isomorphism $\varphi^*(\mathrm{Ad}(P)) \cong \mathrm{Ad}(\varphi^* P)$. Thus, for calculating the Hodge star of $\overline{(\vartheta^*\tilde{\Omega})}$ with respect to the metric h, it is enough to apply it to $\varphi^*\overline{\Omega}$. Using that φ is a conformal orientation preserving diffeomorphism, we obtain

$$
\begin{aligned}
\varphi^*(*_{\mathsf{g}}\overline{\Omega}) &= \varphi^*(\mathsf{g}^{-1}(\overline{\Omega}) \lrcorner\, \mathsf{v}_{\mathsf{g}}) \\
&= \big(\varphi_*^{-1} \circ \mathsf{g}^{-1}(\overline{\Omega})\big) \lrcorner\, \varphi^*\mathsf{v}_{\mathsf{g}} \\
&= \big((\varphi^* \circ \mathsf{g} \circ \varphi_*)^{-1}(\varphi^*\overline{\Omega})\big) \lrcorner\, \mathsf{v}_{\varphi^*\mathsf{g}}\,.
\end{aligned}
$$

But $\varphi^* \circ \mathsf{g} \circ \varphi_* : \mathfrak{X}(N) \to \Omega^1(N)$ is the isomorphism defined by the pullback metric $\varphi^*\mathsf{g}$. Using this and Lemma 6.2.6, we obtain

$$
\pm\varphi^*\overline{\Omega} = \varphi^*(*_{\mathsf{g}}\overline{\Omega}) = (\varphi^*\mathsf{g})^{-1}(\varphi^*\overline{\Omega}) \lrcorner\, \mathsf{v}_{\varphi^*\mathsf{g}} = *_{\mathsf{h}}\varphi^*\overline{\Omega}\,.
$$

■

Remark 6.2.8 From Proposition 6.2.7 we conclude that, in particular, (anti-)self-duality of a connection is a property which is invariant under gauge transformations.

Next, we will prove that (anti-)self-dual connections correspond to absolute minima of the Yang–Mills action. For that purpose, let us assume that G is compact and simple and, for the Ad-invariant scalar product on $\mathfrak{g}$, let us choose the negative of the Killing form,

$$\langle A, B\rangle_{\mathfrak{g}} = -\operatorname{tr}(\mathrm{ad}A \circ \mathrm{ad}B)\,,$$

cf. Sect. 5.4 of Volume I. Then,

$$\| \,\Omega\, \|^2 = -\int_M \operatorname{tr}(\mathrm{ad}\Omega \wedge *\mathrm{ad}\Omega)\,.$$

Recall from Chap. 4 the first Pontryagin class $\mathsf{p}_1(\mathrm{Ad}(P)) \in H^4_{\mathrm{dR}}(M)$ and the corresponding first Pontryagin index. By Corollary 4.6.17,

$$\mathfrak{p}_1(\mathrm{Ad}(P)) = \int_M \mathsf{p}_1(\mathrm{Ad}(P))\,.$$

Proposition 6.2.9 *Let G be a compact Lie group and let P be a principal G-bundle over a 4-dimensional oriented compact Riemannian manifold. Then, the following lower bound for the Yang–Mills action holds:*

$$S(\omega) \geq 4\pi^2 |\mathfrak{p}_1(\mathrm{Ad}(P))|\,.$$

Proof According to Example 4.6.22,

$$\mathsf{p}_1(\mathrm{Ad}(P)) = -\frac{1}{8\pi^2}\operatorname{tr}(\mathrm{ad}\Omega \wedge \mathrm{ad}\Omega)\,.$$

Decomposing the curvature form according to (2.8.8) as $\Omega = \Omega_+ + \Omega_-$, using (2.7.3) and the (anti-)self-duality of $\Omega_\pm$, and integrating over M, we obtain

$$\begin{aligned} 8\pi^2 \mathfrak{p}_1(\mathrm{Ad}(P)) &= \langle \Omega, *\Omega\rangle_{L^2} \\ &= \langle \Omega_+ + \Omega_-, \Omega_+ - \Omega_-\rangle_{L^2} \\ &= \| \,\Omega_+\, \|^2 - \| \,\Omega_-\, \|^2\,. \end{aligned} \tag{6.2.10}$$

On the other hand, we have

$$S(\omega) = \tfrac{1}{2} \| \,\Omega\, \|^2 = \tfrac{1}{2}\langle \Omega_+ + \Omega_-, \Omega_+ + \Omega_-\rangle_{L^2} = \tfrac{1}{2}(\| \,\Omega_+\, \|^2 + \| \,\Omega_-\, \|^2)\,. \tag{6.2.11}$$

Taking the sum and the difference of Eqs. (6.2.10) and (6.2.11), we obtain

$$-4\pi^2 \mathfrak{p}_1(\mathrm{Ad}(P)) + \| \,\Omega_+\, \|^2 = S(\omega) = 4\pi^2 \mathfrak{p}_1(\mathrm{Ad}(P)) + \| \,\Omega_-\, \|^2\,. \tag{6.2.12}$$

This yields the assertion. ■

Formula (6.2.12) implies the following corollary, which shows that (anti-)self-dual connections correspond to absolute minima of the Yang–Mills action.

Corollary 6.2.10 *For $\mathfrak{p}_1(\mathrm{Ad}(P)) > 0$, we have $S(\omega) \geq 4\pi^2\mathfrak{p}_1(\mathrm{Ad}(P))$ and equality if $\Omega_- = 0$, that is, if ω is self-dual. For $\mathfrak{p}_1(\mathrm{Ad}(P)) < 0$, we have $S(\omega) \geq -4\pi^2\mathfrak{p}_1(\mathrm{Ad}(P))$ and equality if $\Omega_+ = 0$, that is, if ω is anti-self-dual.* ■

From (6.2.10) we note that, for a self-dual connection, $\mathfrak{p}_1(\mathrm{Ad}(P)) > 0$. Correspondingly, for an anti-self-dual connection, we have $\mathfrak{p}_1(\mathrm{Ad}(P)) < 0$.

In the sequel, an (anti-)self-dual connection on a 4-dimensional Riemannian manifold will be called an (anti-)instanton. In the next sections, we will systematically discuss the theory of this important class of solutions.

Exercises

6.2.1 Prove formula (6.2.7).

6.2.2 Prove the statement of Remark 6.2.3/3.

6.3 The BPST Instanton Family

Here, we discuss the so-called BPST-(anti-)instantons, that is, the (anti-)self-dual solutions to the Yang–Mills equation on S^4 with instanton number ± 1 for the gauge group $\mathrm{Sp}(1)$. Here, BPST stands for Belavin, Polyakov, Schwartz and Tyupkin, see [64]. We describe these solutions in the bundle language, characterize them topologically and discuss their local description. Finally, we construct further (anti-)self-dual solutions by using the conformal symmetry of S^4. We use the notation of Examples 1.1.22, 1.1.24 and 1.3.22.

We will use the diffeomorphism $\mathrm{S}^4 \cong \mathbb{H}\mathrm{P}^1$ given by (B.1). To be consistent with standard formulae in gauge theory, we choose the orientation of $\mathbb{H}\mathrm{P}^1$ so that this diffeomorphism is compatible with the standard orientation of S^4, cf. Remark 4.5.4. Recall that the stereographic projection mappings (U_s, φ_s) and $(U_n, \overline{\varphi_n})$ constitute an oriented atlas of S^4. Choosing one of them, say φ_s, and extending it to a diffeomorphism

$$\varphi_s : \mathrm{S}^4 \cong \mathbb{H}\mathrm{P}^1 \to \mathbb{H} \cup \{\infty\} \tag{6.3.1}$$

by sending the southpole $-\mathbf{e}_0$ to $\{\infty\}$, one obtains a conformal identification.

Now, consider the block-diagonal embedding of the closed subgroup[9] $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$ and its action by right translations on $\mathrm{Sp}(2)$. Here, the first and the second component of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ are identified with the upper and lower diagonal block, respectively. By Example 1.1.4/4, this action defines a principal $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-bundle P over

$$\mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \cong G_{\mathbb{H}}(1,2) \cong \mathbb{H}\mathrm{P}^1 .$$

[9] Since we use the language of quaternions, we consistently write $\mathrm{Sp}(1)$. Recall that $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ as real Lie groups.

By Examples 5.2.11 and 5.4.9, $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is the spin group in four dimensions and P coincides with the spin structure $S(\mathrm{S}^4)$. Using the left actions

$$\sigma_{\mp} : (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1)\,, \quad \sigma_{\mp}(h)(g) := \lambda_{\mp}(h)g\,,$$

defined in Example 5.5.7, we build the following associated bundles:

$$P_- := P \times_{\mathrm{Sp}(1)\times\mathrm{Sp}(1),\sigma_-} \mathrm{Sp}(1)\,, \quad P_+ := P \times_{\mathrm{Sp}(1)\times\mathrm{Sp}(1),\sigma_+} \mathrm{Sp}(1)\,. \tag{6.3.2}$$

Clearly, both of these bundles are principal $\mathrm{Sp}(1)$-bundles over $\mathbb{H}\mathrm{P}^1$ with the right $\mathrm{Sp}(1)$-action given by right translation on the typical fibre $\mathrm{Sp}(1)$. The canonical projections in P and $P_{\mp}$ are denoted by π and $\pi_{\mp}$, respectively.

For our purposes, we need an explicit matrix description of these bundles. This is provided by the following remark.

Remark 6.3.1

1. We use the following parameterization of the Lie groups involved:

$$\mathrm{Sp}(2) = \left\{ \begin{bmatrix} \mathbf{q}_1 & \mathbf{p}_1 \\ \mathbf{q}_2 & \mathbf{p}_2 \end{bmatrix} : \|\mathbf{q}_1\|^2 + \|\mathbf{q}_2\|^2 = 1\,,\ \|\mathbf{p}_1\|^2 + \|\mathbf{p}_2\|^2 = 1\,,\ \overline{\mathbf{q}_1}\,\mathbf{p}_1 + \overline{\mathbf{q}_2}\,\mathbf{p}_2 = 0 \right\},$$

where $\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2 \in \mathbb{H}$. Then,

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) = \left\{ \begin{bmatrix} \mathbf{u}_1 & 0 \\ 0 & \mathbf{u}_2 \end{bmatrix} : \|\mathbf{u}_1\| = 1 = \|\mathbf{u}_2\|\,,\ \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{H} \right\}.$$

In this parameterization, the diffeomorphism (5.4.8) is given by

$$\mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \to \mathbb{H}\mathrm{P}^1\,, \quad \left[\begin{bmatrix} \mathbf{q}_1 & \mathbf{p}_1 \\ \mathbf{q}_2 & \mathbf{p}_2 \end{bmatrix}\right] \mapsto \left[\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}\right].$$

Now, using this formula, together with (B.2), and denoting $\varphi_s(\mathbf{z}) = \mathbf{x}$, we may write down useful (equivalent) representations of points on $\mathrm{S}^4 \setminus \{-\mathbf{e}_0\}$:

$$\mathbf{x} \mapsto (1 + \|\mathbf{x}\|^2)^{-\frac{1}{2}} \left[\begin{bmatrix} \mathbf{1} \\ \mathbf{x} \end{bmatrix}\right] \mapsto (1 + \|\mathbf{x}\|^2)^{-\frac{1}{2}} \left[\begin{bmatrix} \mathbf{1} & -\overline{\mathbf{x}} \\ \mathbf{x} & \mathbf{1} \end{bmatrix}\right]. \tag{6.3.3}$$

2. In the above parameterization, points of $P_{\mp}$ are represented as

$$[(k, \mathbf{u})]\,, \quad k = \begin{bmatrix} \mathbf{q}_1 & \mathbf{p}_1 \\ \mathbf{q}_2 & \mathbf{p}_2 \end{bmatrix} \in \mathrm{Sp}(2)\,,\ \mathbf{u} \in \mathrm{Sp}(1)\,,$$

with the defining equivalence relation given by

$$(k, \mathbf{u}) \sim (kh, \sigma_{\mp}(h^{-1})\mathbf{u})\,, \quad h = \begin{bmatrix} \mathbf{u}_1 & 0 \\ 0 & \mathbf{u}_2 \end{bmatrix} \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)\,.$$

Since the actions $\sigma_\mp$ are transitive, we may choose the following parameterizations of $[(k, \mathbf{u})]$:

$$\left(\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\\ \mathbf{q}_2 & \mathbf{p}_2\end{bmatrix}\begin{bmatrix}\mathbf{u} & 0\\ 0 & \mathbf{u}_2\end{bmatrix}, \mathbf{1}\right) = \left(\begin{bmatrix}\mathbf{q}_1\mathbf{u} & \mathbf{p}_1\\ \mathbf{q}_2\mathbf{u} & \mathbf{p}_2\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{u}_2\end{bmatrix}, \mathbf{1}\right) \quad \text{for} \quad P_-\,,$$

$$\left(\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\\ \mathbf{q}_2 & \mathbf{p}_2\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 & 0\\ 0 & \mathbf{u}\end{bmatrix}, \mathbf{1}\right) = \left(\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\mathbf{u}\\ \mathbf{q}_2 & \mathbf{p}_2\mathbf{u}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 & 0\\ 0 & \mathbf{1}\end{bmatrix}, \mathbf{1}\right) \quad \text{for} \quad P_+\,.$$

Thus, we may identify

$$P_- \to \mathrm{Sp}(2)/\lambda_+(\mathrm{Sp}(1)\times\mathrm{Sp}(1))\,, \quad [(k,\mathbf{u})] \mapsto \left[\begin{bmatrix}\mathbf{q}_1\mathbf{u} & \mathbf{p}_1\\ \mathbf{q}_2\mathbf{u} & \mathbf{p}_2\end{bmatrix}\right],$$

$$P_+ \to \mathrm{Sp}(2)/\lambda_-(\mathrm{Sp}(1)\times\mathrm{Sp}(1))\,, \quad [(k,\mathbf{u})] \mapsto \left[\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\mathbf{u}\\ \mathbf{q}_2 & \mathbf{p}_2\mathbf{u}\end{bmatrix}\right].$$

Clearly, these mappings define principal Sp(1)-bundle isomorphisms.
By Remark 1.1.25, P_- coincides with the Stiefel bundle $S_{\mathbb{H}}(1,2) \to G_{\mathbb{H}}(1,2)$ and, thus, with the quaternionic Hopf bundle $P_{\mathbb{H}}$. To make contact with the original definition of $P_{\mathbb{H}}$, given in Example 1.1.22, one easily shows (Exercise 6.3.2) that, in the above parameterization, elements of $S_{\mathbb{H}}(1,2) \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)$ may be represented as follows:

$$\begin{bmatrix}\mathbf{q}_1 & -\frac{\mathbf{q}_1\overline{\mathbf{q}_2}}{\|\mathbf{q}_1\|}\\ \mathbf{q}_2 & \|\mathbf{q}_1\|\end{bmatrix}, \quad \|\mathbf{q}_1\|^2 + \|\mathbf{q}_2\|^2 = 1\,, \tag{6.3.4}$$

that is, by elements $(\mathbf{q}_1, \mathbf{q}_2) \in S^7 \subset \mathbb{H}^2$. This describes the isomorphism (1.1.12) for $\mathbb{K} = \mathbb{H}$ and $n = 2$ explicitly.

3. In the parameterization given in point 1, we have a natural system $\{(U_{s,n}, \chi_{s,n})\}$ of local trivializations of P. In the standard notation $\chi_{s,n} = \pi \times \kappa_{s,n}$, it is given by

$$\kappa_s\left(\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\\ \mathbf{q}_2 & \mathbf{p}_2\end{bmatrix}\right) := \begin{bmatrix}\frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} & 0\\ 0 & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|}\end{bmatrix}, \ \kappa_n\left(\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\\ \mathbf{q}_2 & \mathbf{p}_2\end{bmatrix}\right) := \begin{bmatrix}\frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} & 0\\ 0 & \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|}\end{bmatrix}. \tag{6.3.5}$$

The corresponding transition mapping $\rho_{s,n} := \kappa_s \cdot \kappa_n^{-1} : U_s \cap U_n \to \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ reads

$$\rho_{s,n}\left(\pi\begin{bmatrix}\mathbf{q}_1 & \mathbf{p}_1\\ \mathbf{q}_2 & \mathbf{p}_2\end{bmatrix}\right) = \begin{bmatrix}\frac{\mathbf{q}_1\,\overline{\mathbf{q}_2}}{\|\mathbf{q}_1\|\|\mathbf{q}_2\|} & 0\\ 0 & \frac{\mathbf{p}_2\,\overline{\mathbf{p}_1}}{\|\mathbf{p}_2\|\|\mathbf{p}_1\|}\end{bmatrix}. \tag{6.3.6}$$

Clearly, $\{(U_{s,n}, \chi_{s,n})\}$ induces systems of local trivializations $\{(U_{s,n}, \chi^{\mp}_{s,n})\}$ in $P_\mp$.

♦

Next recall that, by Example 1.3.19, P carries a canonical connection ω^0 given by (1.3.17). Since for $k \in \mathrm{Sp}(2)$ we have $k^{-1} = \overline{k}$, in the above parameterization, formula (1.3.17) reads[10]

$$\omega^0 = \begin{bmatrix} \overline{\mathbf{q}_1}\,\mathrm{d}\mathbf{q}_1 + \overline{\mathbf{q}_2}\,\mathrm{d}\mathbf{q}_2 & 0 \\ 0 & \overline{\mathbf{p}_1}\,\mathrm{d}\mathbf{p}_1 + \overline{\mathbf{p}_2}\,\mathrm{d}\mathbf{p}_2 \end{bmatrix} . \tag{6.3.7}$$

By definition, ω^0 is invariant under left Sp(2)-translations. Clearly, ω^0 induces Sp(2)-invariant connection forms on the principal bundles $P_\mp$:

$$\omega^- = \overline{\mathbf{q}_1}\,\mathrm{d}\mathbf{q}_1 + \overline{\mathbf{q}_2}\,\mathrm{d}\mathbf{q}_2 \,, \quad \omega^+ = \overline{\mathbf{p}_1}\,\mathrm{d}\mathbf{p}_1 + \overline{\mathbf{p}_2}\,\mathrm{d}\mathbf{p}_2 \,. \tag{6.3.8}$$

Under the identification $P_- \cong P_{\mathbb{H}}$, ω^- coincides with the canonical connection of the quaternionic Hopf bundle, cf. formula (1.3.21). The above splitting of ω^0 has a deep geometric meaning which will be explained in Remark 6.5.10.

Proposition 6.3.2 *The connection forms ω^+ and ω^- are self-dual and anti-self-dual, respectively.*

Proof Since ω^0 is Sp(2)-invariant and since Sp(2) acts transitively on the bundle space, it is enough to prove (anti-)self-duality at one point of P_- and P_+, respectively. We choose the point corresponding to the unit element $\mathbb{1} \in \mathrm{Sp}(2)$. By the defining relations of Sp(2), at this point we have $\mathrm{d}\overline{\mathbf{q}_1} = -\mathrm{d}\mathbf{q}_1$ and $\mathrm{d}\overline{\mathbf{p}_2} = -\mathrm{d}\mathbf{p}_2$. Thus, by the Structure Equation, the curvature form of ω^0 at $\mathbb{1}$ reads

$$\Omega^0_{\mathbb{1}} = \begin{bmatrix} \mathrm{d}\overline{\mathbf{q}_2} \wedge \mathrm{d}\mathbf{q}_2 & 0 \\ 0 & \mathrm{d}\overline{\mathbf{p}_1} \wedge \mathrm{d}\mathbf{p}_1 \end{bmatrix} . \tag{6.3.9}$$

To find the local representative of $\Omega^0_{\mathbb{1}}$ at $\pi(\mathbb{1})$, we use the chart (U_s, φ_s). Then, by (B.2) and by the Local Reconstruction Formula (1.4.18),

$$\Omega^0_{\mathbb{1}} = (\pi^* \mathscr{F}^0_s)_{\mathbb{1}} = \big((\varphi_s \circ \pi)^* \circ (\varphi_s^{-1})^* \mathscr{F}^0_s\big)_{\mathbb{1}} \equiv \big((\varphi_s \circ \pi)^* \mathbb{F}^0_s\big)_{\mathbb{1}} \ ,$$

where $\mathscr{F}^0_s$ is the local representative of Ω^0 on U_s and $\mathbb{F}^0_s$ is its pullback under the chart mapping φ_s to $\varphi_s(U_s) = \mathbb{R}^4$. By (6.3.3), we obtain

$$\mathbb{F}^0_s(0) = \begin{bmatrix} \mathrm{d}\overline{\mathbf{x}} \wedge \mathrm{d}\mathbf{x} & 0 \\ 0 & \mathrm{d}\mathbf{x} \wedge \mathrm{d}\overline{\mathbf{x}} \end{bmatrix} .$$

We claim that the $\mathfrak{sp}(1)$-valued 2-forms $\mathbb{F}^+_s(0) = \mathrm{d}\mathbf{x} \wedge \mathrm{d}\overline{\mathbf{x}}$ and $\mathbb{F}^-_s(0) = \mathrm{d}\overline{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}$ are self-dual and anti-self-dual, respectively, with respect to the Euclidean metric on $\mathbb{R}^4$. In standard coordinates $\{x^i\}$ on $\mathbb{R}^4$, the action of the Hodge star operator on 2-forms is given by

[10] Since this expression is given in terms of the associative multiplication in $\mathbb{H}$, in the sequel it will be worthwhile to work with the associative exterior calculus, cf. Remark 1.4.8 /1.

$$*_{\mathbb{R}^4}\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) = \frac{1}{2}\varepsilon^{ij}{}_{kl}\,\mathrm{d}x^k \wedge \mathrm{d}x^l\,. \tag{6.3.10}$$

We calculate

$$\begin{aligned}\mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x} &= (\mathrm{d}x^1\mathbf{1} - \mathrm{d}x^2\mathbf{i} - \mathrm{d}x^3\mathbf{j} - \mathrm{d}x^4\mathbf{k}) \wedge (\mathrm{d}x^1\mathbf{1} + \mathrm{d}x^2\mathbf{i} + \mathrm{d}x^3\mathbf{j} + \mathrm{d}x^4\mathbf{k})\\ &= 2(\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4)\mathbf{i} + 2(\mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^4 \wedge \mathrm{d}x^2)\mathbf{j}\\ &\quad + 2(\mathrm{d}x^1 \wedge \mathrm{d}x^4 - \mathrm{d}x^2 \wedge \mathrm{d}x^3)\mathbf{k}\,.\end{aligned}$$

On the other hand, from (6.3.10), we read off

$$*_{\mathbb{R}^4}(\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4) = -(\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4)$$

and analogous formulae for the second and the third term. Thus,

$$*_{\mathbb{R}4}(\mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}) = -\mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x},$$

that is, $*_{\mathbb{R}^4}\mathbb{F}_s^-(0) = -\mathbb{F}_s^-(0)$. By Lemma B.1, φ_s is an orientation preserving conformal diffeomorphism from $U_s \subset \mathrm{S}^4$ to $\mathbb{R}^4$ and, thus, using Proposition 6.2.7 we obtain:

$$\mathscr{F}_s^- = \varphi_s^*\mathbb{F}_s^- = -\varphi_s^*\left(*_{\mathbb{R}^4}\mathbb{F}_s^-\right) = -*_{\mathrm{S}^4}\mathscr{F}_s^-\,.$$

In the same way, one shows that $\mathscr{F}_s^+$ is self-dual. ∎

Thus, the canonical connection on P yields both a self-dual and an anti-self-dual Yang–Mills connection. To make contact with the physics literature, let us describe these solutions in terms of their local representatives. We present the calculation for ω^- using the conformal identification (B.4). For clearness of presentation, in our notation we skip the stereographic projection mapping, thus, identifying the local representatives $\mathscr{A}_{s,n}^-$ of ω^- for the system of local trivializations $\{(U_{s,n}, \chi_{s,n}^-\}$ with their counterparts $\mathbb{A}_{s,n}^- := (\varphi_s^{-1})^*\mathscr{A}_{s,n}^-$ on $\mathbb{H} \cong \mathbb{R}^4$. By (6.3.5), the mapping $\kappa_s^- : \pi^{-1}(U_s) \to \mathrm{Sp}(1)$, associated with χ_s^-, is given by

$$\kappa_s^-(\mathbf{q}_1, \mathbf{q}_2) = \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|}\,.$$

Then, the local section σ_s, defined by κ_s^- via $\kappa_s^-(\sigma_s(\mathbf{x})) = \mathbb{1}$, reads as follows:

$$\sigma_s(\mathbf{x}) = \frac{1}{\sqrt{1 + \|\mathbf{x}\|^2}}\begin{bmatrix}\mathbf{1}\\ \mathbf{x}\end{bmatrix}.$$

Thus,

$$\sigma_s^*\omega^-(\mathbf{x}) = \frac{\mathbf{1}}{\sqrt{1+\|\mathbf{x}\|^2}}\,\mathrm{d}\left(\frac{\mathbf{1}}{\sqrt{1+\|\mathbf{x}\|^2}}\right) + \frac{\bar{\mathbf{x}}}{\sqrt{1+\|\mathbf{x}\|^2}}\,\mathrm{d}\left(\frac{\mathbf{x}}{\sqrt{1+\|\mathbf{x}\|^2}}\right).$$

Denoting $\mathbb{A}_s^-(\mathbf{x}) := \sigma_s^* \omega^-(\mathbf{x})$, we obtain

$$\mathbb{A}_s^-(\mathbf{x}) = \frac{1}{2} \frac{\bar{\mathbf{x}}\, \mathrm{d}\mathbf{x} - \mathrm{d}\bar{\mathbf{x}}\, \mathbf{x}}{1 + \|\mathbf{x}\|^2} \equiv \mathrm{Im} \left\{ \frac{\bar{\mathbf{x}}\, \mathrm{d}\mathbf{x}}{1 + \|\mathbf{x}\|^2} \right\} . \tag{6.3.11}$$

Next, let us calculate the local representative $\mathbb{F}_s^-$ of the curvature. By Remark 1.4.8/1, we have

$$\mathbb{F}_s^- = \mathrm{d}\mathbb{A}_s^- + \mathbb{A}_s^- \wedge \mathbb{A}_s^- .$$

Since $\mathrm{d}\bar{\mathbf{x}}\, \mathbf{x} + \bar{\mathbf{x}}\, \mathrm{d}\mathbf{x} = \mathrm{d}\|\mathbf{x}\|^2$, we may write

$$\mathbb{A}_s^-(\mathbf{x}) = \frac{\bar{\mathbf{x}}\, \mathrm{d}\mathbf{x}}{1 + \|\mathbf{x}\|^2} - \frac{1}{2} \frac{\mathrm{d}\|\mathbf{x}\|^2}{1 + \|\mathbf{x}\|^2} .$$

Using this, one easily calculates

$$\mathrm{d}\mathbb{A}_s^-(\mathbf{x}) = \frac{\mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}}{(1 + \|\mathbf{x}\|^2)^2} - \frac{\bar{\mathbf{x}}\mathrm{d}\mathbf{x} \wedge \bar{\mathbf{x}}\, \mathrm{d}\mathbf{x}}{(1 + \|\mathbf{x}\|^2)^2}$$

and

$$(\mathbb{A}_s^- \wedge \mathbb{A}_s^-)(\mathbf{x}) = \frac{\bar{\mathbf{x}}\, \mathrm{d}\mathbf{x} \wedge \bar{\mathbf{x}}\, \mathrm{d}\mathbf{x}}{(1 + \|\mathbf{x}\|^2)^2} .$$

Thus,

$$\mathbb{F}_s^-(\mathbf{x}) = \frac{\mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}}{(1 + \|\mathbf{x}\|^2)^2} . \tag{6.3.12}$$

Note that $\mathbb{F}_s^-(0) = \mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}$, indeed. A completely analogous calculation yields the local representative

$$\mathbb{A}_s^+(\mathbf{x}) = (\varphi_s^{-1})^* \mathscr{A}_s^+(\mathbf{x}) = \mathrm{Im} \left\{ \frac{\mathbf{x}\, \mathrm{d}\bar{\mathbf{x}}}{1 + \|\mathbf{x}\|^2} \right\} \tag{6.3.13}$$

of ω^+. Thus,

$$\mathbb{F}_s^+(\mathbf{x}) = \frac{\mathrm{d}\mathbf{x} \wedge \mathrm{d}\bar{\mathbf{x}}}{(1 + \|\mathbf{x}\|^2)^2} . \tag{6.3.14}$$

Remark 6.3.3

1. By Proposition 6.2.7, the $\mathfrak{sp}(1)$-valued 1-forms $\mathbb{A}_s^+$ and $\mathbb{A}_s^-$ may be viewed as the global representatives of a self-dual and an anti-self-dual connection form on the trivial principal $\mathrm{Sp}(1)$-bundles $(\varphi_s^{-1})^* P_+$ and $(\varphi_s^{-1})^* P_-$ over $\mathbb{R}^4$, respectively. The solutions (6.3.11) and (6.3.13) have first been found by Belavin, Polyakov, Schwartz and Tyupkin, see [64]. Therefore, they are called the BPST instanton and the BPST anti-instanton on $\mathbb{R}^4$, respectively. Correspondingly, the connection forms ω^+ and ω^- are called the BPST instanton and BPST anti-instanton on S^4,

respectively. In the mathematics literature, they are often referred to as the basic (anti-)instantons.

2. There is a fundamental theorem of K. Uhlenbeck [636] which states the following: let ω be a self-dual connection on a bundle P over $M \setminus \{m_1, \ldots, m_k\}$ such that its Yang–Mills action (6.2.1) is finite. Then, (P, ω) extends smoothly to M, that is, both the bundle and the connection extend smoothly across each of the points m_i. This result is usually referred to as the Removable Singularity Theorem. As an application, there is a natural one-to-one correspondence between self-dual connections over $\mathbb{R}^4$ having a finite action and self-dual connections on bundles over S^4. In the Euclidean context under consideration, it is reasonable to refer to the finite-action property as to finite energy. In the sequel, we adopt this terminology. ♦

Next, let us characterize the BPST (anti-)instanton on S^4 topologically. By Theorem 4.8.8, principal Sp(1)-bundles over S^4 are classified by their second Chern class. Thus, topologically, the BPST (anti-)instantons are completely characterized by the second Chern indices of the bundles P_- and P_+. By Remark 4.5.4, we have

$$\mathfrak{c}_2(P_-) = \int_{\mathrm{S}^4} \mathrm{c}_2(P_-) = 1\,, \quad \mathfrak{c}_2(P_+) = \int_{\mathrm{S}^4} \mathrm{c}_2(P_+)) = -1\,.$$

The following yields interesting additional insight: as a consequence of Theorem 1.1.11, principal bundles over S^n with connected structure group G are classified by elements of $\pi_{n-1}(G)$. After bringing the bundle to a normal form, this equivalence is provided by the restriction of one of the transition mappings, say $\rho_{s,n} : U_s \cap U_n \to G$, to the equator S^{n-1} of S^n. Thus, principal bundles with structure group $G = \mathrm{Sp}(1) \cong \mathrm{S}^3$ over S^4 are classified by elements of $\pi_3(\mathrm{Sp}(1))$, that is, by homotopy classes of mappings $\mathrm{S}^3 \to \mathrm{S}^3$. These, in turn, are labeled by their mapping degree. By (6.3.6) and (B.1), for the bundles P_- and P_+ we obtain

$$(\rho^-_{s,n})_{\restriction \mathrm{S}^3}(\mathbf{x}) = \overline{\mathbf{x}}\,, \quad (\rho^+_{s,n})_{\restriction \mathrm{S}^3}(\mathbf{x}) = \mathbf{x}\,. \tag{6.3.15}$$

The first mapping has degree -1 and the second one has degree $+1$ (Exercise 6.3.3).[11] Thus, up to the sign, the first Chern index and the mapping degree distinguishing an element of $\pi_3(\mathrm{Sp}(1))$ coincide.

Again, let us make contact with the description in terms of local representatives. We show how the above mapping degree characterizes the corresponding self-dual connections on $\mathbb{R}^4$ with finite energy. Let $\mathbb{A}$ be such a connection. Then, first, the finite energy requirement ensures that the curvature form $\mathbb{F}$ of $\mathbb{A}$ is square integrable. This implies that $\mathbb{F}$ must be asymptotically flat, that is, $\mathbb{F} \to 0$ for $\|\mathbf{x}\| \to \infty$. This, in turn, means that $\mathbb{A}$ must be asymptotically a pure gauge, $\mathbb{A} \mapsto g^{-1}dg$ for $\|\mathbf{x}\| \to \infty$. Clearly, the mapping g is, in general, only defined outside of a ball with radius $R > 0$ centered at 0. In general, it cannot be extended continuously to all of $\mathbb{R}^4$, because its restriction to $\mathrm{S}^3_R := \{\mathbf{x} \in \mathbb{R}^4 : \|\mathbf{x}\| = R\}$,

[11] Choosing, instead, the transition mapping $\rho_{n,s}$ results in a change of sign of these mapping degrees.

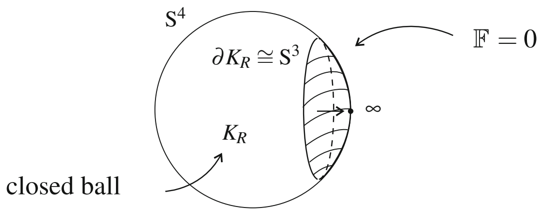

Fig. 6.1 The closed ball K_R in the proof of Proposition 6.3.4

$$g_{\restriction_{S^3_R}} : S^3_R \to \mathrm{Sp}(1) \cong S^3 , \tag{6.3.16}$$

may have a nontrivial mapping degree.

Proposition 6.3.4 *Let ω be a self-dual connection on a principal* $\mathrm{Sp}(1)$*-bundle P over* S^4 *and let $\mathbb{A}$ be its representative on $\mathbb{R}^4$ given by one of the stereographic projection mappings. Then, the degree of the mapping* (6.3.16) *characterizing $\mathbb{A}$ coincides, up to the sign, with the second Chern index of P.*

Proof Let Ω be the curvature form of ω and let $\mathbb{F}$ be its local representative with respect to the chosen stereographic projection mapping, say (U_s, φ_s).[12] We wish to express the second Chern index

$$\int_{S^4} \mathsf{c}_2(P) = \frac{1}{8\pi^2} \int_{S^4} \mathrm{tr}(\Omega \wedge \Omega)$$

in terms of the mapping degree characterizing $\mathbb{A}$. Clearly, $\mathbb{A}$ may be modified without changing the degree of the mapping (6.3.16) in such a way that $\mathbb{F}$ vanishes not only at infinity, but outside of a closed ball K_R of radius R and on its boundary $\partial K_R \cong S^3$, for sufficiently large R, see Fig. 6.1. As usual, the boundary ∂K_R is endowed with the orientation corresponding to the coorientation pointing outwards. Then,

$$\int_{S^4} \mathsf{c}_2(P) = \frac{1}{8\pi^2} \int_{K_R} \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) .$$

As a 4-form on a contractible subset of $\mathbb{R}^4$, $\mathrm{tr}(\mathbb{F} \wedge \mathbb{F})$ is closed and thus, by the Poincaré Lemma, exact. The following Lemma yields a potential.

Lemma 6.3.5 *The* 3*-form* $Q_3(\mathbb{A}) = \mathrm{tr}(\mathbb{A} \wedge \mathrm{d}\mathbb{A} + \frac{2}{3}\,\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})$ *fulfils*

$$\mathrm{d}Q_3(\mathbb{A}) = \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) .$$

The form Q_3 is called the Chern-Simons 3-form.[13]

---

[12] Then, under the identification $S^4 = \mathbb{R}^4 \cup \{\infty\}$, infinity corresponds to the south pole $-\mathbf{e}_0$.
[13] See [130].

Proof In the associative calculus, the Structure Equation yields

$$\mathbb{F} \wedge \mathbb{F} = \mathrm{d}\mathbb{A} \wedge \mathrm{d}\mathbb{A} + \mathrm{d}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} + \mathbb{A} \wedge \mathbb{A} \wedge \mathrm{d}\mathbb{A} + \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\,.$$

Using the cyclicity of the trace, we obtain

$$\mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}) = 0\,, \quad \mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathrm{d}\mathbb{A}) = \mathrm{tr}(\mathrm{d}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})\,.$$

Thus,

$$\mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = \mathrm{tr}(\mathrm{d}\mathbb{A} \wedge \mathrm{d}\mathbb{A}) + 2\,\mathrm{tr}(\mathrm{d}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})\,.$$

Since

$$\mathrm{tr}(\mathrm{d}\mathbb{A} \wedge \mathrm{d}\mathbb{A}) = \mathrm{d}(\mathrm{tr}(\mathbb{A} \wedge \mathrm{d}\mathbb{A}))\,,$$

and, again by the cyclicity of the trace,

$$\mathrm{d}(\mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})) = 3\,\mathrm{tr}(\mathrm{d}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})\,,$$

we obtain

$$\mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = \mathrm{d}(\mathrm{tr}(\mathbb{A} \wedge \mathrm{d}\mathbb{A})) + \frac{2}{3}\,\mathrm{d}(\mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})) = \mathrm{d}Q_3\,.$$

This proves the lemma.

We continue with the proof of the proposition. Using Lemma 6.3.5 and Stokes' Theorem, we obtain

$$\int_{K_R} \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = \int_{K_R} \mathrm{d}Q_3 = \int_{\partial K_R} Q_3\,. \tag{6.3.17}$$

Since $\mathbb{F}_{\restriction \partial K_R} = 0$, we have

$$\begin{aligned}\int_{\partial K_R} Q_3 &= \int_{\partial K_R} \mathrm{tr}\left(\mathbb{A} \wedge \mathrm{d}\mathbb{A} + \frac{2}{3}\,\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\right)\\ &= \int_{\partial K_R} \mathrm{tr}\left(\mathbb{A} \wedge (\mathbb{F} - \mathbb{A} \wedge \mathbb{A}) + \frac{2}{3}\,\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\right)\\ &= -\frac{1}{3} \int_{\partial K_R} \mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})\,,\end{aligned}$$

that is,

$$\int_{K_R} \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = -\frac{1}{3} \int_{\partial K_R} \mathrm{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A})\,. \tag{6.3.18}$$

Denoting $h = g_{\restriction \partial K_R}$ we have

$$\mathbb{A}_{\restriction \partial K_R} = h^{-1}\mathrm{d}h = h^*(\theta)\,,$$

where θ is the Maurer–Cartan form on Sp(1). Thus, using Remark I/4.3.6/4, we obtain

$$\int_{K_R} \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = -\frac{1}{3} \int_{\partial K_R} h^* \left(\mathrm{tr}(\theta \wedge \theta \wedge \theta)\right) = -\frac{1}{3}\, \deg(h) \int_{\mathrm{Sp}(1)} \mathrm{tr}(\theta \wedge \theta \wedge \theta)\,,$$

where $\deg(h)$ denotes the degree of the mapping $h : \partial K_R \cong \mathrm{S}^3 \to \mathrm{Sp}(1) \cong \mathrm{S}^3$. Finally, a simple calculation (Exercise 6.3.1) yields

$$\int_{\mathrm{Sp}(1)} \mathrm{tr}(\theta \wedge \theta \wedge \theta) = 24\pi^2\,. \tag{6.3.19}$$

Thus,

$$\int_{\mathrm{S}^4} \mathsf{c}_2(P) = \frac{1}{8\pi^2} \int_{K_R} \mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) = -\deg(h)\,.$$

■

Remark 6.3.6

1. In the sequel, the mapping degree $\deg(h)$ or, equivalently, minus the second Chern index of P will be called the instanton number. It will be denoted by k(P).
2. For the BPST (anti-)instanton on S^4, the statement of Proposition 6.3.4 can be seen by direct inspection. Consider ω^-. As above, let us represent infinity by the south pole $-\mathbf{e}_0$ and let us study the asymptotic behaviour of $\mathbb{A}_s^-$ given by (6.3.11) by taking the limit $\|\mathbf{x}\| \to \infty$:

$$\mathbb{A}_s^-(\mathbf{x}) \stackrel{\|\mathbf{x}\|\to\infty}{\longrightarrow} \left(\frac{\bar{\mathbf{x}}}{\|\mathbf{x}\|}\right)^{-1} \mathrm{d}\left(\frac{\bar{\mathbf{x}}}{\|\mathbf{x}\|}\right).$$

   Thus, the mapping (6.3.16) coincides with the restriction of the transition mapping ρ_s^- to the equator of S^4, cf. Eq. (6.3.15). ♦

In the remainder of this section we show how to construct further instanton solutions by using the conformal invariance of the equation $*\mathbb{F} = \pm\mathbb{F}$. By Appendix B, under the conformal identification, $\mathrm{S}^4 = \mathbb{H}\mathrm{P}^1 \cong \mathbb{H} \cup \{\infty\}$ the proper (that is, orientation preserving) conformal group of S^4 is given by

$$\mathrm{C}_0(\mathrm{S}^4, [\mathsf{g}_0]) = \mathrm{SL}(2, \mathbb{H})/\{\pm\mathbf{1}\}\,. \tag{6.3.20}$$

Clearly, its universal covering group is $\widetilde{\mathrm{C}}_0(\mathrm{S}^4, [\mathsf{g}_0]) = \mathrm{SL}(2, \mathbb{H})$. For concreteness, consider the canonical (anti-self-dual) solution ω^- on P_-. View P_- as the quaternionic Hopf bundle, cf. Remark 6.3.1/2.

Proposition 6.3.7 *The action of the conformal group of* S^4 *lifts naturally to an action of* $\mathrm{SL}(2,\mathbb{H})$ *on* P_- *by automorphisms.*

Proof It is easy to show (Exercise 6.3.4) that the mapping $\tilde{\Psi} : \mathrm{SL}(2,\mathbb{H}) \times \mathrm{S}^7 \to \mathrm{S}^7$ given by

$$\tilde{\Psi}\left(\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}, \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}\right) := \left(\| \mathbf{a}\mathbf{q}_1 + \mathbf{b}\mathbf{q}_2 \|^2 + \| \mathbf{c}\mathbf{q}_1 + \mathbf{d}\mathbf{q}_2 \|^2\right)^{-\frac{1}{2}} \begin{bmatrix} \mathbf{a}\mathbf{q}_1 + \mathbf{b}\mathbf{q}_2 \\ \mathbf{c}\mathbf{q}_1 + \mathbf{d}\mathbf{q}_2 \end{bmatrix}$$

defines a left smooth action on the bundle space $\mathrm{S}^7 \subset \mathbb{H}^2$. This mapping obviously commutes with the right principal action of $\mathrm{Sp}(1)$ and it projects onto the conformal action on S^4, cf. Appendix B. ∎

Clearly, the conformal group lifts to P_+ in the same way. Combining this proposition with Proposition 6.2.7, we conclude that $\tilde{\Psi}_k^* \omega^-$ is again an anti-self-dual connection form on P_-, for any $k \in \mathrm{SL}(2,\mathbb{H})$. On the other hand, by construction, ω^- is $\mathrm{Sp}(2)$-invariant and $\mathrm{Sp}(2) \subset \mathrm{SL}(2,\mathbb{H})$ is the full symmetry group of ω^-. Thus, the orbit of ω^- under the action of the conformal group is $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$. It turns out that, for $G = \mathrm{Sp}(1)$, all anti-instantons on S^4 with instanton number $\mathrm{k}(P) = -1$ are obtained in this way. This will be shown in Sect. 6.5.

To describe the family of anti-self-dual solutions obtained by conformal transformations explicitly, we need an explicit parameterization of the above homogeneous space. Since $\mathrm{Sp}(2)$ is the maximal compact subgroup of the semisimple Lie group $\mathrm{SL}(2,\mathbb{H})$, this is easily achieved by using the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{H})$. For convenience, we write it in the inverse order $\mathrm{SL}(2,\mathbb{H}) = NAK$, where

$$K = \mathrm{Sp}(2)\,, \quad A = \left\{ \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} : \lambda \in \mathbb{R}_+ \right\}, \quad N = \left\{ \begin{bmatrix} 1 & 0 \\ -\mathbf{s} & 1 \end{bmatrix} : \mathbf{s} \in \mathbb{H} \right\}.$$

Then, elements of $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$ are (globally) parameterized as follows:

$$\mathbb{R}_+ \times \mathbb{H} \to \mathfrak{M}_{-1} \cong \mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)\,, \quad (\lambda, \mathbf{s}) \mapsto \begin{bmatrix} \sqrt{\lambda} & 0 \\ -\sqrt{\lambda}\mathbf{s} & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \cdot \mathrm{Sp}(2)\,. \tag{6.3.21}$$

After putting $\mathbf{x}_0 = \lambda \mathbf{s}$, from (B.9) we read off the following family of conformal transformations

$$\mathbf{x} \mapsto \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}_0)\,. \tag{6.3.22}$$

Applying this transformation to (6.3.11), we obtain a 5-parameter family of anti-instantons with $\mathrm{k}(P) = -1$:

$$\mathbb{A}(\mathbf{x}; \lambda, \mathbf{x}_0) = \mathrm{Im}\left\{ \frac{\overline{(\mathbf{x} - \mathbf{x}_0)}\, \mathrm{d}\mathbf{x}}{\lambda^2 + \|\mathbf{x} - \mathbf{x}_0\|^2} \right\}. \tag{6.3.23}$$

Correspondingly, for the curvature we get

$$\mathbb{F}(\mathbf{x}; \lambda, \mathbf{x}_0) = \operatorname{Im}\left\{\frac{\lambda^2\, \mathrm{d}\bar{\mathbf{x}} \wedge \mathrm{d}\mathbf{x}}{(\lambda^2 + \|\mathbf{x} - \mathbf{x}_0\|^2)^2}\right\}. \tag{6.3.24}$$

Note that the curvature is centered at $\mathbf{x}_0$ and it is spread over a region of magnitude λ. Therefore, λ is called the scale and $\mathbf{x}_0$ is called the centre of the instanton.

In the same way, from (6.3.13), we may create a 5-parameter family of instantons with $\mathrm{k}(P) = 1$.

Remark 6.3.8 Over the years, the relevance of instantons in quantum field theory has been investigated. We refer to [569] for an introduction to this problem on a sound mathematical basis. The basic observation is that instantons interpolate between topologically inequivalent vacua of the quantum theory. This is often referred to as the tunneling effect. Here, we only explain the classical counterpart of this effect. Starting from a classical gauge potential $\mathbb{A}$ on Minkowski space, we choose a gauge such that $A_0 = 0$ and consider only static configurations, that is, configurations fulfilling $A_k = A_k(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$. A classical vacuum is characterized by $\mathbb{F}(\mathbf{x}) = 0$. Thus, the corresponding potential must be a pure gauge,

$$A_k(\mathbf{x}) = h^{-1}(\mathbf{x})\, \partial_k h(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^3$. We assume that the limit

$$\lim_{\mathbf{x}\to\infty} h(\mathbf{x}) = h_0$$

exists. Then, h may be extended to $\mathrm{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ and we obtain a classification of classical vacua in terms of the degree of the mapping $h\colon \mathrm{S}^3 \to \mathrm{Sp}(1) \cong \mathrm{S}^3$. Now, let $(\mathbf{x}, t) \mapsto g(\mathbf{x}, t)$ be the mapping obtained from the instanton asymptotics $A_\mu(\mathbf{x}, t) \sim g^{-1}(\mathbf{x}, t)\, \partial_\mu g(\mathbf{x}, t)$. By choosing an appropriate gauge, one can fulfil the following conditions

1. $g(\mathbf{x}, t) \to 1$ for $\mathbf{x} \in \mathbb{R}^3$ and $t \to -\infty$,
2. $g(\mathbf{x}, t) \to 1$ for $t \in \mathbb{R}$ and $\mathbf{x} \to \infty$,
3. $g(\mathbf{x}, t) \to h(\mathbf{x})$ for $t \to \infty$.

We see that g interpolates between $h \equiv 1$ und $h = h(\mathbf{x})$, that is, g interpolates between the classical vacua $A_k(\mathbf{x}) = 0$ and $A_k(\mathbf{x}) = h^{-1}(\mathbf{x})\, \partial_k h(\mathbf{x})$. ♦

Exercises

6.3.1 Prove formula (6.3.19).

6.3.2 Prove that $S_{\mathbb{H}}(1, 2)$ may be parameterized by the matrices given in formula (6.3.4).

6.3.3 Prove that the mapping $f : \mathrm{S}^3 \to \mathrm{S}^3$, $f(\mathbf{x}) = \bar{\mathbf{x}}$, has degree -1.

6.3.4 Prove that the mapping $\tilde{\Psi}$ defined in Proposition 6.3.7 is a smooth left action of $\mathrm{SL}(2, \mathbb{H})$ on S^7.

6.4 The ADHM Construction

In this section, we construct all (anti-)self-dual Sp(1)-connections on S^4 with arbitrary instanton number $\mathrm{k}(P)$. This construction goes back to Atiyah, Drinfeld, Hitchin and Manin [35] and is, therefore, called the ADHM construction. In our presentation we follow the strategy outlined at the end of Sect. II/3 of [30]. For that purpose, we recall from Theorem 3.4.10 that the quaternionic Stiefel bundle[14]

$$\pi^c : S_{\mathbb{H}}(1, k+1) \cong \mathrm{S}^{4k+3} \to G_{\mathbb{H}}(1, k+1) \cong \mathbb{H}\mathrm{P}^k$$

is k-classifying for the principal Sp(1)-bundles $P \to \mathrm{S}^4 \cong \mathbb{H}\mathrm{P}^1$. Now, the ADHM construction may be summarized as follows: take the canonical Sp(1)-connection[15]

$$\omega^c = \mathbf{q}^\dagger \mathrm{d}\mathbf{q} \tag{6.4.1}$$

on the quaternionic Stiefel bundle $S_{\mathbb{H}}(1, k+1)$ and pull it back via a family of classifying mappings $f : \mathrm{S}^4 \to \mathbb{H}\mathrm{P}^k$. If this family is suitable, this yields a family of (anti-)self-dual Sp(1)-connections on P. Here, $\mathbf{q} \in \mathrm{S}^{4k+3}$, that is,

$$\mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_k) \in \mathbb{H}^{k+1} \setminus \{0\}, \quad \| \mathbf{q} \| = 1 .$$

Recall that for a classifying mapping $f : \mathrm{S}^4 \to \mathbb{H}\mathrm{P}^k$, the pullback bundle is given by

$$P \equiv f^*(S_{\mathbb{H}}(1, k+1)) = \left\{([(\mathbf{x}_1, \mathbf{x}_2)], \mathbf{q}) \in \mathbb{H}\mathrm{P}^1 \times \mathrm{S}^{4k+3} : f([(\mathbf{x}_1, \mathbf{x}_2)]) = \pi^c(\mathbf{q})\right\} .$$

The pullback of ω^c reads

$$\omega = \mathbf{f}^* \omega^c = \bar{\mathbf{f}}\, \mathrm{d}\mathbf{f} , \tag{6.4.2}$$

with the bundle morphism

$$\mathbf{f} = \mathrm{pr}_2 \circ i_P , \tag{6.4.3}$$

where pr_2 is the projection onto the second factor in $\mathbb{H}\mathrm{P}^1 \times \mathrm{S}^{4k+3}$ and $i_P : P \to \mathbb{H}\mathrm{P}^1 \times \mathrm{S}^{4k+3}$ denotes the natural inclusion mapping. To summarize, we have the commutative diagram

[14] Cf. Example 1.1.24 and Remark 1.1.25.

[15] Cf. Example 1.3.19 for further details.

$$
\begin{tikzcd}
P \arrow[r, "\mathbf{f}"] \arrow[d, "\pi"'] & S^{4k+3} \arrow[d, "\pi^c"] \\
S^4 \arrow[r, "f"] & \mathbb{H}P^k
\end{tikzcd}
$$

Now, the basic idea of the authors of [35] was to consider the following smooth family of linear mappings

$$
\mathrm{v} : \mathbb{H}^2 \to L(\mathbb{H}^k, \mathbb{H}^{k+1}) , \quad \mathrm{v}(\mathbf{x}_1, \mathbf{x}_2) := C\mathbf{x}_1 + D\mathbf{x}_2 , \tag{6.4.4}
$$

where C and D are constant $((k+1)\times k)$-matrices with quaternionic entries, fulfilling

(a) $\mathrm{rank}_{\mathbb{H}} \mathrm{v}(\mathbf{x}_1, \mathbf{x}_2) = k$ for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{H}^2 \setminus \{0\}$,
(b) $\mathrm{v}^\dagger(\mathbf{x}_1, \mathbf{x}_2)\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2)$ is real for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{H}^2$.

By property (a), the image $\mathrm{im}(\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2))$ of the linear mapping

$$
\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2) : \mathbb{H}^k \to \mathbb{H}^{k+1} \tag{6.4.5}
$$

is a k-dimensional subspace of $\mathbb{H}^{k+1}$ which clearly depends on $[(\mathbf{x}_1, \mathbf{x}_2)] \in \mathbb{H}P^1$ only. Thus, it defines a vector subbundle

$$
E := \bigcup_{[(\mathbf{x}_1,\mathbf{x}_2)]\in\mathbb{H}P^1} \mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)
$$

of rank k of the trivial quaternionic vector bundle

$$
E_0 = \mathbb{H}P^1 \times \mathbb{H}^{k+1} \to \mathbb{H}P^1 .
$$

By construction, E is the direct sum of quaternionic line bundles, defined by the columns of v. Next, let $\mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)^\perp \cong \mathrm{coker}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)$ be the (one-dimensional) quaternionic orthogonal complement of $\mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)$ in $\mathbb{H}^{k+1}$. Clearly,

$$
L := \bigcup_{[(\mathbf{x}_1,\mathbf{x}_2)]\in\mathbb{H}P^1} \mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)^\perp \tag{6.4.6}
$$

is a vector subbundle of E_0 of rank 1, that is, L is a quaternionic line bundle over $\mathbb{H}P^1$. By construction, E and L are complementary in E_0,

$$
E_0 = E \oplus L .
$$

Let us denote the orthogonal projectors corresponding to this splitting by

$$
\mathbb{Q}[(\mathbf{x}_1, \mathbf{x}_2)] : \mathbb{H}^{k+1} \to \mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right) , \quad \mathbb{P}[(\mathbf{x}_1, \mathbf{x}_2)] : \mathbb{H}^{k+1} \to \mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)^\perp .
$$

Remark 6.4.1 By Example 1.2.9/2, L is associated with the bundle of orthonormal frames $O(L)$ of L. This is a principal Sp(1)-bundle over $\mathbb{H}\mathrm{P}^1$ whose fibre over $[(\mathbf{x}_1, \mathbf{x}_2)]$ may be identified with the vectors $\mathbf{e}([(\mathbf{x}_1, \mathbf{x}_2)]) \in \mathbb{H}^{k+1}$ fulfilling

$$\mathbf{e}([(\mathbf{x}_1, \mathbf{x}_2)])^\dagger \mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)]) = 0\,, \quad \mathbf{e}([(\mathbf{x}_1, \mathbf{x}_2)])^\dagger \mathbf{e}([(\mathbf{x}_1, \mathbf{x}_2)]) = 1\,. \tag{6.4.7}$$

♦

The mapping v defines a smooth classifying mapping

$$\mathrm{u} : \mathbb{H}\mathrm{P}^1 \to G_{\mathbb{H}}(1, k+1) \cong \mathbb{H}\mathrm{P}^k\,, \quad \mathrm{u}([(\mathbf{x}_1, \mathbf{x}_2)]) := \mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)^\perp.$$

According to the idea spelled out at the beginning, we take the pullback bundle $P = \mathrm{u}^*(S_{\mathbb{H}}(1, k+1))$ and the corresponding pullback of the canonical connection via the induced mapping $\mathbf{u} : P \to \mathrm{S}^{4k+3}$,

$$\omega = \mathbf{u}^* \omega^c = \mathbf{u}^\dagger\, \mathrm{d}\mathbf{u}\,. \tag{6.4.8}$$

Then, the curvature of ω is given by

$$\Omega = \mathrm{d}\mathbf{u}^\dagger \wedge \mathrm{d}\mathbf{u} + \mathbf{u}^\dagger \mathrm{d}\mathbf{u} \wedge \mathbf{u}^\dagger \mathrm{d}\mathbf{u}\,. \tag{6.4.9}$$

On the other hand, by definition of P, the elements $([(\mathbf{x}_1, \mathbf{x}_2)], \mathbf{q}) \in P$ are exactly those fulfilling $\mathbf{q} \in \mathrm{S}^{4k+3} \cap \left(\mathrm{im}\left(\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)])\right)^\perp\right)$, that is, $\mathbf{q}$ is an orthonormal frame in L. Thus, we have $P \cong O(L)$ and, consequently, an isomorphism

$$P \times_{\mathrm{Sp}(1)} \mathbb{H} \mapsto L\,, \quad \left[\left(([(\mathbf{x}_1, \mathbf{x}_2)], \mathbf{q}), \mathbf{a}\right)\right] \mapsto \left([(\mathbf{x}_1, \mathbf{x}_2)], \mathbf{q}\mathbf{a}\right). \tag{6.4.10}$$

By (6.4.3), under the identification $P \cong O(L)$ the mapping $\mathbf{u}$ becomes the identity onto its image, that is, it sends a point $p \in P$, viewed as an orthonormal frame $\mathbf{e}$ on L, onto itself as an element of S^{4k+3}. Thus, we can write

$$\omega_{\mathbf{e}} = \mathbf{e}^\dagger \mathrm{d}\mathbf{e}\,, \quad \Omega_{\mathbf{e}} = \mathrm{d}\mathbf{e}^\dagger \wedge \mathrm{d}\mathbf{e} + \mathbf{e}^\dagger \mathrm{d}\mathbf{e} \wedge \mathbf{e}^\dagger \mathrm{d}\mathbf{e}\,. \tag{6.4.11}$$

Moreover, under this identification, the projectors $\mathbb{Q}$ and $\mathbb{P}$ lift to orthogonal projection-valued mappings on $O(L)$,

$$\hat{\mathbb{Q}}(\mathbf{e}) = \mathbb{1} - \mathbf{e}\,\mathbf{e}^\dagger\,, \quad \hat{\mathbb{P}}(\mathbf{e}) = \mathbf{e}\,\mathbf{e}^\dagger\,. \tag{6.4.12}$$

In this picture, the covariant derivative defined by ω is given as follows. Using (1.2.11), (1.4.2) and the isomorphism (6.4.10), we obtain

$$(\nabla\Phi)(\pi(\mathbf{e})) = \mathbf{e}(\mathrm{d}\tilde{\Phi} + \mathbf{e}^\dagger \mathrm{d}\mathbf{e}\,\tilde{\Phi}) = \mathbf{e}\,\mathbf{e}^\dagger \mathrm{d}(\mathbf{e}\tilde{\Phi}) = \mathbb{P}\mathrm{d}\Phi\,.$$

Thus,

$$\nabla = \mathbb{P} \circ \mathrm{d} \,. \tag{6.4.13}$$

This formula has a nice geometric interpretation: we take the covariant derivative d in E_0, corresponding to the trivial flat connection, and project it onto L.

Lemma 6.4.2 *We have*

$$\Omega_{\mathbf{e}} = \mathbf{e}^\dagger(\hat{\mathbb{P}}\mathrm{d}\hat{\mathbb{P}} \wedge \mathrm{d}\hat{\mathbb{P}}\hat{\mathbb{P}})\,\mathbf{e}\,.$$

Proof Since $\mathbf{e}^\dagger\mathbf{e} = \mathbb{1}$, we get

$$(\mathrm{d}\mathbf{e}^\dagger)\,\mathbf{e} + \mathbf{e}^\dagger\mathrm{d}\mathbf{e} = 0\,.$$

Using this, we calculate

$$\begin{aligned}\hat{\mathbb{P}}\mathrm{d}\hat{\mathbb{P}} \wedge \mathrm{d}\hat{\mathbb{P}}\hat{\mathbb{P}} &= \mathbf{e}\left(\mathbf{e}^\dagger\mathrm{d}(\mathbf{e}\,\mathbf{e}^\dagger) \wedge \mathrm{d}(\mathbf{e}\,\mathbf{e}^\dagger)\mathbf{e}\right)\mathbf{e}^\dagger\\ &= \mathbf{e}\left(\mathbf{e}^\dagger\mathrm{d}\mathbf{e} \wedge \mathbf{e}^\dagger\mathrm{d}\mathbf{e} + \mathrm{d}\mathbf{e}^\dagger \wedge \mathrm{d}\mathbf{e} + (\mathrm{d}\mathbf{e}^\dagger)\mathbf{e} \wedge (\mathrm{d}\mathbf{e}^\dagger)\mathbf{e} + \mathbf{e}^\dagger\mathrm{d}\mathbf{e} \wedge (\mathrm{d}\mathbf{e}^\dagger)\mathbf{e}\right)\mathbf{e}^\dagger\\ &= \mathbf{e}\left(\mathbf{e}^\dagger\mathrm{d}\mathbf{e} \wedge \mathbf{e}^\dagger\mathrm{d}\mathbf{e} + \mathrm{d}\mathbf{e}^\dagger \wedge \mathrm{d}\mathbf{e}\right)\mathbf{e}^\dagger\,.\end{aligned}$$

In view of (6.4.11), this yields the assertion. ■

Comparing with (1.5.13), we see that the curvature endomorphism form R^∇ acting on L associated with Ω is given by

$$\mathsf{R}^\nabla = \mathbb{P}\mathrm{d}\mathbb{P} \wedge \mathrm{d}\mathbb{P}\mathbb{P}\,. \tag{6.4.14}$$

The proof of the following proposition can be found in various (similar) versions in the literature, see [30], [138], [135] and further references therein.

Proposition 6.4.3 *The connection ω on P is self-dual and has the instanton number* $\mathrm{k}(P) = k$.

Proof By condition (b) above, the mapping $R : \mathbb{H}\mathrm{P}^1 \to \mathrm{Aut}(\mathbb{H}^k)$ defined by

$$R([(\mathbf{x}_1, \mathbf{x}_2)]) := \mathrm{v}^\dagger([(\mathbf{x}_1, \mathbf{x}_2)])\mathrm{v}([(\mathbf{x}_1, \mathbf{x}_2)]) \tag{6.4.15}$$

has real-valued entries, that is, the entries are proportional to $\mathbf{1} \in \mathbb{H}$. Now, by the first equation in (6.4.7), we have $\mathrm{v}^\dagger\mathbf{e} = 0$ and, thus, also $\mathrm{v}R^{-1}\mathrm{v}^\dagger\mathbf{e} = 0$. But, by (6.4.15),

$$\mathrm{v}R^{-1}\mathrm{v}^\dagger = \mathrm{v}R^{-1}\mathrm{v}^\dagger\mathrm{v}R^{-1}\mathrm{v}^\dagger\,,$$

that is, $\mathrm{v}R^{-1}\mathrm{v}^\dagger$ projects onto the subspace orthogonal to $\mathbf{e}$. Thus, it must coincide with $\mathbb{Q}$ and we obtain pointwise

$$\mathbb{1} - \mathbb{P} = \mathbb{Q} = \mathrm{v}R^{-1}\mathrm{v}^\dagger\,. \tag{6.4.16}$$

Calculating

$$\mathrm{d}\mathbb{P} = -\mathrm{d}\mathbb{Q} = -(\mathrm{dv})R^{-1}\mathrm{v}^\dagger - \mathrm{v}(\mathrm{d}R^{-1})\mathrm{v}^\dagger - \mathrm{v}R^{-1}\mathrm{dv}^\dagger ,$$

and, using $\mathbb{P}\mathrm{v} = 0$, we get

$$\mathbb{P}\mathrm{d}\mathbb{P} = -\mathbb{P}(\mathrm{dv})R^{-1}\mathrm{v}^\dagger .$$

Correspondingly,

$$\mathrm{d}\mathbb{P}\mathbb{P} = -\mathrm{v}R^{-1}(\mathrm{dv}^\dagger)\mathbb{P} .$$

Inserting these formulae into (6.4.14), we obtain

$$\mathsf{R}^\nabla = \mathbb{P}(\mathrm{dv})R^{-1} \wedge (\mathrm{dv}^\dagger)\mathbb{P} . \tag{6.4.17}$$

Now, under the conformal identification $\mathbb{H}\mathrm{P}^1 \cong \mathbb{H} \cup \{\infty\}$ given by the stereographic projection chart φ_s, elements $[(\mathbf{x}_1, \mathbf{x}_2)] \in \mathbb{H}\mathrm{P}^1 \setminus \{\infty\}$ are represented by the homogeneous coordinate $[(\mathbb{1}, \mathbf{x}_2)] \equiv \mathbf{x} \in \mathbb{H}$. This yields

$$\mathrm{v}(\mathbf{x}) = \mathrm{C} + \mathrm{D}\mathbf{x} \tag{6.4.18}$$

and, thus, $\mathrm{dv} = D\,\mathrm{d}\mathbf{x}$. Finally, using the fact that R commutes with dv, we see that R^∇ is proportional to $\mathrm{d}\mathbf{x} \wedge \mathrm{d}\bar{\mathbf{x}}$, that is, ω is self-dual.

The second statement follows immediately from the Whitney Sum Formula, cf. Theorem 4.3.2. In more detail, to prove that the second Chern class of L is equal to k, it is enough to show that the second Chern class of E is equal to $-k$. But, by construction, E is the direct sum of k quaternionic line bundles corresponding to the k column vectors of v. Each of them may be identified with the standard line bundle over $\mathbb{H}\mathrm{P}^1$ having the second Chern class -1. ■

Thus, the above construction yields instantons. We get anti-instantons, if we choose instead

$$\mathrm{v}(\mathbf{x}) = C + D\,\bar{\mathbf{x}} , \tag{6.4.19}$$

see also Remark 6.4.6 below.

Our next task is to count the number of independent solutions. For that purpose we bring v into a normal form. Without loss of generality, we may assume that v is given by (6.4.18).

Proposition 6.4.4 *The following transformations yield isomorphic bundles and, consequently, isomorphic self-dual connections:*

$$C \to QCK , \quad D \to QDK , \tag{6.4.20}$$

where $Q \in \mathrm{Sp}(k+1)$ *and* $K \in \mathrm{GL}(k, \mathbb{R})$. *Using these transformations,* $\mathbf{e} \to Q\mathbf{e}$ *and the mapping* v *can be brought to the following canonical form:*

$$\mathrm{v}(\mathbf{x}) = \begin{bmatrix} \lambda \\ B - \mathbf{x}\mathbb{1} \end{bmatrix}, \tag{6.4.21}$$

where λ *is a quaternionic* $(1 \times k)$*-vector and* B *is a quaternionic symmetric* $(k \times k)$*-matrix. The canonical form* (6.4.21) *is preserved by the transformations* (6.4.20) *fulfilling*

$$Q = \begin{bmatrix} \mathbf{a} & 0 \\ 0 & R \end{bmatrix}, \quad K = R^{\mathrm{T}}, \quad \textit{where } \mathbf{a} \in \mathrm{Sp}(1), \ R \in \mathrm{O}(k). \tag{6.4.22}$$

Proof First, restricting to constant matrices is necessary to respect the linear structure of the construction. Next, by the first equation of (6.4.7), v and $\mathbf{e}$ can be multiplied from the left by the same matrix Q only. The second equation in (6.4.7) implies that Q must belong to $\mathrm{Sp}(k+1)$. Finally, to preserve the reality condition (b), v can be multiplied only by a matrix $K \in \mathrm{GL}(k, \mathbb{R})$. To prove that these transformations yield isomorphic bundles and connections is a simple exercise left to the reader.

Next, we bring v to a normal form. First, the real symmetric matrix $D^\dagger D$ transforms under (6.4.20) to $K^T(D^\dagger D)K$. Thus, we can use K to diagonalize $D^\dagger D$ and afterwards rescale the diagonal entries to 1. This yields $D^\dagger D = \mathbb{1}_k$. Clearly, this condition is invariant under any transformation $D \mapsto DK$, with $K \in \mathrm{O}(k)$. Next, one easily shows (Exercise 6.4.1) that for any D fulfilling $D^\dagger D = \mathbb{1}_k$ there exists an element $Q \in \mathrm{Sp}(k+1)$ such that

$$D = Q^\dagger \begin{bmatrix} 0 \\ -\mathbb{1}_k \end{bmatrix}. \tag{6.4.23}$$

Moreover, writing

$$C = Q^\dagger \begin{bmatrix} \lambda \\ B \end{bmatrix},$$

where λ is a $(1 \times k)$- and B is a $(k \times k)$-block, we have $D^\dagger C = -B$. But, by condition (b), $D^\dagger C$ is symmetric and, thus, B must be symmetric, too. ■

The conditions (a) and (b) now read

(a) $\mathrm{rank}_{\mathbb{H}} \begin{bmatrix} \lambda \\ B - \mathbf{x}\mathbb{1} \end{bmatrix} = k$ for every $\mathbf{x} \in \mathbb{H}$,

(b) $\lambda^\dagger \lambda + B^\dagger B$ is real.

Let us denote

$$V_k := \left\{ \begin{bmatrix} \lambda \\ B \end{bmatrix} \in \mathbb{H}^{k \times (k+1)} : B = B^{\mathrm{T}}, \ \mathrm{rank}_{\mathbb{H}} \begin{bmatrix} \lambda \\ B - \mathbf{x}\mathbb{1} \end{bmatrix} = k, \ \lambda^\dagger \lambda + B^\dagger B \text{ real} \right\}.$$

Now, we can calculate the number of free parameters labelling the ADHM solutions modulo the transformations (6.4.20), that is, the number of free real parameters in $V_k/(\mathrm{Sp}(1) \times \mathrm{O}(k))$ with the action given by (6.4.22). Since the stabilizer of this action is

$$\{(\mathbb{1}, \mathbb{1}), (-\mathbb{1}, -\mathbb{1})\} \cong \mathbb{Z}_2\,,$$

this number is given by $\dim_{\mathbb{R}}(V_k) - \dim\big(\mathrm{Sp}(1) \times \mathrm{O}(k)\big)$. The vectors λ contain $4k$ real parameters and the matrices B, being symmetric, contain $4\frac{k(k+1)}{2}$ real parameters. Since the matrix $\lambda^\dagger\lambda + B^\dagger B$ is self-adjoint and has positive diagonal entries, the condition that it be real gives rise to $3\frac{k(k-1)}{2}$ independent equations. Finally, the property of maximal rank is generic. Thus, altogether, for the number of free real parameters we obtain

$$\left(4k + 4\frac{k(k+1)}{2} - 3\frac{k(k-1)}{2}\right) - \left(3 + \frac{k(k-1)}{2}\right) = 8k - 3\,. \tag{6.4.24}$$

Thus, we have the following.

Corollary 6.4.5 *The space $\mathfrak{M}_{\mathrm{k}}$ of* $\mathrm{Sp}(1)$*-instanton solutions on* S^4 *obtained via the ADHM construction may be identified with V_k factorized with respect to the free action of* $(\mathrm{Sp}(1) \times \mathrm{O}(k))/\mathbb{Z}_2$*. It is a smooth* $(8k-3)$*-dimensional manifold.*

Remark 6.4.6 It is obvious from the above presentation, that the ADHM-construction immediately generalizes to any $\mathrm{Sp}(n)$, $n > 1$. As already outlined in the original paper [35], it can be adapted to the case of the classical groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ as well, see also [162], [138], [99] and [442] for details. ♦

Now, we can solve the first equation in (6.4.7) for $\mathbf{e}$ and we can, in principle, find the explicit k-instanton solution. For that purpose, we parameterize the local section $\mathbf{x} \to \mathbf{e}(\mathbf{x})$ as follows:

$$\mathbf{e}(\mathbf{x}) = \frac{1}{\sqrt{\rho(\mathbf{x})}} \begin{bmatrix} -1 \\ U(\mathbf{x}) \end{bmatrix},$$

where U is a quaternionic $(k \times 1)$-block and $\rho(\mathbf{x}) = 1 + \|U(\mathbf{x})\|^2$. Then, the first equation of (6.4.7) implies

$$U(\mathbf{x}) = \left(\lambda(B - \mathbf{x}\mathbb{1})^{-1}\right)^\dagger\,. \tag{6.4.25}$$

Inserting this into the first equation of (6.4.11), we obtain the following k-instanton solution on $\mathbb{H} = \mathbb{R}^4$:

$$\mathbb{A}(\mathbf{x}) = \frac{\mathrm{Im}\left(U^\dagger(\mathbf{x})\mathrm{d}U(\mathbf{x})\right)}{1 + \|U(\mathbf{x})\|^2}\,. \tag{6.4.26}$$

Clearly, it may be difficult to calculate the inverse matrix $(B - \mathbf{x}\mathbb{1})^{-1}$ for large n explicitly. Moreover, we note that $\mathbf{e}$ and, thus, $\mathbb{A}$ may have apparent singularities. However, these singularities may be removed (shifted to infinity) by appropriate gauge transformations. Behind, there is a standard procedure in algebraic geometry, see e.g. [259]. For the case under consideration, see also [191], [244] and the examples below.

The calculations in the following examples are left to the reader (Exercise 6.4.2).

Example 6.4.7

1. For $k = 1$, denoting $B = \mathbf{x}_0$ and choosing λ to be a positive scalar, formula (6.4.26) yields

$$\mathbb{A}(\mathbf{x};\lambda,\mathbf{x}_0) = -\frac{\lambda^2}{(\lambda^2 + \|\mathbf{x}-\mathbf{x}_0\|^2)}\mathrm{Im}\left\{\mathrm{d}\overline{\mathbf{x}}\left(\overline{\mathbf{x}-\mathbf{x}_0}\right)^{-1}\right\}.$$

   The apparent singularity at $\mathbf{x} = \mathbf{x}_0$ may be removed by the gauge transformation $\mathbf{x} \to g(\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}_0}{\|\mathbf{x}-\mathbf{x}_0\|}$. The gauge transformed potential reads (Exercise 6.4.4):

$$\mathbb{A}(\mathbf{x};\lambda,\mathbf{x}_0) = \mathrm{Im}\left\{\frac{(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}\overline{\mathbf{x}}}{\lambda^2 + \|\mathbf{x}-\mathbf{x}_0\|^2}\right\}. \tag{6.4.27}$$

   This is the $k = +1$-counterpart of (6.3.23). Setting $\lambda = 1$ and $\mathbf{x}_0 = 0$ we get the BPST-instanton.
2. If we choose $B = \mathrm{diag}(\mathbf{x}_1,\ldots,\mathbf{x}_k)$, where $\mathbf{x}_0,\ldots,\mathbf{x}_k$ are distinct points in $\mathbb{H}$, and $\lambda = (\lambda_1\mathbf{1},\ldots,\lambda_k\mathbf{1})$, with $\lambda_i > 0$, then we obtain the 't Hooft multi instanton solutions [627] in the singular gauge:

$$\mathbb{A}(\mathbf{x};\lambda_i,\mathbf{x}_i) = -\sum_{i=1}^{k}\frac{\lambda_i^2}{\|\mathbf{x}-\mathbf{x}_i\|^4\rho(\mathbf{x})}\mathrm{Im}\left\{\mathrm{d}\overline{\mathbf{x}}(\mathbf{x}-\mathbf{x}_i)\right\}, \tag{6.4.28}$$

   where

$$\rho(\mathbf{x}) = 1 + \sum_{i=1}^{k}\frac{\lambda_i^2}{\|\mathbf{x}-\mathbf{x}_i\|^2}.$$

   Clearly, this is a $5k$-parameter family of self-dual solutions.
3. From the family of 't Hooft solutions one may generate further solutions via conformal transformations. This way, a $(5k+4)$-parameter family of solutions was obtained by Jackiw, Nohl and Rebbi [342]:

$$\mathbb{A}(\mathbf{x};\lambda_i,\mathbf{x}_i) = -\sum_{i=0}^{k}\frac{\lambda_i^2}{\|\mathbf{x}-\mathbf{x}_i\|^4\rho(\mathbf{x})}\mathrm{Im}\left\{\mathrm{d}\overline{\mathbf{x}}(\mathbf{x}-\mathbf{x}_i)\right\}, \tag{6.4.29}$$

   where $\mathbf{x}_0,\ldots,\mathbf{x}_k$ are distinct points in $\mathbb{H}$, $\lambda_0,\ldots,\lambda_k$ are positive numbers and

$$\rho(\mathbf{x}) = 1 + \sum_{i=0}^{k}\frac{\lambda_i^2}{\|\mathbf{x}-\mathbf{x}_i\|^2}.$$

♦

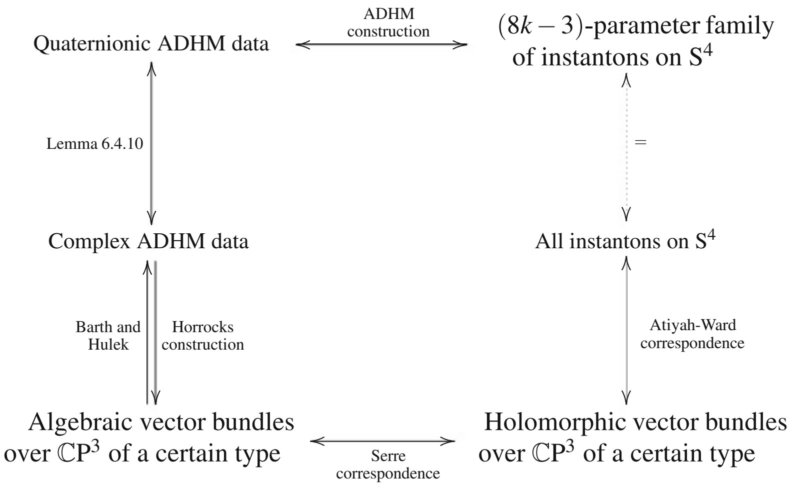

Fig. 6.2 Equivalences used in the proof that the ADHM construction yields all instantons on S^4

It turns out that $\mathfrak{M}_k$ is the full moduli space of k-instantons, that is, by the ADHM-construction, all instantons on S^4 are obtained. The proof of this fact rests on the following deep results:

1. One reformulates the ADHM-construction in terms of complex geometry on the twistor space $\mathbb{C}P^3$. Then, it appears as the Horrocks construction [311], [312] from algebraic geometry yielding algebraic[16] and, thus, holomorphic vector bundles over $\mathbb{C}P^3$ of a special type.
2. By the Atiyah–Ward correspondence, holomorphic vector bundles over $\mathbb{C}P^3$ of this type are in one-to-one correspondence with instantons on S^4, see [42], [37] and [30]. We also refer to [58] for a detailed proof.
3. Using results of Barth and Hulek [55–57], one shows that all algebraic vector bundles over $\mathbb{C}P^3$ of this special type are obtained via the Horrocks construction.

Figure 6.2 shows the logic of the proof schematically.
We explain points 1 and 2 in some detail. Point 3 is beyond the scope of this book. As before, we limit our attention to the gauge group $Sp(1) \cong SU(2)$. First, we need some algebraic preliminaries. As explained in Appendix A, we identify $\mathbb{C}$ with $\mathrm{span}\{\mathbf{1}, \mathbf{i}\} \subset \mathbb{H}$ and $\mathbb{H}$ with $\mathbb{C}^2$ by writing quaternions in the form $z_1 + \mathbf{j} z_2$, for any $z_1, z_2 \in \mathbb{C}$. This implies a complex isomorphism $\mathbb{H}^k \cong \mathbb{C}^k \oplus \mathbf{j}\,\mathbb{C}^k$ and, identifying $\mathbf{z}_1 + \mathbf{j}\,\mathbf{z}_2 = (\mathbf{z}_1, \mathbf{z}_2)$, we get

$$\mathbb{H}^k \cong \mathbb{C}^k \oplus \mathbf{j}\,\mathbb{C}^k \cong \mathbb{C}^{2k}\,. \tag{6.4.30}$$

[16] That is, the transition functions may be chosen to be rational functions of the complex coordinates.

Let us denote the standard scalar products on $\mathbb{H}^k$ and $\mathbb{C}^{2k}$ by k and h, respectively, and let us choose the following skew form on $\mathbb{C}^{2k}$:

$$\mathbb{J} = \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}. \tag{6.4.31}$$

We have

$$\mathsf{k}(\mathbf{q}_1, \mathbf{q}_2) = \mathbf{q}_1^\dagger \mathbf{q}_2 , \quad \mathsf{h}(\mathbf{z}, \mathbf{w}) = \mathbf{z}^\dagger \mathbf{w} , \quad \mathbb{J}(\mathbf{z}, \mathbf{w}) = \mathbf{z}^{\mathrm{T}} \mathbb{J} \mathbf{w} ,$$

where $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{z}$ and $\mathbf{w}$ are viewed as column vectors. These structures are related as follows:

$$\mathsf{k}(\mathbf{q}_1, \mathbf{q}_2) = \mathsf{h}(\mathbf{z}, \mathbf{w}) + \mathbf{j}\, \mathbb{J}(\mathbf{z}, \mathbf{w}) , \tag{6.4.32}$$

where $\mathbf{q}_1 = \mathbf{z}_1 + \mathbf{j}\, \mathbf{z}_2$, $\mathbf{q}_2 = \mathbf{w}_1 + \mathbf{j}\, \mathbf{w}_2$ and $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$, $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$.

Next, let $\sigma : \mathbb{H}^k \to \mathbb{H}^k$ be the complex anti-linear isomorphism induced from right multiplication on $\mathbb{H}^k$ by $\mathbf{j}$. Then, $\sigma^2 = -\operatorname{id}$ and, under the isomorphism (6.4.30),

$$\sigma(\mathbf{z}_1 + \mathbf{j}\, \mathbf{z}_2) = -\bar{\mathbf{z}}_2 + \mathbf{j}\, \bar{\mathbf{z}}_1 = (-\bar{\mathbf{z}}_2, \bar{\mathbf{z}}_1) . \tag{6.4.33}$$

Thus, in the above bases, viewing $\mathbf{z} \in \mathbb{C}^{2k}$ as a row vector,

$$\sigma(\mathbf{z}) = \bar{\mathbf{z}} \mathbb{J} . \tag{6.4.34}$$

Finally, we note that σ relates h and $\mathbb{J}$ as follows

$$\mathsf{h}(\sigma(\mathbf{z}), \mathbf{w}) = \mathbb{J}(\mathbf{z}, \mathbf{w}) . \tag{6.4.35}$$

Remark 6.4.8 In the sequel, given a complex vector space V, an anti-linear isomorphism $\sigma : V \to V$ fulfilling $\sigma^2 = \operatorname{id}$ or $\sigma^2 = -\operatorname{id}$ will be called, respectively, a real or a symplectic structure[17] of V, cf. also Sect. 5.3. ♦

Now let us consider the isomorphism (6.4.30) for $k = 2$, that is, $\mathbb{H}^2 \cong \mathbb{C}^4$ together with the corresponding right projective spaces $\mathbb{H}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^3$. Using the above conventions, for $\mathbf{z}, \mathbf{z}' \in \mathbb{C}^2$, we write $\mathbf{z} + \mathbf{j}\, \mathbf{z}' = (\mathbf{z}, \mathbf{z}')$ and thus, denoting $\mathbf{z} = (z_1, z_2)$ and $\mathbf{z}' = (z_3, z_4)$, elements of $\mathbb{C}^4$ are parameterized as follows:

$$(z_1 + \mathbf{j}\, z_3, z_2 + \mathbf{j}\, z_4) = (z_1, z_2, z_3, z_4) .$$

Clearly, any complex line is contained in a quaternionic line. Thus, we obtain the following fibre bundle

$$\pi : \mathbb{C}\mathrm{P}^3 \to \mathbb{H}\mathrm{P}^1 . \tag{6.4.36}$$

[17] This is the terminology of complex geometry. Instead, we could call σ a quaternionic structure in that case.

The mapping π is called the (projective) Penrose twistor transformation [508] for S^4 and the bundle is called the projective twistor bundle. Consider a quaternionic line and view it as a copy of $\mathbb{C}^2$. Then, all complex lines in it form a copy of $\mathbb{C}P^1$. Thus, the fibres of π are copies of $\mathbb{C}P^1$. In terms of the above coordinates, the projection π is given by

$$\pi([(z_1, z_2, z_3, z_4)]) = [(z_1 + \mathbf{j}\, z_3, z_2 + \mathbf{j}\, z_4)] \,. \tag{6.4.37}$$

The symplectic structure σ on $\mathbb{C}^4$ descends to a mapping of $\mathbb{C}P^3$, denoted by the same symbol,

$$\sigma : \mathbb{C}P^3 \to \mathbb{C}P^3 \,, \quad \sigma([(z_1, z_2, z_3, z_4)]) = [(-\overline{z_3}, -\overline{z_4}, \bar{z}_1, \bar{z}_2)] \,, \tag{6.4.38}$$

which is anti-linear in homogeneous coordinates and fulfils $\sigma^2 = \mathrm{id}$. It is common to call such a mapping a real structure on $\mathbb{C}P^3$. By definition, σ acts trivially on $\mathbb{H}P^1$ and, thus, it preserves the fibre structure.

Under the above identification $\mathbb{H}^2 \cong \mathbb{C}^4$, the natural action of $SL(2, \mathbb{H})$ on $\mathbb{H}^2$ descends to an action on $\mathbb{C}P^3$ preserving the fibration (6.4.36) and projecting onto the conformal action on $\mathbb{H}P^1$, see Appendix B. Thus, the maximal compact subgroup $Sp(2) \subset SL(2, \mathbb{H})$ acts transitively on $\mathbb{C}P^3$ preserving the natural metrics on $\mathbb{C}P^3$ and $\mathbb{H}P^1$. In coordinates, writing

$$[(z_1 + \mathbf{j}\, z_3, z_2 + \mathbf{j}\, z_4)] = [(\mathbf{1}, (z_2 + \mathbf{j}\, z_4)(z_1 + \mathbf{j}\, z_3)^{-1}] \equiv [(\mathbf{1}, \mathbf{x})] \,,$$

and calculating $(z_2 + \mathbf{j}\, z_4)\overline{(z_1 + \mathbf{j}\, z_3)} = (\overline{z_1} z_2 + z_3 \overline{z_4}) + \mathbf{j}\, (\overline{z_1} z_4 - z_3 \overline{z_2})$, we find the following presentation of the fibre $\pi^{-1}([(\mathbf{1}, \mathbf{x})])$ over $[(\mathbf{1}, \mathbf{x})]$: it consists of elements $[(z_1, \dots, z_4)] \in \mathbb{C}P^3$ fulfilling the conditions

$$\zeta = \frac{\overline{z_1} z_2 + z_3 \overline{z_4}}{|z_1|^2 + |z_3|^2} \,, \quad \xi = \frac{\overline{z_1} z_4 - z_3 \overline{z_2}}{|z_1|^2 + |z_3|^2} \,, \quad \mathbf{x} = \zeta + \mathbf{j} \xi \,. \tag{6.4.39}$$

Thus, for the points in the fibre $\pi^{-1}([(\mathbf{1}, 0)])$, we read off the stabilizer $U(1) \times Sp(1) \subset Sp(2)$. By a similar calculation, for the fibre $\pi^{-1}([(0, \mathbf{1})])$, we obtain the stabilizer $Sp(1) \times U(1)$. This yields the following presentations of $\mathbb{C}P^3$ as a homogeneous space:

$$\mathbb{C}P^3 \cong Sp(2)/(U(1) \times Sp(1)) \,, \quad \mathbb{C}P^3 \cong Sp(2)/(Sp(1) \times U(1)) \,. \tag{6.4.40}$$

Remark 6.4.9

1. By Example 5.4.9, $\mathbb{H}P^1 \cong S^4 \cong Sp(2)/(Sp(1) \times Sp(1))$. Thus, the homogeneous presentation (6.4.40) of $\mathbb{C}P^3$ explicitly shows that the fibres of π are copies of

$$Sp(1)/U(1) \cong \mathbb{C}P^1 \,.$$

Identifying $\mathbb{C}\mathrm{P}^1 \cong \mathrm{S}^2$ in the standard way[18] via $(\zeta, \xi) \mapsto (2\bar{\zeta}\xi, |\zeta|^2 - |\xi|^2)$, we read off that σ acts on the fibres via

$$\sigma(\zeta, \xi) \mapsto (-2\bar{\zeta}\xi, |\xi|^2 - |\zeta|^2),$$

that is, it sends any point to its antipodal point. Recall that a projective line in $\mathbb{C}\mathrm{P}^3$ is the image of a 2-dimensional subspace of $\mathbb{C}^4$. By definition, a projective line is said to be a real line if it is invariant under σ. Thus the fibres of π are exactly the real lines in $\mathbb{C}\mathrm{P}^3$ and S^4 may be viewed as the parameter space of the real lines.

2. Comparing with Example 5.5.14, we see that $\mathbb{C}\mathrm{P}^3$ may be naturally identified with the negative projective spinor bundle $\mathrm{P}^-(\mathrm{S}^4)$.[19] Via this identification, it obtains a natural complex structure such that the orientation induced on S^4 is opposite to the original orientation of S^4, cf. Remark 5.5.8.[20] In the sequel, we assume that $\mathbb{C}\mathrm{P}^3$ is endowed with this complex structure. ♦

Now, recall that the ADHM-data are given by mappings

$$\mathrm{v} : \mathbb{H}^2 \to L(\mathbb{H}^k, \mathbb{H}^{k+1}), \quad \mathrm{v}(\mathbf{x}_1, \mathbf{x}_2) := C\mathbf{x}_1 + D\mathbf{x}_2,$$

cf. (6.4.4), where C and D are constant $(k+1) \times k$-matrices with quaternionic entries, fulfilling conditions (a) and (b). Using (6.4.30) for $k = 2$, we may view v as a mapping

$$\mathrm{v} : \mathbb{C}^4 \to L(\mathbb{H}^k, \mathbb{H}^{k+1}).$$

Explicitly, writing $\mathbf{x}_1 = z_1 + \mathbf{j}\, z_3$ and $\mathbf{x}_2 = z_2 + \mathbf{j}\, z_4$, we obtain

$$\mathrm{v}(\mathbf{z}) = Cz_1 + C\,\mathbf{j}\,z_3 + Dz_2 + D\,\mathbf{j}\,z_4 \equiv \mathrm{v}_1 z_1 + \mathrm{v}_2 z_2 + \mathrm{v}_3 z_3 + \mathrm{v}_4 z_4,$$

with

$$C = \frac{1}{2}(\mathrm{v}_1 - \mathrm{v}_3\,\mathbf{j}), \quad D = \frac{1}{2}(\mathrm{v}_2 - \mathrm{v}_4\,\mathbf{j}), \tag{6.4.41}$$

and

$$\mathrm{v}_1 + \mathrm{v}_3\mathbf{j} = 0, \quad \mathrm{v}_2 + \mathrm{v}_4\mathbf{j} = 0. \tag{6.4.42}$$

Decomposing

$$\mathrm{v}_\alpha = A'_\alpha + \mathbf{j}A''_\alpha, \quad \alpha = 1, \ldots, 4, \tag{6.4.43}$$

into matrices with complex entries and building the $(2k+2) \times k$-matrices

---

[18] Cf. Remark 1.1.21.

[19] Clearly, it may also be identified with the positive projective spinor bundle.

[20] If we adopt this point of view, Theorem 4.1 of [37] cited in Remark 5.5.8 guarantees the integrability of the almost complex structure constructed there. Clearly, given the homogeneous presentation (6.4.40) one can define the almost complex structure in terms of the corresponding Lie algebra decomposition. Then, checking the integrability is a purely algebraic task, see [218] for details.

$$A_\alpha := \begin{bmatrix} A'_\alpha \\ A''_\alpha \end{bmatrix} ,$$

we obtain a mapping

$$A : \mathbb{C}^4 \to L(W, V) , \quad A(\mathbf{z}) = A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 , \tag{6.4.44}$$

where $W = \mathbb{C}^k$ and $V = \mathbb{C}^{2k+2}$. In this presentation, the conditions (6.4.42) take the form

$$\mathbb{J}\overline{A}_3 = A_1 , \quad \mathbb{J}\overline{A}_4 = A_2 , \tag{6.4.45}$$

where $\mathbb{J}$ is the skew form on V given by (6.4.31). We endow V with the symplectic structure σ given by (6.4.34) and W with the real structure given by complex conjugation.

Lemma 6.4.10 *The quaternionic ADHM mappings given by* (6.4.4) *and fulfilling conditions (a) and (b) are in one-to-one correspondence with mappings* (6.4.44) *fulfilling the following conditions:*

$$\sigma(A(\mathbf{z})\mathbf{w}) = A(\sigma(\mathbf{z}))\overline{\mathbf{w}} , \quad \mathbf{w} \in W , \tag{6.4.46}$$

$$\dim_{\mathbb{C}} (\operatorname{im} A(\mathbf{z})) = k , \quad \mathbf{z} \neq 0 , \tag{6.4.47}$$

$$A(\mathbf{z})^{\mathrm{T}} \mathbb{J} A(\mathbf{z}) = 0 . \tag{6.4.48}$$

Mappings A fulfilling the conditions (6.4.46)–(6.4.48) will be referred to as complex ADHM data.

Proof To show (6.4.46), using (6.4.34) and (6.4.45), we calculate

$$\sigma(A(\mathbf{z})\mathbf{w}) = -\mathbb{J}\overline{A(\mathbf{z})\mathbf{w}} = (A_3\overline{z_1} + A_4\overline{z_2} - A_1\overline{z_3} - A_2\overline{z_4})\overline{\mathbf{w}} = A(\sigma(\mathbf{z}))\overline{\mathbf{w}} .$$

Next, (6.4.47) is an immediate consequence of condition (a). Finally, we analyze condition (b). For that purpose, consider any pair (i, l) of columns of $\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2)$ and decompose them according to (6.4.43),

$$\big(\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2)\big)_i = \Big(\sum_\alpha A'_\alpha z_\alpha\Big)_i + \mathbf{j}\Big(\sum_\alpha A''_\alpha z_\alpha\Big)_i \equiv \mathbf{A}'_i(\mathbf{z}) + \mathbf{j}\,\mathbf{A}''_i(\mathbf{z}) ,$$

and $\big(\mathrm{v}(\mathbf{x}_1, \mathbf{x}_2)\big)_l$ correspondingly. Then,

$$\mathbf{A}_i(\mathbf{z}) = \begin{bmatrix} \mathbf{A}'_i(\mathbf{z}) \\ \mathbf{A}''_i(\mathbf{z}) \end{bmatrix} , \quad \mathbf{A}_l(\mathbf{z}) = \begin{bmatrix} \mathbf{A}'_l(\mathbf{z}) \\ \mathbf{A}''_l(\mathbf{z}) \end{bmatrix}$$

are, respectively, the i-th and the l-th columns of $A(\mathbf{z})$. Now, (6.4.32) implies

$$\mathsf{k}(\mathbf{A}_i, \mathbf{A}_l) = \mathsf{h}(\mathbf{A}_i, \mathbf{A}_l) + \mathbf{j}\,\mathbb{J}(\mathbf{A}_i, \mathbf{A}_l) ,$$

and the reality condition (b) yields

$$\mathbb{J}(\mathbf{A}_i, \mathbf{A}_l) = 0\,,$$

for any pair (i, l). This is equivalent to (6.4.48).

Inverting the above reformulation yields the converse statement. ■

Remark 6.4.11 Condition (6.4.47) is equivalent to the statement that $\operatorname{im} A(\mathbf{z})$ is an isotropic subspace of V with respect to $\mathbb{J}$,[21]

$$\operatorname{im} A(\mathbf{z}) \subset (\operatorname{im} A(\mathbf{z}))^{\mathbb{J}}\ , \quad \mathbf{z} \neq 0\,. \tag{6.4.49}$$

Here, $(\operatorname{im} A(\mathbf{z}))^{\mathbb{J}}$ is the $\mathbb{J}$-orthogonal complement. ♦

Definition 6.4.12 A holomorphic symplectic involution on a holomorphic vector bundle $\mathscr{L}$ over $\mathbb{C}\mathrm{P}^3$ is a holomorphic isomorphism[22] $\tau : \mathscr{L} \to \sigma^*\overline{\mathscr{L}}$ with $\tau^2 = -\operatorname{id}$, where $\tau^2 := \sigma^*(\overline{\tau}) \circ \tau$.

Remark 6.4.13 We explain the bundle $\sigma^*\overline{\mathscr{L}}$ in some detail. For the canonical covering $\{U_\alpha\}$ of $\mathbb{C}\mathrm{P}^3$ by homogeneous coordinates, defined by $U_\alpha := \{[\mathbf{z}] \in \mathbb{C}\mathrm{P}^3 : z_\alpha \neq 0\}$, we denote $\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 1$ and $\sigma(4) = 2$. Then, $\sigma^{-1}(U_\alpha) = U_{\sigma(\alpha)}$. Now, given a holomorphic cocycle $\{g_{\alpha\beta}\}$ associated with the covering $\{U_\alpha\}$, the bundle $\sigma^*\overline{\mathscr{L}}$ has the holomorphic cocycle $\{g^\sigma_{\alpha\beta}\}$ defined by

$$g^\sigma_{\alpha\beta} := \overline{g}_{\sigma(\alpha)\sigma(\beta)} \circ \sigma\,.$$

Correspondingly, there is an anti-linear bundle isomorphism $\mathscr{L} \cong \sigma^*\overline{\mathscr{L}}$. ♦

The following construction is due to Horrocks [311], [312].

Proposition 6.4.14 (Horrocks construction) *Any linear mapping $A : W \to V$, fulfilling the conditions* (6.4.46)–(6.4.48)*, gives rise to a holomorphic vector bundle $\mathscr{L}$ of rank 2 over $\mathbb{C}\mathrm{P}^3$ with the following properties.*

1. *$\mathscr{L}$ is holomorphically trivial over each fibre of π.*
2. *There exists a holomorphic symplectic involution on $\mathscr{L}$.*

A holomorphic vector bundle $\mathscr{L}$ over $\mathbb{C}\mathrm{P}^3$ with the properties 1 and 2 is usually referred to as an instanton bundle.

Proof Let there be given a linear mapping $A : W \to V$, fulfilling the conditions (6.4.46)–(6.4.48). Recall that V is endowed with the skew form $\mathbb{J}$ given by (6.4.31), with the symplectic structure σ and with the natural Hermitean form h, fulfilling the compatibility condition (6.4.35). Take the vector spaces

[21] Cf. Definition I/7.2.2.

[22] $\overline{\mathscr{L}}$ denotes the bundle conjugate to $\mathscr{L}$. Note that $\sigma^*\overline{\mathscr{L}}$ is a holomorphic bundle, because σ is anti-holomorphic, and $\sigma^*\overline{\sigma^*\overline{\mathscr{L}}} = \mathscr{L}$.

$$\mathscr{E}_{\mathbf{z}} := \operatorname{im} A(\mathbf{z}) , \quad \mathscr{E}^0_{\mathbf{z}} := (\operatorname{im} A(\mathbf{z}))^{\mathbb{J}} \tag{6.4.50}$$

and the quotient space

$$\mathscr{L}_{\mathbf{z}} := (\operatorname{im} A(\mathbf{z}))^{\mathbb{J}} / \operatorname{im} A(\mathbf{z}) . \tag{6.4.51}$$

Since $\dim_{\mathbb{C}} (\operatorname{im} A(\mathbf{z})) = k$, we have $\dim_{\mathbb{C}} (\operatorname{im} A(\mathbf{z}))^{\mathbb{J}} = k + 2$ and, thus, $\dim_{\mathbb{C}} \mathscr{L}_{\mathbf{z}} = 2$. Clearly, $\mathscr{L}_{\mathbf{z}}$ inherits a non-degenerate skew form from $\mathbb{J}$. By construction, the subspaces $\mathscr{E}_{\mathbf{z}}$, $\mathscr{E}^0_{\mathbf{z}}$ and $\mathscr{L}_{\mathbf{z}}$ depend on $[\mathbf{z}] \in \mathbb{C}\mathrm{P}^3$ only. Thus, the subspaces $\mathscr{E}_{[\mathbf{z}]}$ and $\mathscr{E}^0_{[\mathbf{z}]}$ combine to vertical subbundles

$$\mathscr{E} := \bigcup_{[\mathbf{z}] \in \mathbb{C}\mathrm{P}^3} \mathscr{E}_{[\mathbf{z}]} , \quad \mathscr{E}^0 := \bigcup_{[\mathbf{z}] \in \mathbb{C}\mathrm{P}^3} \mathscr{E}^0_{[\mathbf{z}]}$$

of the trivial holomorphic vector bundle $\underline{V} := \mathbb{C}\mathrm{P}^3 \times V$ endowed with the Hermitean fibre metric h and the skew form $\mathbb{J}$ inherited from V. Consequently, the quotient spaces $\mathscr{L}_{[\mathbf{z}]}$ combine to the quotient vector bundle

$$\mathscr{L} := \bigcup_{[\mathbf{z}] \in \mathbb{C}\mathrm{P}^3} \mathscr{L}_{[\mathbf{z}]} = \mathscr{E}^0 / \mathscr{E} . \tag{6.4.52}$$

We may identify $\mathscr{L}$ with the orthogonal complement of $\mathscr{E}$ in $\mathscr{E}^0 \subset \underline{V}$, which we also denote by $\mathscr{L}$. By general arguments [583], as an algebraic vector bundle, $\mathscr{L}$ carries a holomorphic structure.

Next, by (6.4.35), the orthogonal complement $\mathscr{E}^{\perp}_{[\mathbf{z}]}$ of $\mathscr{E}_{[\mathbf{z}]} \subset V$ coincides with $(\sigma(\mathscr{E}_{[\mathbf{z}]}))^0$ and by (6.4.46), we have

$$\sigma(\mathscr{E}_{[\mathbf{z}]}) = \mathscr{E}_{\sigma([\mathbf{z}])} . \tag{6.4.53}$$

Thus,

$$\mathscr{E}^0_{[\mathbf{z}]} = \mathscr{E}^{\perp}_{\sigma([\mathbf{z}])} , \tag{6.4.54}$$

and, by the positive definiteness of the inner product, $\mathscr{E}^0_{[\mathbf{z}]} \cap \mathscr{E}_{\sigma([\mathbf{z}])} = 0$. Thus,

$$V = \mathscr{E}_{\sigma([\mathbf{z}])} \oplus \mathscr{E}^{\perp}_{\sigma([\mathbf{z}])} = \mathscr{E}_{\sigma([\mathbf{z}])} \oplus \mathscr{E}^0_{[\mathbf{z}]}$$

and, viewing $\mathscr{L}_{[\mathbf{z}]}$ as the orthogonal complement of $\mathscr{E}_{[\mathbf{z}]}$ in $\mathscr{E}^0_{[\mathbf{z}]}$, we obtain the following orthogonal direct sum decomposition

$$V = \mathscr{E}_{[\mathbf{z}]} \oplus \mathscr{L}_{[\mathbf{z}]} \oplus \mathscr{E}_{\sigma([\mathbf{z}])} , \tag{6.4.55}$$

together with the corresponding splitting of the trivial bundle $\underline{V}$. Thus,

$$\mathscr{L}_{[\mathbf{z}]} = \{\mathbf{v} \in V : \mathsf{h}(\mathbf{v}, \mathbf{u}) = 0 , \ \mathbb{J}(\mathbf{v}, \mathbf{u}) = 0 , \ \text{for all } \mathbf{u} \in \operatorname{im}(A(\mathbf{z}))\} , \tag{6.4.56}$$

that is,

$$\mathscr{L}_{[\mathbf{z}]} = \mathscr{E}_{[\mathbf{z}]}^{\perp} \cap \mathscr{E}_{\sigma([\mathbf{z}])}^{\perp} = \mathscr{E}_{\sigma([\mathbf{z}])}^{0} \cap \mathscr{E}_{[\mathbf{z}]}^{0} .$$

We show that $\mathscr{L}_{[\mathbf{z}]}$ depends only on $\mathbf{x} = \pi([\mathbf{z}]) \in \mathbb{H}\mathrm{P}^1$, that is, on the fibre through $[\mathbf{z}]$. According to Remark 6.4.9, the latter coincides with the real line $l_{\mathbf{x}}$ through $[\mathbf{z}]$ and $\sigma([\mathbf{z}])$. Let $[\mathbf{w}]$ be any point on $l_{\mathbf{x}}$ and let $L_{[\mathbf{w}]}$, $L_{[\mathbf{z}]}$ and $L_{\sigma([\mathbf{z}])}$ be the complex lines through zero in $\mathbb{C}^4$ corresponding to $[\mathbf{w}]$, $[\mathbf{z}]$ and $\sigma([\mathbf{z}])$, respectively. Any vector $\mathbf{w} \in L_{[\mathbf{w}]}$ is a linear combination of a vector in $L_{[\mathbf{z}]}$ and a vector in $L_{\sigma([\mathbf{z}])}$, because $L_{[\mathbf{z}]}$ and $L_{\sigma([\mathbf{z}])}$ span a two-dimensional plane (containing zero) in $\mathbb{C}^4$ and $L_{[\mathbf{w}]}$ rotates from $L_{[\mathbf{z}]}$ to $L_{\sigma([\mathbf{z}])}$ when $[\mathbf{w}]$ runs from $[\mathbf{z}]$ to $\sigma([\mathbf{z}])$. Thus, since $A(\mathbf{w})$ depends linearly on $\mathbf{w}$, we obtain

$$\mathscr{E}_{[\mathbf{w}]}^{0} \cap \mathscr{E}_{[\mathbf{z}]}^{0} = \mathscr{E}_{[\mathbf{w}]}^{0} \cap \mathscr{E}_{\sigma([\mathbf{z}])}^{0} = \mathscr{E}_{\sigma([\mathbf{z}])}^{0} \cap \mathscr{E}_{[\mathbf{z}]}^{0} .$$

As a result, the two-dimensional subspace

$$\mathscr{R}_{\mathbf{x}} = \mathscr{E}_{\sigma([\mathbf{z}])}^{0} \cap \mathscr{E}_{[\mathbf{z}]}^{0} \subset V$$

is the complement of $\mathscr{E}_{[\mathbf{w}]}$ in $\mathscr{E}_{[\mathbf{w}]}^{0}$ for any $[\mathbf{w}] \in l_{\mathbf{x}}$. Thus, the restriction of $\mathscr{L}$ to $l_{\mathbf{x}}$ is trivial with the fibre given by $\mathscr{R}_{\mathbf{x}}$ and the holomorphic structure induced from $\mathscr{R}_{\mathbf{x}}$.

Finally, the anti-linear automorphism σ of $\mathbb{C}^{2k+2}$ defines an anti-holomorphic vector bundle automorphism of $\underline{V}$ covering $\sigma : \mathbb{C}\mathrm{P}^3 \to \mathbb{C}\mathrm{P}^3$ by

$$\sigma : \underline{V} \to \underline{V} , \quad \sigma([\mathbf{z}], \mathbf{v}) := \big(\sigma([\mathbf{z}]), \sigma(\mathbf{v})\big) . \tag{6.4.57}$$

Thus, by (6.4.53) and (6.4.55), σ induces an anti-holomorphic vector bundle automorphism of $\mathscr{L}$ covering σ, which we denote by the same symbol:

$$\sigma : \mathscr{L} \to \mathscr{L} , \quad \sigma([\mathbf{z}], \mathbf{v}) := (\sigma([\mathbf{z}]), \sigma(\mathbf{v})) , \quad \mathbf{v} \in \mathscr{R}_{\pi([\mathbf{z}])} .$$

Now, the desired holomorphic symplectic involution of $\mathscr{L}$ is obtained by combining this automorphism with the anti-linear bundle isomorphism $\sigma^*\overline{\mathscr{L}} \cong \mathscr{L}$ explained in Remark 6.4.13. ∎

Remark 6.4.15

1. Since (6.4.55) is an orthogonal direct sum decomposition, $\mathscr{R}_{\mathbf{x}}$ inherits a positive Hermitean inner product from h on V. Identifying the restriction of $\mathscr{L}$ to a real line $l_{\mathbf{x}}$ with $\mathscr{R}_{\mathbf{x}}$, we obtain a positive Hermitean inner product on the space of sections.
2. Property 1 of $\mathscr{L}$ can be interpreted in terms of characteristic classes. By a theorem of Grothendieck [265], every holomorphic vector bundle of rank n over $\mathbb{C}\mathrm{P}^1$ is isomorphic to a direct sum of line bundles $L^{k_i} = L \otimes \ldots \otimes L$ (k_i times), where L denotes the (unique up to isomorphisms) holomorphic line bundle over $\mathbb{C}\mathrm{P}^1$and the integers $(k_1, \ldots, k_n)$ are unique up to permutation. These integers

are holomorphic but not topological invariants. Only their sum is a topological invariant. Thus, since $\mathscr{L}$ is of rank 2, restricted to a real line it is isomorphic to a direct sum $L^{k_1} \oplus L^{k_2}$ of holomorphic line bundles. Now, property 1 implies $k_1 = k_2 = 0$. Thus, in particular, $k_1 + k_2 = 0$, that is, the first Chern class of $\mathscr{L}$ vanishes.[23] As a consequence, the instanton number k (the second Chern class) is the only topological invariant of $\mathscr{L}$. For further details, we refer to [30].

3. Given a bundle $\mathscr{L}$ obtained from complex ADHM data via the Horrocks construction, we can work back through Lemma 6.4.10 to recover the explicit construction of instantons in terms of the quaternionic data. Indeed, by (6.4.56) and (6.4.32), $\mathscr{R}_{\mathbf{x}}$ coincides with $\mathrm{im}(\mathrm{v}(\pi(\mathbf{z})))^{\perp}$. Thus,

$$\mathscr{L} = \pi^*(L)\,, \tag{6.4.58}$$

and the orthogonal projector in V onto $\mathscr{L}_{\mathbf{z}}$ coincides, under the identification $V \cong \mathbb{H}^{k+1}$, with the orthogonal projector $\mathbb{P}$ in $\mathbb{H}^{k+1}$ onto $\mathrm{im}\big(\mathrm{v}\big(\pi(\mathbf{z})\big)\big)^{\perp}$. This implies that the canonical connection $\tilde{\omega}$ on $\mathscr{L}$ obtained from projecting the trivial connection on $\underline{V}$ onto $\mathscr{L}$ is the pullback of the canonical connection ω given by (6.4.8),

$$\tilde{\omega} = \pi^*\omega\,. \tag{6.4.59}$$

4. We briefly comment on the algebro-geometric background. For a compact complex manifold M, a monad (in the sense of Horrocks) is a complex

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$$

of algebraic vector bundles over M fulfilling $\beta \circ \alpha = 0$. The algebraic vector bundle $\ker \beta / \operatorname{im} \alpha$ is called the cohomology of the monad. In our case, $M = \mathbb{C}\mathrm{P}^3$ and we have the monad

$$0 \longrightarrow \mathscr{E} \xrightarrow{j} \underline{V} \xrightarrow{j^* \circ \mathbb{J}} \mathscr{E}^* \longrightarrow 0\,,$$

where j is the natural inclusion mapping and $\mathbb{J}$ is viewed as a homomorphism $V \to V^*$. Since $\ker\big(j^* \circ \mathbb{J}\big) = (\operatorname{im} j)^{\mathbb{J}}$, we find that the instanton bundle $\mathscr{L}$ coincides with the cohomology of this monad. This is the approriate language for accomplishing the proof of point 3 in the introduction. For details, see Chap. VII in [30]. ♦

Next, consider a self-dual connection ω on a principal Sp(1)-bundle P over $\mathbb{H}\mathrm{P}^1$ given in terms of its quaternionic ADHM data. Let L be the associated quaternionic line bundle given by the basic representation and let ∇ be the covariant derivative of ω. Then, L carries a fibre metric induced from the quaternionic scalar product on $\mathbb{H}$ which is compatible with ∇. By field restriction, L becomes a complex Hermitean

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[23] See also [42] for a semicontinuity argument. Alternatively, one may deduce $\mathrm{c}_1(\mathscr{L}) = 0$ from property 1 of $\mathscr{L}$ by observing that any fibre of π represents a generator of $H_2(\mathbb{C}\mathrm{P}^3)$.

vector bundle with structure group $\mathrm{SU}(2)$ over S^4. The following theorem covers the more general case of a Hermitean vector bundle of arbitrary rank.

Theorem 6.4.16 (Atiyah–Ward) *Let* (L, h) *be a Hermitean vector bundle over* S^4 *endowed with a self-dual metric connection* ∇ *and let* $\pi : \mathbb{C}\mathrm{P}^3 \to \mathrm{S}^4$ *be the projective twistor bundle. Let* $\mathbb{C}\mathrm{P}^3$ *be endowed with the complex structure induced via the identification with the negative projective spinor bundle of* S^4 *and let* σ *be the real structure on* $\mathbb{C}\mathrm{P}^3$ *given by* (6.4.38). *Then, the pullback bundle* $\mathscr{L} := \pi^* L$ *carries a natural holomorphic structure and a holomorphic isomorphism* $\tau : \sigma^*\overline{\mathscr{L}} \to \mathscr{L}^*$ *fulfilling:*

1. $\mathscr{L}$ *is holomorphically trivial on each fibre of* π.
2. *The holomorphic isomorphism* τ *induces a positive definite Hermitean structure on the space of holomorphic sections of* $\mathscr{L}$ *over each fibre of* π.

Conversely, every such bundle over $\mathbb{C}\mathrm{P}^3$ *is the pullback of a bundle L with self-dual connection over* S^4.

Proof The Hermitean fibre metric h of L induces a Hermitean fibre metric $\tilde{\mathsf{h}}$ on $\mathscr{L}$ and, with respect to this fibre metric, $\tilde{\nabla} = \pi^*\nabla$ is a Hermitean connection on $\mathscr{L}$. If ω and Ω are the connection form and the curvature of ∇, then $\tilde{\omega} = \pi^*\omega$ and $\tilde{\Omega} = \pi^*\Omega$ are the connection and the curvature of $\tilde{\nabla}$, respectively. By Corollary 2.8.3, any 2-form on $\mathbb{R}^4$ is anti-self-dual iff it is of type $(1, 1)$ for some (and hence for all) compatible complex structures. Combining this with the fact that the complex structure chosen on $\mathbb{C}\mathrm{P}^3$ reverses the orientation of S^4, we conclude that $\tilde{\Omega}$ is of type $(1, 1)$. Now, Theorem 2.6.12 implies that $\mathscr{L}$ admits a holomorphic structure such that $\tilde{\nabla}$ is the canonical connection, that is, $\tilde{\nabla}$ is of type $(1, 0)$.

We show that $\mathscr{L}$ is holomorphically trivial over each fibre of π. Thus, let $\mathbf{x} \in \mathrm{S}^4$. Every basis $\{\mathbf{e}_\alpha\}$, $\alpha = 1, \dots, k$, of the fibre $L_{\mathbf{x}}$ induces a frame $\{\tilde{\mathbf{e}}_\alpha\}$ in $\mathscr{L}_{\restriction \pi^{-1}(\mathbf{x})}$ via $[\mathbf{z}] \mapsto \tilde{\mathbf{e}}_\alpha([\mathbf{z}]) := ([\mathbf{z}], \mathbf{e}_\alpha)$. It is enough to prove that the sections $\tilde{\mathbf{e}}_\alpha$ are holomorphic. Since $\tilde{\omega}$ is the pullback of ω under π, the elements of the induced frame are covariantly constant along $\pi^{-1}(\mathbf{x})$. Indeed, by Proposition 1.5.3,

$$\tilde{\nabla}\tilde{\mathbf{e}}_\alpha = \tilde{\mathscr{A}}_\alpha^\beta \tilde{\mathbf{e}}_\beta = \left(\pi^*\mathscr{A}_\alpha^\beta\right)\tilde{\mathbf{e}}_\beta \,,$$

where $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are the local representatives of ω and $\tilde{\omega}$, respectively. Thus, $\tilde{\nabla}\tilde{\mathbf{e}}_\alpha$ is annihilated by any vector tangent to $\pi^{-1}(\mathbf{x})$. Now, decomposing $\tilde{\mathbf{e}}_\alpha = \sum_\beta a_{\alpha\beta}\, \mathbf{h}_\beta$ in a local holomorphic frame $\{\mathbf{h}_\alpha\}$ in $\mathscr{L}_{\restriction \pi^{-1}(\mathbf{x})}$, we have

$$\tilde{\nabla}\tilde{\mathbf{e}}_\alpha = \sum_\beta (\mathrm{d}a_{\alpha\beta})\mathbf{h}_\beta + \sum_\beta a_{\alpha\beta}\tilde{\nabla}\mathbf{h}_\beta = 0 \,.$$

Decomposing the above sum into its $(1, 0)$ and $(0, 1)$-parts and using that $\tilde{\nabla}$ is a $(1, 0)$-connection, we read off that the $(0, 1)$-component is $\sum_\beta(\bar{\partial} a_{\alpha\beta})\mathbf{h}_\beta$. Now, vanishing of this quantity implies that the functions $a_{\alpha\beta}$ must be holomorphic.

Next, the anti-linear involution σ on $\mathbb{C}\mathrm{P}^3$ and the Hermitean fibre metric $\tilde{\mathsf{h}}$ on $\mathscr{L}$ yield a bundle isomorphism[24]

$$\tau : \sigma^*\overline{\mathscr{L}} \to \mathscr{L}^* : \quad \tau\Big(\sum_\alpha \overline{w}_\alpha \tilde{\mathbf{e}}_\alpha(\sigma([\mathbf{z}]))\Big) := \sum_\alpha w_\alpha \tilde{\mathbf{e}}^*_\alpha([\mathbf{z}])\,. \tag{6.4.60}$$

Here, $\{\tilde{\mathbf{e}}_\alpha\}$ is a local orthonormal frame with respect to $\tilde{\mathsf{h}}$ of $\mathscr{L}$ obtained via pullback from a local orthonormal frame $\{\mathbf{e}_\alpha\}$ of L and $\{\tilde{\mathbf{e}}^*_\alpha\}$ is the dual coframe. To prove that this isomorphism is holomorphic, we must show that τ maps $(1,0)$-forms on $\sigma^*\overline{\mathscr{L}}$ to $(1,0)$-forms on $\mathscr{L}^*$. By the proof of Theorem 2.6.12, the complex structure on $\mathscr{L}^*$ is locally defined by the forms

$$(\mathrm{d}z^j, \mathrm{d}w^\alpha + \mathscr{B}^\alpha{}_\beta w^\beta)\,, \quad \mathscr{B} := \tilde{\mathscr{A}}^{\,0,1}\,.$$

Since the forms $\mathrm{d}w^\alpha + \mathscr{B}^\alpha{}_\beta w^\beta$ are pullbacks under π, they are invariant under σ. Thus, the complex structure on $\sigma^*\overline{\mathscr{L}}$ is given by $(\mathrm{d}z^j, \mathrm{d}\overline{w}^\alpha + \overline{\mathscr{B}}^\alpha{}_\beta\,\overline{w}^\beta)$. Using the Hermiticity condition $\overline{\mathscr{B}}_{\alpha\beta} = -\mathscr{B}_{\beta\alpha}$, it reads

$$(\mathrm{d}z^j, \mathrm{d}\overline{w}^\alpha - \mathscr{B}_\beta{}^\alpha\,\overline{w}^\beta).$$

Now we must apply τ. Using $\tau(\mathrm{d}\overline{w}^\alpha) = \mathrm{d}w^\alpha$ and $\tau(\lambda \mathrm{d}z_j) = \overline{\lambda}\mathrm{d}z_j$, we get

$$(\mathrm{d}z^j, \mathrm{d}w^\alpha - \mathscr{B}_\beta{}^\alpha\, w^\beta)$$

which coincides with the complex structure of $\mathscr{L}^*$, because the induced connection on the dual bundle is given by the negative transpose.

Finally, since $\mathscr{L}$ is holomorphically trivial over each fibre, we may use τ to define a Hermitean structure on the space of holomorphic sections of $\mathscr{L}$ over each fibre:

$$\langle s_1, s_2\rangle([\mathbf{z}]) := \tau\big(s_2(\sigma([\mathbf{z}]))\big)\big(s_1([\mathbf{z}])\big)\,,$$

where $\mathbf{z} \in \pi^{-1}(\mathbf{x})$ and s_1 and s_2 are holomorphic sections over $\pi^{-1}(\mathbf{x})$. Then, by Definition (6.4.60), we have $\langle s_1, s_2\rangle(\mathbf{z}) = \tilde{\mathsf{h}}\big(s_1([\mathbf{z}]), s_2([\mathbf{z}])\big)$, showing that $\langle\cdot,\cdot\rangle$ is positive definite and Hermitean.

For the proof of the converse statement, we refer to the proof of Theorem 5.2 of [37]. ■

Remark 6.4.17 Theorem 6.4.16 is one way of spelling out what usually is referred to as the Atiyah–Ward correspondence [42]. It generalizes immediately to Hermitean vector bundles with self-dual connection over any self-dual 4-manifold [37]. Then, $\mathbb{C}\mathrm{P}^3$ must be replaced by the projective spinor bundle $\mathrm{P}^-(M)$, cf. Remark 5.5.8. ♦

Now, by a theorem of Serre [583], [582], any holomorphic vector bundle over a complex algebraic variety in a projective space is algebraic and, thus, combining the

[24] Remember that we may identify $\sigma^*\overline{\mathscr{L}}$ with $\mathscr{L}$, cf. Remark 6.4.13.

results presented above with point 3 of the programme outlined at the beginning, we obtain that the ADHM construction yields all instantons on S^4. Thus, we get the following fundamental theorem.

Theorem 6.4.18 (Atiyah-Drinfeld-Hitchin-Manin) *For a Yang–Mills theory on* S^4 *with gauge group* $\mathrm{Sp}(1)$*, every k-instanton arises from the parameters* (λ, B) *satisfying conditions (a) and (b). In an asymptotic gauge, using the conformal identification* $S^4 \cong \mathbb{H} \cup \{\infty\}$*, the solution is given by formula* (6.4.26) *with U defined by* (6.4.25)*. Gauge equivalent potentials are described by transformations* (6.4.20) *fulfilling* (6.4.22).

For the full proof we refer to [35], [162], [163] and to [30] for a detailed presentation.

Remark 6.4.19 This classification result generalizes to any classical compact Lie group, see [164] for details. There, first the group $G = \mathrm{O}(n)$ was treated. Then, the instantons for the groups $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ were viewed as $\mathrm{O}(2n)$- and $\mathrm{O}(4n)$-instantons, respectively, equipped with an additional structure. ♦

Exercises

6.4.1 Show that for any quaternionic $((k+1)\times k)$-matrix D fulfilling $D^\dagger D = \mathbb{1}_k$ there exists a matrix $Q \in \mathrm{Sp}(k+1)$ such that (6.4.23) holds. *Hint.* Decompose D into blocks of dimension $(1\times k)$ and $(k\times k)$ and B into blocks of dimension (1×1), $(1\times k)$, $(k\times 1)$ and $(k\times k)$ and convince yourself that (6.4.23) fixes the $(1\times k)$- and the $(k\times k)$-block of Q. Show that $D^\dagger D = \mathbb{1}_k$ guarantees that this fixing is compatible with the requirement that Q be an element of $\mathrm{Sp}(k+1)$.

6.4.2 Verify the formulae given in Example 6.4.7.

6.4.3 Prove formula (6.4.46).

6.5 The Instanton Moduli Space

In this section, we study the moduli space of all instanton solutions. For a given principal bundle $P(M,G)$ with instanton number $\mathrm{k} > 0$, it is defined as

$$\mathfrak{M}_{\mathrm{k}} := \{[\omega] \in \mathscr{M}(P) : *\Omega^\omega = \Omega^\omega\} . \tag{6.5.1}$$

This definition makes sense, because local gauge transformations map (anti-)self-dual connections to (anti-)self-dual connections, cf. Remark 6.2.8. Correspondingly, we write $\mathfrak{M}_{-\mathrm{k}}$ for anti-instantons.

In the first part, we present general results holding for any compact, self-dual oriented Riemannian manifolds M having an additional property to be specified later. First, we limit our attention to the case of irreducible connections. We will see in Chap. 8 that the latter constitute an open set in the space of all connections. Next, we will concentrate on Sp(1)-connections on S^4. For that case, the moduli space will be described in detail. Finally, we will discuss the role of the reducible connections.

First, we wish to find a good candidate for the tangent space of the moduli space. For that purpose, let $\mathrm{p}_- : \bigwedge^2 M \otimes \mathrm{Ad}(P) \to \bigwedge^2_- M \otimes \mathrm{Ad}(P)$ be the projection with respect to the decomposition $\bigwedge^2 M = \bigwedge^2_+ M \oplus \bigwedge^2_- M$.

Lemma 6.5.1 *Let ω be a self-dual connection on P. Then, each 1-parameter family $t \mapsto \omega_t$ of self-dual connections on P, fulfilling $\omega_0 = \omega$, defines an element of*

$$\ker(\mathrm{p}_- \circ \mathrm{d}^1_\omega)\,.$$

Proof Denoting $\tau_t = \omega_t - \omega \in \mathscr{T}$, by the Structure Equation, we have

$$\Omega_t = \Omega + \mathrm{d}_\omega \tau_t + \frac{1}{2}[\tau_t, \tau_t]\,,$$

and, by the self-duality requirement,

$$\mathrm{p}_-(\mathrm{d}_\omega \tau_t + \frac{1}{2}[\tau_t, \tau_t]) = 0\,. \tag{6.5.2}$$

Differentiating this equation with respect to t at $t = 0$ and using $\tau_0 = 0$ yields $p_-(\mathrm{d}_\omega \dot{\tau}) = 0$, that is, $\dot{\tau} \in \ker(\mathrm{p}_- \circ \mathrm{d}^1_\omega)$. ■

Now, by (6.1.28), we conclude that $\ker(\mathrm{p}_- \circ \mathrm{d}^1_\omega)/\operatorname{im}(\mathrm{d}^0_\omega)$ is a good candidate for the tangent space to the moduli space. Thus, for a Yang–Mills theory on the principal bundle $P(M, G)$ endowed with an irreducible self-dual connection ω, we are led to consider the sequence defined by the differential operators

$$\mathrm{d}_0 := \mathrm{d}^0_\omega\,, \quad \mathrm{d}_1 := \mathrm{p}_- \circ \mathrm{d}^1_\omega\,.$$

Lemma 6.5.2 *The sequence*

$$0 \longrightarrow \Omega^0(M, \mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_0} \Omega^1(M, \mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_1} \Omega^2_-(M, \mathrm{Ad}(P)) \longrightarrow 0 \tag{6.5.3}$$

is an elliptic complex of first order differential operators.

We denote the elliptic complex (6.5.3) by $\mathfrak{E}_{\mathrm{YM}}$ and call it the Yang–Mills complex.

Proof Since ω is self-dual, using (1.4.12) and (1.5.13), we obtain

$$\mathrm{d}_1 \circ \mathrm{d}_0 = p_- \circ \mathrm{d}^1_\omega \circ \mathrm{d}^0_\omega = p_-\big(\mathsf{R}^\nabla\big) = 0\,,$$

that is, (6.5.3) defines a complex. To prove that it is elliptic, we have to show that its (reduced) sequence of symbol mappings[25]

[25]Cf. Sect. 5.7. Note that $\mathrm{Ad}(P)$ is redundant here.

$$0 \longrightarrow \mathbb{R} \xrightarrow{\sigma_0(\xi)} \mathrm{T}^*_m M \xrightarrow{\sigma_1(\xi)} \bigwedge^2_- \mathrm{T}^*_m M \longrightarrow 0$$

is exact for all $m \in M$ and all $\xi \in \mathrm{T}^*_m M$. Here, $\sigma_0(\xi)(t) = t\xi$ and $\sigma_1(\xi)(\alpha) = p_-(\xi \wedge \alpha)$. Clearly, σ_0 is injective and $\operatorname{im}\sigma_0 \subset \ker\sigma_1$. We show that, conversely, $\ker\sigma_1 \subset \operatorname{im}\sigma_0$. Let $\vartheta^1, \dots, \vartheta^4$ be a basis of $\mathrm{T}^*_m M$ such that $\vartheta^1 = \xi$ and let

$$\alpha = \sum_i \alpha_i\, \vartheta^i \in \ker\sigma_1\,.$$

Then, $p_-(\alpha_2\, \vartheta^1 \wedge \vartheta^2 + \alpha_3\, \vartheta^1 \wedge \vartheta^3 + \alpha_4\, \vartheta^1 \wedge \vartheta^4) = 0$. Passing to the basis $\{\varphi_i^\pm\}$ defined in Remark 2.8.1, we have

$$\vartheta^1 \wedge \vartheta^2 = \tfrac{1}{\sqrt{2}}(\varphi_1^+ + \varphi_1^-)\,,\ \ \vartheta^1 \wedge \vartheta^3 = \tfrac{1}{\sqrt{2}}(\varphi_2^+ + \varphi_2^-)\,,\ \ \vartheta^1 \wedge \vartheta^4 = \tfrac{1}{\sqrt{2}}(\varphi_3^+ + \varphi_3^-)\,,$$

where $\varphi_i^\pm$ denote the basis vectors in $\bigwedge^2_\pm M$, respectively. This yields

$$\alpha_2\, \varphi_1^- + \alpha_3\, \varphi_2^- + \alpha_4\, \varphi_3^- = 0\,,$$

that is, $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and, thus, $\alpha = \alpha_1 \vartheta^1$. In particular, we obtain

$$\dim(\ker\sigma_1) = 1\,.$$

This implies that σ_1 is surjective. ■

The cohomology groups of the complex (6.5.3) are

$$H^0_\omega = \ker(\mathrm{d}_0)\,, \quad H^1_\omega = \ker(\mathrm{d}_1)/\operatorname{im}(\mathrm{d}_0)\,, \quad H^2_\omega = \Omega^2_-(M)/\operatorname{im}(\mathrm{d}_1)\,. \tag{6.5.4}$$

By ellipticity, they are all finite-dimensional. Clearly, the adjoint of d_1 coincides with the restriction of d^* to $\Omega^2_-(M)$. By ellipticity, each of the Hodge-Laplace operators

$$\Box_0 = \mathrm{d}_1^* \circ \mathrm{d}_0\,, \quad \Box_1 = \mathrm{d}_1^* \circ \mathrm{d}_1 + \mathrm{d}_0 \circ \mathrm{d}_1^*\,, \quad \Box_2 = \mathrm{d}_1 \circ \mathrm{d}_1^*\,, \tag{6.5.5}$$

is elliptic and has a finite-dimensional kernel[26]

$$\mathscr{H}^p_\omega = \left\{\alpha \in \Omega^p(M, \mathrm{Ad}(P)) : \Box_p \alpha = 0\right\}\,, \quad p = 0, 1, 2\,.$$

Moreover, the Hodge Decomposition Theorem 2.7.2 holds,

$$\Omega^p(M, \mathrm{Ad}(P)) = \mathscr{H}^p_\omega \oplus \operatorname{im}(\mathrm{d}_\omega) \oplus \operatorname{im}(\mathrm{d}^*_\omega)\,.$$

Thus,

[26] As a consequence of the regularity of solutions to elliptic equations, these spaces remain unchanged after completing $\Omega^p(M, \mathrm{Ad}(P))$ with respect to any Sobolev norm.

$$H^p_\omega \cong \mathscr{H}^p_\omega \,, \quad p = 0, 1, 2 \,.$$

Denote $h^p_\omega := \dim(H^p_\omega)$. Comparing the second equation in (6.5.4) with Lemma 6.5.1, we see that the first cohomology H^1_ω should serve as a model for the tangent space of the moduli space. The basic idea consists now in showing that $h^0_\omega = 0 = h^2_\omega$. Then, the Atiyah–Singer Index Theorem 5.8.14 for the complex $\mathfrak{E}_{\mathrm{YM}}$ will provide us with a formula for h^1_ω and thus, eventually, for the (virtual) dimension of the moduli space.

Lemma 6.5.3 *For an irreducible self-dual connection ω on a principal bundle $P(M, G)$ with G being compact and semi-simple, we have*

$$h^0_\omega = 0 \,.$$

Proof By Theorem 6.1.5, $H^0_\omega = \ker(\mathrm{d}_0)$ coincides with the Lie algebra of the stabilizer of ω and, thus, with the Lie algebra of the centralizer of the holonomy group of ω in $\mathfrak{g}$. By the irreducibility assumption, the centralizer of the holonomy group coincides with the center of G which, by the assumption of semi-simplicity of G, is finite. Thus, its Lie algebra is zero-dimensional. ∎

Lemma 6.5.4 *Let $P(M, G)$ be a principal bundle with a compact and semi-simple structure group G over a 4-dimensional self-dual compact Riemannian manifold with positive scalar curvature. Then, for any irreducible self-dual connection ω on P, we have*

$$h^2_\omega = 0 \,.$$

Proof Since $\Box_2 = \mathrm{d}_1 \circ \mathrm{d}_1^*$, we have $H^2_\omega \cong \ker(\mathrm{d}_1 \circ \mathrm{d}_1^*)$. Thus, we have to calculate

$$\mathrm{d}_1 \circ \mathrm{d}_1^* = \mathrm{p}_- \circ \mathrm{d}_\omega \circ \mathrm{d}_\omega^* \circ \iota_- \,,$$

where $\iota_- : \Omega^2_-(M, \mathrm{Ad}(P)) \to \Omega^2(M, \mathrm{Ad}(P))$ is the natural inclusion mapping. Let $\alpha \in \Omega^2_-(M, \mathrm{Ad}(P))$. Then, using $*\alpha = -\alpha$ and $\mathrm{d}_\omega^* = - * \circ \mathrm{d}_\omega \circ *$, we obtain

$$\langle \mathrm{d}_\omega \circ \mathrm{d}_\omega^* \alpha, \alpha \rangle = \langle \mathrm{d}_\omega^* \alpha, \mathrm{d}_\omega^* \alpha \rangle = \langle \mathrm{d}_\omega \alpha, \mathrm{d}_\omega \alpha \rangle = \langle \mathrm{d}_\omega^* \circ \mathrm{d}_\omega \alpha, \alpha \rangle \,.$$

Thus,

$$\mathrm{d}_1 \circ \mathrm{d}_1^* = \frac{1}{2} \mathrm{p}_- \circ \Box_\omega \circ \iota_- \,,$$

and we may apply the Weitzenboeck Formula (2.7.63),

$$\Box_\omega \alpha \; = \; \left(\nabla^{(\omega^0 + \omega)}\right)^* \nabla^{(\omega^0 + \omega)} \alpha + \alpha \circ (\mathsf{R} + \mathsf{Ric} \wedge \mathrm{id}) + \mathfrak{R}^{\nabla^\omega}(\alpha) \,, \tag{6.5.6}$$

where ω^0 is the Levi-Civita connection of M. The last term in (6.5.6) vanishes, because the curvature endomorphism of a self-dual connection acts trivially on $\Omega^2_-(M, \mathrm{Ad}(P))$. Thus, it remains to calculate the second term of this sum. This is most easily done in a local orthonormal frame $\{e_i\}$ on M. Using (2.8.26), we obtain:

$$\begin{aligned}\big(\alpha\circ(\mathsf{R}+\mathsf{Ric}\wedge\mathrm{id})\big)(e_i,e_j) &= \eta^{kl}\alpha(e_k,\mathsf{R}(e_i,e_j)e_l)+\alpha(\mathsf{Ric}(e_i),e_j)-\alpha(\mathsf{Ric}(e_j),e_i)\\ &= \mathsf{R}_{klij}\alpha^{kl}+\eta^{kl}(\mathsf{R}_{li}\alpha_{lk}-\mathsf{R}_{lj}\alpha_{ki})\\ &= \frac{\mathsf{Sc}}{3}\alpha_{ij}+\mathsf{W}_{klij}\alpha^{kl}\,,\end{aligned}$$

where Sc is the scalar curvature of M and W is the Weyl tensor. Since M is self-dual, $\mathsf{W}_-=0$ and, thus, for $\alpha\in\Omega^2_-(M,\mathrm{Ad}(P))$,

$$\Box_\omega\alpha\;=\;\big(\nabla^{(\omega^0+\omega)}\big)^*\nabla^{(\omega^0+\omega)}\alpha+\frac{\mathsf{Sc}}{3}\alpha\,.$$

This implies

$$2\int_M\langle \mathrm{d}_1\circ\mathrm{d}_1^*\alpha,\alpha\rangle\,\mathsf{v_g}=\int_M|\nabla^{(\omega^0+\omega)}\alpha|^2\mathsf{v_g}+\int_M\frac{\mathsf{Sc}}{3}|\alpha|^2\mathsf{v_g}\,.$$

Since Sc is positive, we conclude $h^2_\omega=\dim(\ker(\mathrm{d}_1\circ\mathrm{d}_1^*))=0$. ■

Now, since $h^0_\omega=0=h^2_\omega$ for the type of manifolds under consideration, the dimension h^1_ω coincides with (minus) the analytical index of the elliptic complex $\mathfrak{E}_{\mathrm{YM}}$ given by (6.5.3), with $\mathrm{Ad}(P)$ replaced by its complexification. Thus, we may apply the Atiyah–Singer Index Theorem 5.8.14, to calculate the dimension h^1_ω.

Lemma 6.5.5 *The topological index of the elliptic complex $\mathfrak{E}_{\mathrm{YM}}$ is given by*

$$\mathrm{ind}(\mathfrak{E}_{\mathrm{YM}})=-2\mathfrak{p}_1(\mathrm{Ad}(P))+\frac{1}{2}\dim G(\chi(M)-\sigma(M))\,, \tag{6.5.7}$$

where $\mathfrak{p}_1(\mathrm{Ad}(P))$ is the Pontryagin index of $\mathrm{Ad}(P)$ and $\chi(M)$ and $\sigma(M)$ are, respectively, the Euler number and the signature of M.

Proof Our proof is along the lines of [246]. According to (5.7.42) and (5.7.44), it suffices to compute the index of the assembled complex

$$\Omega^0(M,\mathrm{Ad}(P)_{\mathbb{C}})\oplus\Omega^2_-(M,\mathrm{Ad}(P)_{\mathbb{C}})\xrightarrow{\mathrm{d}_0+\mathrm{d}_1^*}\Omega^1(M,\mathrm{Ad}(P)_{\mathbb{C}})\,, \tag{6.5.8}$$

which we denote by $\mathfrak{E}$. Let τ denote the grading operator (5.7.46) obtained via the isomorphism $Cl(M)\cong\bigwedge^*\mathrm{T}^*M$ from the chirality operator. Decompose

$$\textstyle\bigwedge^*\mathrm{T}^*_{\mathbb{C}}M=\bigwedge^+_e\mathrm{T}^*M\oplus\bigwedge^-_e\mathrm{T}^*M\oplus\bigwedge^+_o\mathrm{T}^*M\oplus\bigwedge^-_o\mathrm{T}^*M\,,$$

where $\pm$ refer to the eigenvalues of τ and e,o refer to even and odd form degree. This decomposition induces the following complexes:

$$P_e^+ : \Omega_e^+(M, \mathrm{Ad}(P)_{\mathbb{C}}) \to \Omega_o^-(M, \mathrm{Ad}(P)_{\mathbb{C}}) ,$$
$$P_o^+ : \Omega_o^+(M, \mathrm{Ad}(P)_{\mathbb{C}}) \to \Omega_e^-(M, \mathrm{Ad}(P)_{\mathbb{C}}) ,$$

denoted by $\mathfrak{E}_e^+$ and $\mathfrak{E}_o^+$, respectively, and

$$P_o^- : \Omega_o^-(M, \mathrm{Ad}(P)_{\mathbb{C}}) \to \Omega_e^+(M, \mathrm{Ad}(P)_{\mathbb{C}}) ,$$
$$P_e^- : \Omega_e^-(M, \mathrm{Ad}(P)_{\mathbb{C}}) \to \Omega_o^+(M, \mathrm{Ad}(P)_{\mathbb{C}}) ,$$

denoted by $\mathfrak{E}_o^-$ and $\mathfrak{E}_e^-$. Here, $P_{e,o}^{\pm}$ is obtained from $\mathrm{d}_\omega + \mathrm{d}_\omega^*$ by restriction. Note that the projections

$$\bigwedge^1 \mathrm{T}^*_{\mathbb{C}} M \to \bigwedge_o^+ \mathrm{T}^* M \quad \text{and} \quad \bigwedge^0 \mathrm{T}^*_{\mathbb{C}} M \oplus \bigwedge_-^2 \mathrm{T}^*_{\mathbb{C}} M \to \bigwedge_e^- \mathrm{T}^* M$$

are isomorphisms which identify the bundles of $\mathfrak{E}$ with those of $\mathfrak{E}_e^-$. One can check that the principal symbols of $\mathrm{d}_0 + \mathrm{d}_1^*$ and P_e^- coincide (Exercise 6.5.2). Thus,

$$\mathrm{ind}(\mathfrak{E}) = \mathrm{ind}(\mathfrak{E}_e^-) .$$

Let $\mathfrak{E}_{\mathrm{dR}}(M, \mathrm{Ad}(P)_{\mathbb{C}})$ and $\mathfrak{E}_{\mathrm{sgn}}(M, \mathrm{Ad}(P)_{\mathbb{C}})$ denote the de Rham complex and the signature complex, respectively,[27] twisted with $\mathrm{Ad}(P)_{\mathbb{C}}$. Using

$$\mathrm{ind}(\mathfrak{E}_e^-) = -\,\mathrm{ind}(\mathfrak{E}_o^+)$$

and the additivity of the index, we obtain

$$\mathrm{ind}\left(\mathfrak{E}_{\mathrm{dR}}(M, \mathrm{Ad}(P)_{\mathbb{C}})\right) = \mathrm{ind}(\mathfrak{E}_e^+) + \mathrm{ind}(\mathfrak{E}_e^-) ,$$
$$\mathrm{ind}\left(\mathfrak{E}_{\mathrm{sgn}}(M, \mathrm{Ad}(P)_{\mathbb{C}})\right) = \mathrm{ind}(\mathfrak{E}_e^+) + \mathrm{ind}(\mathfrak{E}_o^+) .$$

Thus,

$$\mathrm{ind}(\mathfrak{E}_e^-) = \frac{1}{2}\Big(\mathrm{ind}\left(\mathfrak{E}_{\mathrm{dR}}(M, \mathrm{Ad}(P)_{\mathbb{C}})\right) - \mathrm{ind}\left(\mathfrak{E}_{\mathrm{sgn}}(M, \mathrm{Ad}(P)_{\mathbb{C}})\right)\Big) .$$

Now, the assertion follows from the formulae (5.9.13) and (5.9.17), because in our case

$$\mathrm{ch}_2(\mathrm{Ad}(P)_{\mathbb{C}}) = -\mathsf{c}_2(\mathrm{Ad}(P)_{\mathbb{C}}) = \mathsf{p}_1(\mathrm{Ad}(P)) .$$

■

Now, the idea will be to write down a local model $\mathfrak{C}_\omega$ for the moduli space in the neighbourhood of a chosen irreducible self-dual connection ω and to prove that it yields local coordinates on the global moduli space (endowed with the appropriate topology)

---

[27] Cf. Examples 5.7.22 and 5.7.23.

$$\mathfrak{M} = \mathscr{C}^{+}/\mathscr{G} \tag{6.5.9}$$

in the neighbourhood of $[\omega]$, cf. (6.5.1). Here, $\mathscr{C}^{+}$ is the space of all irreducible self-dual connections on P. Finally, an atlas on $\mathfrak{M}$ is constructed using local charts of this type. From Lemma 6.5.1 we know that $H^1_\omega = \ker(\mathrm{d}_1)/\operatorname{im}(\mathrm{d}_0)$ is a candidate for $\mathrm{T}_{[\omega]}\mathfrak{M}$. Since $\operatorname{im}(\mathrm{d}_0)$ is generated by local gauge transformations, as a local model near ω we can take $\ker(\mathrm{d}_1)$ and intersect it with a local slice fixing the gauge. An appropriate choice is $\mathrm{d}^*_\omega\tau = 0$. Thus, we consider the following subset of $\mathscr{T}$:

$$\mathfrak{C}_\omega := \left\{\tau \in \mathscr{T} : \mathrm{d}_1\tau + \tfrac{1}{2}p_-([\tau,\tau]) = 0\,,\ \ \mathrm{d}_0^*\tau = 0\right\}\,. \tag{6.5.10}$$

Now, up to some analytical technicalities,[28] we will prove the following fundamental theorem.

Theorem 6.5.6 (Atiyah–Hitchin–Singer) *Let $P(M,G)$ be a principal bundle with a compact and semi-simple structure group G over a 4-dimensional self-dual compact Riemannian manifold with positive scalar curvature. Then, the moduli space of irreducible self-dual connections on P is either empty[29] or a manifold of dimension*

$$\dim\mathfrak{M} = 2\mathfrak{p}_1(\mathrm{Ad}(P)) - \frac{1}{2}\dim G(\chi(M) - \sigma(M))\,. \tag{6.5.11}$$

Proof Let ω be an irreducible self-dual connection on P. In the first step, we prove that $\mathfrak{C}_\omega$ is an h^1_ω-dimensional manifold with tangent space H^1_ω. For that purpose, let G_p be the Green's operators and let H_p be the orthogonal projectors onto the harmonic subspaces $\mathscr{H}^p_\omega$ of the elliptic complex $\mathfrak{E}_{\mathrm{YM}}$. Then,

$$H_p + G_p \circ \Box_p = \mathrm{id}\,,\quad p = 1,2,3\,,$$

with the Hodge-Laplace operators given by (6.5.5). Recall that the Green's operators commute with d_0 and d_1 as well as with their adjoints. By Lemmas 6.5.3 and 6.5.4, we have $h^0_\omega = 0 = h^2_\omega$ and, thus, $H_0 = 0 = H_2$. Consider the following mapping

$$\Phi : \Omega^1(M,\mathrm{Ad}(P)) \to \Omega^1(M,\mathrm{Ad}(P))\,,\quad \Phi(\tau) := \tau + \frac{1}{2}G_1 \circ \mathrm{d}_1^*\left(\mathrm{p}_-([\tau,\tau])\right)\,.$$

Denoting $\alpha = \frac{1}{2}\mathrm{p}_-([\tau,\tau])$ and using $H_2(\alpha) = 0$, we calculate

$$\begin{aligned}
\mathrm{d}_1\Phi(\tau) &= \mathrm{d}_1\tau + \mathrm{d}_1 \circ G_1 \circ \mathrm{d}_1^*\alpha\\
&= \mathrm{d}_1\tau + G_2 \circ \mathrm{d}_1 \circ \mathrm{d}_1^*\alpha\\
&= \mathrm{d}_1\tau + G_2 \circ \Box_2\alpha\\
&= \mathrm{d}_1\tau + (\mathrm{id} - H_2)(\alpha)
\end{aligned}$$

[28] For a detailed presentation of the Sobolev-type arguments involved, we refer to Part IV in [83].

[29] Of course, from the previous sections, we know already that self-dual connections exist.

$$= \mathrm{d}_1\tau + \frac{1}{2}\mathrm{p}_-([\tau, \tau]) .$$

Similarly, using $\mathrm{d}_1 \circ \mathrm{d}_0 = 0$, we get

$$\mathrm{d}_0^*\Phi(\tau) = \mathrm{d}_0^*\tau + \mathrm{d}_0^* \circ G_1 \circ \mathrm{d}_1^*\alpha = \mathrm{d}_0^*\tau + G_1 \circ \mathrm{d}_0^* \circ \mathrm{d}_1^*\alpha = \mathrm{d}_0^*\tau .$$

We conclude that $\Phi(\tau)$ is harmonic iff $\tau \in \mathfrak{C}_\omega$, that is, Φ maps $\mathfrak{C}_\omega$ onto $\mathscr{H}_\omega^1 \cong H_\omega^1$. Clearly, the differential of Φ at $\tau = 0$ is the identity. Thus, after an appropriate Sobolev completion of $\Omega^1(M, \mathrm{Ad}(P))$ as discussed in Sect. 6.1, we may extend Φ to this completion and we may apply the Inverse Function Theorem for Banach space mappings to conclude that Φ is invertible on C^∞-sections and that Φ^{-1} yields local coordinates on $\mathfrak{C}_\omega$.

In the next step, we show that a neighbourhood of the origin in $\mathfrak{C}_\omega$ contains, up to local gauge transformations, all self-dual connections which are sufficiently close to ω, that is, such a neighbourhood yields a local model of the moduli space.[30] More precisely, we will prove that there exists a neighbourhood U of 0 in $\Omega^1(M, \mathrm{Ad}(P))$ and a neighbourhood W of 0 in $\Omega^0(M, \mathrm{Ad}(P))$ such that for any $\tau \in U$, there exists a unique $X \in W$ fulfilling

$$\mathrm{d}_0^*\big((\omega + \tau)^{(\exp X)} - \omega\big) = 0 . \tag{6.5.12}$$

By (6.1.2), we have

$$(\omega + \tau)^{(\exp X)} - \omega = \mathrm{d}_0 X + \tau + r(X, \tau) ,$$

where $r(tX, t\tau) = t^2 r(X, \tau, t)$ and $r(X, \tau, t)$ is locally defined and smooth. Thus,

$$\mathrm{d}_0^*\big((\omega + \tau)^{(\exp X)} - \omega\big) = \mathrm{d}_0^* \circ \mathrm{d}_0 X + \mathrm{d}_0^*\tau + \mathrm{d}_0^* r(X, \tau) .$$

Applying G_0 to this quantity and using $H_0 = 0$, we obtain

$$G_0 \circ \mathrm{d}_0^*\big((\omega + \tau)^{(\exp X)} - \omega\big) = X + G_0 \circ \mathrm{d}_0^*\tau + G_0 \circ \mathrm{d}_0^* r(X, \tau) .$$

We conclude that (6.5.12) is fulfilled iff

$$X + G_0 \circ \mathrm{d}_0^*\tau + G_0 \circ \mathrm{d}_0^* r(X, \tau) = 0 . \tag{6.5.13}$$

Now, we choose neighbourhoods $U_1 \subset \Omega^1(M, \mathrm{Ad}(P))$ and $W_1 \subset \Omega^0(M, \mathrm{Ad}(P))$ of the origin and consider the mapping

$$\Psi : U_1 \times W_1 \to \Omega^0(M, \mathrm{Ad}(P)) , \quad \Psi(\tau, X) := X + G_0 \circ \mathrm{d}_0^*\tau + G_0 \circ \mathrm{d}_0^* r(X, \tau) .$$

[30] This will follow from $H_0 = 0$, that is, in particular, the assumption that ω be irreducible is essential here.

Again, by standard Sobolev-type arguments, Ψ may be extended to a suitable Sobolev completion and the Implicit Function Theorem for Banach spaces may be applied yielding that, for sufficiently small U and W, for any $\tau \in U$ there exists a unique $X(\tau) \in W$ such that (6.5.12) holds. From elliptic regularity, one then concludes that $X(\tau)$ is C^∞ if τ is C^∞. In particular, if $\omega + \tau$ is self-dual and sufficiently close to ω, then there exists a gauge transformation $u = \exp X$ such that $(\omega + \tau)^{(u)}$ belongs to $\omega + \mathfrak{C}_\omega$. Moreover, by the uniquess of $X(\tau)$, no two self-dual connections in $\omega + \mathfrak{C}_\omega$ sufficiently close to ω can be equivalent under a small gauge transformation.

Finally, we must endow the global moduli space $\mathfrak{M} = \mathscr{C}^+/\mathscr{G}$ with a manifold structure. By standard arguments, $\mathfrak{M}$ is a topological Hausdorff space. We show that, in a neighbourhood of any $[\omega] \in \mathfrak{M}$, the local model $\mathfrak{C}_\omega$ yields a local chart, that is, for a sufficiently small neighbourhood $U \subset \omega + \mathfrak{C}_\omega$ of the origin, the natural projection to $\mathfrak{M}$ is injective. For that purpose, let $\omega + \tau$, with $\tau \in U$, be another self-dual connection and assume that it is gauge equivalent to ω under an (arbitrarily large) gauge transformation $u \in \mathscr{G}$. Viewing the latter as a section of $\mathrm{End}(\mathrm{Ad}(P))$, by (6.1.8),

$$u^{-1} \mathrm{d}_\omega u = \tau \, . \tag{6.5.14}$$

Now, take the component $\mathfrak{h}_0 \subset \mathrm{End}(\mathfrak{g})$ consisting of the endomorphisms invariant under the natural action[31] of G and decompose $\mathrm{End}(\mathfrak{g}) = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1$ is the orthogonal complement with respect to the scalar product induced from the Ad-invariant scalar product on $\mathfrak{g}$. Take the corresponding orthogonal direct sum decomposition $\mathrm{End}(\mathrm{Ad}(P)) = E_0 \oplus E_1$. It is easy to show (Exercise 6.5.1) that the irreducibility of ω implies

$$\ker \left\{ \mathrm{d}_\omega : \Gamma^\infty(E_1) \to \Omega^1(M, E_1) \right\} = 0 \, . \tag{6.5.15}$$

Thus, the smallest eigenvalue λ of the positive self-adjoint elliptic operator

$$\Box_\omega = \mathrm{d}_\omega^* \mathrm{d}_\omega : \Gamma^\infty(E_1) \to \Gamma^\infty(E_1)$$

is positive and, for any $u_1 \in \Gamma^\infty(E_1)$, we obtain:

$$\| \mathrm{d}_\omega u_1 \|^2 = \langle \Box_\omega u_1, u_1 \rangle \geq \lambda \| u_1 \|^2 \ .$$

Inserting the decomposition $u = u_0 + u_1$ with respect to the above orthogonal splitting of $\mathrm{End}(\mathrm{Ad}(P))$ into (6.5.14) and using that u, as a section of $\mathrm{End}(\mathrm{Ad}(P))$, is isometric, we obtain

$$\| \tau \|^2 = \| \mathrm{d}_\omega u \|^2 \geq \| \mathrm{d}_\omega u_1 \|^2 \geq \lambda \| u_1 \|^2 \ .$$

[31] The adjoint representation induces a natural representation $T = \mathrm{Ad} \otimes \mathrm{Ad}^* : G \to \mathrm{Aut}(\mathrm{End}(\mathfrak{g}))$ via $T(g)(\eta) := \mathrm{Ad}(g) \circ \eta \circ \mathrm{Ad}(g^{-1})$.

Up to some further analytical arguments, this shows that, for small enough τ, the gauge transformation $u = u_0 + u_1$ will be (uniformly) arbitrarily close to the subspace $\Gamma^\infty(E_0)$. Hence, by definition of E_0, u will be close to a constant mapping with values in the centre of G, the latter belonging to the centralizer of ω.

It remains to show that the transition mappings are smooth. For this purely technical exercise we refer to [83]. Finally, the dimension formula (6.5.11) follows from Lemma 6.5.5. ■

Example 6.5.7 For $M = \mathrm{S}^4$, we have $\chi(M) = 2$ and $\sigma(M) = 0$ (Exercise 6.5.3). Then, (6.5.11) reduces to

$$\dim \mathfrak{M} = 2\mathfrak{p}_1(\mathrm{Ad}(P)) - \dim G\,.$$

By (4.3.21), for $G = \mathrm{SU}(2) \cong \mathrm{Sp}(1)$, we have

$$\mathfrak{p}_1(\mathrm{Ad}(P)) = -4\mathfrak{c}_2(P) = 4\mathrm{k}(P)\,.$$

Thus, we obtain

$$\dim \mathfrak{M} = 8\mathrm{k}(P) - 3\,, \tag{6.5.16}$$

cf. formula (6.4.24). This number has been found earlier by Schwarz [568] and Jackiw and Rebbi [343]. It can be easily seen that, using an orientation-reversing diffeomorphism of S^4, one obtains the same statement for $\mathrm{k}(P) < 0$, with $\mathrm{k}(P)$ replaced by $-\mathrm{k}(P)$. For a detailed analysis of all the classical groups in this context, we refer to [37]. ♦

Next, we study the moduli space of $\mathrm{Sp}(1)$-instantons on S^4 with instanton number $\mathrm{k}(P) = \pm 1$ in some detail. As already mentioned in Sect. 6.3, it coincides with the homogeneous space $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$. Here, we give the proof of this fact. It is enough to consider one case, say $\mathrm{k}(P) = -1$, the other one being obtained by an orientation-reversing diffeomorphism of S^4. As a first check, comparing with formula (6.5.16), we have $\dim(\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)) = 5 = 8|\mathrm{k}(P)| - 3$, indeed.

Lemma 6.5.8 1. *Within the isomorphism class of principal* $\mathrm{Sp}(1)$*-bundles over* S^4 *defined by the instanton number* $\mathrm{k}(P) = -1$*, the quaternionic Hopf bundle* P_- *is the unique element admitting a lift of* $\mathrm{Sp}(2)$ *to automorphisms.*

2. *The canonical connection* ω^- *is the unique* $\mathrm{Sp}(2)$*-invariant connection on* P_-*.*

Proof 1. Denote $K = \mathrm{Sp}(2)$, $H = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $G = \mathrm{Sp}(1)$. By Remark 1.9.7/1, since K acts transitively on $K/H \cong \mathrm{S}^4$, principal G-bundles over K/H admitting a lift of K are labeled by Lie group homomorphisms $\lambda : H \to G$ and have the structure

$$P_\lambda = K \times_H G\,.$$

We claim that λ is surjective. Assume, on the contrary, that it is not. Then, by Corollary 5.3.7 and Proposition 5.1.7 in Part I, the induced Lie algebra homomorphism $\mathrm{d}\lambda : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{sp}(1)$ is not surjective. As a consequence, $\mathrm{im}(\mathrm{d}\lambda)$ is either trivial or a $\mathrm{u}(1)$-subalgebra of $\mathfrak{sp}(1)$. Since $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is connected, $\mathrm{im}(\lambda)$ is trivial or a $\mathrm{U}(1)$-subgroup of $\mathrm{Sp}(1)$. Now, clearly, P_λ admits a reduction Q to the subgroup $\mathrm{im}(\lambda)$. In case $\mathrm{im}(\lambda)$ trivial, Q provides a global section of P. In case $\mathrm{im}(\lambda)$ a $\mathrm{U}(1)$-subgroup, Theorem 4.8.1 and $H^2_{\mathbb{Z}}(\mathrm{S}^4) = 0$ imply that Q is trivial. In either case, we conclude that P is trivial, which is a contradiction. Thus, λ must be surjective. But the only surjective homomorphisms from H to G are given by projection onto the first or second component of H, respectively. Now, the condition $\mathrm{k}(P) = -1$ selects the projection onto the first component. By Remark 6.3.1/2, P_λ is isomorphic to P_-.

2. Denote the Lie algebras of K, H and G by $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{g}$, respectively, and consider the reductive decomposition

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m} .$$

By point 1, we may identify P_- with $K \times_H G$. By Remark 1.9.12/4, the K-invariant connections on $K \times_H G$ are classified by H-equivariant mappings $\tilde{\Phi} : \mathfrak{m} \to \mathfrak{g}$, that is,

$$\tilde{\Phi} \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda(h)) \circ \tilde{\Phi} , \quad h \in H .$$

As noted in this Remark, $\tilde{\Phi}$ may be viewed as an operator intertwining the representations $\mathrm{Ad}(H)_{\restriction \mathfrak{m}}$ and $\mathrm{Ad}(\lambda(H))$. Now, decomposing these representations into irreducible components and using Schur's Lemma, one may construct all solutions $\tilde{\Phi}$ explicitly. Here, the only solution is $\tilde{\Phi} = 0$, because $\mathrm{Ad}(H)_{\restriction \mathfrak{m}}$ coincides with the vector representation of $\mathrm{SO}(4)$ and $\mathrm{Ad}(\lambda(H))$ is the adjoint representation of G. We conclude that on the above bundle we have a unique K-invariant connection form $\tilde{\omega}$. It is given by formula (1.9.41), with $\tilde{\Phi} = 0$. In the terminology introduced in Remark 1.9.14/2, $\tilde{\omega}$ coincides with the canonical connection on $K \times_H G$. Note that in the present case one may choose representatives in such a way that this formula reduces to

$$\tilde{\omega}_p(Z) = (\mathrm{pr}_1)'(A_{\mathfrak{h}}) .$$

Now, it is easy to check that $\tilde{\omega}$ coincides with the pullback of ω^- under the isomorphism $P_- \to \mathrm{Sp}(2)/\lambda_+(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$ given in Remark 6.3.1/2 (Exercise 6.5.4). ∎

Theorem 6.5.9 (Atiyah–Hitchin–Singer) *The moduli space $\mathfrak{M}_{-1}$ of anti-self-dual connections on P_- with instanton number -1 is diffeomorphic to* $\mathrm{SL}(2, \mathbb{H})/\mathrm{Sp}(2)$.

Proof Consider the action $\tilde{\Psi}$ of the conformal covering group $\tilde{\mathrm{C}}_0(\mathrm{S}^4) = \mathrm{SL}(2, \mathbb{H})$ on P_- given by Proposition 6.3.7. By Proposition 6.2.7, $\tilde{\mathrm{C}}_0(\mathrm{S}^4)$ acts on the space of (anti-)self-dual connections and thus, by Remark 6.2.8, it acts on $\mathfrak{M}_{-1}$. We must prove that this action is transitive with stabilizer $\mathrm{Sp}(2)$.

Let $[\omega] \in \mathfrak{M}_{-1}$ and let $K \subset \mathrm{SL}(2, \mathbb{H})$ be its stabilizer. Since $\dim \mathrm{SL}(2, \mathbb{H}) = 15$ and, by (6.5.16), $\dim \mathfrak{M}_{-1} = 5$, we have $\dim K \geq 10$. Let $\omega \in [\omega]$ be a K-invariant representative. Since ω is anti-self-dual, by (6.2.10),

$$-\mathsf{p}_1(\mathrm{Ad}(P)) = \frac{1}{8\pi^2} \parallel \Omega^\omega \parallel^2 \mathsf{v}_{\mathsf{g}_0} .$$

Thus, $\parallel \Omega^\omega \parallel$ is a non-negative function, which is K-invariant and non-vanishing on a K-invariant open subset $U \subset \mathrm{S}^4$. This, in turn, defines a K-invariant Riemannian metric g on U via

$$\mathsf{g} = \parallel \Omega^\omega \parallel \mathsf{g}_0$$

belonging to the conformal class of the standard metric g_0. By construction, K acts on the Riemannian manifold (U, g) by isometries. Now, by Theorem 2.2.18, the isometry group of an n-dimensional Riemannian manifold has dimension at most $\frac{1}{2}n(n+1)$ and if the dimension is maximal, then the manifold is a space of constant curvature. This implies $\dim K \le 10$. We conclude that $\dim K = 10$ and that g must be a metric of constant curvature. Since $\parallel \Omega^\omega \parallel$ is finite, Theorem 1 in Note 10 of [383]/Part I implies that g must be a metric of positive constant curvature on S^4 isometric to g_0. That is, there exists an isometry $c \in \mathrm{C}_0(\mathrm{S}^4)$ such that

$$\Psi_c^* \mathsf{g} = \mathsf{g}_0 .$$

The transformation Ψ_c lifts to a transformation $\tilde{\Psi}_{\tilde{c}}$, $\tilde{c} \in \mathrm{SL}(2, \mathbb{H})$, of P_- and we have $\tilde{c}^{-1} K \tilde{c} = \mathrm{Sp}(2)$, because g_0 is $\mathrm{Sp}(2)$-invariant. Thus, $\tilde{\Psi}_{\tilde{c}}^* \omega$ is $\mathrm{Sp}(2)$-invariant and, by Lemma 6.5.8, it must coincide with the unique $\mathrm{Sp}(2)$-invariant connection ω^- on P_-,

$$\tilde{\Psi}_{\tilde{c}}^* \omega = \omega^- .$$

This shows that $\mathrm{SL}(2, \mathbb{H})$ acts transitively on $\mathfrak{M}_{-1}$ with stabilizer $\mathrm{Sp}(2)$. ■

The second part of the above proof is along the lines of [357]. It differs completely from the original proof in [37]. There, a vanishing argument based on the Weitzenboeck formula for the Dirac operator was used. However, the idea to use the theory of invariant connections was already mentioned in [37].

Remark 6.5.10

1. By Example 5.2.11,[32]

$$\mathrm{SL}(2, \mathbb{H})/\{\pm \mathbf{1}\} \cong \mathrm{SO}_+(1,5) , \quad \mathrm{Sp}(2)/\{\pm \mathbf{1}\} \cong \mathrm{SO}(5) . \tag{6.5.17}$$

   Thus, $\mathrm{C}_0(\mathrm{S}^4)$ may be identified with $\mathrm{SO}_+(1, 5)$ and

$$\mathfrak{M}_{-1} \cong \mathrm{SL}(2, \mathbb{H})/\mathrm{Sp}(2) = \mathrm{SO}_+(1,5)/\mathrm{SO}(5) .$$

   Recall from point 5 of Example 2.5.27 that the latter homogeneous space is symmetric and may be identified with the 5-dimensional hyperbolic hypersurface $H_+(1, 5)$ in $(\mathbb{R}^6, \eta)$.

[32] Recall that $\mathrm{Spin}_{r,s} = \mathrm{Spin}_{s,r}$, that is, we could also take $\mathrm{SO}_+(5, 1)$ below.

Here, η is the pseudo-Euclidean metric in 6 dimensions, with the signature convention $(-, +\dots, +)$. Now, given the parameterization (6.3.21) and viewing $\mathbf{x}_0$ as an element $\mathbf{z}_0 \in \mathrm{S}^4 \subset \mathbb{R}^5$ via the stereographic projection mapping φ_s, the mapping

$$(0, 1) \times \mathrm{S}^4 \to \mathring{\mathrm{D}}^5 \setminus \{0\}, \quad (\lambda, \mathbf{z}_0) \mapsto (1-\lambda)\mathbf{z}_0$$

yields a diffeomorphism of $\mathfrak{M}_{-1}$ onto the punctured open ball in $\mathbb{R}^5$.[33] The BPST anti-instanton is obtained by taking the limit $\lambda \to 1$, that is, it sits in the centre. For each pair $(\lambda, \mathbf{z}_0)$, in the limit $\lambda \to 0$, one approaches $\mathbf{z}_0 \in \mathrm{S}^4$, that is, the original manifold S^4 may be viewed as the boundary of the open ball thus yielding its compactification. Note that in this limit, the curvature becomes more and more concentrated at $\mathbf{z}_0$. Also note that we have a collar

$$[0, \lambda_0) \times \mathrm{S}^4 = \left\{(1-\lambda)\mathbf{z}_0 \in \mathring{\mathrm{D}}^5 : \lambda < \lambda_0\right\} \cup \mathrm{S}^4, \quad \lambda_0 < 1.$$

We will see below that this characterization of the moduli space near its boundary generalizes to any compact, simply connected and oriented 4-manifold satisfying a certain topological condition.

2. In a series of papers [277], [262], [156], [432], the Riemannian metric of the moduli spaces $\mathfrak{M}_{\pm 1}$ (inherited from the L^2-metric on the space of connections) has been studied. It was shown that this metric is conformally flat, rotationally invariant and incomplete. The volume defined by this metric is finite.
3. From the proof of Theorem 6.5.9, it should be clear that there is a deep relation between (anti-)self-dual Yang–Mills connections on S^4 and the (anti-)self-dual parts of the Levi-Civita connection of the standard metric on S^4. Indeed, by Example 1.1.18, the bundle of oriented orthonormal frames $O_+(\mathrm{S}^4)$ coincides with SO(5) viewed as a principal SO(4)-bundle over S^4 and, by Proposition 2.5.10 and Remark 2.5.28, the Levi-Civita connection of the standard Riemannian metric on S^4 coincides with the SO(5)-invariant connection on this bundle. By Example 5.4.9, the (unique) spin bundle $S(\mathrm{S}^4)$ coincides with Sp(2) viewed as a principal (Sp(1) × Sp(1))-bundle over S^4. Thus, the spin connection on $S(\mathrm{S}^4)$ coincides with the Sp(2)-invariant connection ω^0 on Sp(2) defined by (6.3.7). Now, consider the decomposition

$$\bigwedge^2 \mathrm{TS}^4 = \bigwedge^2_+ \mathrm{TS}^4 \oplus \bigwedge^2_- \mathrm{TS}^4$$

into self-dual and anti-self-dual elements corresponding to the eigenvalues ± 1 of the Hodge star operator of g_0, cf. (2.8.8). By the discussion in Sect. 2.8, this is an SO(4)-invariant splitting corresponding to the Lie algebra decomposition

[33] Clearly, this is the Poincaré model of the hyperbolic 5-space.

$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. It induces principal bundle morphisms[34] $O_+(S^4) \to O(\bigwedge^2_\pm TS^4)$ onto the principal SO(3)-bundles of (positive and negative) orthonormal frames of $\bigwedge^2_+ TS^4$ and $\bigwedge^2_- TS^4$, respectively. Clearly, the unique lifts of $O(\bigwedge^2_\pm TS^4)$ to the Sp(1)-principal spin bundles $S(\bigwedge^2_\pm TS^4)$ coincide with the bundles $P_\pm$ defined by (6.3.2), cf. also Example 5.4.11. Thus, the induced (anti-)self-dual connections $\omega^\pm$ on $P_\pm$ defined by (6.3.8) coincide with the $S(\bigwedge^2_\pm TS^4)$-components of the spin connection on S^4.
This relation generalizes as follows, see Proposition 2.2 in [37]: for any 4-dimensional manifold endowed with an Einstein metric,[35] the induced connections on the bundles $O(\bigwedge^2_\pm TS^4)$ and $S(\bigwedge^2_\pm TS^4)$ are (anti-)self-dual. Conversely, if the induced connections on $O(\bigwedge^2_+ TS^4)$ and $S(\bigwedge^2_+ TS^4)$ are self-dual, then the metric is Einstein. ♦

We close this section by discussing how reducible self-dual connections modify the above picture, leading to a full understanding of the structure of the moduli space $\mathfrak{M}$ of self-dual SU(2)-connection of instanton number 1 over four-manifolds fulfilling conditions to be described below. Some points are beyond the scope of this book, so that we must refer to the original work of Donaldson [157] and to the textbooks of Freed and Uhlenbeck [213], Lawson [406] and Donaldson and Kronheimer [159].

By Theorem 6.1.5, the stabilizer of a connection is given by the centralizer of its holonomy group in the structure group. For reducible connections, the holonomy group is a proper subgroup of the structure group and, thus, in this case we obtain nontrivial stabilizers leading to a nontrivial stratified structure of the full gauge orbit space. In this picture, the reducible connections correspond to the singular strata. The resulting stratification will be discussed in detail in Chap. 8. Here, we are interested in reducible SU(2)-connections which are self-dual. In that case, by the Ambrose-Singer Theorem 1.7.15, discrete subgroups give a vanishing curvature and may therefore be excluded. The only proper subgroups giving a non-vanishing curvature are copies of U(1). Any U(1)-subgroup is conjugate to the standard embedding

$$H := \left\{ \begin{bmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{bmatrix} \in G : \vartheta \in \mathbb{R} \right\} . \tag{6.5.18}$$

Next, we make the following assumptions on M.

(a) M is simply connected.
(b) The intersection form[36] s_M of M is positive definite.

Note that we do not assume that M be self-dual. Thus, in general, $H^2_\omega \neq 0$, cf. the proof of Lemma 6.5.4. Now, let ω be a reducible connection on $P(M, G)$ and let Ω

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[34]These morphisms are induced from the group homomorphism $SO(4) \to SO(4)/\mathbb{Z}_2 = SO(3) \times SO(3)$ combined with the canonical projections onto the first and the second SO(3)-component, respectively.

[35]See Definition 2.3.12.

[36]Cf. Definition 5.7.56.

be its curvature. Then, the restriction of Ω to a holonomy bundle $P(\omega)$ is a 2-form with values in the Lie algebra $i\mathbb{R}$ of $\mathrm{U}(1)$ and, thus, it is given by a 2-form $i\mathscr{F}$ on M. By Theorem 4.6.11, the corresponding de Rham cohomology class

$$[-(2\pi)^{-1}\mathscr{F}] \in H^2_{\mathrm{dR}}(M)$$

coincides with the first Chern class $\mathsf{c}_1(P(\omega))$. Moreover, by the Bianchi identity, we have $\mathrm{d}\mathscr{F} = 0$. Now, let us assume that ω is self-dual with instanton number $\mathrm{k}(P) = 1$. Then,

$$\mathrm{d}^*\mathscr{F} = * \circ \mathrm{d} \circ *\mathscr{F} = *\mathrm{d}\mathscr{F} = 0\,,$$

that is, $\mathscr{F}$ is harmonic. Conversely, if $\mathscr{F}$ is harmonic, then using (5.7.56) we have

$$\|\mathscr{F}_-\|^2 = -\mathsf{s}_M(\mathscr{F}_-, \mathscr{F}_-)$$

and, therefore, assumption (b) implies $\mathscr{F}_- = 0$ showing that ω is self-dual. Moreover,

$$\begin{aligned}
\mathsf{s}_M(\mathscr{F}, \mathscr{F}) &= \int_M \mathscr{F} \wedge \mathscr{F} \\
&= -\frac{1}{2}\int_M \operatorname{tr}\left(\begin{bmatrix} \mathrm{i}\mathscr{F} & 0 \\ 0 & -\mathrm{i}\mathscr{F} \end{bmatrix} \wedge \begin{bmatrix} \mathrm{i}\mathscr{F} & 0 \\ 0 & -\mathrm{i}\mathscr{F} \end{bmatrix}\right) \\
&= -4\pi^2 \mathfrak{c}_1(P) \\
&= 4\pi^2\,.
\end{aligned}$$

Thus, $\mathsf{c}_1(P(\omega)) = [-(2\pi)^{-1}\mathscr{F}]$ fulfils

$$\mathsf{s}_M(\mathsf{c}_1(P(\omega)), \mathsf{c}_1(P(\omega))) = 1\,. \tag{6.5.19}$$

It is also easily seen (Exercise 6.5.5) that $\mathscr{F}$ is the same for any element of the gauge-equivalence class defined by ω. Note, however, that $\mathscr{F}$ is not invariant under conjugation with elements of the form

$$\begin{bmatrix} 0 & \mathrm{e}^{i\varphi} \\ \mathrm{e}^{-i\varphi} & 0 \end{bmatrix} \in G\,.$$

Under such a transformation, elements of H given by (6.5.18), are transformed into their inverses. On the level of the Lie algebra $i\mathbb{R}$, this means that elements are sent to their negatives. Thus, in particular, $\mathscr{F}$ is sent to $-\mathscr{F}$. To summarize, we have constructed a mapping between gauge-equivalence classes $[\omega]$ of reducible self-dual connections on P and pairs $(u, -u)$ with $u \in H^2_{\mathbb{Z}}(M)$ fulfilling $s_M(u, u) = 1$.

Proposition 6.5.11 *The assignment* $[\omega] \to \pm\mathsf{c}_1(P(\omega))$ *is bijective.*

Proof Since, by assumption (a), we have $H^1_{\mathrm{dR}}(M) = 0$, injectivity is an immediate consequence of Proposition 4.8.1. For the proof of surjectivity, let $u \in H^2_{\mathbb{Z}}(M)$

fulfilling $\mathsf{s}_M(u, u) = 1$. We construct a reducible self-dual connection ω such that $\mathsf{c}_1(P(\omega)) = \pm u$ as follows. Let L be a line bundle with $\mathsf{c}_1(L) = u$ and let $\overline{L}$ be its conjugate bundle. Then, endowing L with a Hermitean fibre metric, $E := L \oplus \overline{L}$ becomes a Hermitean vector bundle. Let $O(E)$ be the associated principal U(2)-bundle of unitary frames. Clearly, $O(E)$ reduces to a principal SU(2)-bundle Q. Using $\mathsf{s}_M(u, u) = 1$, we find

$$\mathsf{c}_2(L \oplus \overline{L}) = \mathsf{c}_1(L) \cup \mathsf{c}_1(\overline{L}) = u \cup (-u) = -1\,,$$

and, thus, Q is isomorphic to P according to Theorem 4.8.8. Let $\hat{Q} \subset Q$ be the subbundle of unitary frames of $L \subset E$. Then, for any connection $\hat{\omega}$ on $\hat{Q}$, the associated 1-form $\mathscr{F}$ of its curvature fulfils $[-(2\pi)^{-1}\mathscr{F}] = u$. Now, by Hodge theory, there exists a 1-form α on M such that $\mathscr{F} + \mathrm{d}\alpha$ is harmonic.[37] As already stated above, by assumption (b), to this harmonic 2-form there corresponds a self-dual reducible connection on P. ■

Let $2\nu(M)$ be the number of elements $u \in H^2_{\mathbb{Z}}(M)$ fulfilling $s_M(u, u) = 1$. Then, under the assumptions of the above proposition, the moduli space $\mathfrak{M}$ contains exactly $\nu(M)$ reducible connections. The structure of the singularities caused by these points have been analyzed in detail, see e.g. Sect. 4 of [213]. The starting point is the following. In the case under consideration, the stabilizer $\mathscr{G}_\omega$ of a reducible self-dual connection ω is isomorphic to S^1 and its Lie algebra is the 1-dimensional kernel of

$$\mathrm{d}_\omega : \Omega^0(M, \mathrm{Ad}(P)) \to \Omega^1(M, \mathrm{Ad}(P))\,.$$

The latter represents the 0-th cohomology of the elliptic complex (6.5.3). Clearly, $\mathscr{G}_\omega$ acts on the cohomology groups of this complex, and the complex is equivariant under this action. Now, as in the proof of Theorem 6.5.6, one can construct local slices of the form (6.5.10), the only difference being that one must factorize with respect to the S^1-action. Moreover, as already mentioned, in general we now have $H^2_\omega \neq 0$. By ellipticity of the complex, the mapping $\mathrm{d}_1 = \mathrm{p}_- \circ \mathrm{d}^1_\omega$ restricted to a slice defined by $\mathrm{d}^*_\omega \alpha = 0$, $\alpha \in \Omega^1(M, \mathrm{Ad}(P))$, is Fredholm. This is the basic fact which makes it possible to calculate the first and the second cohomology of the complex, together with the action of S^1, explicitly. One obtains [213]

$$H^1_\omega \cong \mathbb{C}^q\,, \quad H^2_\omega \cong \mathbb{C}^p \oplus \mathrm{p}_-(H^2_{\mathrm{dR}}(M))\,, \tag{6.5.20}$$

for some integers p and q. Here, $\mathbb{C}^q$ and $\mathbb{C}^p$ are endowed with the standard S^1-action. On $\mathrm{p}_-(H^2_{\mathrm{dR}}(M))$, S^1 acts trivially. Moreover, if

$$\mathrm{p}_-(H^2_{\mathrm{dR}}(M)) = 0\,, \tag{6.5.21}$$

then $p + q = 3$, the latter following from the Atiyah–Singer Index Theorem.

[37] Clearly, we have $[\mathscr{F} + \mathrm{d}\alpha] = u$.

Remark 6.5.12 Recall that the signature of s_M is denoted by (b^+, b^-). Since

$$\mathsf{s}_M(\alpha, \alpha) = \|\alpha_+\|^2 - \|\alpha_-\|^2 \,,$$

$\mathrm{p}_-(H^2_{\mathrm{dR}}(M))$ is the maximal subspace where s_M is negative definite. Thus, the condition (6.5.21) is equivalent to $b^- = 0$, that is, it is equivalent to the condition that s_M be positive definite. ♦

We conclude that if $H^2_\omega = 0$, there exists a small neighbourhood of ω homeomorphic to $\mathbb{C}^3/\mathrm{U}(1)$. The latter may be identified with a cone on $\mathbb{C}\mathrm{P}^2$.

For $H^2_\omega \neq 0$, the situation is much more complicated. In this context, the idea of perturbing the metric of the base manifold M plays a crucial role. One can prove the following [213].

(a) The set of C^k-metrics on M for which the irreducible connections in $\mathfrak{M}$ form a smooth manifold is open and dense.
(b) For an open and dense set of C^k-metrics, H^2_ω vanishes at each singular point in $\mathfrak{M}$.

Remark 6.5.13 By point (b), we see that the above local description of the singular points in terms of cones on $\mathbb{C}\mathrm{P}^2$ holds true in the generic case. Moreover, we obtain a generalization of the dimension formula (6.5.11) of Atiyah, Hitchin and Singer to the case of arbitrary compact 4-manifolds (for an open and dense set of metrics), cf. the proof of Theorem 6.5.6 where, originally, Lemma 6.5.4 and, thus, the self-duality of M was used. ♦

Next, one shows that the manifold $\hat{\mathfrak{M}} \subset \mathfrak{M}$ of irreducible connections is orientable. Finally, using deep analytic results of Taubes [613] on the existence of self-dual connections for the class of manifolds of the above type, one can prove that there exists a collar $(0, 1] \times M \subset \mathfrak{M}$ and that $\mathfrak{M} \cup M$ is a compact manifold with boundary. To summarize, one has the following fundamental theorem.

Theorem 6.5.14 (Donaldson) *Let P be a principal* $\mathrm{SU}(2)$*-bundle with instanton number* $\mathrm{k}(P) = 1$ *over a compact, simply connected, oriented smooth* 4*-manifold with positive definite intersection form. Then, the moduli space $\mathfrak{M}$ has the following structure.*

1. *Let $2\nu(M)$ be the number of solutions to the equation $s_M(u, u) = 1$. Then, for almost all metrics on M, there exist ν points $p_1, \dots, p_{\nu(M)}$ in $\mathfrak{M}$ such that $\mathfrak{M} \setminus \{p_1, \dots, p_{\nu(M)}\}$ is a smooth* 5*-dimensional oriented manifold. The points p_i are in one-to-one correspondence with gauge equivalence classes of reducible self-dual connections.*
2. *Each point p_i admits a neighbourhood of $\mathfrak{M}$ which is homeomorphic to a cone on* $\mathbb{C}\mathrm{P}^2$.
3. *There exists a collar* $(0, 1] \times M \subset \mathfrak{M}$ *and the space $\overline{\mathfrak{M}} = \mathfrak{M} \cup M$ is a compact manifold with boundary.*

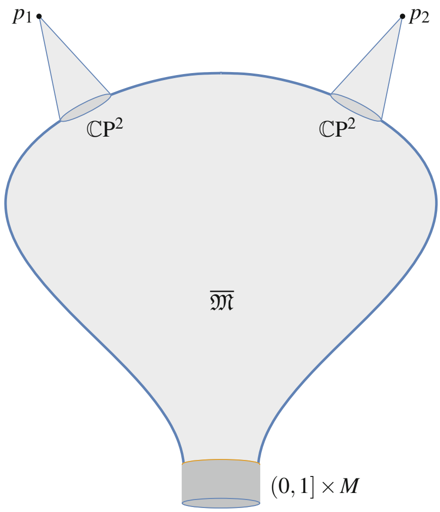

Fig. 6.3 The moduli space $\mathfrak{M}$ of Theorem 6.5.14 for the case $\nu(M) = 2$

This leads to a modification of the shape of the moduli space described under point 1 of Remark 6.5.10, see Fig. 6.3.

Remark 6.5.15 The assumptions in Donaldson's Theorem may be relaxed, see [213] and [159]. In particular, the assumption that s_M be positive definite may be dropped. Then, it is reasonable to rewrite (6.5.11) as

$$\dim \mathfrak{M} = 2\mathfrak{p}_1(\mathrm{Ad}(P)) - \frac{1}{2}\dim G(1 - b_1 + b^-)\,, \tag{6.5.22}$$

where b_1 is the first Betti number and b^- is the second component of the signature of the intersection form s_M. ♦

Exercises

6.5.1 Prove (6.5.15).

6.5.2 Complete the proof of Lemma 6.5.5 by showing that the principal symbols of $\mathrm{d}_0^* + \mathrm{d}_1$ and P_e^- coincide.

6.5.3 Complete the proofs of the statements of Example 6.5.7.

6.5.4 Prove that the invariant connection $\tilde{\omega}$ constructed in the proof of Lemma 6.5.8 coincides (under the identification mentioned in this proof) with ω^- constructed in Sect. 6.3.

6.5.5 Prove that the 1-form $\mathscr{F}$ on M, representing the curvature of a reducible connection ω is the same for any element of the gauge-equivalence class defined by ω.

6.6 Instantons and Smooth 4-Manifolds

In this section, we show that the results of the previous section have deep implications on the theory of differentiable structures on compact simply connected 4-manifolds. We start with recalling some basic topological results without giving proofs. By a fundamental theorem of Whitehead [665], two compact simply connected topological 4-manifolds are homotopy equivalent iff their intersection forms are equivalent. Thus, let M be a compact simply connected 4-manifold. Then, $\mathsf{w}_1(M) = 0$ and hence M is orientable. Let us fix an orientation. If M is not smooth, then the definition of the intersection form s_M given by (5.7.56) has to be generalized as follows. For $u, v \in H^2_{\mathbb{Z}}(M)$, we define

$$\mathsf{s}_M(u, v) := (u \cup v)[M] \,, \tag{6.6.1}$$

where $\cup : H^2_{\mathbb{Z}}(M) \otimes H^2_{\mathbb{Z}}(M) \to H^4_{\mathbb{Z}}(M)$ is the cup-product and $[M] \in H_4(M)$ is the fundamental class of M given by the orientation. Clearly, s_M is a symmetric non-degenerate bilinear form on $H^2_{\mathbb{Z}}(M)$. As before, its signature is denoted by (b^+, b^-), the difference $\sigma(M) := b^+ - b^-$ is called the signature of M and the rank of $H^2_{\mathbb{Z}}(M)$ is denoted by $b(M)$.

By Poincaré duality, the elements u and v of $H^2_{\mathbb{Z}}(M)$ may be represented by cycles μ and ν belonging to $H_2(M)$. Under this identification, one assigns to each intersection point of μ and ν an integer ± 1 and s_M is the sum of these multiplicities. This interpretation explains the name of s_M. It also shows that s_M is unimodular,[38] see [406] for further details.

Example 6.6.1

1. Let $M = \mathrm{S}^4$. We have $H_2(\mathrm{S}^4) = H^2_{\mathbb{Z}}(\mathrm{S}^4) = 0$ and, thus, $\mathsf{s}_M = 0$.
2. Let $M = \mathrm{S}^2 \times \mathrm{S}^2$. Then, $H_2(\mathrm{S}^2 \times \mathrm{S}^2)$ is generated by $u = \mathrm{S}^2 \times \{*\}$ and $v = \{*\} \times \mathrm{S}^2$, where $*$ denotes a chosen point of S^2. Thus, the matrix of s_M in the basis $\{u, v\}$ of $H_2(\mathrm{S}^2 \times \mathrm{S}^2)$ is given by
$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$
3. Let $M = \mathbb{C}\mathrm{P}^2$. Here, the second homology $H_2(\mathbb{C}\mathrm{P}^2)$ has one generator. Thus, the matrix of s_M is given by the 1×1-matrix with entry 1, which in the present context is usually denoted by $\langle 1 \rangle$. ♦

Definition 6.6.2 A unimodular symmetric bilinear form s over $\mathbb{Z}$ is called even (or of type II) if $\mathsf{s}(u, u) \in 2\mathbb{Z}$ for all $u \in H^2_{\mathbb{Z}}(M)$. Otherwise, it is called odd (or of type I).

Equivalently, viewing s as a matrix, it is even if all its diagonal entries are even and odd otherwise.

[38] If s_M is expressed as a matrix with integer entries, then $\det(\mathsf{s}_M) = \pm 1$.

The following facts may be found in [450]. Indefinite unimodular symmetric bilinear forms s are classified by their rank and signature.

(a) For type I, they are given by

$$\mathsf{s} = \langle 1 \rangle \oplus \ldots \oplus \langle 1 \rangle \oplus \langle -1 \rangle \oplus \ldots \oplus \langle -1 \rangle \,,$$

where $\langle 1 \rangle$ and $\langle -1 \rangle$ denote the two possible 1-forms of rank 1.

(b) For type II, they are given by

$$\mathsf{s} = \sigma_1 \oplus \ldots \oplus \sigma_1 \oplus E_8 \oplus \ldots \oplus E_8 \,,$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

The classification of definite unimodular symmetric bilinear forms over $\mathbb{Z}$ is a much more involved task. In Table 2.5 of the book of Lawson [406], the reader can find a list showing that the number of such forms drastically increases with their rank, e.g. for rank 40, there are more than 10^{51} such forms.

Now, by the result of Whitehead cited above, the following questions naturally arise:

(a) Which unimodular symmetric bilinear forms can appear as intersection forms of a compact simply connected 4-manifold?
(b) How many inequivalent manifolds carry the same form?

For topological manifolds, these questions have been answered by Freedman [214] in 1982. The Freedman Theorem states that every unimodular symmetric bilinear form over $\mathbb{Z}$ is the intersection form of a compact simply connected topological 4-manifold. Given such a form s, in the type II case, this manifold is unique, whereas in the type I case there are exactly two distinct manifolds corresponding to s.

Now, let us consider differentiable 4-manifolds. Apart from the classical Rohlin Theorem 5.9.7 stating that, for a compact simply connected[39] 4-manifold with intersection form of type II the signature $\sigma(M)$ is divisible by 16, up until 1982 not much was known. At this point, the work of Donaldson presented in the previous section

[39] If M is simply connected, vanishing of the second Stiefel–Whitney class is equivalent to the signature being of type II.

led to a breakthrough. We will show that Theorem 6.5.14 almost immediately implies the following.[40]

Theorem 6.6.3 (Donaldson) *Let M be a compact simply connected*[41] *oriented differentiable* 4*-manifold whose intersection form s_M is positive definite. Then,*

$$\mathsf{s}_M = \langle 1 \rangle \oplus \ldots \oplus \langle 1 \rangle \,. \tag{6.6.2}$$

Proof By Theorem 6.5.14 and standard cobordism theory, there exists a compact oriented 5-manifold $\mathfrak{M}_0 \subset \mathfrak{M}$ with boundary

$$\partial \mathfrak{M}_0 = M + p\,\mathbb{C}\mathrm{P}^2 + q\,\overline{\mathbb{C}\mathrm{P}^2}\,, \quad p + q = \nu(M)\,,$$

where $\overline{\mathbb{C}\mathrm{P}^2}$ denotes $\mathbb{C}\mathrm{P}^2$ with the opposite orientation. This manifold is obtained by removing, say $M \times (\frac{1}{2}, 1)$, from the collar and by removing neighbourhoods from each of the cone points. Since the signature $\sigma(M)$ is a cobordism invariant, we conclude $\sigma(M) = q - p$. Since the intersection form is positive definite, we have $\sigma(M) = b(M)$. Thus,

$$b(M) = \sigma(M) = q - p \leq q + p = \nu(M)\,. \tag{6.6.3}$$

On the other hand, for any element $u \in H^2_{\mathbb{Z}}(M)$ fulfilling $\mathsf{s}_M(u, u) = 1$, we may take the orthogonal decomposition

$$H^2_{\mathbb{Z}}(M) = \mathbb{Z}u \oplus H^2_{\mathbb{Z}}(M)^{\perp}\,,$$

given by writing

$$w = \mathsf{s}_M(w, u)u + (w - \mathsf{s}_M(w, u)u)\,,$$

for any $w \in H^2_{\mathbb{Z}}(M)$. Thus, for another element $v \in H^2_{\mathbb{Z}}(M)$ fulfilling $\mathsf{s}_M(v, v) = 1$ and such that $v \neq \pm u$, the Schwartz inequality implies $(\mathsf{s}_m(u, v))^2 < 1$ and thus

$$\mathsf{s}_M(u, v) = 0\,,$$

because $\mathsf{s}_M(u, v)$ is an integer. This implies $v \in H^2_{\mathbb{Z}}(M)^{\perp}$. By this procedure, we may exhaust the rank $b(M)$ of $H^2_{\mathbb{Z}}(M)$ iff s_M is diagonalizable over the integers. Consequently, we have

$$\nu(M) \leq b(M)$$

[40] The only additional input we need is elementary knowledge of cobordism theory. For our purposes, the information contained in Appendix B of [213] is sufficient. For a more detailed presentation, see e.g. [104], Sect. 16 of Chap. II.

[41] We present the theorem in its original formulation. The assumption of being simply connected may be dropped, see also the Remark after Theorem 6.5.14.

and $\nu(M) = b(M)$ iff s_M has the form given by (6.6.2). Combining this with (6.6.3) yields the assertion. ■

By this theorem, the answer to the questions (a) and (b) posed above is drastically simplified. All forms differing from (6.6.2) are ruled out. Combining this with the above mentioned result of Freedman and Example 6.6.1, we obtain the following.

Corollary 6.6.4 *Let M be a smooth compact simply-connected oriented 4-manifold. If s_M is positive definite and even, then M is homeomorphic to S^4. If s_M is positive definite and odd, then M is homeomorphic to a connected sum of positively oriented copies of $\mathbb{C}\mathrm{P}^2$.* ■

In the following example we sketch a striking consequence of Donaldson theory: the existence of exotic smooth structures on $\mathbb{R}^4$. For details we refer to [406] and [253].

Example 6.6.5 (Exotic differentiable structure on $\mathbb{R}^4$) Let us consider the compact simply connected topological 4-manifold M with intersection form

$$\mathsf{s}_M = E_8 \oplus \langle 1 \rangle \,.$$

Its existence is guaranteed by the Freedman Theorem. On the other hand, by the Donaldson Theorem, it does not admit a smooth structure. The idea of the construction consists in considering M with a point $p \in M$ removed. By a result of Gompf [253], this manifold is smoothable, that is, there exists a neighbourhood U of p in M such that $U \setminus \{p\}$ is diffeomorphic to $V \setminus \varphi(\mathrm{S}^2)$, where V is a neighbourhood of the image of $\mathrm{S}^2 \cong \mathbb{C}\mathrm{P}^1$ under a homeomorphism φ in $\mathbb{C}\mathrm{P}^2$.[42]

Now, consider the embedding $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2\,,\ [(z_1, z_2)] \mapsto [(z_1, z_2, 0)]\,.$ Then,

$$\mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1 \to \mathbb{C}^2\,, \quad [(z_1, z_2, z_3)] \mapsto (z_1/z_3, z_2/z_3)$$

is a homeomorphism. Thus, for any homeomorphism $\varphi : \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$, we have

$$\mathbb{C}\mathrm{P}^2 \setminus \varphi(\mathbb{C}\mathrm{P}^1) \cong \varphi(\mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1) \cong \mathbb{C}^2 \cong \mathbb{R}^4\,,$$

that is, $\mathbb{C}\mathrm{P}^2 \setminus \varphi(\mathbb{C}\mathrm{P}^1)$ is homeomorphic to $\mathbb{R}^4$. But, $\tilde{\mathbb{R}}^4 = \mathbb{C}\mathrm{P}^2 \setminus \varphi(\mathbb{C}\mathrm{P}^1)$ cannot be diffeomorphic to the ordinary $\mathbb{R}^4$. This follows from the fact that $\tilde{\mathbb{R}}^4$ contains a compact subset which cannot be enclosed by a smoothly embedded 3-sphere.[43] Indeed, choose an open neighbourhood $\tilde{U}$ of $\varphi(\mathrm{S}^2)$ and assume that the compact subset $K = \tilde{\mathbb{R}}^4 \setminus \big(\tilde{U} \setminus \varphi(\mathrm{S}^2)\big)$ can be enclosed by a smoothly embedded $\mathrm{S}^3 \subset \big(\tilde{U} \setminus \varphi(\mathrm{S}^2)\big)$. Then, we could cut along S^3 and attach a 4-disk. This would give a

[42] In such a situation, we say that p is resolvable. By a general theorem of Quinn [527], for any compact topological 4-manifold M whose Kirby-Siebenmann invariant is zero, the following holds: M has a smooth structure defined outside a finite set of singular points such that each of these points is resolvable. The manifold considered in the example fulfils the assumptions of this theorem.

[43] In the ordinary $\mathbb{R}^4$, any compact set can be enclosed by a smoothly embedded 3-sphere.

smoothing of M which, by the Donaldson Theorem, is impossible. For details of this surgery we refer to [406], [213]. ♦

For further examples and a lot of further references, we refer to the textbooks [574] and [28]. Clearly, these exotic structures are only part of a huge field of research initiated by Donaldson. In particular, Donaldson has constructed a set of new differential topological invariants, now called Donaldson invariants, of 4-manifolds, see [159], [574] and [28]. These invariants may be used to distinguish between the diffeomorphism types of certain 4-manifolds, e.g. they allowed for showing that there exist compact 4-manifolds with infinitely many non-equivalent smooth structures.

6.7 Stability

For the discussion of stability of solutions of the Yang–Mills equation we must find the second variational formula for the Yang–Mills functional (6.2.1) at a critical point. Thus, let ω be a critical point. As in Sect. 6.2, we consider $t \mapsto \omega_t = \omega + t\alpha$ with $\alpha \in \mathrm{T}_\omega\mathscr{C} = \mathscr{T}$ and calculate the second variation by expanding $S(\omega_t)$ up to second order. Using

$$\Omega_t = \Omega + t\mathrm{d}_\omega\alpha + \frac{t^2}{2}[\alpha, \alpha]\,, \tag{6.7.1}$$

we get

$$S(\omega_t) = S(\omega) + t\langle\Omega, \mathrm{d}_\omega\alpha\rangle_{L^2} + \frac{t^2}{2}(\langle\Omega, [\alpha, \alpha]\rangle_{L^2} + \langle\mathrm{d}_\omega\alpha, \mathrm{d}_\omega\alpha\rangle_{L^2})\,,$$

and thus

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\restriction_0 S(\omega_t) = \langle[\alpha, \alpha], \Omega\rangle_{L^2} + \langle\mathrm{d}_\omega\alpha, \mathrm{d}_\omega\alpha\rangle_{L^2}\,.$$

By definition of the adjoint, the second term may be rewritten as

$$\langle\mathrm{d}_\omega\alpha, \mathrm{d}_\omega\alpha\rangle_{L^2} = \langle\alpha, \mathrm{d}_\omega^*\mathrm{d}_\omega\alpha\rangle_{L^2}\,.$$

To calculate the first term, we decompose α and Ω in a local coframe $\{\vartheta^i\}$ in T^*M and use the Ad-invariance of the scalar product. Then, by (2.7.49),

$$\begin{aligned}
[\alpha, \alpha]\dot{\wedge} * \Omega &= \eta^{ik}\eta^{jl}\langle[\alpha_i, \alpha_j], \Omega_{kl}\rangle\mathsf{v_g} \\
&= \eta^{jl}\langle\alpha_j, \eta^{ik}[\Omega_{kl}, \alpha_i]\rangle\mathsf{v_g} \\
&= \alpha\dot{\wedge} * \mathfrak{R}^{\nabla^\omega}(\alpha)\,,
\end{aligned}$$

where

$$\mathfrak{R}^{\nabla^\omega}(\alpha) = \eta^{ik}[\Omega_{kl}, \alpha_i]\vartheta^l \tag{6.7.2}$$

is the Weitzenboeck curvature operator for the case $\sigma = \mathrm{Ad}$ acting on 1-forms, cf. Definition 2.7.10. Thus, the Hessian at ω of the Yang–Mills functional S is given by

$$\mathfrak{H}_\omega = \mathrm{d}_\omega^* \mathrm{d}_\omega + \mathfrak{R}^{\nabla^\omega} . \tag{6.7.3}$$

This is the basic object for the study of stability. Clearly, by gauge invariance of the Yang–Mills action, the variational problem we are dealing with may be viewed as a problem on the gauge orbit space $\mathscr{M}$. Thus, in a first step we should get rid of variations along the gauge orbits. This is done by using the decomposition (6.1.28), with the first component representing the subspace tangent to the orbit and the second being a model of the tangent space to the gauge orbit space at $[\omega]$. Thus, by gauge invariance of the Yang–Mills functional, we may restrict the above variational problem to the subspace of variations fulfilling

$$\mathrm{d}_\omega^* \alpha = 0 .$$

If we do so, the Hessian (6.7.3) may be rewritten as

$$\mathfrak{H}_\omega = \Box_\omega + \mathfrak{R}^{\nabla^\omega} . \tag{6.7.4}$$

This object may now be investigated using standard geometric methods. In this analysis, the crucial role is played by the Generalized Weitzenboeck Formula (2.7.61). Applying point 1 of Corollary 2.7.21 to the case $E = \mathrm{Ad}(P)$, we obtain

$$\mathfrak{H}_\omega(\alpha) = \left(\nabla^{(\omega^0+\omega)}\right)^* \nabla^{(\omega^0+\omega)} \alpha + \alpha \circ \mathsf{Ric} + 2\mathfrak{R}^{\nabla^\omega}(\alpha) , \tag{6.7.5}$$

for any $\alpha \in \Omega^1(M, \mathrm{Ad}(P))$ fulfilling $\mathrm{d}_\omega^* \alpha = 0$.

Definition 6.7.1 A Yang–Mills connection ω is said to be stable if

$$\langle \alpha, \mathfrak{H}_\omega(\alpha) \rangle_{L^2} \; > \; 0$$

for all nonzero $\alpha \in \ker \mathrm{d}_\omega^* \subset \Omega^1(M, \mathrm{Ad}(P))$. It is said to be weakly stable if $\langle \alpha, \mathfrak{H}_\omega(\alpha) \rangle_{L^2} \geq 0$.

Remark 6.7.2 By the results of Chap. 5, the operator $\mathfrak{H}_\omega$ is elliptic and self-adjoint. Morever, the Bochner-Laplace operator $\left(\nabla^{(\omega^0+\omega)}\right)^* \nabla^{(\omega^0+\omega)}$ is obviously non-negative. Thus, the restriction of $\mathfrak{H}_\omega$ to $\ker \mathrm{d}_\omega^*$, has eigenvalues $\lambda_1 < \lambda_2 < \ldots$ such that $\lim_{n\to\infty} \lambda_n \to \infty$ and the corresponding eigenspaces E_{λ_i} are finite-dimensional. One defines the index $i(\omega)$ and the nullity $n(\omega)$ of ω by

$$i(\omega) := \dim\left(\oplus_{\lambda<0} E_\lambda\right) , \quad n(\omega) := \dim E_0 .$$

In this Morse theoretic terminology, a solution ω is stable iff $i(\omega) = n(\omega) = 0$. Correspondingly, a solution is weakly stable iff $i(\omega) = 0$. ♦

For the study of stability, we follow the classical paper of Bourguignon and Lawson [95]. To start with, we need the following observations. First, recall the notion of the gradient $\operatorname{grad} f = \mathsf{g}^{-1}(\mathrm{d}f)$ of a function $f \in C^\infty(M)$. A vector field $X \in \mathfrak{X}(M)$ is said to be of gradient type if

$$\mathrm{d}(\mathsf{g}(X)) = 0 .$$

Clearly, this condition implies that, locally, there exists a function f such that $X = \operatorname{grad} f$, that is, $\mathsf{g}(X) = \mathrm{d}f$. This explains the terminology.

Lemma 6.7.3 *Let $X \in \mathfrak{X}(M)$ be of gradient type and let $\beta \in \Omega^2(M, \mathrm{Ad}(P))$ be such that $\mathrm{d}^*_\omega \beta = 0$. Then,*

$$\mathrm{d}^*_\omega (X \lrcorner \beta) = 0 .$$

Proof Setting $\beta = *\alpha$ in (2.7.9) and using (2.7.3), we obtain $X \lrcorner \beta = *(*\beta \wedge \mathsf{g}(X))$. Now, using (2.7.13), (2.7.3), (1.5.9) and once again (2.7.9), we calculate

$$\begin{aligned} \mathrm{d}^*_\omega (X \lrcorner \beta) &= (-1)^n * \mathrm{d}_\omega (*\beta \wedge \mathsf{g}(X)) \\ &= (-1)^n * \big(\mathrm{d}_\omega(*\beta) \wedge \mathsf{g}(X) + (-1)^{(n-2)} (*\beta) \wedge \mathrm{d}(\mathsf{g}(X))\big) \\ &= 0 . \end{aligned}$$

■

Now, let us focus on the case $M = \mathrm{S}^n$. We consider the following finite-dimensional subspace of $\mathfrak{X}(\mathrm{S}^n)$:

$$\mathscr{V} := \left\{ \operatorname{grad} f \in \mathfrak{X}(\mathrm{S}^n) : f = F_{\restriction \mathrm{S}^n} \text{ for some linear } \; F : \mathbb{R}^{n+1} \to \mathbb{R} \right\} . \tag{6.7.6}$$

Note that we have a natural isomorphism

$$\mathbb{R}^{n+1} \to \mathscr{V} , \quad \mathbf{v} \mapsto V(\mathbf{x}) := \mathbf{v} - \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{x} , \quad \mathbf{x} \in \mathrm{S}^n . \tag{6.7.7}$$

It is easy to see (Exercise 6.7.1) that

$$V = \operatorname{grad} f , \quad \text{where } \; f(\mathbf{x}) = \langle \mathbf{v}, \mathbf{x} \rangle . \tag{6.7.8}$$

Given a submanifold $M \subset \mathbb{R}^k$, the orthogonal projector $\mathbb{P}$ onto $\mathrm{T}M$ along the orthogonal complement of $\mathrm{T}M$ in $\mathbb{R}^k$ defines a connection ∇^0 on M called the induced Euclidean connection, cf. formula (6.4.13). It can be shown that ∇^0 coincides with the covariant derivative ∇^{ω^0} of the Levi-Civita connection defined by the natural induced Riemannian metric on M. We apply this concept to the case of $\mathrm{S}^n \subset \mathbb{R}^{n+1}$.

Lemma 6.7.4 *Let ∇^0 be the induced Euclidean connection on S^n. For any $V \in \mathscr{V}$,*

$$\nabla^0_Y V = -fY , \quad (\nabla^0)^* \nabla^0 V = V , \tag{6.7.9}$$

where $Y \in \mathfrak{X}(\mathrm{S}^n)$.

Proof To prove the first equation, for $Y \in \mathrm{T}_{\mathbf{x}}\mathrm{S}^n$, we calculate

$$\big(\nabla^0_Y V\big)(\mathbf{x}) = \mathbb{P} \circ \nabla_Y(\mathbf{v} - \langle \mathbf{v}, \mathbf{x}\rangle \mathbf{x}) = \mathbb{P}(-\langle \mathbf{v}, \mathbf{x}\rangle Y) = -f(\mathbf{x})Y\,.$$

To prove the second equation, we choose an orthonormal frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ on a neighbourhood of $\mathbf{x} \in \mathrm{S}^n$ and use formula (2.7.31). Then,

$$(\nabla^0)^*\nabla^0 V = -\sum_i \big(\nabla^0_{\mathbf{e}_i}\nabla^0_{\mathbf{e}_i} - \nabla^0_{\nabla^0_{\mathbf{e}_i}\mathbf{e}_i}\big)V = -\sum_i \nabla^0_{\mathbf{e}_i}\big(\mathbb{P} \circ \nabla_{\mathbf{e}_i} V\big)\,.$$

Thus, using the first equation, we get

$$(\nabla^0)^*\nabla^0 V = -\sum_i \mathbb{P} \circ \nabla_{\mathbf{e}_i}\big(-f\mathbf{e}_i\big) = \sum_i \mathbb{P}\big(\mathbf{e}_i(f)\mathbf{e}_i\big) = \sum_i \mathbb{P}\big(\langle \mathbf{v}, \mathbf{e}_i\rangle \mathbf{e}_i\big) = \mathbb{P}(\mathbf{v}) = V\,.$$

■

For the study of stability, the following family of quadratic forms on $\mathscr{V}$ will be crucial. For any $\beta \in \Omega^2(M, \mathrm{Ad}(P))$, we put

$$Q_\beta(V) := \langle i_V\beta, \mathfrak{H}_\omega(i_V\beta)\rangle_{L^2}\,. \tag{6.7.10}$$

Proposition 6.7.5 (Bourguignon–Lawson) *Let P be a principal G-bundle over S^n and let ω be a Yang–Mills connection. Let $\beta \in \Omega^2(M, \mathrm{Ad}(P))$ be harmonic, that is,*

$$\mathrm{d}^*_\omega\beta = 0\,, \quad \mathrm{d}_\omega\beta = 0\,.$$

Then, the trace of the quadratic form Q_β is given by

$$\mathrm{tr}(Q_\beta) = 2(4-n)\|\beta\|^2\,.$$

Proof For simplicity, in this proof we write ∇ for $\nabla^{(\omega^0+\omega)}$ and ∇^0 for ∇^{ω^0}.

By Lemma 6.7.3, we have $\mathrm{d}^*_\omega(i_V\beta) = 0$ for any $V \in \mathscr{V}$. Thus, the Hessian acting on $i_V\beta$ is given by formula (6.7.5),

$$\mathfrak{H}_\omega(i_V\beta) = \nabla^*\nabla(i_V\beta) + i_V\beta \circ \mathsf{Ric} + 2\mathfrak{R}^{\nabla^\omega}(i_V\beta)\,. \tag{6.7.11}$$

First, we calculate $\mathfrak{H}_\omega(i_V\beta)$ at a point $\mathbf{x} \in \mathrm{S}^n$. For that purpose, we choose an orthonormal basis $\{\varepsilon_0, \dots, \varepsilon_n\}$ of $\mathscr{V}$ such that
(a) under the isomorphism (6.7.7), $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ correspond to $\mathbf{x}, \mathbf{e}_1, \dots, \mathbf{e}_n$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ form an orthonormal basis of $\mathrm{T}_{\mathbf{x}}\mathrm{S}^n$. Then,

$$\varepsilon_0(\mathbf{x}) = 0\,, \quad \varepsilon_1(\mathbf{x}) = \mathbf{e}_1\,, \quad \dots\,, \quad \varepsilon_n(\mathbf{x}) = \mathbf{e}_n\,.$$

(b) the vector fields $\varepsilon_1, \dots, \varepsilon_n$ are parallel at $\mathbf{x}$,

$$\nabla^0 \varepsilon_j(\mathbf{x}) = 0\,, \quad j = 1, \ldots, n\,.$$

Now, using (2.7.31), for any vector field X we calculate at $\mathbf{x}$:

$$\begin{aligned}
\big(\nabla^*\nabla(i_V\beta)\big)(X) &= -\sum_j \big(\nabla_{\varepsilon_j}\nabla_{\varepsilon_j}(i_V\beta) - \nabla_{\nabla_{\varepsilon_j}\varepsilon_j}(i_V\beta)\big)(X) \\
&= -\sum_j \big(\nabla_{\varepsilon_j}\nabla_{\varepsilon_j}(i_V\beta)\big)(X) \\
&= -\sum_j \nabla_{\varepsilon_j}\Big\{\nabla_{\varepsilon_j}\big((i_V\beta)(X)\big) - (i_V\beta)\big(\nabla^0_{\varepsilon_j}X\big)\Big\} \\
&= -\sum_j \nabla_{\varepsilon_j}\Big\{\big(\nabla_{\varepsilon_j}\beta\big)(V,X) + \beta(\nabla^0_{\varepsilon_j}V, X)\Big\}\,.
\end{aligned}$$

We may choose $X = \sum_j a_j\varepsilon_j$ with constant coefficients a_j. Then, $\nabla^0_{\varepsilon_j}X = 0$ and, thus,

$$\begin{aligned}
&\big(\nabla^*\nabla(i_V\beta)\big)(X) \\
&\quad= -\sum_j \big(\nabla_{\varepsilon_j}\nabla_{\varepsilon_j}\beta\big)(V,X) - 2\sum_j \big(\nabla_{\varepsilon_j}\beta\big)\big(\nabla^0_{\varepsilon_j}V, X\big) - \beta\Big(\sum_j \nabla^0_{\varepsilon_j}\nabla^0_{\varepsilon_j}V, X\Big) \\
&\quad= \big(\nabla^*\nabla\beta\big)(V,X) - 2\sum_j \big(\nabla_{\varepsilon_j}\beta\big)\big(\nabla^0_{\varepsilon_j}V, X\big) + \beta\big(\nabla^{0*}\nabla^0 V, X\big)\,.
\end{aligned}$$

By (2.7.25), $\mathrm{d}^*_\omega\beta(X) = 0$ means $\sum_j(\nabla_{\varepsilon_j}\beta)(\varepsilon_j, X) = 0$. Thus, using the first equation of (6.7.9), we have

$$\sum_j(\nabla_{\varepsilon_j}\beta)(\nabla^0_{\varepsilon_j}V, X) = -\sum_j(\nabla_{\varepsilon_j}\beta)(f\varepsilon_j, X) = 0\,.$$

Together with the second equation of (6.7.9), this implies

$$\big(\nabla^*\nabla(i_V\beta)\big)(X) = \big(\nabla^*\nabla\beta\big)(V,X) + \beta(V,X)\,.$$

Inserting this result into (6.7.11), using Example 2.7.13 and formula (6.7.2), we obtain at $\mathbf{x}$:

$$\mathfrak{H}_\omega(i_V\beta) = \big(\nabla^*\nabla\beta)(V,\cdot) + n\beta(V,\cdot) + 2\sum_{i=1}^n[\Omega(\mathbf{e}_i,\cdot), \beta(V,\mathbf{e}_i)]\,. \tag{6.7.12}$$

Finally, we apply the Generalized Weitzenboeck Formula (2.7.61) to the 2-form β. Using (2.7.45) and $\Box_\omega\beta = 0$, we obtain

$$\big(\nabla^*\nabla\beta)(V,\cdot) = -2(n-2)\beta(V,\cdot) - \sum_{i=1}^{n}\{[\Omega(\mathbf{e}_i,V),\beta(\mathbf{e}_i,\cdot)] - [\Omega(\mathbf{e}_i,\cdot),\beta(\mathbf{e}_i,V)]\}\ ,$$

and thus, at $\mathbf{x}$ we have

$$\mathfrak{H}_\omega(i_V\beta) = (4-n)\big(i_V\beta\big) - \sum_{i=1}^{n}\{[\Omega(\mathbf{e}_i,V),\beta(\mathbf{e}_i,\cdot)] + [\Omega(\mathbf{e}_i,\cdot),\beta(\mathbf{e}_i,V)]\}\ . \tag{6.7.13}$$

Now, we can calculate

$$\begin{aligned}\operatorname{tr}\big(Q_\beta\big) &= \sum_{j=0}^{n}\langle i_{\varepsilon_j}\beta, \mathfrak{H}_\omega(i_{\varepsilon_j}\beta)\rangle_{L^2}\\ &= 2(4-n)\sum_{j<k}^{n}\int_{\mathrm{S}^n}\langle\beta(\mathbf{e}_j,\mathbf{e}_k),\beta(\mathbf{e}_j,\mathbf{e}_k)\rangle_{\mathfrak{g}}\mathsf{v}_{\mathsf{g}}\\ &= 2(4-n)\|\beta\|^2\ ,\end{aligned}$$

because the second term in (6.7.13) results in taking the contraction of a symmetric 2-form with an anti-symmetric one. ■

Note that harmonic elements of $\Omega^2(M,\mathrm{Ad}(P))$ certainly exist, e.g. for β we can take the curvature form of ω. Thus, as an immediate consequence of this proposition, we obtain the following.[44]

Corollary 6.7.6 *There are no weakly stable Yang–Mills connections on* S^n *for* $n \geq 5$. ■

Theorem 6.7.7 (Bourguignon–Lawson) *Any weakly stable Yang–Mills connection on* S^4 *with gauge group* $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ *or* $\mathrm{U}(2)$ *is either self-dual or anti-self-dual.*

Proof Assume that ω is a weakly stable solution. Then, $\langle\alpha,\mathfrak{H}_\omega(\alpha)\rangle_{L^2} \geq 0$ for any nonzero $\alpha \in \ker \mathrm{d}^*_\omega \subset \Omega^1(M,\mathrm{Ad}(P))$. Thus,

$$Q_\beta(V) = \langle i_V\beta, \mathfrak{H}_\omega(i_V\beta)\rangle_{L^2} \geq 0\,,$$

for any $V \in \mathscr{V}$ and any $\beta \in \Omega^2(M,\mathrm{Ad}(P))$ fulfilling $\mathrm{d}^*_\omega\beta = 0$. Now, by Proposition (6.7.5), for $n = 4$ we have $\operatorname{tr}\big(Q_\beta\big) = 0$ and thus

$$\mathfrak{H}_\omega(i_V\beta) = 0\,, \tag{6.7.14}$$

for any $V \in \mathscr{V}$ and any harmonic $\beta \in \Omega^2(M,\mathrm{Ad}(P))$. Next, consider the curvature form Ω of ω. Since ω is a Yang–Mills connection, Ω is harmonic and thus fulfils (6.7.14). Let us decompose Ω into its self-dual and anti-self-dual components,

[44] The authors of [95] assign this result to J. Simons.

$$\Omega = \Omega^+ + \Omega^- .$$

It is almost immediate (Exercise 6.7.2) that, on a compact oriented 4-manifold, a vector-valued 2-form is harmonic iff its self-dual and anti-self-dual components are both harmonic. Thus,

$$\mathfrak{H}_\omega(i_V \Omega^\pm) = 0 .$$

Consequently, using (6.7.12), we obtain

$$\left(\nabla^* \nabla \Omega^+\right)(V, \cdot) + 4\Omega^+(V, \cdot) + 2\sum_{i=1}^{4} [\Omega^+(\mathbf{e}_i, \cdot) + \Omega^-(\mathbf{e}_i, \cdot), \Omega^+(V, \mathbf{e}_i)] = 0 .$$

By linearity in V, we may substitute for V any of the frame elements ε_i and then take arbitrary linear combinations of the resulting equations. This yields

$$\begin{aligned} &\left(\nabla^* \nabla \Omega^+\right)(X, Y) + 4\Omega^+(X, Y) + 2\sum_{i=1}^{4} [\Omega^+(\mathbf{e}_i, X), \Omega^+(\mathbf{e}_i, Y)] \\ &\quad = -2\sum_{j=1}^{4} [\Omega^+(\mathbf{e}_i, X), \Omega^-(\mathbf{e}_i, Y)] , \end{aligned} \tag{6.7.15}$$

for any $X, Y \in \mathrm{T}_\mathbf{x}\mathrm{S}^4$. Clearly, the left hand side of this equation is anti-symmetric in X and Y. On the other hand, the right hand side is symmetric. This can be easily checked by direct inspection using the bases $\{\varphi^i_\pm\}$ of $\bigwedge^2_\pm \mathrm{T}^*M$ given under point 1 of Remark 2.8.1.[45] Thus, both sides of (6.7.15) must vanish. Using this fact, again by direct inspection using the bases $\{\varphi^i_\pm\}$, one finds (Exercise 6.7.3)

$$[\Omega^+(X, Y), \Omega^-(Z, W)] = 0 , \quad X, Y, Z, W \in \mathrm{T}_\mathbf{x}\mathrm{S}^4 . \tag{6.7.16}$$

We conclude that the Lie subalgebras $\mathfrak{g}^\pm_\mathbf{x}$ of $\mathfrak{g}$ generated by the curvature transformations $\Omega^\pm(X, Y)$, where $X, Y \in \mathrm{T}_\mathbf{x}\mathrm{S}^4$, commute,

$$[\mathfrak{g}^+_\mathbf{x}, \mathfrak{g}^-_\mathbf{x}] = 0 .$$

Now, by direct inspection of the table giving the classification of regular semisimple Lie subalgebras of a semisimple Lie algebra,[46] we see that if $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{su}(3)$ or $\mathfrak{u}(2)$, then either $\mathfrak{g}^+_\mathbf{x}$ or $\mathfrak{g}^-_\mathbf{x}$ must be Abelian. Therefore, one of the 4-tensors $T^\pm$, defined by

$$T^\pm(X, Y, Z, W) := [\Omega^\pm(X, Y), \Omega^\pm(Z, W)] , \quad X, Y, Z, W \in \mathrm{T}_\mathbf{x}\mathrm{S}^4 ,$$

[45] In more abstract terms, this fact is an immediate consequence of the isomorphism (2.8.11).

[46] See [170], Sect.II.5, n°17.

must vanish at $\mathbf{x}$ and, thus, also in a neighbourhood $U \subset \mathrm{S}^4$ of $\mathbf{x}$. Assume that $T^+ = 0$ on U. Then, by the Aronszajn Theorem [21], since T^+ is an algebraic function of a solution of the elliptic equation $\Box_\omega \Omega^+ = 0$, it admits a unique continuation to the whole of S^4, that is, $T^+ = 0$ on S^4. Then, (6.7.15) implies

$$\nabla^* \nabla \Omega^+ + 4\Omega^+ = 0\,.$$

Since $\nabla^*\nabla \geq 0$, we conclude $\Omega^+ = 0$. If T^- vanishes, an analogous argument yields $\Omega^- = 0$. ■

Remark 6.7.8

1. In [95], the reader can find various extensions of Theorem 6.7.7. First, the case of a real 4-dimensional Riemannian vector bundle E over a Riemannian 4-manifold M with structure group $G = \mathrm{SO}(4)$ is dealt with in detail. In that case, there are two independent characteristic invariants (the first Pontryagin index and the Euler number). This makes the analysis more delicate, but a similar result can be proved, see Theorem 8.11 therein. In more detail, the splittings of $\bigwedge^2 \mathrm{T}M$ and $\bigwedge^2 E$ induced by the Hodge star operator yield a two-fold decomposition of the Riemannian curvature form and, consequently, the stability conditions are spelled out in terms of what is called by the authors a two-fold self-duality. This notion seems to be a reasonable generalization of (anti-)self-duality for the nonsimple group SO(4). The simplest example of this type is the tangent bundle of S^4. Here, the first Pontryagin index vanishes, the Euler number is equal to 2 and the (two-fold self-dual, but not self-dual) Levi-Civita connection yields an absolute minimum. Second, it is quite straightforward to generalize Theorem 6.7.7 to the case of a 4-dimensional compact orientable homogeneous Riemannian manifold. Then, for the gauge group SU(2), any weakly stable Yang–Mills connection is either self-dual, or anti-self-dual, or reduces to an Abelian gauge field, see Theorem 10.1 in [95]. Similar results hold for U(2), SU(3) and SO(4), see [96].
2. We stress that [95] contains another interesting type of results. Using again Weitzenboeck type arguments, Bourguignon and Lawson prove the existence of C^0-neighbourhoods of the minimal Yang–Mills connections which do not contain any other solution. ♦

Exercises

6.7.1 Prove formula (6.7.8).

6.7.2 Prove that, on a compact oriented 4-manifold, a (vector-valued) 2-form β is harmonic iff its components β^+ and β^- are both harmonic.

6.7.3 Prove formula (6.7.16).

6.8 Non-minimal Solutions

In view of the results of the previous section, it is natural to ask whether there exist critical points of the Yang–Mills functional other than absolute or relative minima. It is interesting to address this question, in particular, in the case of bundles over S^4 with structure groups $SU(2)$, $SU(3)$ and $U(2)$. This problem is closely related to the question whether there exist non-(anti-)self-dual solutions on 4-dimensional Riemannian manifolds. In this section, we make some remarks on these problems. In the study of non-minimal solutions, two ingredients play a basic role: first, the theory of invariant connections as developed in Sect. 1.9 and, second, advanced methods of the calculus of variations as developed by Taubes [613–616, 618]. The latter are beyond the scope of this book.

We start with the following observation [339].

Proposition 6.8.1 (Itoh) *Let $M = K/H$ be a compact oriented Riemannian symmetric space and let P be a principal G-bundle admitting a lift of K to automorphisms of P. Then, the curvature of the canonical invariant connection*[47] *on P is parallel and, thus, the canonical invariant connection provides a Yang–Mills connection.*

Proof By Remark 1.9.7/1, principal G-bundles over K/H admitting a lift of K are labeled by Lie group homomorphisms $\lambda : H \to G$ and have the structure

$$P_\lambda = K \times_H G\,.$$

We denote the lift of the K-action on K/H to P_λ by $\varDelta$. Let ω^c be the canonical connection on P_λ, cf. Eq. (1.9.43), and let $\varOmega^c$ be its curvature form. We will prove that

$$\nabla^{(\omega^o+\omega^c)}\varOmega^c = 0\,.$$

Then, the assertion will follow from (2.7.58).

Let $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of K and H, respectively, and let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ be the canonical decomposition defined by the symmetric space structure of M. For any $A \in \mathfrak{m}$, let $\varphi^A_t = \exp(tA)$ and let $\varPhi^A_t$ denote the flow of the Killing vector field A_* on P_λ generated by A via the K-action. This means

$$\varPhi^A_t(p) = \varDelta_{\exp(tA)}(p)\,, \quad (A_*)_p = \varDelta'_p(A)\,,$$

for any $p \in P_\lambda$. Fix a point $p_0 \in P_\lambda$ in the fibre over $[\mathbb{1}_K] \in M$. Then, by (1.9.43), the integral curve $\varPhi^A_t(p_0)$ through p_0 is horizontal with respect to ω^c. Consequently, the corresponding curve

$$t \mapsto \alpha(t) := [(\varPhi^A_t(p_0), B)]\,, \quad B \in \mathfrak{g}\,, \tag{6.8.1}$$

[47] See Remark 1.9.14/2.

through $[(p_0, B)] \in \mathrm{Ad}(P_\lambda)$ is parallel with respect to the induced connection along the curve $t \mapsto \gamma(t) = \varphi_t^A([\mathbb{1}_K])$ through $[\mathbb{1}_K] \in M$.

Since M is symmetric, the Levi-Civita connection ω^0 of the symmetric space is also canonical, see Proposition 2.5.10. Thus, the same arguments apply: taking the canonical lift of the K-action to the frame bundle $L(M)$ and viewing $\mathrm{T}M$ as a bundle associated with $L(M)$, for any frame e_0 at $[\mathbb{1}_K]$ and any $X \in \mathrm{T}_{[\mathbb{1}_K]}M \cong \mathfrak{m}$, we consider the curve[48]

$$t \mapsto \tilde{X}(t) := [(\tilde{\varphi}_t^A(e_0), \iota_{e_0}^{-1}(X))] \tag{6.8.2}$$

in $\mathrm{T}M$ running through X. Here, $t \mapsto \tilde{\varphi}_t^A(e_0)$ is the unique integral curve of the Killing vector field on $L(M)$, generated by A, through $e_0 \in L(M)$. Again, by (1.9.43), $\tilde{X}$ is parallel along γ with respect to ω^0.

Now, for $X, Y \in \mathrm{T}_{[\mathbb{1}_K]}M$, using the parallel extension along γ given by (6.8.2), we calculate at $t = 0$:

$$\begin{aligned}\left(\nabla_{\dot\gamma}^{(\omega^0+\omega^c)}\Omega^c\right)(\tilde{X}, \tilde{Y}) &= \nabla_{\dot\gamma}^{(\omega^0+\omega^c)}\left(\Omega^c(\tilde{X}, \tilde{Y})\right) - \Omega^c\left(\nabla_{\dot\gamma}^{\omega^0}\tilde{X}, \tilde{Y}\right) - \Omega^c\left(\tilde{X}, \nabla_{\dot\gamma}^{\omega^0}\tilde{Y}\right)\\ &= \nabla_{\dot\gamma}^{(\omega^0+\omega^c)}\left(\Omega^c(\tilde{X}, \tilde{Y})\right).\end{aligned}$$

Finally, using the K-invariance of Ω^c, we show that $\nabla_{\dot\gamma}^{(\omega^0+\omega^c)}\left(\Omega^c(\tilde{X}, \tilde{Y})\right) = 0$. For that purpose, we denote the curvature form viewed as a 2-form on P_λ by $\tilde{\Omega}^c$ and use the fact that horizontal lifts of tangent vectors from M to P_λ are given by Killing vectors of the K-action Δ,

$$X_p^h = \Delta_p'(X), \quad X \in \mathfrak{m} \cong \mathrm{T}_{[\mathbb{1}_K]}M.$$

Now, by the obvious identity

$$\Delta'_{\Phi_t^A(p_0)}(X) = \left(\Phi_t^A\right)'\left(X_{p_0}^h\right)$$

and by K-invariance of $\tilde{\Omega}^c$, we have

$$\begin{aligned}\Omega_{\gamma(t)}^c(\tilde{X}(t), \tilde{Y}(t)) &= \left[\left(\Phi_t^A(p_0), \tilde{\Omega}^c_{\Phi_t^A(p_0)}\left(\Delta'_{\Phi_t^A(p_0)}(X), \Delta'_{\Phi_t^A(p_0)}(Y)\right)\right)\right]\\ &= \left[\left(\Phi_t^A(p_0), \left(\left(\Phi_t^A\right)^*\tilde{\Omega}^c\right)_{p_0}\left(X^h, Y^h\right)\right)\right]\\ &= \left[\left(\Phi_t^A(p_0), \tilde{\Omega}^c_{p_0}\left(X^h, Y^h\right)\right)\right],\end{aligned}$$

showing that the curve $t \mapsto \Omega_{\gamma(t)}^c(\tilde{X}(t), \tilde{Y}(t))$ is parallel in $\mathrm{Ad}(P_\lambda)$ along γ. Consequently, its covariant derivative along γ vanishes. ■

[48] For the notation, see Remark 1.2.3.

Clearly, this proposition yields a large class of solutions and we may ask whether this class contains non-minimal solutions. To discuss this issue, recall that by Remark 1.9.14, the curvature of the canonical connection is given by

$$\Omega^c(X, Y) = -\lambda'([X, Y]), \quad X, Y \in \mathfrak{m}. \tag{6.8.3}$$

Now, let $\Lambda \in \mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})$. By (1.9.41), it defines a 1-form $\alpha \in \Omega^1(M, \mathrm{Ad}(P_\lambda))$. By a similar argument as in the proof of Proposition 6.8.1, one shows that α is parallel. Thus,

$$\mathrm{d}_\omega^* \alpha = 0, \quad \mathrm{d}_\omega \alpha = 0, \tag{6.8.4}$$

and we may take α as a variation of ω^c. Thus, we consider $\omega_t = \omega^c + t\alpha$, which by construction is K-invariant for every t. Consequently, Ω^{ω_t} is K-invariant, too.

Proposition 6.8.2 (Itoh) *Let $M = K/H$ be a compact oriented Riemannian symmetric space and let P_λ be a principal G-bundle admitting a lift of K to automorphisms of P_λ. Assume*

$$\dim\big(\mathrm{Hom}_H(\mathfrak{m}, \mathfrak{g})\big) \geq 1.$$

Then, the canonical K-invariant connection ω^c on P_λ is not weakly stable.

Proof We choose an orthonormal basis $e_1, \dots, e_n$ in $\mathfrak{m}$ and denote the induced dual coframe on M by $\vartheta^1, \dots, \vartheta^n$. Using (6.7.3), (6.7.2), (6.8.3), (6.8.4) and (1.9.40), we calculate

$$\begin{aligned}
\langle \alpha, \mathfrak{H}_\omega(\alpha)\rangle_{L^2} &= \langle \alpha, \mathfrak{R}^{\nabla^{\omega^c}}(\alpha)\rangle_{L^2} \\
&= \sum_i \langle \alpha(e_l)\vartheta^l, [\Omega^c(e_i, e_k), \alpha(e_i)]\vartheta^k\rangle_{L^2} \\
&= -\sum_{i,k} \int_M \langle \Lambda(e_k), [\lambda'([e_i, e_k]), \Lambda(e_i)]\rangle_{L^2} \mathsf{v}_g \\
&= \sum_{i,k} \langle \Lambda(e_k), \Lambda\big([e_i, [e_i, e_k]]\big)\rangle_{L^2} \mathrm{vol}(M) \\
&= -\tfrac{1}{2} \sum_i \|\Lambda(e_i)\|^2_{\mathfrak{g}} \, \mathrm{vol}(M).
\end{aligned}$$

The last step is a straightforward calculation using the commutation relations of a symmetric space (Exercise 6.8.2).[49] The minus sign comes from the fact that the Cartan-Killing tensor is negative definite for any semisimple Lie algebra. Thus, for any $\Lambda \neq 0$, we have a negative definite Hessian on directions generated by Λ. ■

In particular, let us consider the canonical connection ω^c on P_λ for the case where G is a compact simple Lie group and

[49] Note that, for a symmetric space, $\sum_i [e_i, [e_i, \cdot]]$ coincides with the second Casimir operator of $\mathrm{ad}(\mathfrak{h})_{\restriction \mathfrak{m}}$.

$$M = \mathrm{S}^4 \cong \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1))\,,$$

where $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is embedded block-diagonally[50] and $\lambda : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to G$. In the notation above, we have $K = \mathrm{Sp}(2)$ and $H = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Using formula (6.8.3), it is easy to analyze (anti-)self-duality of the curvature form Ω^c and, by direct inspection of the conditions on Ω^c obtained this way, one finds (Exercise 6.8.1):

Lemma 6.8.3 *The induced homomorphism $\lambda' : \mathfrak{sp}(1) \times \mathfrak{sp}(1) \to \mathfrak{g}$ is injective iff the canonical connection ω^c is not (anti-)self-dual.* ■

Since for $\mathfrak{g} = \mathfrak{su}(2)$ or $\mathfrak{g} = \mathfrak{su}(3)$ the homomorphisms λ' cannot be injective, in this case, the canonical connection is (anti-)self-dual. On the contrary, by direct inspection of the table giving the classification of regular semisimple Lie subalgebras of a semisimple Lie algebra,[51] if $\mathfrak{g} = \mathfrak{sp}(2)$ or G_2 or such that $\mathrm{rank}(\mathfrak{g}) \geq 3$, then injective homomorphisms λ' exist, that is, in these cases we have solutions to the Yang–Mills equation which are not (anti-)self-dual. A simple example of this type is provided by the $\mathrm{Sp}(2)$-invariant $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$-valued connection ω^0 defined by (6.3.7).[52] Combining these observations with Proposition 6.8.2, we find a large class of non-minimal Yang–Mills connections on S^4 which are not (anti-)self-dual. In [339], the reader can also find a similar analysis for $M = \mathbb{C}\mathrm{P}^2$. In [402] and [384], this line of research has been continued. In particular, for the special case of principal H-bundles $K \to K/H$ with K/H being a compact symmetric space, in [402] the index and the nullity of the canonical connection has been listed for every compact simple K. Moreover, it has been shown there how to analyze the case of an arbitrary homogeneous space, see also [503].

Special attention has been paid to the case of cohomogeneity one,[53] see [638], [88], [504], [549–551] and [44]. Here, the Yang–Mills equation reduces to a system of ordinary second order differential equations for the coefficient functions of the invariant connection on a one-dimensional space. Correspondingly, the self-duality condition is expressed in terms of a first order system. Clearly, in general, there will be nongeneric orbit types giving rise to boundary conditions for the solutions. Now, in each class of invariant connections, one looks for solutions of this system of equations corresponding to minima of the reduced action. The principle of symmetric criticality [501] ensures that these minima correspond to stationary points of the Yang–Mills action in the space of all connections. Thus, in this way one finds solutions of the Yang–Mills equation. Subsequently, one may investigate whether they are minimal or not. Much attention has been paid to the following model class.

Example 6.8.4 (Bor-Montgomery, Sadun–Segert) Let V be the 5-dimensional space of real, symmetric and traceless 3×3 matrices endowed with the inner product given by

[50] Cf. Sect. 6.3 for details.

[51] Cf. the proof of Theorem 6.7.7.

[52] In sharp contrast, the induced connections $\omega^{\pm}$ given by (6.3.8) are (anti-)self-dual.

[53] The cohomogeneity of a G-action is the dimension of the orbit space.

$$\langle Q_1, Q_2 \rangle := \frac{1}{2}\,\mathrm{tr}(Q_1 Q_2\rangle\,.$$

Clearly, SO(3) acts orthogonally on V by conjugation.[54] Identifiying V with $\mathbb{R}^5$ and restricting this action to $S^4 \subset \mathbb{R}^5$ yields an action of SO(3) and, thus, also an action of Sp(1) on S^4. It is easy to see (Exercise 6.8.3) that this action has two orbit types; the principal orbits are 3-dimensional and there are two nongeneric orbits both isomorphic to the real 2-dimensional projective space. The orbit space may be identified with a line segment on S^4

$$I = \left\{ Q_\theta = \cos\theta Q_0 + \sin\theta Q_3 : 0 \leq \theta \leq \frac{\pi}{3} \right\}\,, \tag{6.8.5}$$

where Q_0 and Q_3 are basis elements in the subspace of diagonal matrices. Now, for any gauge group G, one may classify G-bundles over S^4 admitting a lift of the above SO(3)-action and, subsequently, one may classify the SO(3)-invariant connections on such bundles. If we first limit our attention to the interior of I, then we are in the situation described in Remark 1.9.9 and Corollary 1.9.15. Next, we have to extend the classifying objects obtained this way by implementing appropriate smoothness conditions on the boundary of I. This has been explained in detail in [638]. For the case $G = \mathrm{Sp}(1)$, bundles admitting lifts are classified by pairs (n_+, n_-) of integers fulfilling $n_\pm = 1$ modulo 4.[55] These numbers label the admissible boundary values of the classifying homomorphisms λ_θ. It is easy to calculate the second Chern index of a bundle characterized by (n_+, n_-). One obtains

$$\mathfrak{c}_{(n_+,n_-)} = \frac{n_+^2 - n_-^2}{8}\,.$$

For the case $G = \mathrm{Sp}(1)$, Sadun and Segert have shown that minima of the reduced action exist for all $n_+ \neq 1$ and $n_- \neq 1$. Moreover, they have proved that self-dual connections only exist for $n_- = 1$ and anti-self-dual connections only exist for $n_+ = 1$. Thus, this way one obtains non-self-dual solutions for all Chern numbers different from ± 1. The technical details of the existence proof (standard variational techniques in one dimension) are given in [551]. Clearly, these techniques are not constructive. ♦

Finally, let us mention two papers which are not based on the theory of invariant connection, but rather on advanced variational techniques as developed by Taubes. In [587], an inifinite number of SU(2)-solutions invariant with respect to a U(1)-action on S^4 was found. In [648], the existence of an infinite number of non-minimal SU(2)-solutions on $S^2 \times S^2$ and $S^3 \times S^1$ was proved. The latter solutions do not exhibit any symmetry.

[54] This is the irreducible representation of spin 2.

[55] For good reasons, these bundles are called quadrupole bundles, see [44] for an explanation.

Exercises

6.8.1 Prove Lemma 6.8.3.

6.8.2 Perform the final step in the proof of Proposition 6.8.2.

6.8.3 Show that the orbit space of the SO(3)-action on S^4 defined in Example 6.8.4 is given by (6.8.5).

Chapter 7
Matter Fields and Model Building

In this chapter, we include matter fields into our discussion. In Sect. 7.1, we present the general geometric model of matter fields in the fibre bundle language. Then, in Sects. 7.2–7.5, we study various aspects of Yang–Mills–Higgs models. We discuss the Higgs mechanism in detail, present a topological classification of static finite-energy configurations and address the problem of constructing asymptotic as well as exact solutions to the Yang–Mills–Higgs equations. In particular, we focus on magnetic monopole solutions including the discussion of the Bogomolnyi–Prasad–Sommerfield model. Next, in Sect. 7.6, we pass to a U(1)-gauge model coupled to a matter field of spinorial type, the famous Seiberg–Witten model. The latter has attracted much attention over the last two decades, because its moduli space yields deep insight into the differential topology of 4-manifolds. In particular, it yields new, simpler proofs of results earlier obtained via the theory of instantons. We discuss the basic properties of this model in detail and outline some of the topological consequences. Then, in Sect. 7.7, we present the (classical) standard model of elementary particle physics in the geometric language and, in the remaining two sections, we discuss the method of dimensional reduction in the context of gauge theories in some detail. The latter may be viewed as one of the classical unification schemes–a unification in the spirit of Kaluza and Klein.

7.1 Matter Fields

Usually, the spacetime manifold M will be endowed with some additional geometric structures. Depending on the physical context, in the bundle language these structures will be encoded either in the frame bundle $L(M)$ or in the spin structure $S(M)$.[1] To

[1]For instance, below we will always assume that M is endowed with a Riemannian or a pseudo-Riemannian metric, which can be viewed as an equivariant mapping from $L(M)$ to the space of

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unify the notation, below, we write Q both for $L(M)$ and for $S(M)$ and call it the spacetime principal bundle. Its structure group will be denoted by S. By Remark 1.1.9/2, given the spacetime principal bundle Q and the gauge principal bundle P, we may build the fibre product $Q \times_M P$ which is a principal $(S \times G)$-bundle over M. If we deal with both Q and P, then the right actions of S and G will be denoted by Ψ_Q and Ψ_P, respectively, and the canonical projections of these bundles will be denoted by π_Q and π_P, respectively. For the induced right action of $S \times G$ on $Q \times_M P$ we will write Ψ and the canonical projection of $Q \times_M P$ will be denoted by π.

A classical matter field model is described by the following data.

(a) The model is defined on the tensor product

$$E = E_s \otimes E_i \,,$$

where E_s is the bundle of spacetime degrees of freedom and E_i denotes the bundle of internal degrees of freedom. The bundle E_s comes in two fundamentally different versions:

- E_s is a tensor bundle over M, associated with the frame bundle $L(M)$. In that case, we speak of bosonic matter.
- E_s is a spinor bundle associated with the spin structure $S(M)$. Then we speak of fermionic matter.

Thus,

$$E_s = Q \times_S F_s \,,$$

where F_s is a finite-dimensional vector space carrying a representation μ of S. The bundle E_i is associated with the gauge principal bundle P,

$$E_i = P \times_G F_i \,,$$

where F_i is a finite-dimensional vector space carrying a representation σ of the gauge group G. Besides σ, the vector space F_i may carry an additional Lie group representation corresponding to further internal degrees of freedom called flavour,[2] see e.g. Sect. 7.7 for the case of the standard model.
By Remark 1.2.9/2, E is associated with $Q \times_M P$. The typical fibre

$$F = F_s \otimes F_i$$

of E carries the tensor product representation $\mu \otimes \sigma$ of the direct product $S \times G$.

(Footnote 1 continued)
symmetric second-rank covariant tensors on $\mathbb{R}^n$. Equivalently, it is encoded as a torsion-free metric connection (Levi-Civita connection) on $L(M)$.

[2]These are rigid symmetries not giving rise to local gauge transformations.

(b) A matter field of spacetime type μ and of gauge type σ is a section $\Phi \in \Gamma^\infty(E)$.[3] It will be referred to as a matter field of type (μ, σ). By Proposition 1.2.6, it may be equivalently represented by an element $\tilde{\Phi} \in \mathrm{Hom}_{S\times G}(Q \times_M P, F)$,

$$\Phi(m) = [(z, \tilde{\Phi}(z))] \,, \quad m \in M \,, \; z \in \pi^{-1}(m) \,. \tag{7.1.1}$$

By Remark 1.2.15/3, $\tilde{\Phi}$ and Φ have the same local representative $\varphi : U \to F$ in any local trivialization over $U \subset M$.
The coupling with a gauge potential is encoded in the covariant exterior derivative. This is called the principle of minimal coupling. As we know from Chaps. 2 and 5, Q may carry various connections referred to as spacetime connections. By Remark 1.3.17, if ω_Q and ω_P are spacetime and gauge connections, respectively, then they induce a connection form ω on $Q \times_M P$, given by (1.3.16). Omitting the canonical projections to Q and P, we have $\omega = \omega_Q + \omega_P$ and, thus,

$$D_\omega \tilde{\Phi} = \mathrm{d}\tilde{\Phi} + \big(\mu'(\omega_Q) \otimes \mathrm{id}_{F_i} + \mathrm{id}_{F_s} \otimes \sigma'(\omega_P)\big) \circ \tilde{\Phi} \,, \tag{7.1.2}$$

cf. (1.4.2). Clearly, $\mu'(\omega_Q) \otimes \mathrm{id}_{F_i} + \mathrm{id}_{F_s} \otimes \sigma'(\omega_P)$ must be viewed as a 1-form on $Q \times_M P$ with values in $\mathrm{End}(F)$. It is obtained by differentiating the tensor product representation $\mu \otimes \sigma$. The corresponding covariant exterior derivative of Φ is given by (1.5.3),

$$(\nabla^\omega \Phi)_m(X) = \iota_z \circ (D_\omega \tilde{\Phi})_z(Y) \,, \tag{7.1.3}$$

where $X = \pi'(Y)$, $m \in M$ and $z \in \pi^{-1}(m)$. By (1.4.14), the covariant exterior derivative of the local representative φ is given by

$$D\varphi = \mathrm{d}\varphi + \big(\mu'(\mathscr{A}_Q) \otimes \mathrm{id}_{F_i} + \mathrm{id}_{F_s} \otimes \sigma'(\mathscr{A}_P)\big) \circ \varphi \,, \tag{7.1.4}$$

where $\mathscr{A}_Q$ and $\mathscr{A}_P$ are the local representatives of ω_Q and ω_P, respectively.

(c) Let ϑ_Q and ϑ_P be local gauge transformations, that is, vertical automorphisms of the principal bundles Q and P, respectively. Then, $\vartheta = \vartheta_Q \times \vartheta_P$ is a local gauge transformation in $Q \times P$, which induces a local gauge transformation in $Q \times_M P$ denoted by the same letter. In more detail, we have

$$\vartheta : P \times_M Q \to P \times_M Q \,, \quad \vartheta(m, (q, p)) = \big(m, (\vartheta_Q(q), \vartheta_P(p))\big) \,. \tag{7.1.5}$$

Clearly, any vertical automorphism of $Q \times_M P$ is of this type. An active local gauge transformation of the matter field $\tilde{\Phi}$ is given by

$$\tilde{\Phi} \mapsto \vartheta^* \tilde{\Phi} \,. \tag{7.1.6}$$

[3] As already noted under (a), Φ may further carry a certain flavour type. If not otherwise stated, we suppress these rigid degrees of freedom.

Clearly, $\vartheta^*\tilde{\Phi}$ is of the same type as $\tilde{\Phi}$. By Proposition 1.8.3, ϑ is equivalently described by an element $u \in \mathrm{Hom}_{S\times G}(Q \times_M P, S \times G)$, given by

$$\vartheta(z) = \Psi_{u(z)}(z)\,.$$

Thus, using the equivariance of $\tilde{\Phi}$ and decomposing $u = u_S \times u_G$ with respect to the product structure of $S \times G$, for any $z \in Q \times_M P$, we obtain

$$(\vartheta^*\tilde{\Phi})(z) = \tilde{\Phi} \circ \Psi_{u(z)}(z) = \left(\mu_{u_S(z)^{-1}} \otimes \sigma_{u_G(z)^{-1}}\right) \tilde{\Phi}(z)\,. \tag{7.1.7}$$

Via (7.1.1), formula (7.1.6) induces a gauge transformation for Φ,

$$\Phi \mapsto \Phi'\,, \quad \Phi'(m) = [(z, (\vartheta^*\tilde{\Phi})(z))]\,,$$

with $\pi(z) = m$. In terms of the corresponding vertical automorphism $\hat{\vartheta}$ of the associated bundle E provided by Proposition 1.8.4, we obtain

$$\Phi'(m) = \hat{\vartheta}^{-1}(\Phi(m))\,.$$

Finally, the gauge transformation of a local representative φ of Φ is given by

$$\varphi'(m) = \left(\mu_{\rho_S(m)^{-1}} \otimes \sigma_{\rho_G(m)^{-1}}\right) \varphi(m)\,, \tag{7.1.8}$$

where ρ_S and ρ_G are transition functions of Q and P, respectively.

(d) The infinite-dimensional vector space of smooth sections of E will be denoted by $\mathscr{E}$ and will be referred to as the matter configuration space. It can be endowed with the structure of an infinite-dimensional manifold. By point (c) it is acted upon by a (right) representation of the group of local gauge transformations.

As in Sect. 6.1, we assume that

(a) the spacetime manifold M is endowed with a (pseudo-)Riemannian metric g,
(b) the Lie algebra $\mathfrak{g}$ of G carries an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$,
(c) the vector space F carries a scalar product $\langle\cdot,\cdot\rangle_F$ which is invariant with respect to the representation $\mu \otimes \sigma$ of $S \times G$.

Then, we can write down physical actions for matter fields.

(a) Let Φ be a bosonic matter field of type (μ, σ). As explained before, the coupling with a gauge potential ω is described in terms of the covariant derivative $\nabla^\omega\Phi \in \Omega^1(M, E)$, cf. formula (7.1.2). The general gauge-invariant Lagrangian is of the following form:

$$\mathscr{L}(\omega, \Phi) := \frac{1}{2}\nabla^\omega\Phi \mathbin{\dot{\wedge}} *\nabla^\omega\Phi - V(\Phi)\,,$$

where the dot refers to the fibre metric $\langle\cdot,\cdot\rangle_E$ in the tensor product $E = E_s \otimes E_i$. Here, $V : F \to \mathbb{R}$ is a G-invariant function bounded from below which induces a function on E via $V([(p,f)]) = V(f)$. Then,

$$V(\Phi) = V \circ \Phi\,.$$

Correspondingly, we have $V(\tilde{\Phi}) = V \circ \tilde{\Phi} = \pi^* V(\Phi)$. In most of the applications, the potential is of the form

$$V(\Phi) = V(|\Phi|^2)\,,$$

where the norm refers to the fibre metric in E. In that case, V may be viewed as a function on $\mathbb{R}$. The fibre metric in E_s is induced from the metric g and the fibre metric of E_i is given by $\langle\cdot,\cdot\rangle_{F_i}$ via (2.6.4).

(b) Let ψ be a fermionic matter field of type (μ,σ). Here, E is associated with the fibre product bundle $S(M) \times_M P$. In the notation of Chap. 5, the typical gauge-invariant Lagrangian is of the following form:

$$\mathscr{L}(\omega,\psi) := \langle\psi, \not{D}\psi\rangle - V(\psi)\,. \tag{7.1.9}$$

Moreover, there may occur a coupling between bosonic and fermionic matter fields. This will be explained for the case of the standard model in Sect. 7.7.

7.2 Yang–Mills–Higgs Systems

In this section, we introduce one of the fundamental building blocks of the standard model describing the fundamental interactions of elementary particles.

Let (M,g) be an n-dimensional (pseudo-)Riemannian manifold, let G be a compact Lie group and let $P(M,G)$ be a principal G-bundle over M. Let (F,G,σ) be a representation of G and let $E = P \times_G F$ be the corresponding associated bundle. In the terminology introduced in Sect. 7.1, a Higgs field Φ is a bosonic matter field of type (μ,σ), where μ is the trivial representation of $\mathrm{O}(n)$. That is, Φ is a spacetime scalar field. Thus, it may be simply viewed as a section of E or, equivalently, as an element $\tilde{\Phi} \in \mathrm{Hom}_G(P,F)$. As before, we assume that the Lie algebra $\mathfrak{g}$ of G carries an $\mathrm{Ad}(G)$-invariant scalar product which we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ or sometimes simply by k. Moreover, we assume that F is endowed with a scalar product $\langle\cdot,\cdot\rangle_F$ which is invariant under the representation σ.

A Yang–Mills–Higgs configuration is a pair (ω,Φ), where ω is a gauge connection form on P. Thus, the configuration space of a Yang–Mills–Higgs system is $\mathscr{C} \times \mathscr{E}$. In the notation of Sect. 7.1, the action functional is given by

$$S(\omega,\Phi) = \frac{1}{2}\|\Omega\|^2 + \frac{1}{2}\|\nabla^\omega\Phi\|^2 - \int_M V(\Phi)\mathsf{v}_{\mathsf{g}}\,,$$

or, in more detail,

$$S(\omega, \Phi) = \int_M \left\{ \frac{1}{2}\Omega \,\dot{\wedge}\, *\Omega + \frac{1}{2}\nabla^\omega \Phi \,\dot{\wedge}\, *\nabla^\omega \Phi - V(\Phi)\mathsf{v_g} \right\} . \tag{7.2.1}$$

The second term in this formula describes the minimal coupling of the gauge potential with the matter field and V will be referred to as the Higgs potential. It describes the self-interaction of the Higgs field. Its typical form is

$$V(|\Phi|^2) = \frac{1}{2}\mu^2|\Phi|^2 + \frac{1}{4}\lambda|\Phi|^4 , \tag{7.2.2}$$

where $\lambda > 0$. Depending on whether μ^2 is positive or negative, the minimum of V is either given by $\Phi = 0$ or by

$$|\Phi| = \sqrt{-\frac{\mu^2}{\lambda}} . \tag{7.2.3}$$

While in the first case the minimum is invariant under the full gauge group G, in the second case, the minima are only invariant under some subgroup $H \subset G$ and take their value in some orbit $G/H \subset F$. Thus, by an appropriate choice of the parameter μ^2 we may produce a symmetry reduction, that is, we may obtain a classical ground state having a smaller symmetry than the original Lagrangian of the theory. In the physics literature, this is commonly called spontaneous symmetry breaking.[4] It will be explained in detail in the next section.

Next, let us derive the field equations via the variational principle for $S(\omega, \Phi)$. For that purpose, consider a variation of the configuration (ω, Φ) given by

$$\omega_t = \omega + t\alpha , \quad \Phi_t = \Phi + t\tau ,$$

where $\alpha \in \Omega^1(M, \mathrm{Ad}(P))$ and $\tau \in \Gamma^\infty(E)$. Using (2.7.54) and (7.1.2), we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \left(\nabla^{\omega_t} \Phi_t\right) = \nabla^\omega \tau + \sigma'(\alpha)\Phi .$$

By Definition 1.5.2, on 0-forms we have $\mathrm{d}_\omega = \nabla^\omega$. Using this, together with

$$\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} V(\Phi(m) + t\tau(m)) = V'(\Phi(m))(\tau(m)) = \langle V'(\Phi), \tau\rangle_{E_m} ,$$

and recalling the calculation for the pure Yang–Mills case from the beginning of Sect. 6.2, we obtain

[4]Here, we exclusively discuss the breaking of local gauge symmetry. For a discussion of global symmetry breaking, we refer to [3] and references therein.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} S(\omega_t, \Phi_t) = \langle \Omega, \mathrm{d}_\omega \alpha \rangle_{L^2} + \langle \nabla^\omega \Phi, \nabla^\omega \tau + \sigma'(\alpha)\Phi \rangle_{L^2} - \langle V'(\Phi), \tau \rangle_{L^2}$$
$$= \langle \mathrm{d}_\omega^* \Omega - J, \alpha \rangle_{L^2} + \langle \mathrm{d}_\omega^* \circ \mathrm{d}_\omega \Phi - V'(\Phi), \tau \rangle_{L^2} ,$$

where J is the unique 1-form on M with values in $\mathrm{Ad}(P)$ which satisfies

$$\langle J(m), \alpha_m \rangle = -\langle \nabla^\omega \Phi, \sigma'(\alpha_m)\Phi \rangle$$

for all $m \in M$ and $\alpha_m \in \big(\mathrm{T}^*M \otimes \mathrm{Ad}(P)\big)_m$. Here, $\langle \cdot, \cdot \rangle$ on the left hand side denotes the fibre metric of $\mathrm{T}^*M \otimes \mathrm{Ad}(P)$, whereas on the right hand side it denotes the fibre metric of $\mathrm{T}^*M \otimes (P \times_G F)$. The 1-form J is called the current of the Yang–Mills–Higgs system. We may write

$$J = -\big\langle \big(\mathsf{k}^{-1} \otimes \mathrm{id}\big)(\sigma'\Phi), \mathrm{d}_\omega \Phi \big\rangle_F , \tag{7.2.4}$$

where σ' is viewed as an element of $\mathfrak{g}^* \otimes \mathrm{End}(F)$, so that $\sigma'\Phi \in \mathfrak{g}^* \otimes F$ and thus $\big(\mathsf{k}^{-1} \otimes \mathrm{id}\big)\sigma' \in \mathfrak{g} \otimes \mathrm{End}(F)$. Since the L^2-inner products involved are non-degenerate, we conclude that $\delta S = 0$ iff

$$\mathrm{d}_\omega^* \Omega = J , \quad \mathrm{d}_\omega^* \circ \mathrm{d}_\omega \Phi = V'(\Phi) . \tag{7.2.5}$$

This system of nonlinear partial differential equations will be referred to as the Yang–Mills–Higgs equations with potential V. For the sake of completeness, let us also recall Propositions 1.4.11 and 1.4.13. In the case under consideration, they read

$$\mathrm{d}_\omega \Omega = 0 , \quad \mathrm{d}_\omega \circ \mathrm{d}_\omega \Phi = \sigma'(\Omega)\Phi . \tag{7.2.6}$$

Clearly, the first of these equations is the ordinary Bianchi identity for the curvature form. By a slight abuse of terminology, the second identity may be called the Bianchi identity for Φ.

Remark 7.2.1 In applications, (F, G, σ) often coincides with the adjoint representation $(\mathfrak{g}, G, \mathrm{Ad})$. Then, the field equations read

$$\mathrm{d}_\omega^* \Omega = [\nabla^\omega \Phi, \Phi] , \quad \mathrm{d}_\omega^* \circ \mathrm{d}_\omega \Phi = V'(\Phi) , \tag{7.2.7}$$

and the Bianchi identities take the form

$$\mathrm{d}_\omega \Omega = 0 , \quad \mathrm{d}_\omega \circ \mathrm{d}_\omega \Phi = [\Omega, \Phi] . \tag{7.2.8}$$

◆

For the remainder of this section, we assume that M is the 4-dimensional Minkowski space. Let us study the energy functional $E(\omega, \Phi)$ of a Yang–Mills–Higgs system for a chosen space-like hypersurface $\Sigma_0 \subset M$. Let $(\mathbf{e}_0, \dots, \mathbf{e}_3)$ be the standard basis in M and let $x^0, \dots, x^3$ be the corresponding standard coordinates. We take $\Sigma_0 := \{\mathbf{x} \in M : x^0 = \text{const.}\}$ and decompose

$$M = \mathbb{R}\mathbf{e}_0 \times \Sigma_0 . \tag{7.2.9}$$

Then, for any configuration (ω, Φ), we restrict Ω and $\nabla^\omega \Phi$ to Σ_0 and decompose them relative to (7.2.9),

$$\Omega = \Omega_{k0} \mathrm{d}x^k \wedge \mathrm{d}x^0 + \frac{1}{2} \Omega_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l , \quad \nabla^\omega \Phi = \nabla^\omega_0 \Phi \mathrm{d}x^0 + \nabla^\omega_k \Phi \mathrm{d}x^k ,$$

where $k = 1, 2, 3$. We define

$$\Omega^{\mathrm{e}} := \Omega_{k0} \mathrm{d}x^k , \quad \Omega^{\mathrm{m}} := \frac{1}{2} \Omega^{kl} \varepsilon_{klm} \mathrm{d}x^m , \tag{7.2.10}$$

$$\Pi := \nabla^\omega_0 \Phi , \quad \mathscr{D}\Phi := \nabla^\omega_k \Phi \mathrm{d}x^k . \tag{7.2.11}$$

The 1-forms Ω^{e} and Ω^{m} on Σ_0 are referred to as the colour electric and the colour magnetic components of Ω, respectively. Note that (Exercise 7.2.1)

$$* \Omega^{\mathrm{m}} = i^* \Omega , \quad \mathscr{D}\Phi = i^* \nabla^\omega \Phi , \tag{7.2.12}$$

where $i : \Sigma_0 \to M$ is the natural inclusion mapping.

Now, the energy functional is given by the integral over Σ_0 of the component T_{00} of the energy-momentum tensor. For the Yang–Mills–Higgs theory, it reads[5]

$$E(\omega, \Phi) = \frac{1}{2} \int_{\Sigma_0} \left(\Omega^{\mathrm{e}} \dot{\wedge} * \Omega^{\mathrm{e}} + \Omega^{\mathrm{m}} \dot{\wedge} * \Omega^{\mathrm{m}} + \Pi \dot{\wedge} * \Pi + \mathscr{D}\Phi \dot{\wedge} * \mathscr{D}\Phi + V(\Phi) \mathrm{v}_{R^3} \right) ,$$

where $*$ denotes the Hodge star operator on $\mathbb{R}^3$. In short, we may write

$$E(\omega, \Phi) = \frac{1}{2} \left(\|\Omega^{\mathrm{e}}\|^2 + \|\Omega^{\mathrm{m}}\|^2 + \|\Pi\|^2 + \|\mathscr{D}\Phi\|^2 + \int_{\Sigma_0} V(\Phi) \mathrm{v}_{R^3} \right) . \tag{7.2.13}$$

Below, we will consider the static case.

Definition 7.2.2 A configuration (ω, Φ) of a Yang–Mills–Higgs theory on M is called static if it is invariant under time translations, that is, invariant under the action of the additive group $\mathbb{R}$ on M given by

$$\delta : \mathbb{R} \times M \to M , \quad \delta(s, \mathbf{x}) := (x^0 + s, x^1, x^2, x^3) .$$

Clearly, the translation invariant mapping Φ is simply given by its values on Σ_0. By Example 1.9.18, the $\mathbb{R}$-invariant connection ω is uniquely characterized by a

[5] See e.g. [208].

connection $\tilde{\omega}$ on the trivial principal bundle $\tilde{P} = \Sigma_0 \times G$ and by an equivariant mapping ω^0 from $\tilde{P}$ to $L(\mathbb{R}\mathbf{e}_0, \mathfrak{g})$. We see that, in the static case, putting $\omega^0 = 0$ has a geometric meaning. In accordance with the physics literature, we call this the temporal gauge. By making this choice, one restricts the admissible gauge transformations to vertical automorphisms of $\tilde{P}$. For a (global) representative $\mathbb{A}$ of ω, we have

$$\mathbb{A} = A_0 \mathrm{d}x^0 + A_k \mathrm{d}x^k \,.$$

Here, $A_k \mathrm{d}x^k$ and $A_0 \mathrm{d}x^0$ are the representatives of $\tilde{\omega}$ and ω^0, respectively, and the temporal gauge reads $A_0 = 0$. Clearly, we must show that the choice of the temporal gauge is, in the static case, consistent with the field equations: indeed, we then have

$$\Pi = 0\,, \quad J_0 = 0\,, \quad \Omega^{\mathrm{e}} = 0\,,$$

and, thus, the field equations reduce to the following system of equations on Σ_0:

$$\mathrm{d}^*_\omega * \Omega^{\mathrm{m}} = J\,, \quad \mathscr{D}^* \circ \mathscr{D}\Phi = V'(\Phi)\,, \tag{7.2.14}$$

where $J = -\langle (\mathsf{k}^{-1} \otimes \mathrm{id}) \circ \sigma'(\Phi), \mathscr{D}\Phi \rangle_F$. Clearly, (7.2.14) are the field equations of a Yang–Mills–Higgs system on $\mathbb{R}^3$.

We conclude that, for the static theory in the temporal gauge, the energy functional reduces to

$$E(\omega, \Phi) = \frac{1}{2}\left(\|\Omega^{\mathrm{m}}\|^2 + \|\mathscr{D}\Phi\|^2 + \int_{\Sigma_0} V(\Phi) \mathrm{v}_{R^3} \right) . \tag{7.2.15}$$

Thus, for any finite energy configuration (ω, Φ), the differential forms Ω^{m} and $\mathscr{D}\Phi$ must be square integrable and, for $\|\mathbf{x}\| \to \infty$,

$$\|\mathbf{x}\|^2 V(\Phi) \to 0\,. \tag{7.2.16}$$

To analyze these requirements, let us consider condition (7.2.16) under the following assumptions on V. Let $F_{\min} \subset F$ be the set of absolute minima of V. By invariance, $F_{\min}$ is a union of orbits of G. We assume that $F_{\min}$ consists of a single orbit G/H. By possibly shifting V, we may also assume that V vanishes on $F_{\min}$. From (7.2.16), we conclude that, at large distances, the global representative φ of Φ must take values in G/H. More precisely, we get a mapping

$$\varphi_\infty : \mathrm{S}^2 \to G/H\,, \quad \varphi_\infty(\mathbf{x}) := \lim_{r \to \infty} \varphi(r\mathbf{x})\,, \tag{7.2.17}$$

that is, the asymptotic values of φ define an element $[\varphi_\infty] \in \pi_2(G/H)$. Since the mapping degree is a homotopy invariant, $[\varphi_\infty]$ cannot be changed by continuous deformations, that is, the space of finite-energy configurations decomposes into topological sectors labelled by $\pi_2(G/H)$. Note that this statement is obtained without any reference to the field equations.

Remark 7.2.3 While it seems to us that the investigation of finite energy configurations is interesting in itself, the study of these topological sectors has attracted special attention, because in realistic models they characterize magnetic monopole configurations, see [140, 249, 251, 314, 315, 566, 610]. In the literature, by an abuse of language, the topological charges to be constructed below are often called magnetic charges. In order to justify this terminology one must, of course, accommodate the electromagnetic field in a gauge invariant way in the model under consideration. In particular, after symmetry breaking, the residual gauge group H should contain only one U(1)-factor, because there should be only one electromagnetic field. We will discuss an example of this type in Sect. 7.4. ♦

It turns out that, apart from a possible torsion part, $\pi_2(G/H)$ may be characterized in terms of integral invariants induced from closed invariant 2-forms on G/H. Following Horvathy and Rawnsley [315], let us construct these invariants: let us assume that H is connected. Let θ_G be the Maurer–Cartan form on G. Take the standard direct sum decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{h}, \mathfrak{g}]\,, \tag{7.2.18}$$

where $\mathfrak{c}$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ and consider the projection $\mathrm{pr}_{\mathfrak{c}} : \mathfrak{g} \to \mathfrak{c}$ onto the first summand. Then, $\mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G)$ is a closed 2-form on G which is obviously right H-invariant and left G-invariant (Exercise 7.2.2). Thus, it descends to an invariant 2-form η on G/H with values in $\mathfrak{c}$. With $\pi : G \to G/H$ being the canonical projection, we have

$$\pi^*\eta = \mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G)\,. \tag{7.2.19}$$

We define

$$\rho : \pi_2(G/H) \to \mathfrak{c}\,, \quad \rho([\varphi_\infty]) := \frac{1}{2\pi}\int_{S^2} \varphi_\infty^* \eta\,. \tag{7.2.20}$$

Since η is closed, this integral depends only on the homotopy class of φ_∞ and, thus, ρ is correctly defined.

Now,[6] fix a maximal torus in H, denote its Lie algebra by $\mathfrak{t}$ and take the unit lattice

$$\Gamma := \{X \in \mathfrak{h} : \exp(2\pi X) = \mathbb{1}\} \cap \mathfrak{t}\,.$$

Let $\mathfrak{z}$ denote the Lie algebra of the center of H, consider the standard direct sum decomposition

$$\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{h}] \tag{7.2.21}$$

and let $\mathrm{pr}_{\mathfrak{z}} : \mathfrak{h} \to \mathfrak{z}$ be the canonical projection onto the first summand. Choose a $\mathbb{Z}$-basis $\{\zeta_1, \ldots, \zeta_p\}$ of $\mathrm{pr}_{\mathfrak{z}}(\Gamma)$ and extend it to a basis $\{\zeta_1, \ldots, \zeta_p, \ldots, \zeta_q\}$ of $\mathfrak{c}$. Then,

[6] For the basic Lie algebraic notions used in the sequel, we refer to Appendix C or to [329] for more details.

decomposing η with respect to this basis, we get a family $\{\eta^k\}$ of closed invariant 2-forms on G/H:

$$\eta = \sum_{k=1}^{q} \eta^k \zeta_k \,. \tag{7.2.22}$$

Note that $\eta^k = f^k(\eta)$ for the dual basis $\{f^1, \dots, f^q\}$. It can be shown easily (Exercise 7.2.3) that, for $k > p$, the 2-forms η^k are exact. Thus, if we insert the decomposition (7.2.22) into (7.2.20), they do not contribute to the integral in (7.2.20) and we obtain

$$\rho([\varphi_\infty]) = \sum_{k=1}^{p} m_k([\varphi_\infty])\zeta_k \tag{7.2.23}$$

with

$$m_k([\varphi_\infty]) = \frac{1}{2\pi} \int_{S^2} \varphi_\infty^* \eta^k \,, \quad k = 1, \dots, p \,. \tag{7.2.24}$$

In particular, ρ takes values in $\mathfrak{z}$. In Proposition 7.2.5, we will show that $m_1, \dots, m_p$ is a p-tuple of integers. These integers are called the topological charges of the Yang–Mills–Higgs system.

Remark 7.2.4 For the special case of the adjoint representation, η is given by the Kirillov symplectic form, cf. Sect. 8.4 of Part I. In more detail, let $X_0 = \varphi_\infty(\mathbf{x}_0) \in \mathfrak{g}$ be a point with stabilizer H and let $G \cdot X_0 \cong G/H$ be the adjoint orbit through X_0. Then, every $Z \in \mathfrak{z}$ defines a surjective mapping $\pi_Z : G \cdot X_0 \to G \cdot Z$. We set

$$\eta^Z := \pi_Z^* \omega^Z \,, \tag{7.2.25}$$

where ω^Z is the Kirillov form on the orbit through Z. Then, for a chosen $\mathbb{Z}$-basis, formula (7.2.25) yields a family of invariant 2-forms which coincide with the one defined above. This is a simple consequence of the Maurer–Cartan equation (Exercise 7.2.4). ♦

Next, due to $\pi_2(G) = 0$, the long exact homotopy sequence of the principal H-bundle $\pi : G \to G/H$ implies that the connecting homomorphism

$$\delta : \pi_2(G/H) \to \pi_1(H) \tag{7.2.26}$$

is injective. In particular, if G is simply connected, then δ is an isomorphism. The homomorphism δ is defined as follows: choose a covering of S^2 given in spherical coordinates by

$$U_1 = \left\{\mathbf{x}(\vartheta, \phi) \in S^2 : 0 \le \vartheta < \frac{\pi}{2} + \varepsilon\right\} \,, \; U_2 = \left\{\mathbf{x}(\vartheta, \phi) \in S^2 : \frac{\pi}{2} - \varepsilon < \vartheta \le \pi\right\} \,.$$

Since U_1 and U_2 are contractible, there exist smooth mappings $g_i : U_i \to G$, $i = 1, 2$, such that

$$\varphi_\infty(\mathbf{x}) = \sigma(g_i(\mathbf{x}))[\mathbb{1}] \equiv \pi \circ g_i(\mathbf{x}) \,, \tag{7.2.27}$$

where $[\mathbb{1}] \in G/H = F_{\min}$ is a chosen point. Then, on the equator S^1 given by $\theta = \frac{\pi}{2}$,

$$\gamma : \mathrm{S}^1 \to H \,, \quad \gamma(\mathbf{x}) := g_1^{-1}(\mathbf{x}) g_2(\mathbf{x}) \tag{7.2.28}$$

is a loop in H. Let $[\gamma]$ be the corresponding homotopy class. Then,

$$\delta([\varphi_\infty]) = [\gamma] \,. \tag{7.2.29}$$

Now, consider the decomposition (7.2.21). Let H_{ss} be the connected Lie subgroup of H whose Lie algebra is $[\mathfrak{h}, \mathfrak{h}]$. It is compact and semisimple and, since $[\mathfrak{h}, \mathfrak{h}]$ is an ideal, it is also normal. Thus, H/H_{ss} is a compact connected Lie group with Lie algebra $\mathfrak{z}$. Since the latter is Abelian, H/H_{ss} must be a torus and, therefore, $\pi_1(H/H_{ss}) \cong \mathbb{Z}^p$, where $p = \dim \mathfrak{z}$. Moreover, since H_{ss} is compact and semisimple, $\mathbb{T} = \pi_1(H_{ss})$ is a finite Abelian group. Finally, using the homotopy sequence of the fibration $H_{ss} \to H \to H/H_{ss}$ and the fact that $\pi_1(H/H_{ss})$ is free Abelian, we obtain the following structure of the fundamental group of the compact connected Lie group H [315]:

$$\pi_1(H) = \mathbb{Z}^p \oplus \mathbb{T} \,. \tag{7.2.30}$$

The $\mathbb{Z}^p$-part yields a p-tuple of integers which are defined as follows: let θ_H be the Maurer–Cartan form of H. Then, by the Maurer–Cartan equation, $\mathrm{pr}_{\mathfrak{z}}(\theta_H)$ is a closed $\mathfrak{z}$-valued 1-form on H. We define

$$\lambda : \pi_1(H) \to \mathfrak{z} \,, \quad \lambda([\gamma]) := \frac{1}{2\pi} \int_\gamma \mathrm{pr}_{\mathfrak{z}}(\theta_H) \,. \tag{7.2.31}$$

Since $\mathrm{pr}_{\mathfrak{z}}(\theta_H)$ is closed, the integral only depends on the homotopy class of γ, that is, λ is well defined. As above, decomposing it with respect to a $\mathbb{Z}$-basis yields a p-tuple $(m_1([\gamma]), \ldots, m_p([\gamma])) \in \mathbb{Z}^p$. We show that, with respect to the same $\mathbb{Z}$-basis, these integers coincide with the numbers $m_k([\varphi_\infty])$ defined by (7.2.24).

Proposition 7.2.5 (Horvathy–Rawnsley) *For any* $[\varphi_\infty] \in \pi_2(G/H)$, *we have*

$$\rho([\varphi_\infty]) = \lambda \circ \delta([\varphi_\infty]) \,.$$

Proof In the notation above, let S^1 be the equatorial circle of S^2. Using Stokes' Theorem together with (7.2.27), we calculate

$$\begin{aligned}2\pi\rho([\varphi_\infty]) &= \lim_{\varepsilon\to 0}\left\{\int_{U_1} g_1^*\circ\pi^*\eta + \int_{U_2} g_2^*\circ\pi^*\eta\right\}\\ &= \lim_{\varepsilon\to 0}\left\{\int_{U_1} g_1^*\circ \mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G) + \int_{U_2} g_2^*\circ \mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G)\right\}\\ &= \int_{\mathrm{S}^1} \mathrm{pr}_{\mathfrak{c}}\left\{g_2^*\theta_G - g_1^*\theta_G\right\}.\end{aligned}$$

By (7.2.28), on S^1 we have

$$\mathrm{pr}_{\mathfrak{c}}\circ(g_2^*\theta_G - g_1^*\theta_G) = \mathrm{pr}_{\mathfrak{c}}\circ\mathrm{Ad}(\gamma)\circ\gamma^*\theta_H .$$

Then, since $\mathrm{pr}_{\mathfrak{z}}$ is $\mathrm{Ad}(H)$-invariant and coincides with $\mathrm{pr}_{\mathfrak{c}}$ on $\mathfrak{h}$, we obtain

$$2\pi\rho([\varphi_\infty]) = \int_{\mathrm{S}^1}\mathrm{pr}_{\mathfrak{c}}(\gamma^*\theta_H) = \int_\gamma \mathrm{pr}_{\mathfrak{z}}(\theta_H) = 2\pi\lambda([\gamma]) .$$

By (7.2.29), the assertion follows. ■

Since δ is injective, we conclude that the integers $m_k([\varphi_\infty])$ defined by (7.2.24) generate the free part of $\pi_2(G/H)$. Its torsion part coincides with the kernel of ρ. Clearly, this part cannot be determined by means of invariants built from differential forms.

Remark 7.2.6 We note that if the $\mathbb{T}$-part in the decomposition (7.2.30) is nontrivial, then $\mathbb{Z}_n$-charges may occur, see e.g. [249, 653] and further references therein. The simplest example of this type is $H = \mathrm{SO}(3)$ with $\pi_1(H) = \mathbb{Z}_2$. ♦

It remains to analyze the condition

$$\mathscr{D}\varphi = 0 \tag{7.2.32}$$

for $\|\mathbf{x}\|\to\infty$. By Proposition 1.6.10, it implies that ω asymptotically takes values in the Lie algebra $\mathfrak{h}$ of H. This clearly means that asymptotically Ω^{m} takes values in $\mathfrak{h}$, too. We will show that (7.2.32) implies a presentation of the topological invariant (7.2.20) in terms of the curvature. To find it, let $\mathbb{A}$ and $\mathbb{F}$ be global representatives of ω and Ω, respectively. In what follows, let $\mathrm{S}^2_r = \{\mathbf{x}\in\mathbb{R}^3 : \|\mathbf{x}\| = r\}$. We say that a relation holds on S^2_∞ if it holds asymptotically on S^2_r for $r\to\infty$ and we write $\int_{\mathrm{S}^2_\infty}$ for $\lim_{r\to\infty}\int_{\mathrm{S}^2_r}$. For clearness of presentation, we first limit our attention to the case where φ is in the adjoint representation.

Proposition 7.2.7 *For any contractible open subset $U\subset\mathrm{S}^2_\infty$, the following holds:*

$$\varphi\cdot\mathbb{F} = \mathrm{d}(\varphi\cdot\mathbb{A}) + \varphi_0\cdot(\varphi^*\eta) , \tag{7.2.33}$$

where $\varphi_0\in F_{min}$ has stabilizer H and the dot denotes the Killing form.

Proof By (7.2.32), $\mathrm{d}\varphi = -[\mathbb{A}, \varphi]$. Using this and the Structure Equation, we find

$$\begin{aligned}\varphi \cdot \mathbb{F} &= \varphi \cdot \mathrm{d}\mathbb{A} + \tfrac{1}{2}\varphi \cdot [\mathbb{A}, \mathbb{A}] \\ &= \mathrm{d}(\varphi \cdot \mathbb{A}) - \mathrm{d}\varphi \overset{\cdot}{\wedge} \mathbb{A} + \tfrac{1}{2}\varphi \cdot [\mathbb{A}, \mathbb{A}] \\ &= \mathrm{d}(\varphi \cdot \mathbb{A}) - \varphi \cdot [\mathbb{A}, \mathbb{A}] + \tfrac{1}{2}\varphi \cdot [\mathbb{A}, \mathbb{A}] \\ &= \mathrm{d}(\varphi \cdot \mathbb{A}) - \tfrac{1}{2}\varphi \cdot [\mathbb{A}, \mathbb{A}] .\end{aligned}$$

Thus, it remains to analyze $\varphi \cdot [\mathbb{A}, \mathbb{A}]$ on U. Since U is contractible and φ takes values in a single orbit $F_{\min}$, there exists a mapping $g : U \to G$ such that

$$\varphi_0 = \mathrm{Ad}(g^{-1})\varphi ,$$

for a chosen vector $\varphi_0 \in \mathfrak{g}$ with stabilizer H, that is, $\varphi_0 \in \mathfrak{c}$. We put

$$\widetilde{\mathbb{A}} = \mathrm{Ad}(g^{-1})\mathbb{A} + g^*\theta_G .$$

Then, $\mathrm{d}\varphi_0 = 0$ and, by (7.2.32), $[\widetilde{\mathbb{A}}, \varphi_0] = 0$. Thus, $\widetilde{\mathbb{A}}$ takes values in $\mathfrak{h}$. Using this and the fact that the decomposition (7.2.18) is orthogonal, together with the Ad-invariance of the Killing form and the Maurer–Cartan equation, we get

$$\varphi \cdot [\mathbb{A}, \mathbb{A}] = \varphi_0 \cdot [\widetilde{\mathbb{A}} - g^*\theta_G, \widetilde{\mathbb{A}} - g^*\theta_G] = g^*\left(\varphi_0 \cdot [\theta_G, \theta_G]\right) = -2g^*\left(\varphi_0 \cdot \mathrm{d}\theta_G\right) .$$

Since $\varphi_0 \in \mathfrak{c}$, comparing with (7.2.19), we finally obtain

$$g^*\left(\varphi_0 \cdot \mathrm{d}\theta_G\right) = g^*\big(\varphi_0 \cdot \mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G)\big) = \varphi_0 \cdot (\varphi^*\eta) .$$

■

Integrating the identity (7.2.33) over S^2_∞, we obtain the following formula.

Corollary 7.2.8

$$\frac{1}{2\pi}\int_{\mathrm{S}^2_\infty} \varphi \cdot \mathbb{F} = \varphi_0 \cdot \rho([\varphi_\infty]) . \tag{7.2.34}$$

■

Remark 7.2.9

1. Using (7.2.22) and (7.2.23), from (7.2.34) we read off

$$\int_{\mathrm{S}^2_\infty} \varphi \cdot \mathbb{F} = 2\pi \sum_{k=1}^{p} m_k([\varphi_\infty])\, \varphi_0 \cdot \zeta_k . \tag{7.2.35}$$

2. Proposition 7.2.7 admits various generalizations. First, one may consider generalized invariants [610]

$$I^n = \int_{S^2_\infty} \varphi^n \cdot \mathbb{F} \, ,$$

where powers are taken in the universal enveloping algebra. Even more generally, analogous invariants may be built with any invariant polynomial on the Lie algebra [314]. Second, the above proposition immediately generalizes from the adjoint representation to an arbitrary representation σ. Then, the Killing form is replaced by a bilinear invariant function $f : \mathfrak{g} \times F \to \mathbb{R}$, that is, a function fulfilling

$$f(\mathrm{Ad}(g)X, \sigma(g)x) = f(X, x) \, , \quad X \in \mathfrak{g} \, , \; x \in F \, , \; g \in G \, .$$

Then, by the same arguments, one obtains [315]

$$f(\mathbb{F}, \varphi) = \mathrm{d}(f(\mathbb{A}, \varphi)) + \varphi^* \langle f_0, \eta \rangle \, ,$$

where $f_0 \in \mathfrak{g}^*$ is given by $f_0 = f(\varphi_0, \cdot)$. ♦

Finally, it is of course interesting to look for finite energy solutions of the system (7.2.14). Under the additional assumption $V = 0$, this question will be addressed in Sect. 7.5. On the other hand, for the issues to be discussed below, it is illuminating to find the asymptotic solutions of (7.2.14). In this case, the finite energy condition implies that the system decouples and, as can be easily seen, the second equation just yields a fall-off law for the radial dependence of Φ. Thus, within each topological sector defined by $[\varphi] \in \pi_2(G/H)$, we are left with the pure Yang–Mills equation on $\mathbb{R}^3$ with gauge group H. To study its asymptotic solution, we choose a representative $\mathbb{A}$ of ω and choose the radial gauge[7] $\mathbf{x} \cdot \mathbb{A}(\mathbf{x}) = 0$. Then, asymptotically, (7.2.14) reduces to the Yang–Mills equation on the 2-sphere S^2. This equation has been solved by Atiyah and Bott in the general context of an arbitrary Riemannian surface M, see Theorem 6.7 in [33]. We also refer to Friedrich and Habermann [220], who worked out the case of the two-sphere in detail. Following their paper, we present the proof for this case here.

Assume that H is a compact connected Lie group. We first observe that any homomorphism $\tau : \mathrm{U}(1) \to H$ defines a Yang–Mills connection over S^2 as follows: we take the complex Hopf bundle $S^3(S^2, \mathrm{U}(1))$, cf. Example 1.1.20, endowed with the canonical connection ω^c, cf. Example 1.3.20, and build the associated principal H-bundle

$$P_\tau := S^3 \times_{\mathrm{U}(1)} H \, .$$

Recall the natural injective bundle morphism $\iota : S^3 \to P_\tau$ given by $\iota(\mathbf{x}) := [(\mathbf{x}, \mathbb{1}_H)]$. By Proposition 1.3.13, the transport of ω^c under ι yields a unique connection ω_τ on P_τ fulfilling $\iota^* \omega_\tau = \mathrm{d}\tau \circ \omega^c$. Since the curvature of ω^c is given by[8] $\tilde{\Omega}^c = \frac{i}{2} \otimes \pi^* \mathrm{v}_{S^2}$,

[7] For a rigorous existence proof we refer to [637]. This gauge is also used in the physics literature.

[8] For clearness, here we have denoted the curvature, viewed as a horizontal 2-form on the bundle, by $\tilde{\Omega}^c$. Since we deal with a 2-dimensional base space, by (1.2.14), we have $\tilde{\Omega}^c = \widetilde{*\Omega^c} \otimes \pi^* \mathrm{v}_{S^2}$

where v_{S^2} is the canonical volume form of S^2, ω^c is a Yang–Mills connection. Since $\iota^*\Omega_\tau = \mathrm{d}\tau \circ \Omega^c$, the induced connection ω_τ is Yang–Mills, too. To summarize, denoting by $\mathrm{YM}(S^2, H)$ the set of pairs (P, ω), where P is a principal H-bundle over S^2 and ω is a Yang–Mills connection on P, we obtain a mapping

$$\chi : \mathrm{Hom}(\mathrm{U}(1), H) \to \mathrm{YM}(S^2, H), \quad \chi(\tau) := (P_\tau, \omega_\tau). \tag{7.2.36}$$

The following is a simple exercise which we leave to the reader (Exercise 7.2.5).

Lemma 7.2.10 *Let τ and $\tilde{\tau}$ belong to* $\mathrm{Hom}(\mathrm{U}(1), H)$. *Then, τ and $\tilde{\tau}$ are conjugate under inner automorphisms of H iff the corresponding pairs (P_τ, ω_τ) and $(P_{\tilde{\tau}}, \omega_{\tilde{\tau}})$ are equivalent, that is, if there exists a vertical isomorphism $\vartheta : P_\tau \to P_{\tilde{\tau}}$ such that $\vartheta^*\omega_{\tilde{\tau}} = \omega_\tau$.*

Using this lemma, by passing to equivalence classes, we obtain the following injective mapping:

$$\tilde{\chi} : \widetilde{\mathrm{Hom}}(\mathrm{U}(1), H) \to \widetilde{\mathrm{YM}}(S^2, H), \quad \tilde{\chi}([\tau]) := [(P_\tau, \omega_\tau)]. \tag{7.2.37}$$

Lemma 7.2.11 *Let (P, S^2, H) be a principal fibre bundle and let Γ be a Yang–Mills connection on P. Then, the holonomy group of Γ is either trivial or* $\mathrm{U}(1)$.

Proof Let $\mathscr{H}_{p_0}(\Gamma)$ be the holonomy group and let $P_{p_0}(\Gamma)$ be the holonomy bundle of Γ with respect to a chosen point $p_0 \in P$. Let ω be the connection form of Γ and let Ω be its curvature form. As a 2-form on S^2, Ω necessarily has the form $\Omega = B\,\mathsf{v}_{S^2}$, where $B = *\Omega$ is a section of $\mathrm{Ad}(P)$. In terms of B, the Yang–Mills equation reads

$$\mathrm{d}_\omega B = 0, \tag{7.2.38}$$

that is, B is covariantly constant. Thus, the corresponding equivariant mapping $\tilde{B} : P \to \mathfrak{g}$ fulfils $\tilde{B}(p) = \mathbb{Q}$ for some fixed $\mathbb{Q} \in \mathfrak{h}$ and all $p \in P_{p_0}(\Gamma)$. Now, by Proposition 1.7.12, Γ is reducible to $P_{p_0}(\Gamma)$ and, by the Ambrose–Singer Theorem 1.7.15, the Lie algebra of $\mathscr{H}_{p_0}(\Gamma)$ is spanned by $\mathbb{Q}$. Therefore, $\mathscr{H}_{p_0}(\Gamma)$ is discrete or 1-dimensional. But, by Remark 1.7.11, $\mathscr{H}_{p_0}(\Gamma)$ is connected and, thus, it is either trivial or $\mathrm{U}(1)$ or $\mathbb{R}$. Suppose $\mathscr{H}_{p_0}(\Gamma) = \mathbb{R}$. Then, $P_{p_0}(\Gamma)$ is trivial and, for a chosen global section $s : S^2 \to P_{p_0}(\Gamma)$, the Yang–Mills equation (7.2.38) for the reduced (Abelian) connection $\hat{\omega}$ with curvature $\hat{\Omega}$ yields

$$s^*\hat{\Omega} = \mathrm{d}(s^*\hat{\omega}) = c\,\mathsf{v}_{S^2}, \quad c \in \mathbb{R}.$$

Integrating this equation over S^2 and using Stokes' Theorem, we find $c = 0$ which contradicts the irreducibility of $\hat{\omega}$. ■

(Footnote 8 continued)
with $\pi : S^3 \to S^2$ denoting the canonical projection. Below, usually the curvature will be viewed as a 2-form with values in $\mathrm{Ad}(P)$.

From the proof, we conclude that the local representative of the curvature with respect to a local section in the holonomy bundle is given by

$$\mathbb{F} = \mathbb{Q}\mathsf{v}_{\mathrm{S}^2} \,. \tag{7.2.39}$$

Theorem 7.2.12 (Atiyah–Bott) *The mapping* $\tilde{\chi}$ *given by* (7.2.37) *yields a one-to-one correspondence between conjugacy classes of homomorphisms* $\mathrm{U}(1) \to H$ *and equivalence classes of Yang–Mills connections over* S^2.

Proof It remains to show that $\tilde{\chi}$ is surjective, that is, given any $(P, \omega) \in \mathrm{YM}(\mathrm{S}^2, H)$ we must construct a homomorphism $\tau : \mathrm{U}(1) \to H$ such that (P_τ, ω_τ) is equivalent to (P, ω).

1. Consider the case $H = \mathrm{U}(1)$. Then, in the same notation as above, the Yang–Mills equation implies that on $P_{p_0}(\Gamma)$ we have $\tilde{B} = \widetilde{*\Omega} = ic$ for some $c \in \mathbb{R}$. Since the first Chern index

$$\mathsf{c}_1(P) = \int_{\mathrm{S}^2} \mathsf{c}_1(P) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{S}^2} \mathrm{tr}(\Omega) = -\frac{\mathrm{i}c}{2\pi \mathrm{i}} 4\pi = -2c \tag{7.2.40}$$

is an integer, the mapping

$$\tau : \mathrm{U}(1) \to \mathrm{U}(1)\,, \quad \tau(\mathrm{e}^{i2\pi t}) := \mathrm{e}^{i4\pi tc}$$

is a homomorphism. We show that the induced pair (P_τ, ω_τ) is equivalent to (P, ω). Since the adjoint bundle of a principal $\mathrm{U}(1)$-bundle over S^2 is trivial, we may view the curvatures of the connections under consideration as $i\mathbb{R}$-valued 2-forms on S^2. In this sense, we obtain

$$\Omega_\tau = \mathrm{d}\tau \circ \Omega^c = \mathrm{d}\tau \left(\frac{i}{2}\right) \mathsf{v}_{\mathrm{S}^2} = ic\mathsf{v}_{\mathrm{S}^2} = \Omega \,.$$

Thus, we have $\mathsf{c}_1(P) = \mathsf{c}_1(P_\tau)$, that is, by Theorem 4.8.1, there exists a vertical isomorphism $\vartheta_1 : P_\tau \to P$ of principal $\mathrm{U}(1)$-bundles. The curvature of $\vartheta_1^* \omega$ coincides with Ω and thus with the curvature of ω_τ. Since the curvatures of $\vartheta_1^*\omega$ and ω_τ coincide, there exists a vertical automorphism of P_τ transforming $\vartheta_1^*\omega$ to ω_τ. Indeed, since the adjoint bundle is trivial, there exists an $i\mathbb{R}$-valued 1-form α on S^2 such that $\vartheta_1^*\omega - \omega_\tau = \pi_\tau^*\alpha$. By equality of the curvatures, we get $\mathrm{d}\alpha = 0$. Now, vanishing of the first de Rham cohomology group of S^2 implies the existence of a potential λ of α. This proves the assertion for $G = \mathrm{U}(1)$.

2. Now, let H be an arbitrary compact connected Lie group. Then, ω reduces to a connection form $\hat{\omega}$ on the holonomy bundle $P_{p_0}(\Gamma)$, where Γ is the connection corresponding to ω. By Lemma 7.2.11, the holonomy group $\mathscr{H}_{p_0}(\Gamma)$ is either trivial or $\mathrm{U}(1)$. Thus, by point 1, we have a homomorphism $\hat{\tau} : \mathrm{U}(1) \to \mathscr{H}_{p_0}(\Gamma)$ and, by Theorem 1.7.9, $\hat{\tau}$ yields a homomorphism

$$\tau : \mathrm{U}(1) \to H\,, \quad \tau(\mathrm{e}^{i2\pi t}) = \exp(4\pi \mathbb{Q} t)\,, \tag{7.2.41}$$

where $\mathbb{Q}$ is the generator of $\mathscr{H}_{p_0}(\Gamma)$ given by (7.2.39). Also by point 1, there exists an isomorphism $\hat{\vartheta} : P_{\hat{\tau}} \to P_{p_0}(\Gamma)$ such that $\hat{\vartheta}^* \hat{\omega} = \omega_{\hat{\tau}}$ which obviously can be extended to an isomorphism $\vartheta : P_\tau \to P$ yielding the equivalence of (P, ω) and (P_τ, ω_τ). ■

Remark 7.2.13

1. According to Theorem 7.2.12, finite energy asymptotic solutions of the Yang–Mills–Higgs system are classified by conjugacy classes of elements $\mathbb{Q} \in \mathfrak{h}$ satisfying the following quantization condition

$$\exp(4\pi \mathbb{Q}) = \mathbb{1}_H \,. \tag{7.2.42}$$

   Inserting (7.2.41) into (7.2.31) and using (7.2.29), we obtain

$$\rho([\varphi_\infty]) = \mathrm{pr}_{\mathfrak{z}}(2\mathbb{Q}) \,, \tag{7.2.43}$$

   that is, $\mathbb{Q}$ determines the invariants discussed before completely. As a consequence, $2\mathbb{Q}$ defines a topological charge.
2. By (7.2.39), the curvature $\mathbb{F}$ of any solution $\mathbb{A}$ fulfils $\mathbb{F} = \mathbb{Q}\mathrm{v}_{\mathrm{S}^2}$. In spherical coordinates (ϑ, ϕ) on S^2, the solution $\mathbb{A}$ is given by

$$A_\vartheta = 0 \,, \quad A_\phi = \pm(1 \mp \cos\vartheta)\mathbb{Q} \,. \tag{7.2.44}$$

   Note that $\mathbb{A}$, extended to a gauge potential on $\mathbb{R}^3$, has a singularity at the origin and, thus, an infinite energy. Also note that these solutions are spherically symmetric, cf. Example 1.9.17 for the case $H = \mathrm{SU}(2)$ or [424] for general H. We will come back to these solutions in Sect. 7.4.
3. By studying the Hessian in the same spirit as in Sect. 6.7, the stability of the above Yang–Mills connections can be analyzed,[9] see [33] for the general case of a Riemannian surface M and [100, 220, 313, 316] for $M = \mathrm{S}^2$. We also refer to [691] for a pedagogical presentation. The number of negative modes of the Hessian turns out to be equal to

$$n = 2\sum_q (2|q| - 1)$$

   where the half integers q are the nonzero eigenvalues of $\mathbb{Q}$ with respect to a chosen root system. Thus, stability only holds if $\mathbb{Q}$ has eigenvalues $0, \pm\frac{1}{2}$.
4. It can be shown that critical points of the Yang–Mills functional on S^2 correspond to critical points of the energy functional on the loop space ΩH, see [220]. ♦

[9] Since, here, we are on S^2, methods of complex analysis can be applied.

Exercises

7.2.1 Prove formula (7.2.12).

7.2.2 Let θ_G be the Maurer–Cartan form on G and let $\mathrm{pr}_{\mathfrak{c}} : \mathfrak{g} \to \mathfrak{c}$ be the projection defined by (7.2.18). Prove that $\mathrm{pr}_{\mathfrak{c}}(\mathrm{d}\theta_G)$ is a closed 2-form on G which is right H-invariant and left G-invariant.

7.2.3 Show that the 2-forms η^k defined by (7.2.22) are exact for all $k > p$.

7.2.4 Using the Maurer–Cartan equation for θ_G, prove that formula (7.2.25) is a special case of (7.2.19).

7.2.5 Prove Lemma 7.2.10.

7.3 The Higgs Mechanism

In this section, we explain the announced spontaneous symmetry breaking induced by the Higgs potential V. Assume $\mu^2 < 0$. Recall that Φ is equivalently described by an element $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$.

Definition 7.3.1 Let $F_{\mathrm{min}} \subset F$ be the set of absolute minima of the potential V. A Higgs field $\tilde{\Phi} \in \mathrm{Hom}_G(P, F)$ is called a Higgs vacuum if $\tilde{\Phi}(P) \subset F_{\mathrm{min}}$.

In the sequel, Higgs vacua will be denoted by $\tilde{\Phi}_v$. Clearly, F_{min} is a level set of the smooth function V. In the sequel, we assume that V' is nowhere vanishing on F_{min}. Then, by the Level Set Theorem, F_{min} is an embedded submanifold of F. Since V is invariant under the representation σ, the level set F_{min} is a union of orbits of σ.

Proposition 7.3.2 *Assume that F_{min} consists of a single orbit of σ. Let $H \subset G$ be the stabilizer of some point $f \in F_{min}$. Then, Higgs vacua are in one-to-one correspondence with reductions of P to the structure group H.*

Proof Since, by assumption, F_{min} is a transitive G-manifold, the assertion is an immediate consequence of Proposition 1.6.2. ■

The subbundle defined by a Higgs vacuum $\tilde{\Phi}_v$ is given by (1.6.2),

$$Q_f = \{p \in P : \tilde{\Phi}_v(p) = f\} .$$

Remark 7.3.3

1. Again, let $f \in F_{\mathrm{min}}$ and $H = G_f$. Then, viewed as a section $\Phi_v \in \Gamma^\infty(E)$, a Higgs vacuum takes values in the subbundle $P \times_G G/H \subset E$ defined by the embedding $[(p, gH)] \mapsto [(p, \sigma(g)f)]$. Thus, Proposition 7.3.2 also follows from Corollary 1.6.5.

2. The existence of Higgs vacua or, equivalently, the existence of reductions of P to suitable subgroups of G depends on the topology of P. If P is trivial, then Higgs vacua always exist.
3. Let ϑ be a vertical automorphism of P. By gauge invariance of V, if $\tilde{\Phi}_v$ is a Higgs vacuum, then the gauge transformed field $\vartheta^* \tilde{\Phi}_v$ is also a Higgs vacuum. By Proposition 1.6.4, Higgs vacua are gauge equivalent iff the corresponding bundle reductions are equivalent. Thus, gauge equivalence classes of Higgs vacua are in one-to-one correspondence with equivalence classes of reductions of P to the subgroup H.
4. A similar characterization of Higgs vacua may be obtained under the following weaker assumptions:

   (a) F_{min} consists of orbits belonging to the same orbit type.
   (b) The projection $F_{\mathrm{min}} \to F_{\mathrm{min}}/G$ is trivial, that is, there exists a submanifold $\Sigma \subset F_{\mathrm{min}}$ which is intersected by each orbit in F_{min} exactly once.

   Under these assumptions, one may choose Σ in such a way that all its elements have the same stabilizer. Then,

   $$Q = \{p \in P : \tilde{\Phi}_v(p) \in \Sigma\}$$

   is a reduction of P to the subgroup H. Conversely, given such a reduction, it obviously defines a Higgs vacuum. ♦

Now, let there be given a Higgs vacuum $\tilde{\Phi}_v$ and assume, as before, that F_{min} consists of a single orbit. Let $f \in F_{\mathrm{min}}$, let $H = G_f$ and let $i : Q_f \to P$ be the corresponding bundle reduction to the structure group H. As usual, denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Since G is compact, we can choose a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{7.3.1}$$

which is orthogonal with respect to the Ad-invariant scalar product. Then, this decomposition is automatically reductive. In the sequel, we will call $(\omega, \tilde{\Phi})$ a configuration of type $[H]$ if ω is irreducible and $\tilde{\Phi}$ takes values in the stratum $F_{[H]}$. Clearly, if $(\omega, \tilde{\Phi})$ is of type $[H]$, then every gauge equivalent configuration is of type $[H]$, too.

Proposition 7.3.4 *Assume that the representation σ acts transitively on the unit sphere in F. Let $\tilde{\Phi}_v$ be a Higgs vacuum, let $f \in F_{min}$ and let $H = G_f$. Let Q_f be the H-reduction of P defined by $\tilde{\Phi}_v$. Then, there is a one-to-one correspondence between gauge equivalence classes of configurations $[(\omega, \tilde{\Phi})]$ of type $[H]$ and triples $\big([(\hat{\omega}, \tau)], \eta\big)$, where*

1. *$\hat{\omega}$ is a connection form on Q_f,*
2. *τ is is a horizontal 1-form of type $\mathrm{Ad}(H)\mathfrak{m}$ on Q_f,*
3. *η is a function on Q_f with values in the $[H]$-stratum of the orbit space F/G.*

The equivalence class $[(\hat{\omega}, \tau)]$ is taken with respect to gauge transformations on Q_f.

Proof By the assumption on σ, for any representative $(\omega, \tilde{\Phi}) \in [(\omega, \tilde{\Phi})]$, we can find a gauge transformation ϑ on P such that

$$\frac{\vartheta^* \tilde{\Phi}}{\|\vartheta^* \tilde{\Phi}\|} = \frac{f}{\|f\|} . \tag{7.3.2}$$

That is, for any $[(\omega, \tilde{\Phi})]$, we may limit our attention to representatives fulfilling this condition. Let $(\omega, \tilde{\Phi})$ be one such representative. Consider its restriction $(i^*\omega, i^*\tilde{\Phi})$ to Q_f, where $i : Q_f \to P$ is the canonical inclusion mapping. Using (7.3.1), decompose

$$i^*\omega = i^*\omega_{\mathfrak{h}} + i^*\omega_{\mathfrak{m}} , \quad i^*\Phi = f + \tilde{\phi} . \tag{7.3.3}$$

Denote $\hat{\omega} = i^*\omega_{\mathfrak{h}}$ and $\tau = i^*\omega_{\mathfrak{m}}$. Then, the following hold.

(a) By Proposition 1.6.8, $\hat{\omega}$ is a connection form on Q_f and τ is a horizontal 1-form of type $\mathrm{Ad}(H)\mathfrak{m}$ on Q_f.
(b) By construction, $\tilde{\phi} : Q_f \to F$ is equivariant with respect to the H-action. By (7.3.2) and (7.3.3), we have $\tilde{\phi} = \|\tilde{\phi}\| \frac{f}{\|f\|}$. Thus, $\tilde{\phi}$ is completely characterized by the gauge-invariant matter field

$$\eta : Q_f \to F/G \equiv \mathbb{R}_+ , \quad \eta(q) := \|\tilde{\phi}\| ,$$

taking values in the stratum of type $[H]$ of the orbit space F/G.

Thus, the configuration $(\omega, \tilde{\Phi})$ fulfilling (7.3.2) is characterized by the triple $(\hat{\omega}, \tau, \eta)$ of geometric objects living on Q_f. Next, consider another representative $(\omega', \tilde{\Phi}')$ of $[(\omega, \tilde{\Phi})]$, also fulfilling (7.3.2). Then, since the norm of $\tilde{\Phi}$ is gauge invariant, we have $\tilde{\Phi}' = \tilde{\Phi}$ and ω' is the image of ω under an H-valued gauge transformation. Consequently, the construction based on the decomposition (7.3.3) yields a configuration $(\hat{\omega}', \tau')$ which is equivalent to $(\hat{\omega}, \tau)$ under vertical automorphisms of Q_f.

Conversely, by standard arguments,[10] given a triple $\big((\hat{\omega}, \tau), \eta\big)$, the configuration $(\omega, \tilde{\Phi})$ may be reconstructed uniquely up to a gauge transformation. ■

To our knowledge, the fact that the symmetry breaking mechanism is related to principal bundle reductions was first observed by Trautman [631], see also [227, 362, 540].

Remark 7.3.5

1. The connection $\hat{\omega}$ is the gauge potential corresponding to the broken symmetry. According to the terminology in Sect. 7.1, τ is a bosonic matter field. In the sequel, it will be called an intermediate vector boson of gauge type $\mathrm{Ad}(H)\mathfrak{m}$. Finally, η will be referred to as the surviving Higgs field.

[10] In particular, recall from Proposition 1.6.7 that the associated vector bundles $P \times_G F$ and $Q \times_H F$ are isomorphic.

2. If $\tilde{\Phi}$ takes values in more than one orbit type, then interesting topological effects may occur. In the next section, we will discuss the case when Φ vanishes on some subset of M. This may give rise to magnetic monopoles. In a general setting, one speaks in this context of defects related to a broken symmetry, see [444]. ♦

Let us calculate the action functional (7.2.1) after the symmetry breaking, that is, in terms of the classifying objects given by Proposition 7.3.4. Denoting $\hat{f} = \frac{f}{\|f\|}$ and $\eta_v = \|f\|$, we calculate

$$i^*\Omega = \Omega^{\hat{\omega}} + D_{\hat{\omega}}\tau + \frac{1}{2}[\tau, \tau], \quad i^*\big(D_\omega\tilde{\Phi}\big) = \big(\mathrm{d}\eta + (\eta_v + \eta)\sigma'(\tau)\big)\hat{f},$$

and

$$i^*(V(\tilde{\Phi})) = \frac{1}{2}\mu^2(\eta_v + \eta)^2 + \frac{1}{4}\lambda(\eta_v + \eta)^4 .$$

Thus, using the orthogonality of the decomposition (7.3.1), we obtain the following reduced action functional

$$\begin{aligned}\hat{S}(\hat{\omega}, \tau, \eta) =& \frac{1}{2}\|\Omega^{\hat{\omega}} + \frac{1}{2}[\tau, \tau]_{\mathfrak{h}}\|^2 + \frac{1}{2}\|D_{\hat{\omega}}\tau + \frac{1}{2}[\tau, \tau]_{\mathfrak{m}}\|^2 + \frac{1}{2}\|\mathrm{d}\eta\|^2 \\ &+ \frac{1}{2}(\eta_v + \eta)^2\|\sigma'(\tau)\hat{f}\|^2 + V(\eta),\end{aligned} \tag{7.3.4}$$

where the indices $\mathfrak{h}$ and $\mathfrak{m}$ denote the projection onto $\mathfrak{h}$ and $\mathfrak{m}$, respectively. The physical interpretation of the terms occurring in (7.3.4) is as follows: the first term gives the Yang–Mills functional for the reduced gauge potential $\hat{\omega}$ (modified by an additional summand), the second and the third term are standard kinetic action functionals for the matter fields τ and η, the fourth term contains a typical mass contribution for the intermediate vector boson τ, together with a contribution describing the interaction of τ and η and the last term is a self-interaction term of the surviving Higgs field η. In particular, it contains a mass term. To summarize, we see that in the process of spontaneous symmetry breaking, the intermediate vector bosons acquire a mass. This is referred to as the Higgs mechanism,[11] see [106, 186, 273, 274, 298–300, 364] for the classical literature.

Remark 7.3.6 Since $\sigma'(\mathrm{Ad}(h)A) = \sigma(h) \circ \sigma'(A) \circ \sigma(h^{-1})$, for any $h \in H$ and any $A \in \mathfrak{m}$, the bilinear form

$$\langle\cdot,\cdot\rangle_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad \langle A, B\rangle_{\mathfrak{m}} := \langle\sigma'(A)\hat{f}, \sigma'(B)\hat{f}\rangle$$

is an H-invariant scalar product on $\mathfrak{m}$. Thus,

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[11] In a realistic model, like the standard model of elementary particle physics, a large number of fermionic fields describing matter occur. All these fields, except for the neutrino field, also acquire a mass via the Higgs mechanism. This will be explained in Sect. 7.7.

$$\|\sigma'(\tau)\hat{f}\|^2 = \int_M \tau \mathbin{\dot{\wedge}} *\tau\,, \tag{7.3.5}$$

with the dot defined by $\langle\cdot,\cdot\rangle_{\mathfrak{m}}$. ♦

Let us illustrate the Higgs mechanism for a toy model with gauge group $\mathrm{SU}(2)$.

Example 7.3.7 (*Georgi–Glashow model*) Consider the trivial principal $\mathrm{SU}(2)$-bundle $P = M \times \mathrm{SU}(2)$ over Minkowski space M and the associated bundle $E \equiv \mathrm{Ad}(P) = P \times_{\mathrm{SU}(2)} \mathfrak{su}(2)$. This model is often called the Georgi–Glashow model of electroweak interactions. Using the Lie algebra isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathbb{R}^3$ and the identification of the adjoint representation of $\mathrm{SO}(3)$ with its defining representation on $\mathbb{R}^3$, cf. Examples I/5.2.8 and I/5.4.7, we obtain

$$E \cong P \times_{\mathrm{SO}(3)} \mathbb{R}^3\,. \tag{7.3.6}$$

Since the bundles are trivial, we may describe any configuration (ω, Φ) in terms of its (global) representatives $(\mathbb{A}, \varphi)$ on M. In the standard basis $\{\mathbf{e}_a\}$, $a = 1, 2, 3$, on $\mathbb{R}^3$, we write $\varphi = \varphi^a \mathbf{e}_a$. Now, consider the action functional (7.2.1), with the Higgs potential given by (7.2.2). Then, for $\mu^2 < 0$, the minimum of V is given by

$$\|\varphi\|^2 \equiv \varphi^a \varphi_a = -\frac{\mu^2}{\lambda}\,. \tag{7.3.7}$$

Thus, $F_{\min}$ coincides with a 2-sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ of radius $\eta_v = \sqrt{-\frac{\mu^2}{\lambda}}$. In particular, it consists of a single orbit. We choose

$$\mathbf{f} = \eta_v \mathbf{e}_3 \equiv \begin{bmatrix} 0 \\ 0 \\ \eta_v \end{bmatrix}.$$

The stabilizer of $\mathbf{f}$ is

$$H := \left\{ \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{SO}(3) : R \in \mathrm{SO}(2) \right\}. \tag{7.3.8}$$

Since P is trivial, it admits a reduction to the subgroup H and, thus, there exists a Higgs vacuum φ_v. Since M is topologically trivial, any such reduction is again trivial and thus equivalent to the subbundle $Q_{\mathbf{f}} = M \times \mathrm{SO}(2)$ with the embedding $i : Q_{\mathbf{f}} \to P$ defined by (7.3.8). Clearly, the Higgs vacuum corresponding to this reduction is given by

$$\varphi_v(\mathbf{x}) = \mathbf{f}\,,$$

for all $\mathbf{x} \in M$. The orthogonal reductive decomposition (7.3.1) has the form

$$\mathfrak{so}(3) = \mathfrak{so}(2) \oplus \mathfrak{m}\,, \tag{7.3.9}$$

where $\mathfrak{m} \cong \mathbb{R}^2$. Note that under this identification, the restriction of $H = \mathrm{SO}(2)$ to $\mathfrak{m}$ coincides with the defining representation of SO(2).

Now, consider a configuration of type [SO(2)]. Denoting the global representatives of $\hat{\omega}$ and τ by $\hat{\mathbb{A}}$ and $\mathbb{V}$, respectively, we find

$$\widehat{\mathbb{A}} = \mathbb{A}^3\,, \quad \mathbb{V} = \begin{bmatrix} \mathbb{A}^1 \\ \mathbb{A}^2 \end{bmatrix}.$$

Here, $\widehat{\mathbb{A}}$ is an $\mathfrak{so}(2)$- or $\mathfrak{u}(1)$-valued gauge potential, which may be viewed as a model of the photon field. The intermediate vector boson $\mathbb{V}$ is an $\mathbb{R}^2$-valued covector field carrying the defining representation of SO(2). Alternatively, we may view it as a complex-valued covector field $\mathbb{V} = \mathbb{A}^1 + i\mathbb{A}^2$ carrying the defining representation of U(1). Since $F/G = \mathbb{R}^3/\mathrm{SO}(3) \cong \mathbb{R}_+ \cup \{0\}$, for a configuration of type [SO(2)], the surviving Higgs field η is a function on M with values in $\mathbb{R}_+$.

Finally, let us examine in detail the covariant derivative $i^*\big(D_\omega\tilde{\Phi}\big)$ which is responsible for the mass generation in the reduced action (7.3.4). With $\varphi = (\eta_v + \eta)\mathbf{e}_3$, $D\varphi = \mathrm{d}\varphi + [\mathbb{A}, \varphi]$ and, thus, $D\varphi^a = \mathrm{d}\varphi^a + \varepsilon^a{}_{bc}\, \mathbb{A}^b\varphi^c$, we obtain

$$(D\varphi)^1 = (\eta_v + \eta)\mathbb{A}^2\,, \quad (D\varphi)^2 = -(\eta_v + \eta)\mathbb{A}^1\,, \quad (D\varphi)^3 = \mathrm{d}\eta\,,$$

and, thus, in the standard basis $\{\mathbf{e}_\mu\}$, $\mu = 0, \ldots, 3$, of M,

$$\|D\varphi\|^2 = \int_M \big(\partial_\mu\eta\partial^\mu\eta + (\eta_v + \eta)^2\big(A_\mu{}^1 A^{\mu\,1} + A_\mu{}^2 A^{\mu\,2}\big)\big)\, \mathsf{v}_M\,,$$

cf. formula (7.3.4). We see that the mass of the intermediate vector boson $\mathbb{V}$ is simply given by η_v. ♦

Up until now, we have merely assumed that an absolute minimum of the Higgs potential exists and we have drawn consequences from this fact. In the remainder of this section, we will briefly discuss the existence problem in a model independent way. There is a huge literature on this subject which, on the mathematical side, is related to modern equivariant bifurcation theory [196, 197, 252] and algebraic geometry. The classical papers on this subject are by Michel, Radicati, Abud and Sartori, see [5, 6, 443–446, 557, 558] and further references therein. For further developments, including an extension to the Yang–Mills functional, see also [229–231]. It should be noted that mechanisms of spontaneous symmetry breaking play a role in various branches of physics, or, even more generally of natural sciences. Among the classical papers cited above, [444] gives a nice overview. Our focus is rather on gauge theories only.

As already mentioned at the beginning, for a given gauge-invariant Higgs potential V, the set of absolute minima F_{min} is necessarily a union of orbits. Thus, we are rather dealing with a variational problem on the orbit space F/G. If we want to depart from a concrete model, we should allow V to be an arbitrary gauge invariant function on F or, equivalently, a function on F/G. Thus, the appropriate mathematical tools for the discussion of our variational problem are the theory of Lie group actions as developed in Chap. 6 of Part I and, in close relation, the classical invariant theory, see [301, 523, 570, 571].

Thus, let G be a compact, connected Lie group and let (F, G, σ) be a real representation which is orthogonal with respect to a chosen scalar product h. Consider an orbit type $[H]$ of σ and the corresponding stratum $F_{[H]} \subset F$.[12] Let $f \in F_{[H]}$, let $G_f = H$ be its stabilizer and let $G \cdot f$ be the orbit through f. Since G is compact, there exists a tubular neighbourhood of $G \cdot f$. Let N_f be the corresponding (linear) slice through f, see Sect. 6.4 of Part I. Recall that H acts orthogonally and reducibly on $\mathrm{T}_f F$. The subspaces $\mathrm{T}_f\,(G \cdot f)$, $\mathrm{T}_f F_{[H]}$ and N_f are invariant under this action. By the results of Sect. 6.6 of Part I, we have the orthogonal direct sum decomposition

$$\mathrm{T}_f F = \mathrm{T}_f\,(G \cdot f) \oplus N_f\,.$$

Moreover, defining $N_f^0 := N_f \cap \mathrm{T}_f F_{[H]}$ and $N_f^1 := \left(\mathrm{T}_f F_{[H]}\right)^{\perp} \subset N_f$, we obtain the following direct sum decompositions

$$\mathrm{T}_f F = \mathrm{T}_f\,(G \cdot f) \oplus N_f^0 \oplus N_f^1\,, \quad \mathrm{T}_f F_{[H]} = \mathrm{T}_f\,(G \cdot f) \oplus N_f^0\,. \tag{7.3.10}$$

Note that N_f^0 is the maximal subspace of N_f where H acts trivially. This means, in particular, that under the canonical projection $F \to F/G$, it is identified with the tangent space to the $[H]$-stratum of the orbit space. Also note the following.

(a) For the principal stratum we have $N_f^1 = 0$.
(b) If $N_f^0 = 0$, the orbit $G \cdot f$ is isolated in its stratum, that is, there exists a G-invariant neighbourhood U of $G \cdot f$ which contains no other orbit of the same type.

Next, let us consider the algebra $C^\infty(F)^G$ of G-invariant functions on F. Since the representation σ is real, we can identify F with a finite dimensional Euclidean space and G with a subgroup of the orthogonal group. In this situation, the classical invariant theory is directly applicable: the ring $\mathbb{P}^G(F)$ of G-invariant polynomial functions on F is finitely generated and any set of generators $\rho_1, \ldots, \rho_p$ defines a mapping

$$\rho = (\rho_1, \ldots, \rho_p) : F/G \to \mathbb{R}^p\,,$$

which is a homeomorphism onto its image. By [570], any element of $C^\infty(F)^G$ can be presented as a smooth function of the generators $\rho_1, \ldots, \rho_p$. This implies that any set of generators separates orbits, that is, any such set may be used to parameterize the points of the orbit space F/G. In general, the generators ρ_i fulfil a number of equations and inequalities keeping track of their ranges. Thus, the image $\mathscr{S}$ of ρ is

[12]For simplicity, assume that $F_{[H]}$ is connected. Otherwise, we would have to introduce a second index labeling the connected components, see Chap. 6 of Part I.

a closed semialgebraic variety of $\mathbb{R}^p$. Moreover, ρ maps the connected components of the strata of F/G bijectively onto the primary strata[13] of the variety $\mathscr{S}$.

The set $\{\rho_1, \ldots, \rho_p\}$ and the mapping ρ are called a Hilbert basis and a Hilbert mapping for σ, respectively. One says that an orbit O of σ is critical if any G-invariant function is stationary on O. The following theorem was shown in [443].

Theorem 7.3.8 (Michel) *Under the above assumptions, an orbit is critical for σ iff it is isolated in its stratum.*

Proof Let f_0 be a point in the orbit under consideration and let $V \in C^\infty(F)^G$. Since V and h are G-invariant, the gradient vector field $\nabla V = \mathsf{h}^{-1}(\mathrm{d}V)$ is invariant, too. Moreover, it must be orthogonal to $G \cdot f_0$ at every point $f \in G \cdot f_0$. On the other hand, by the discussion in Sect. 6.7 of Part I, the flow of any invariant vector field leaves the strata invariant and, thus, $(\nabla V)_f \in \mathrm{T}_f F_{[H]}$ for any $f \in G \cdot f_0$. Thus, $(\nabla V)_f \in N_f^0$. Consequently, if $N_f^0 = 0$, then $(\mathrm{d}V)(f) = 0$ for any $V \in C^\infty(F)^G$ and every $f \in G \cdot f_0$.

Conversely, let $G \cdot f_0$ be a critical orbit, that is, $(\mathrm{d}V)(f) = 0$ for any $V \in C^\infty(F)^G$ and every $f \in G \cdot f_0$. It was shown in [557] that the subspace N_f^0 is spanned by the gradients of elements of the Hilbert basis. If they all vanish, then clearly $N_f^0 = 0$. ■

From the above proof, we note the following basic facts:

(a) For every $V \in C^\infty(F)^G$, we have $(\nabla V)_f \in N_f^0$.
(b) N_f^0 is spanned by the gradients of the elements of a Hilbert basis.

These observations can be taken as a starting point for a general model-independent analysis of the variational problem under consideration. For a given stratum, one has to determine those generators which are functionally independent on that stratum. Next, their gradients may be used as a basis of N_f^0 and, then, the gradient of any G-invariant function may be expanded in this basis. Now, the equation $\nabla V = 0$ may be analyzed in terms of the coefficient functions with respect to the chosen basis. Finally, the Hessian has to be studied as well. For a detailed analysis of this approach we refer to [6, 558].

Remark 7.3.9 If the potential V depends on a number of parameters, the location of the stationary points of V will in general depend on these parameters. Varying their values may result in shifting the absolute minimum to a different stratum, thus leading to a different residual symmetry. This gives rise to bifurcation phenomena which, together with the related phase transitions of the physical states, may also be discussed using the above described framework. For a nice illustration, see Example 1 in Sect. 5.4 of Ref. [558]. ♦

[13] See [668, 669] or [571] for this notion. In short, a primary stratification of $\mathscr{S}$ is a locally finite collection E_i of connected semi-analytic submanifolds of $\mathbb{R}^p$, called strata, such that $\mathscr{S} = \cup_i E_i$ and such that, for each i, $\overline{E}_i \setminus E_i$ is a union of lower-dimensional strata.

7.4 Magnetic Monopoles

In this section, we take up the discussion from Sect. 7.2. First, we recall the classical Dirac monopole and, next, we consider the non-Abelian model of Georgi and Glashow introduced in Example 7.3.7 in detail. We discuss the following points:

(a) the identification of the electromagnetic field,
(b) the local characterization in terms of the Poincaré–Hopf index and the global magnetic charge conservation law,
(c) the charge quantization,
(d) the search for exact finite energy solutions exhibiting a magnetic monopole.

First, to recall the classical Dirac monopole, consider a static electromagnetic field in the absence of a magnetic current. Then, the Maxwell equations[14] read

$$\nabla \cdot \mathbf{D} = 4\pi\rho \quad , \quad \nabla \cdot \mathbf{B} = 0 \tag{7.4.1}$$

$$\nabla \times \mathbf{E} = 0 \quad , \quad \nabla \times \mathbf{H} = 0\,. \tag{7.4.2}$$

In an attempt to reconcile magnetically charged particles with quantum mechanics, Dirac [153, 154] considered an electron in the field of a magnetic charge. Postulating the single-valuedness of the wave function, he found that the electric and the magnetic charges must be related by a certain quantization condition, see below. Thus, if a magnetic monopole existed, this would explain the quantization of electric charge.[15]

In the presence of a hypothetical single monopole of strength g, the second equation in (7.4.1) takes the form

$$\nabla \cdot \mathbf{B} = 4\pi g \delta^3(\mathbf{x})\,, \tag{7.4.3}$$

where $g\delta^3(\mathbf{x})$ stands for the magnetic monopole charge density. For the boundary condition $\mathbf{B}(\mathbf{x}) \to 0$ as $\|\mathbf{x}\| \to \infty$, the unique solution to this equation reads

$$\mathbf{B}(\mathbf{x}) = g\,\frac{\mathbf{x}}{\|\mathbf{x}\|^3}\,. \tag{7.4.4}$$

By the Gauß law, the magnetic flux of this field through the surface S^2 of a ball centered at zero is equal to the magnetic charge Q_{m} inside the ball,

$$\int_{\mathrm{S}^2} \mathbf{B} \cdot \mathbf{dS} = Q_{\mathrm{m}}\,.$$

Consider the restriction of the solution (7.4.4) to $\mathrm{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$:

---

[14]In the CGS-system.

[15]For a number of interesting historical references considering the possibility of magnetic monopoles we refer to [112]. The list provided in this paper dates back to the letter of Petrus Peregrinus from 1269.

$$\mathbf{B}_{\restriction_{S^2}}(\mathbf{x}) = g\,\mathbf{x}\,. \tag{7.4.5}$$

Note that it is well defined on all of S^2. Thus, it defines a smooth 2-form $\mathbb{F} = (\mathbf{B} \lrcorner \mathsf{v}_{\mathbb{R}^3})_{\restriction_{S^2}}$ on S^2. In spherical coordinates, we have[16]

$$\mathbf{B}_{\restriction_{S^2}}(\mathbf{x}) = g\,(\cos\varphi\sin\theta\,,\ \sin\varphi\sin\theta\,,\ \cos\theta) \tag{7.4.6}$$

and, thus,

$$\mathbb{F} = -g\,\mathrm{d}\,(\cos\theta)\wedge \mathrm{d}\varphi\,. \tag{7.4.7}$$

This entails

$$Q_{\mathrm{m}} = \int_{S^2}\mathbb{F} = 4\pi g\,. \tag{7.4.8}$$

Since in this discussion the physical constants play a crucial role, we switch over to the physical representation $\mathbb{F} \mapsto \frac{ie}{\hbar c}\mathbb{F}$, see Remark 6.1.1.

Let us now make the following natural assumption: $\frac{ie}{\hbar c}\mathbb{F}$ is the representative of the curvature of a connection form on a principal U(1)-bundle P over S^2. Then, the first Chern class of P (which, by Theorem 4.8.1, determines P uniquely up to isomorphisms) is

$$\mathsf{c}_1(P) = -\frac{1}{2\pi i}\frac{ie}{\hbar c}\mathbb{F} = -\frac{e}{2\pi\hbar c}\mathbb{F}$$

and, thus, the first Chern index reads

$$\mathfrak{c}_1(P) = \int_{S^2}\mathsf{c}_1(P) = -\frac{2eg}{\hbar c}\,. \tag{7.4.9}$$

Since the first Chern index is integer-valued, we obtain the following quantization condition:

$$\frac{2eg}{\hbar c} = m \in \mathbb{Z}\,. \tag{7.4.10}$$

This is the famous Dirac quantization condition. Dirac obtained it by the requirement that the electron wave function be single-valued. We see that, equivalently, it follows from the above requirement that $\frac{ie}{\hbar c}\mathbb{F}$ be the representative of the curvature of a connection form on a principal U(1)-bundle over S^2. This becomes even more transparent by the following.

Remark 7.4.1 Note that

$$\mathbb{A}_+ = g(1-\cos\vartheta)\,d\varphi \quad \text{and} \quad \mathbb{A}_- = -g(1+\cos\vartheta)\,d\varphi \tag{7.4.11}$$

are potentials for $\mathbb{F}$ on $S^2 \setminus \{-\mathbf{e}_0\}$ and $S^2 \setminus \{\mathbf{e}_0\}$, respectively. Thus,

[16] Everywhere, except for the north and the south poles $\mathbf{e}_0$ and $-\mathbf{e}_0$, respectively.

$$i\mathbb{A}_+ = i\mathbb{A}_- + \mathrm{d}(2ig\varphi)$$

on $\mathrm{S}^2 \setminus \big(\{\mathbf{e}_0\} \cup \{-\mathbf{e}_0\}\big)$, that is, by restriction to the equator $\mathrm{S}^1 \subset \mathrm{S}^2$ we obtain a mapping

$$\rho : \mathrm{S}^1 \to \mathrm{U}(1) \cong \mathrm{S}^1\,, \quad \rho(\varphi) = \mathrm{e}^{2ig\varphi}\,,$$

or, in the physical representation, using the quantization condition,

$$\rho(\varphi) = \mathrm{e}^{i\frac{2eg}{\hbar c}\varphi} = \mathrm{e}^{im\varphi}\,.$$

We see that the quantization condition ensures that this function is single-valued and, thus, it may be viewed as a transition function of a principal U(1)-bundle. It then defines transition functions in all associated bundles, in particular, in the complex line bundle whose sections model the electron wave function. Clearly, for $m = 1$, this line bundle is associated with the complex Hopf bundle of Example 1.1.20. It is easy to check that in that case the connection form defined by (7.4.11) coincides with the canonical connection (1.3.20) (Exercise 7.4.1).

Finally, note that m coincides with the mapping degree of ρ and that there is a similar relation to the Chern index as in Proposition 6.3.4. ♦

Now, let us consider the Georgi–Glashow model introduced in Example 7.3.7. We use the same notation and, in particular, we use the identification given by (7.3.6). Let (ω, Φ) be a configuration of this model, let $\tilde{\Phi}$ be the equivariant mapping associated with Φ and let $(\mathbb{A}, \varphi)$ be a (global) representative of this configuration. We define

$$\Gamma := \{\mathbf{x} \in M \,:\, \Phi(\mathbf{x}) = 0\}\,, \quad M_0 := M \setminus \Gamma\,. \tag{7.4.12}$$

Recall that M is the 4-dimensional Minkowski space and that Φ takes values in $\mathbb{R}^3$. Thus, generically, Γ is a 1-dimensional submanifold of M. Let us assume that Γ consists of a union of (disjoint) curves each of which intersects every hypersurface in M defined by $x^0 = \mathrm{const.}$ exactly once. Such a submanifold Γ will be called generic. Below, only this case will be considered.

Let us denote the restrictions of the bundles P and E to $M_0 \subset M$ by P_0 and E_0, respectively. Since P and E are trivial, P_0 and E_0 are trivial, too. The corresponding restriction of (ω, Φ) will be denoted by the same symbols. Consider the mapping

$$\hat{\Phi} : P_0 \to \mathrm{S}^2 \subset \mathbb{R}^3\,, \quad \hat{\Phi} := \tilde{\Phi} \cdot \|\tilde{\Phi}\|^{-1}\,. \tag{7.4.13}$$

By definition, $\hat{\Phi}$ is G-equivariant and, thus, defines a section of the subbundle $\hat{E}_0 = P_0 \times_{\mathrm{SO}(3)} \mathrm{S}^2$ of E_0. Note that $\hat{\Phi}$ induces a decomposition of ω on P_0:

$$\omega = \omega^{\|} + \omega^{\perp}\,, \quad \omega^{\|} = \hat{\Phi}(\hat{\Phi} \cdot \omega)\,, \quad \omega^{\perp} = \omega - \hat{\Phi}(\hat{\Phi} \cdot \omega)\,, \tag{7.4.14}$$

where $\cdot$ denotes the Euclidean scalar product on $\mathbb{R}^3$.[17] Clearly, $\hat{\Phi} \cdot \omega^\perp = 0$.

Let us calculate the projection of the curvature of ω to $\hat{\Phi}$.

Lemma 7.4.2 *For any configuration (ω, Φ) on P_0, the following identity holds:*

$$\hat{\Phi} \cdot \Omega = \mathrm{d}(\hat{\Phi} \cdot \omega) - \frac{1}{2}\hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}] + \frac{1}{2}\hat{\Phi} \cdot [D\hat{\Phi}, D\hat{\Phi}]\,, \tag{7.4.15}$$

where Ω is the curvature form of ω.

Below, depending on whether ω and $\hat{\Phi}$ are viewed as mappings with values in $\mathfrak{so}(3)$ or $\mathbb{R}^3$, respectively, the bracket $[\cdot, \cdot]$ denotes either the commutator in $\mathfrak{so}(3)$ or the cross product in $\mathbb{R}^3$.

Proof Since $D\hat{\Phi} = \mathrm{d}\hat{\Phi} + [\omega, \hat{\Phi}]$, we have

$$\begin{aligned}
[\hat{\Phi}, D\hat{\Phi} - d\hat{\Phi}] &= [\hat{\Phi}, [\omega, \hat{\Phi}]] \\
&= [\hat{\Phi}, [\omega - \hat{\Phi}(\hat{\Phi} \cdot \omega), \hat{\Phi}] + [\hat{\Phi}(\hat{\Phi} \cdot \omega), \hat{\Phi}]] \\
&= [\hat{\Phi}, [\omega^\perp, \hat{\Phi}]] \\
&= \omega^\perp(\hat{\Phi} \cdot \hat{\Phi}) - \hat{\Phi}(\hat{\Phi} \cdot \omega^\perp)\,,
\end{aligned}$$

that is,

$$\omega^\perp = [\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}]\,.$$

On the other hand, using the standard basis $\{\mathbf{e}_a\}$ in $\mathfrak{so}(3) \cong \mathbb{R}^3$, we calculate

$$\begin{aligned}
[D\hat{\Phi} - \mathrm{d}\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}] &= [[\omega, \hat{\Phi}], [\omega, \hat{\Phi}]] \\
&= \mathbf{e}_a \varepsilon^a{}_{bc} \varepsilon^b{}_{kl} \omega^k \hat{\Phi}^l \wedge \varepsilon^c{}_{mn} \omega^m \hat{\Phi}^n \\
&= \mathbf{e}_a (\delta^a{}_m \delta_{bn} - \delta^a{}_n \delta_{bm}) \varepsilon^b{}_{kl} \hat{\Phi}^l \hat{\Phi}^n \omega^k \wedge \omega^m \\
&= \mathbf{e}_m \varepsilon_{nkl} \hat{\Phi}^l \hat{\Phi}^n \omega^k \wedge \omega^m - \mathbf{e}_n \varepsilon_{mkl} \hat{\Phi}^l \hat{\Phi}^n \omega^k \wedge \omega^m \\
&= \hat{\Phi}(\hat{\Phi} \cdot [\omega, \omega])\,.
\end{aligned}$$

Thus,

$$\hat{\Phi} \cdot [\omega, \omega] = \hat{\Phi} \cdot [D\hat{\Phi} - \mathrm{d}\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}]\,,$$

and, therefore, using $\mathrm{d}\hat{\Phi} \cdot \hat{\Phi} = 0$, we obtain

---

[17] It coincides, up to a factor $\frac{1}{2}$, with the negative Killing form of $\mathfrak{so}(3)$.

$$
\begin{aligned}
\hat{\Phi} \cdot \Omega &= \hat{\Phi} \cdot \mathrm{d}\omega + \frac{1}{2}\hat{\Phi} \cdot [\omega, \omega] \\
&= \mathrm{d}(\hat{\Phi} \cdot \omega) - \mathrm{d}\hat{\Phi} \,\dot{\wedge}\, \omega^{\perp} + \frac{1}{2}\hat{\Phi} \cdot [\omega, \omega] \\
&= \mathrm{d}(\hat{\Phi} \cdot \omega) - \mathrm{d}\hat{\Phi} \,\dot{\wedge}\, [\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}] + \frac{1}{2}\hat{\Phi} \cdot [D\hat{\Phi} - \mathrm{d}\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}]\,.
\end{aligned}
$$

But,

$$
\begin{aligned}
\mathrm{d}\hat{\Phi} \,\dot{\wedge}\, [\hat{\Phi}, D\hat{\Phi} - \mathrm{d}\hat{\Phi}] &= \mathrm{d}\hat{\Phi}_a \wedge \varepsilon^a{}_{bc}\hat{\Phi}^b(D\hat{\Phi}^c - \mathrm{d}\hat{\Phi}^c) \\
&= -\hat{\Phi} \cdot [D\hat{\Phi}, \mathrm{d}\hat{\Phi}] + \hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}]\,,
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\hat{\Phi} \cdot \Omega &= \mathrm{d}(\hat{\Phi} \cdot \omega) + \hat{\Phi} \cdot [D\hat{\Phi}, \mathrm{d}\hat{\Phi}] - \hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}] \\
&\quad + \frac{1}{2}\,\hat{\Phi} \cdot [D\hat{\Phi}, D\hat{\Phi}] - \hat{\Phi} \cdot [D\hat{\Phi}, \mathrm{d}\hat{\Phi}] + \frac{1}{2}\,\hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}] \\
&= \mathrm{d}(\hat{\Phi} \cdot \omega) + \frac{1}{2}\,\hat{\Phi} \cdot [D\hat{\Phi}, D\hat{\Phi}] - \frac{1}{2}\,\hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}]\,.
\end{aligned}
$$

■

Let us rewrite Eq. (7.4.15) as follows:

$$
\mathrm{d}(\hat{\Phi} \cdot \omega) - \frac{1}{2}\,\hat{\Phi} \cdot [\mathrm{d}\hat{\Phi}, \mathrm{d}\hat{\Phi}] = \hat{\Phi} \cdot \Omega - \frac{1}{2}\,\hat{\Phi} \cdot [D\hat{\Phi}, D\hat{\Phi}]\,. \tag{7.4.16}
$$

The gauge invariant 2-form

$$
\mathbb{F}_{\mathrm{em}} := \hat{\Phi} \cdot \Omega - \frac{1}{2}\,\hat{\Phi} \cdot [D\hat{\Phi}, D\hat{\Phi}] \tag{7.4.17}
$$

is called the ’t Hooft electromagnetic field strength [623]. A priori, this is a 2-form on the (trivial) bundle P_0, but, since both Ω and $D\hat{\Phi}$ may be viewed as 2-forms on M_0 with values in the adjoint bundle, $\mathbb{F}_{\mathrm{em}}$ may be viewed as an $\mathbb{R}$-valued 2-form on M_0. Note that, separately, the two summands on the left hand side of (7.4.16) are neither gauge invariant, nor may they be interpreted as 2-forms on M_0. But, clearly, their sum must be a gauge invariant 2-form on M_0, too, and thus for any global representative $(\mathbb{A}, \hat{\varphi})$ of $(\omega, \hat{\Phi})$ on M_0, we have

$$
\mathbb{F}_{\mathrm{em}} = \mathrm{d}(\hat{\varphi} \cdot \mathbb{A}) - \frac{1}{2}\,\hat{\varphi} \cdot [\mathrm{d}\hat{\varphi}, \mathrm{d}\hat{\varphi}]\,. \tag{7.4.18}
$$

To justify the name for $\mathbb{F}_{\mathrm{em}}$, we first show the following.

Proposition 7.4.3 *The 2-form $\mathbb{F}_{\mathrm{em}}$ is closed,*

$$\mathrm{d}\mathbb{F}_{\mathrm{em}} = 0\,. \tag{7.4.19}$$

Proof By (7.4.18), we must prove that $\mathrm{d}\big(\hat{\varphi}\cdot[\mathrm{d}\hat{\varphi},\mathrm{d}\hat{\varphi}]\big) = 0$. Since $\hat{\varphi}^2 = 1$, we have $\hat{\varphi}\cdot\mathrm{d}\hat{\varphi} = 0$. Thus, for any vector $X \in \mathrm{T}_{\mathbf{x}}M_0$, the vector $\mathrm{d}\hat{\varphi}(X) \in \mathbb{R}^3$ lies in the plane orthogonal to $\hat{\varphi}(\mathbf{x})$. Consequently, the vector $[\mathrm{d}\hat{\varphi}(Y),\mathrm{d}\hat{\varphi}(Z)]$ is parallel to $\hat{\varphi}(\mathbf{x})$, for any pair of tangent vectors $Y, Z \in \mathrm{T}_{\mathbf{x}}M_0$. This implies $\mathrm{d}\hat{\varphi}(X)\cdot[\mathrm{d}\hat{\varphi}(Y),\mathrm{d}\hat{\varphi}(Z)] = 0$ for any triple of tangent vectors. Thus $\mathrm{d}\hat{\varphi} \stackrel{\cdot}{\wedge} [\mathrm{d}\hat{\varphi},\mathrm{d}\hat{\varphi}]$ vanishes identically. This yields the assertion. ■

Now, choose a hypersurface $\Sigma_0 := \{\mathbf{x} \in M : x^0 = \text{const.}\}$. Assume that the submanifold Γ given by (7.4.12) is generic. Label the curves constituting Γ by Γ_i. By assumption, each Γ_i intersects Σ_0 in an isolated point $\mathbf{x}_i$. Take a family of open balls K_i of radius ε_i centered at $\mathbf{x}_i$ and consider a 'big' open ball K_R of radius R in Σ_0 containing all $\overline{K}_i$. Denote the boundary 2-spheres by S^2_i and S^2_R, respectively, and choose on each of these spheres the orientation pointing outwards. By the Theorem of Stokes and by Proposition 7.4.3, the total magnetic charge contained in K_R is given by

$$Q_{\mathrm{m}} = \int_{\mathrm{S}^2_R}\mathbb{F}_{\mathrm{em}} = \int_{K_R\setminus\cup_i K_i}\mathrm{d}\mathbb{F}_{\mathrm{em}} + \sum_i\int_{\mathrm{S}^2_i}\mathbb{F}_{\mathrm{em}} = \sum_i\int_{\mathrm{S}^2_i}\mathbb{F}_{\mathrm{em}} = \sum_i Q^i_{\mathrm{m}}\,, \tag{7.4.20}$$

that is, it is given by the sum of magnetic charges living on the curves Γ_i. To make the construction independent of φ and, thus, to include any generic Γ, we take the limit

$$Q_{\mathrm{m}} = \lim_{R\to\infty}\int_{\mathrm{S}^2_R}\mathbb{F}_{\mathrm{em}}\,. \tag{7.4.21}$$

As in Sect. 7.2, we will write $\int_{\mathrm{S}^2_\infty}$ for $\lim_{R\to\infty}\int_{\mathrm{S}^2_R}$.

To calculate the flux of $\mathbb{F}$, we must study the behaviour of the second term on the right hand side of (7.4.18). Let S^2_ε be a 2-sphere of radius ε which is not contractible in M_0 and consider the mapping

$$\psi := \hat{\varphi}_{\restriction_{\mathrm{S}^2_\varepsilon}} : \mathrm{S}^2_\varepsilon \to \mathrm{S}^2 \subset \mathbb{R}^3\,. \tag{7.4.22}$$

Lemma 7.4.4 *The mapping ψ fulfils*

$$\frac{1}{2}\,\psi\cdot[d\psi, d\psi] = \psi^*(\mathsf{v}_{\mathrm{S}^2})\,, \tag{7.4.23}$$

where $\mathsf{v}_{\mathrm{S}^2}$ denotes the canonical volume form on S^2.

The proof is by a direct calculation, e.g. using spherical coordinates, and is thus left to the reader (Exercise 7.4.2).

Proposition 7.4.5 *The magnetic charges Q_{m} and Q^i_{m} are given by*

$$Q_{\mathrm{m}} = -4\pi \deg(\psi)\,, \quad Q^i_{\mathrm{m}} = -4\pi \deg(\psi_i)\,, \tag{7.4.24}$$

where $\psi : \mathrm{S}^2_\infty \to \mathrm{S}^2$ and $\psi_i : \mathrm{S}^2_i \to \mathrm{S}^2$, respectively.

Proof Using (7.4.18), the Theorem of Stokes and Lemma 7.4.4, we obtain

$$Q^i_{\mathrm{m}} = \int_{\mathrm{S}^2_i} \mathbb{F}_{\mathrm{em}} = -\int_{\mathrm{S}^2_i} \psi_i^*(\mathrm{v}_{\mathrm{S}^2}) = -4\pi \deg(\psi_i)\,.$$

The same argument applies to $\psi : \mathrm{S}^2_R \to \mathrm{S}^2$ for R such that all singularities are contained in K_R. ■

Remark 7.4.6

1. Since the mapping degree is a homotopy invariant, the mapping ψ defines an element of the second homotopy group $\pi_2(\mathrm{S}^2)$. Viewing $\mathrm{S}^2 \subset \mathbb{R}^3$ as the homogeneous space G/H, with $G = \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$, we recover the topological characterization of φ in terms of an element $[\varphi_\infty] \in \pi_2(G/H)$ found in Sect. 7.2. The degree of the mapping $\psi_i : \mathrm{S}^2_i \to \mathrm{S}^2$ is often called the Poincaré–Hopf index of the zero $\mathbf{x}_i$. For a detailed discussion of the various equivalent topological characterizations we refer to [20].
2. Since Φ vanishes on Γ, $\mathbb{F}_{\mathrm{em}}$ is singular on Γ and thus cannot be continuously extended to the whole of M. Nonetheless, we may consider the 3-form [20]

$$j_{\mathrm{m}} := \mathrm{d}\mathbb{F}_{\mathrm{em}} \tag{7.4.25}$$

on M in the sense of distributions. Since on M_0 we have $\mathrm{d}\mathbb{F}_{\mathrm{em}} = 0$, j_{m} has obviously support on Γ. The distribution-valued 3-form j_{m} is called the magnetic current form. By (7.4.18),

$$j_{\mathrm{m}} = -\frac{1}{2}\mathrm{d}\hat{\varphi} \overset{\cdot}{\wedge} [\mathrm{d}\hat{\varphi}, \mathrm{d}\hat{\varphi}]\,. \tag{7.4.26}$$

In terms of j_{m}, the magnetic charge contained in K_R is given by

$$Q_{\mathrm{m}} = \int_{\mathrm{S}^2_R} \mathbb{F}_{\mathrm{em}} = \int_{K_R} \mathrm{d}\mathbb{F}_{\mathrm{em}} = \int_{K_R} j_{\mathrm{m}}\,.$$

Since, by definition, j_{m} fulfils the continuity equation $\mathrm{d}\,j_{\mathrm{m}} = 0$, the magnetic charge is conserved.[18] ♦

Example 7.4.7 Consider the matter field of the form

[18] We note that although Q_{m} is a purely topological quantity, it does not generate a symmetry, see [20] for a detailed discussion.

$$\varphi : M \to \mathbb{R}^3 , \quad \varphi(\mathbf{x}) = \begin{bmatrix} \varphi_1(\mathbf{x}) \\ \varphi_2(\mathbf{x}) \\ \varphi_3(\mathbf{x}) \end{bmatrix} .$$

1. Let
$$(\varphi_1 + i\varphi_2)(\mathbf{x}) = (ax_1 + ibx_2)^n , \quad \varphi_3(\mathbf{x}) = cx_3 , \quad a, b, c \in \mathbb{R} .$$
One can show that φ carries a magnetic monopole of strength n (Exercise 7.4.3).
2. Let
$$\varphi_1(\mathbf{x}) = 2ax_1 f(\mathbf{x}) , \quad \varphi_2(\mathbf{x}) = 2ax_2 f(\mathbf{x}) , \quad \varphi_3(\mathbf{x}) = (\|\mathbf{x}\|^2 - a^2) f(\mathbf{x}) ,$$
where $a \in \mathbb{R}$ and f is a nowhere vanishing smooth function. One can show that φ carries two monopoles with opposite strengths ± 1 separated by a distance $2a$ (Exercise 7.4.3). ♦

The above analysis shows that the information about the magnetic charges is encoded in the topological behaviour of the Higgs field. Since $\hat{\varphi}$ is defined everywhere on M_0, one often speaks of a description in a non-singular gauge. Next, let us present an alternative picture. Choose a point $\hat{\Phi}_0 \in \mathrm{S}^2$. Let $H \cong \mathrm{SO}(2)$ be the stabilizer of $\hat{\Phi}_0$ and let Q_0 be the reduction of P_0 to H induced by $\hat{\Phi}_0$. Then,

$$Q_0 = \{p \in P_0 \colon \hat{\Phi}(p) = \hat{\Phi}_0\} .$$

Let $i_0 : Q_0 \to P_0$ be the natural inclusion mapping. Then, as in the proof of Proposition 7.3.4, pulling back ω to Q_0 via i_0 and decomposing it with respect to (7.3.9), we obtain

$$\hat{\omega}_0 := i_0^* \omega_{\mathfrak{h}} , \quad \hat{\tau}_0 := i_0^* \omega_{\mathfrak{m}} , \tag{7.4.27}$$

where $\hat{\omega}_0$ is an $\mathfrak{so}(2) \cong \mathbb{R}$-valued connection form and $\hat{\tau}_0$ is a horizontal 1-form of type $\mathrm{Ad}(H)\mathfrak{m}$ on Q_0. Next, let us see what becomes of the electromagnetic field strength $\mathbb{F}_{\mathrm{em}}$ given by (7.4.17). For that purpose, we take the pullback of the identity (7.4.16) to Q_0 under the inclusion mapping i_0. Comparing with (7.4.14), we have

$$i_0^* (\omega^{\|}) = \hat{\omega}_0 . \tag{7.4.28}$$

Using this, together with $i_0^* (\mathrm{d}\hat{\Phi}) = \mathrm{d}\hat{\Phi}_0 = 0$, from (7.4.16) we read off

$$\mathbb{F}_{\mathrm{em}} = \mathrm{d}(\hat{\Phi}_0 \cdot \omega) = \mathrm{d}\hat{\omega}_0 = \hat{\Omega}_0 , \tag{7.4.29}$$

that is, $\mathbb{F}_{\mathrm{em}}$ coincides with the curvature of the reduced connection form on Q_0. Now, Proposition 7.4.5 immediately implies the following.[19]

[19] Here, we view Q_0 as a principal U(1)-bundle.

Corollary 7.4.8 *The first Chern index of the restriction of Q_0 to S^2_∞ is given by*

$$\int_{S^2_\infty} \mathsf{c}_1(Q_0) = 2\deg(\psi)\,. \tag{7.4.30}$$

This observation should be compared with an analogous result in the theory of instantons, see Proposition 6.3.4. In this picture, the magnetic charges are encoded in the nontrivial topology of the reduced bundle Q_0. Now, instead of the global formula (7.4.18), we obtain[20]

$$\mathbb{F}_{\mathrm{em}} = \mathrm{d}\mathbb{A}\,,$$

with $\mathbb{A}$ being a local representative of $\hat{\omega}_0$. Clearly, if Q_0 is nontrivial no global representative exists. In other words, if one insisted in working with a single potential, it would necessarily have singularities. Therefore, one sometimes calls this the description in a singular gauge, where the magnetic monopoles are carried by the singularities of $\mathbb{A}$.

Remark 7.4.9 In the physical representation used in the analysis of the Dirac monopole, we obtain

$$\int_{S^2_R} \mathsf{c}_1(Q_0) = -\frac{1}{2\pi i}\frac{e}{\hbar c}\int_{S^2_\varepsilon} i\mathbb{F}_{\mathrm{em}} = 2\deg(\psi)\,.$$

Thus, denoting in this representation

$$g = \frac{1}{4\pi}\int_{S^2_R}\mathbb{F}_{\mathrm{em}}\,,$$

we read off a quantization condition similar to (7.4.9),

$$\frac{2eg}{\hbar c} = -\deg(\psi)\,.$$

♦

We still stick to the model under consideration and look for an exact static solution of the field equations exhibiting a magnetic monopole with finite energy. By the results of Sect. 7.2, finite energy configurations $(\mathbb{A},\varphi)$ are labelled by elements of $\pi_2(G/H)$, where H is the residual gauge group after symmetry breaking, and asymptotic solutions are characterized by the charge $2\mathbb{Q} \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H, cf. Theorem 7.2.12 and Remark 7.2.13. Explicitly, in spherical coordinates, the asymptotic solutions read

$$A_\vartheta = 0\,,\quad A_\phi = \pm(1 \mp \cos\vartheta)\mathbb{Q}\,.$$

[20] Note that $\mathbb{F}_{\mathrm{em}}$ may still be viewed as a 2-form on M_0, because the adjoint bundle of a principal $\mathrm{U}(1)$-bundle is necessarily trivial.

They are of course supplemented by an appropriate fall-off law of φ for $\|\mathbf{x}\| \to \infty$. As noted before, these solutions are spherically symmetric.

Here, we have $H = \mathrm{SO}(2)$ and, thus, $2\mathbb{Q}$ is simply an integer $2c \in \mathbb{Z}$, cf. point 1 of the proof of Theorem 7.2.12. Then, (7.4.24), (7.4.30) and (7.2.40) imply the following expression for the magnetic charge in terms of the topological charge

$$Q_m = \int_{\mathrm{S}^2_\infty} \mathbb{F}_{\mathrm{em}} = -4\pi \deg(\psi) = -2\pi \int_{\mathrm{S}^2_\infty} \mathsf{c}_1(Q_0) = 4\pi c\,. \tag{7.4.31}$$

For the model under consideration, the above asymptotic solutions were first found by 't Hooft [623] and Polyakov [514]. Therefore, they are called the 't Hooft-Polyakov monopole solutions. In [623] also the energy functional was analyzed in detail, and the mass of the magnetic monopole was calculated. Given these asymptotic solutions, one may wish to extend them to finite energy solutions on all of $\mathbb{R}^3$. This is a very complicated task even for the model under consideration. It was Schwarz [566] who gave a rigorous proof that, for this model, an exact solution fulfilling the imposed boundary conditions exists. However, it is impossible to express this solution in terms of elementary functions. Its numerical behaviour is as follows:

$$\varphi^a(\mathbf{x}) = \frac{x^a}{r^2} H(\xi)\,, \quad A^a_k(\mathbf{x}) = -\frac{\varepsilon_k{}^{ab} x_b}{r^2}(1 - K(\xi))\,, \quad \xi = \eta \cdot r\,.$$

Here, η is the Higgs vacuum and H und K are functions whose qualitative behaviour is shown in Fig. 7.1. The review [250] of Goddard and Olive contains a lot of further comments and references. For a status report concerning the experimental search for magnetic monopoles we refer to [71].

Exercises

7.4.1 Prove that, for $2g = 1$ the gauge potentials given by (7.4.11) are the local representatives of the canonical connection (1.3.20) on the complex Stiefel bundle.

7.4.2 Prove Lemma 7.4.4.

7.4.3 Work out the details of Example 7.4.7.

7.4.4 Write down the canonical connection given by (1.9.43) for the case considered by 't Hooft and Polyakov both in the singular and in the non-singular gauge.

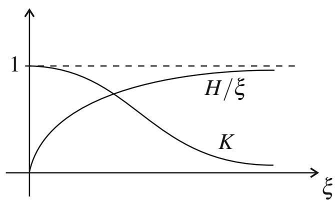

Fig. 7.1 Qualitative behaviour of the functions H and K

7.5 The Bogomolnyi–Prasad–Sommerfield Model

Now, let us try to find the absolute minima of the energy functional $E(\omega, \Phi)$ of a Yang–Mills–Higgs system with the matter field being in the adjoint representation. Recall from the discussion in Sect. 7.2 that, for the static theory in the temporal gauge, the energy functional reduces to

$$E(\omega, \Phi) = \frac{1}{2}\left(\|\Omega^{\mathrm{m}}\|^2 + \|\mathscr{D}\Phi\|^2 + \int_{\Sigma_0} V(\Phi)\mathsf{v}_{R^3}\right). \tag{7.5.1}$$

Since both Ω^{m} and $\mathscr{D}\Phi$ take values in the Lie algebra $\mathfrak{g}$, the energy functional may be rewritten as follows[21]:

$$E(\omega, \Phi) = \frac{1}{2}\left(\|\Omega^{\mathrm{m}} \mp \mathscr{D}\Phi\|^2 + \int_{\Sigma_0} V(\Phi)\mathsf{v}_{R^3}\right) \pm \int_{\Sigma_0} \Omega^{\mathrm{m}} \,\dot{\wedge}\, *\mathscr{D}\Phi\,.$$

This entails a lower bound:

$$E(\omega, \Phi) \geq |\langle \Omega^{\mathrm{m}}, \mathscr{D}\Phi\rangle_{L^2}|\,. \tag{7.5.2}$$

Using (7.2.12) and the Bianchi identity for Ω, we calculate on the space-like hypersurface Σ_0 defined by $x^0 = 0$:

$$\mathrm{d}(\Phi\cdot(*\Omega^{\mathrm{m}})) = \mathrm{d}_\omega\big(\Phi\cdot(*\Omega^{\mathrm{m}})\big) = \mathscr{D}\Phi \,\dot{\wedge}\, (*\Omega^{\mathrm{m}}) + \Phi\cdot(\mathrm{d}_\omega *\Omega^{\mathrm{m}}) = \mathscr{D}\Phi \,\dot{\wedge}\, (*\Omega^{\mathrm{m}})\,.$$

By Stokes' Theorem,

$$\langle \Omega^{\mathrm{m}}, \mathscr{D}\Phi\rangle_{L^2} = \int_{\Sigma_0} \mathrm{d}\big(\Phi\cdot(*\Omega^{\mathrm{m}})\big) = \int_{\mathrm{S}^2_\infty} \Phi\cdot(*\Omega^{\mathrm{m}})\,,$$

and thus,

$$E(\omega, \Phi) \geq \left|\int_{\mathrm{S}^2_\infty} \Phi\cdot(*\Omega^{\mathrm{m}})\right|. \tag{7.5.3}$$

This inequality is called the Bogomolnyi bound [84]. It is the starting point for the search of stable solutions of the Yang–Mills–Higgs system. Clearly, (ω, Φ) is an absolute minimum of the energy functional if this bound is saturated, that is, if

$$V(\Phi) = 0\,, \quad \Omega^{\mathrm{m}} = \pm\mathscr{D}\Phi\,. \tag{7.5.4}$$

Moreover, to guarantee finiteness of the bound (7.5.2), for solutions we must require[22]

---

[21] Recall that V is shifted so that it is non-negative.

[22] Recall Remark 6.2.1 for the notation.

$$|\mathscr{D}\Phi| \to 0\,, \quad |\Omega^{\mathrm{m}}| \to 0\,, \tag{7.5.5}$$

for $\|\mathbf{x}\| \to \infty$. Additionally, we also require

$$|\Phi| \to 1\,, \tag{7.5.6}$$

for $\|\mathbf{x}\| \to \infty$. This may be viewed as a relic of the Higgs potential. The limit $V \to 0$ is often referred to as the Prasad–Sommerfield limit [522]. Clearly, for analytical estimates, these requirements must be made more precise [610]. E.g., the first condition in (7.5.5) should be formulated as follows: for some $\delta > 0$,

$$\|\mathbf{x}\|^{1+\delta}|\mathscr{D}\Phi| \leq \mathrm{const.} \tag{7.5.7}$$

Remark 7.5.1 In the Georgi–Glashow model, conditions (7.5.5) and (7.5.6) imply

$$\left|\int_{S^2_R} \hat{\Phi}\cdot[\mathscr{D}\hat{\Phi}, \mathscr{D}\hat{\Phi}]\right| \leq \int_{S^2_R} |\hat{\Phi}|\,|[\mathscr{D}\hat{\Phi}, \mathscr{D}\hat{\Phi}]| R^2\, \mathrm{d}\sigma \;\to\; 0\,.$$

Using this, together with (7.4.17), (7.2.12) and (7.4.21), we read off the Bogomolnyi bound in the Prasad–Sommerfield limit,

$$E(\omega, \Phi) \geq \left|\int_{S^2_\infty} \mathbb{F}_{\mathrm{em}}\right| = |Q_{\mathrm{m}}|\,. \tag{7.5.8}$$

Thus, in the Prasad–Sommerfield limit, the energy functional of the Georgi–Glashow model is bounded from below by the total magnetic charge. ♦

Now, consider the field equations (7.2.14) on $\Sigma_0 = \mathbb{R}^3$. In the adjoint representation and, under the assumption that $V = 0$, they read

$$*\,\mathrm{d}_\omega\Omega^{\mathrm{m}} = [\mathscr{D}\Phi, \Phi]\,, \quad \mathscr{D}^* \circ \mathscr{D}\Phi = 0\,. \tag{7.5.9}$$

Correspondingly, the Bianchi identities (7.2.6) take the form (Exercise 7.5.1)

$$\mathrm{d}_\omega * \Omega^{\mathrm{m}} = 0\,, \quad \mathscr{D} \circ \mathscr{D}\Phi = [*\Omega^{\mathrm{m}}, \Phi]\,. \tag{7.5.10}$$

If we now require the second equation in (7.5.4) to hold,

$$\Omega^{\mathrm{m}} = \pm\mathscr{D}\Phi\,, \tag{7.5.11}$$

we see that the field equations (7.5.9) reduce to the Bianchi identities (7.5.10). Thus, any exact solution of (7.5.11) entails an exact solution of the Yang–Mills–Higgs system in the Prasad–Sommerfield limit. Equation (7.5.11) is called the Bogomolnyi equation.

Let us study this equation. By the above discussion, any solution of this equation yields an absolute minimum of the energy functional. Consider the decomposition of the Euclidean space

$$\mathbb{R}^4 = \mathbb{R}\mathbf{e}_0 \times \mathbb{R}^3 \tag{7.5.12}$$

and write pr_i , $i = 1, 2$, for the canonical projections onto the first and the second component of (7.5.12), respectively. For $\tilde{\mathbf{x}} \in \mathbb{R}^4$, denote $\mathrm{pr}_1(\tilde{\mathbf{x}}) = x^0$ and $\mathrm{pr}_2(\tilde{\mathbf{x}}) = \mathbf{x}$. In this notation, the action of the Abelian group $\mathbb{R}$ by translations on the first factor is given by

$$\delta : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4 , \quad \delta(s, (x^0, \mathbf{x})) = (x^0 + s, \mathbf{x}) .$$

Proposition 7.5.2 *Solutions to the Bogomolnyi equation are in bijective correspondence with (anti-)self-dual, $\mathbb{R}$-invariant connections on the Euclidean space $\mathbb{R}^4$.*

Proof Let (ω, Φ) be a solution of the Bogomolnyi equation, where ω is a connection form on a principal G-bundle $\pi : P \to \mathbb{R}^3$ and Φ is a section of $\mathrm{Ad}(P)$. Since P is (necessarily) trivial, the pullback bundle $\tilde{P} = \mathrm{pr}_2^* P$ over $\mathbb{R}^4$ is also trivial and thus, as a manifold, diffeomorphic to $\mathbb{R} \times P$, with the diffeomorphism given by

$$\chi : \mathbb{R} \times (\mathbb{R}^3 \times G) \to \tilde{P} , \quad \chi\big(x^0, (\mathbf{x}, g)\big) := \big((x^0, \mathbf{x}), (\mathbf{x}, g)\big) .$$

Note that $\mathrm{pr}_2(x^0, \mathbf{x}) = \pi(\mathbf{x}, g)$, indeed. Under this identification, $\tilde{P}$ carries a natural lift Δ of the $\mathbb{R}$-action δ, given by translations on the $\mathbb{R}$-component.

Now, we may apply the theory of invariant connections from Sect. 1.9. By Example 1.9.18, principal G-bundles over $\mathbb{R}^4$ admitting a lift of the action δ have the form[23] $\tilde{P} = \mathbb{R} \times P$ and $\mathbb{R}$-invariant connections $\tilde{\omega}$ on $\tilde{P}$ are in one-to-one correspondence with pairs (ω, Φ) where ω is a connection form on P and $\Phi \in \Gamma^\infty(\mathrm{Ad}(P))$. It remains to show that (ω, Φ) is a solution of the Bogomolnyi equation iff $\tilde{\omega}$ is (anti-)self-dual. As in Example 1.9.18, we extend $\Phi \otimes \mathbf{e}_0{}^*$ to a 1-form on $\mathbb{R}\mathbf{e}_0$ with values in $\Gamma^\infty(\mathrm{Ad}(P))$ via the $\mathbb{R}$-action and use the natural isomorphism

$$\Omega^1(\mathbb{R}^4, \mathrm{Ad}(P)) \cong \Omega^1_{\mathrm{Ad,hor}}(\tilde{P}, \mathfrak{g}) ,$$

to obtain a horizontal 1-form $\tilde{\tau}$ of type Ad on $\tilde{P}$. Under this identification,

$$\tilde{\omega} = \omega + \tilde{\tau} . \tag{7.5.13}$$

Since the bundles P and $\tilde{P}$ are trivial we can use global representatives $(\mathbb{A}, \varphi)$ of (ω, Φ) and $\tilde{\mathbb{A}}$ of $\tilde{\omega}$, respectively. Denote the representatives of the curvature forms of ω and $\tilde{\omega}$ by $\mathbb{F}$ and $\tilde{\mathbb{F}}$, respectively. Then, by (7.5.13),

$$\tilde{\mathbb{A}} = \mathbb{A} + \varphi \mathrm{d}x^0$$

[23] Note that the roles of P and $\tilde{P}$ are interchanged here.

and, thus, by the Structure Equation,

$$\tilde{\mathbb{F}} = \mathbb{F} + \mathscr{D}\varphi \wedge \mathrm{d}x^0 \,.$$

Let $\mathbb{B}$ be the (global) representative of Ω^{m}. Then, by (7.2.12), $\mathbb{F} = *_{\mathbb{R}^3}\mathbb{B}$ and, using $*_{\mathbb{R}^4}(\alpha \wedge \mathrm{d}x^0) = - *_{\mathbb{R}^3} \alpha$, for any 1-form α on $\mathbb{R}^3$ we calculate

$$\begin{aligned} *_{\mathbb{R}^4}\tilde{\mathbb{F}} &= *_{\mathbb{R}^4}\left(\mathscr{D}\varphi \wedge \mathrm{d}x^0\right) + *_{\mathbb{R}^4}\mathbb{F} \\ &= - *_{\mathbb{R}^3} \left(\mathscr{D}\varphi\right) + *_{\mathbb{R}^4}\left(*_{\mathbb{R}^3} \mathbb{B}\right) \\ &= - *_{\mathbb{R}^3} \left(\mathscr{D}\varphi\right) - \mathbb{B} \wedge \mathrm{d}x^0 \,. \end{aligned}$$

Comparing with $\tilde{\mathbb{F}} = *_{\mathbb{R}^3}\mathbb{B} + \mathscr{D}\varphi \wedge \mathrm{d}x^0$, we see that $\tilde{\mathbb{F}}$ is self-dual iff $\mathbb{B} = -\mathscr{D}\varphi$ and anti-self-dual iff $\mathbb{B} = \mathscr{D}\varphi$. ■

Example 7.5.3 (*The BPS monopole*) Let $G = \mathrm{SU}(2)$. Viewing $\mathbf{x} \in \mathbb{R}^3$ as a quaternion via $\mathbf{x} = x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$, we put

$$\mathbb{A}(\mathbf{x}) = \frac{1}{2}\left(\frac{1}{\|\mathbf{x}\|} - \frac{1}{\sinh \|\mathbf{x}\|}\right) \mathrm{Im}\left(\frac{\mathrm{d}\mathbf{x} \cdot \mathbf{x}}{\|\mathbf{x}\|}\right) , \tag{7.5.14}$$

$$\varphi(\mathbf{x}) = \pm\frac{1}{2}\left(\frac{1}{\|\mathbf{x}\|} - \frac{1}{\tanh \|\mathbf{x}\|}\right) \mathrm{Im}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) . \tag{7.5.15}$$

The reader can check by a straightforward calculation that this a solution of the Bogomolnyi equation with magnetic charge $\pm 4\pi$, that is, with mapping degree $k = \pm 1$ (Exercise 7.5.2). It is called the BPS monopole after Bogomolnyi [84], Prasad and Sommerfield [522]. ♦

Remark 7.5.4

1. It was a challenge to find monopole solutions of higher charge. The first existence proof was presented by Taubes [609, 617]. His method is based on the idea that a charge k monopole should be obtained by gluing together k charge 1 monopoles. However, to find explicit solutions, other techniques had to be developed. The correspondence established in Proposition 7.5.2 suggests that methods from the theory of instantons should be applicable. Indeed, the same sequence of ansätze from [42] led to the construction of multi monopole solutions with gauge group $\mathrm{SU}(2)$, see [139, 520, 521, 650, 651]. Hitchin [306] proved that all $\mathrm{SU}(2)$ monopoles can be obtained this way. A different approach, also related to instanton theory, is due to Nahm [469–471]. He developed an infinite-dimensional version of the ADHM construction to obtain multi monopole solutions. Next, it was again Hitchin [307] who proved that, via the Nahm construction, all $\mathrm{SU}(2)$-monopoles are obtained. This way, an equivalence between the two approaches was established. Later, Hurtubise and Murray [334] extended this result to the case of arbitrary classical groups.

2. As in the case of instantons, it is interesting to study the moduli space $\mathfrak{M}_k$ of charge k monopole solutions. For $G = \mathrm{SU}(2)$, this problem has been solved by Donaldson [158]. He has proved that $\mathfrak{M}_k \cong \mathfrak{R}_k / \sim$, where $\mathfrak{R}_k$ is the complex manifold of rational functions f of degree k on the Riemann sphere $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ fulfilling $f(\infty) = 0$, and $\sim$ denotes factorization with respect to the circle action $f \mapsto \mathrm{e}^{i\vartheta} f$. The proof of this statement is based on the variant of the ADHM construction of Nahm cited above. Given the above isomorphism, one gains a nice intuitive picture of how a general solution looks like: an arbitrary element of $\mathfrak{R}_k$ is given by

$$f(z) = \sum_{i=1}^{k} \frac{a_i}{z - z_i}, \quad a_i \in \mathbb{C}.$$

In particular, we read off that $\dim(\mathfrak{M}_k) = 4k - 1$. Thus, for $k = 1$, we obtain a 3-dimensional moduli space. In the parameterization of Example 7.5.3, any solution is obtained from the BPS monopole via a translation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{x}_0$.
Following the ideas developed by Donaldson and using the results of [334], Hurtubise [333] has found the moduli spaces for arbitrary classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$. In all cases, the moduli spaces are equivalent to spaces of holomorphic mappings from $\mathbb{C}\mathrm{P}^1$ into flag manifolds. In [36], the dynamics of monopoles has been studied in terms of geodesic motion on the moduli space. This goes back to an idea of Manton [425], who suggested that the geodesics of the metric on the moduli space should correspond to scattering of slowly moving monopoles. If one takes this idea seriously, one should study the metric of the moduli space. This has been done for $\mathrm{SU}(2)$-monopoles with special symmetries, see [317, 318] and further references therein, and in special cases also for other gauge groups, see [463] and references therein.
3. In [31], Atiyah proposed to study the Bogomolnyi equation on the hyperbolic 3-space. He showed that hyperbolic monopoles may be regarded as S^1-invariant instantons on S^4. This variant of the theory is still an active field of research. In [464, 465], the twistor approach to this theory has been worked out. Moreover, there is a large number of attempts to construct (or prove the existence of) solutions, see [426, 586, 606] and further references therein. The geometry of the corresponding moduli space has not been clarified up until now, see [481], [482] for attempts in this direction.
4. By the above discussion, the critical set of absolute minima of the Yang–Mills–Higgs action functional consists of the solutions to the Bogomolnyi equation. It was shown by Taubes that there exist smooth, finite action solutions to the $\mathrm{SU}(2)$ Yang–Mills–Higgs equations in the Prasad–Sommerfield limit which do not satisfy the Bogomolnyi equation. In [614], Taubes proved that they are all unstable. It is interesting to ask whether such non-minimal solutions exist if one requires spherical symmetry. For the case of $\mathrm{SU}(2)$, the answer is negative [419]. If one allows for gauge groups with rank larger than 2, then such solutions exist [111]. ♦

For a systematic study of the theory of monopoles we refer to the monographs [36, 346, 585].

Exercises

7.5.1 Prove formulae (7.5.9) and (7.5.10).

7.5.2 Prove that (7.5.14) and (7.5.15) define a solution of the Bogomolnyi equation with magnetic charge $\pm 4\pi$.

7.6 The Seiberg–Witten Model

In 1994, Seiberg and Witten published two papers where they studied the vacuum structure of $N = 2$ supersymmetric Yang–Mills theory [576, 577]. In this context, they found an Abelian gauge model coupled to a spinor field which, according to some heuristic arguments taken from quantum field theory, had to contain the same topological information as the Yang–Mills theory [675].[24] Indeed, within a few months, many of the results obtained via instanton theory, were re-proved within the new theory. In this section, we give an introduction to this fascinating model. By now, there exists a considerable textbook literature on the subject, see [180, 219, 428, 459, 460, 487, 553], to which we refer for an exhaustive presentation.

Consider an oriented compact 4-dimensional Riemannian manifold (M, g) carrying a Spin^c-structure $S^c(M)$. Let $\pi : P \to M$ be the corresponding fundamental U(1)-bundle and let L be the associated determinant line bundle given by (5.4.11). Let ω be the Levi-Civita connection on $O_+(M)$ and let τ be a connection on P. Via the two-fold covering $S^c(M) \to O_+(E) \times_M P$, these connections define a unique connection ω^τ on $S^c(M)$. Let $\Omega_\tau = \mathrm{d}\tau \in \Omega^2(M) \otimes i\mathbb{R}$ be the curvature[25] of τ and let

$$\mathscr{S}^c(M) = S^c(M) \times_{\mathrm{Spin}^c(4)} \Delta_4$$

be the associated canonical spinor bundle[26] endowed with the Dirac operator D_τ defined by ω^τ. By Remark 5.5.6, we have the natural splitting

[24] Roughly speaking, according to Witten the two theories should be viewed as two different asymptotic limits of a single theory which are getting interchanged via S-duality. Under this symmetry, electrically charged states are exchanged with magnetic monopoles, see Remark 7.6.7 below. Up to our knowledge, these quantum field theoretic arguments have never been made mathematically precise up until now, but there exists a research programme for accomplishing this goal, see [428] for a further discussion.

[25] Since the adjoint action of U(1) is trivial, Ad(P) is a trivial bundle.

[26] Cf. formula (5.5.12).

$$\mathscr{S}^c(M) = \mathscr{S}^c_+(M) \oplus \mathscr{S}^c_-(M)\,, \tag{7.6.1}$$

induced from the spinor module splitting $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$. On the other hand, by (2.8.8), we have the decomposition

$$\bigwedge{}^2 \mathrm{T}^* M = \bigwedge{}^2{}_+ \mathrm{T}^* M \oplus \bigwedge{}^2{}_- \mathrm{T}^* M\,, \tag{7.6.2}$$

induced from the Hodge star operator of g. There is a deep relation between these splittings given by (2.8.10),

$$\bigwedge{}^2{}_\pm T^* \cong S^2 V_\pm\,. \tag{7.6.3}$$

Here, T is the basic SO(4)-module and $V_\pm = \Delta_4^\pm$ are the basic modules of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. These isomorphisms are given by the quantization mapping (5.1.11). Explicitly, by point 1 of Remark 2.8.1, in terms of the standard basis $\{\mathbf{e}_i\}$ the space $\bigwedge{}^2{}_\pm T^*$ is spanned by

$$\vartheta^1 \wedge \vartheta^2 \pm \vartheta^3 \wedge \vartheta^4\,, \quad \vartheta^1 \wedge \vartheta^3 \pm \vartheta^4 \wedge \vartheta^2\,, \quad \vartheta^1 \wedge \vartheta^4 \pm \vartheta^2 \wedge \vartheta^3\,,$$

and, thus, $S^2\Delta_4^\pm$ is spanned by $\mathbf{e}_1\mathbf{e}_2 \pm \mathbf{e}_3\mathbf{e}_4$, $\mathbf{e}_1\mathbf{e}_3 \pm \mathbf{e}_4\mathbf{e}_2$ and $\mathbf{e}_1\mathbf{e}_4 \pm \mathbf{e}_2\mathbf{e}_3$. Thus, using the presentation given by (5.1.26), for the generators of $S^2\Delta_4^+$ we obtain

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 = 2i\sigma_1\,, \quad \mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_4\mathbf{e}_2 = 2i\sigma_2\,, \quad \mathbf{e}_1\mathbf{e}_4 + \mathbf{e}_2\mathbf{e}_3 = 2i\sigma_3\,, \tag{7.6.4}$$

as endomorphisms of $\Delta_4^+ \cong \mathbb{C}^2$. This gives an explicit identification of the space of real-valued self-dual forms on $\mathbb{R}^4$ with the space of traceless skew-Hermitean endomorphisms of Δ_4^+. Complexifying these isomorphisms, in particular, we obtain an identification of imaginary-valued self-dual forms with traceless Hermitean endomorphisms. Passing to the bundle level, we obtain natural bundle isomorphisms

$$\bigwedge{}^2{}_\pm \mathrm{T}^*_{\mathbb{C}} M \cong \mathrm{End}_0(\mathscr{S}^c_\pm(M))\,, \tag{7.6.5}$$

where $\mathrm{End}_0(\mathscr{S}^c_\pm(M))$ denote the bundles of traceless endomorphisms.

Remark 7.6.1 Below we will need a scalar product on the space of endomorphisms of a Hermitean vector space $(V, \langle\cdot,\cdot\rangle)$.[27] We define:

$$\langle T_1, T_2\rangle := \frac{1}{2}\,\mathrm{tr}(T_1^* T_2)\,, \quad T_1, T_2 \in \mathrm{End}(V)\,, \tag{7.6.6}$$

where T^* denotes the adjoint endomorphism, $\langle T^* w, v\rangle = \langle w, Tv\rangle$. Now, let $\alpha = \sum_{i<j} \alpha_{ij}\vartheta^i \wedge \vartheta^j \in \Omega^2(M, \mathbb{C})$. Using the quantization mapping c, we calculate

[27] Here, we use the convention that the Hermitean scalar product is anti-linear in the first and linear in the second entry.

$$
\begin{aligned}
|\mathsf{c}(\alpha)|^2 &= |\sum_{i<j} \alpha_{ij} c_i c_j|^2 \\
&= \frac{1}{2} \operatorname{tr}\big(\sum_{i<j}\sum_{k<l} \overline{\alpha_{ij}} \alpha_{kl} c_j c_i c_k c_l\big) \\
&= \frac{1}{2} \operatorname{tr}\big(\sum_{i<j} |\alpha_{ij}|^2 \mathbb{1}_4\big) \\
&= 2 \sum_{i<j} |\alpha_{ij}|^2 .
\end{aligned}
$$

On the other hand, on $\bigwedge^2{}_{\pm}\mathrm{T}^*_{\mathbb{C}}M$ the natural fibre norm is given by

$$
|\alpha|^2 = \sum_{i<j} |\alpha_{ij}|^2 .
$$

Thus, endowing $\mathrm{End}_0(\mathscr{S}^c_{\pm}(M))$ with the fibre metric defined by (7.6.6), we have

$$
|\mathsf{c}(\alpha)|^2 = 2|\alpha|^2 . \tag{7.6.7}
$$

♦

Now, we can formulate the Seiberg–Witten model. Let $\Phi \in \Gamma^\infty(\mathscr{S}^c_+(M))$. Fibrewise orthogonal projection to Φ defines a Hermitean endomorphism $\Phi\Phi^* \in \mathrm{End}(\mathscr{S}^c_+(M))$ by

$$
\Phi\Phi^*(\varphi) := \Phi\langle\Phi, \varphi\rangle , \quad \varphi \in \Gamma^\infty(\mathscr{S}^c_+(M)) .
$$

Its traceless part $\mathsf{q}(\Phi) := (\Phi\Phi^*)_0$ is given by

$$
\mathsf{q}(\Phi)(\varphi) = \Phi\langle\Phi, \varphi\rangle - \frac{1}{2}|\Phi|^2\varphi . \tag{7.6.8}
$$

The proof of the following Lemma is left to the reader (Exercise 7.6.1).

Lemma 7.6.2 *The identities*

$$
|\mathsf{q}(\Phi)|^2 = \frac{1}{4}|\Phi|^4 , \quad \langle T, \mathsf{q}(\Phi)\rangle = \frac{1}{2}\langle T\Phi, \Phi\rangle \tag{7.6.9}
$$

hold for any tranceless Hermitean endomorphism T. ■

Next, for any $X, Y \in \mathrm{T}M$ we define

$$
\beta^\Phi(X, Y) := \frac{1}{4}\left(\langle\Phi, X \cdot Y \cdot \Phi\rangle - \mathsf{g}(X, Y)|\Phi|^2\right) . \tag{7.6.10}
$$

Lemma 7.6.3 *We have* $\beta^{\Phi} \in \Omega^2_+(M, i\mathbb{R})$ *and*

$$\mathsf{c}(\beta^{\Phi}) = -\mathsf{q}(\Phi) . \tag{7.6.11}$$

Proof That β^{Φ} is an imaginary-valued 2-form follows immediately from the Clifford algebra relation $XY + YX = 2\mathsf{g}(X, Y)$ and from the fact that the Clifford multiplication is a Hermitean operator. We prove (7.6.11). Then, in particular, the self-duality of β^{Φ} follows. Let $\{e_i\}$ be a g-orthonormal local frame on M, let $\{\vartheta^j\}$ be the dual coframe and let Φ^A be the components of Φ with respect to the induced local frame in $\mathscr{S}^c_+(M)$. Then,

$$\beta^{\Phi} = \frac{1}{4} \sum_{i<j} \langle \Phi, e_i e_j \Phi \rangle \vartheta^i \wedge \vartheta^j ,$$

and, by (7.6.4), the coefficients of β^{Φ} are given by

$$\begin{aligned} \langle \Phi, e_1 e_2 \Phi \rangle &= \langle \Phi, e_3 e_4 \Phi \rangle = i(\overline{\Phi_1}\Phi_2 + \Phi_1 \overline{\Phi_2}) , \\ \langle \Phi, e_1 e_3 \Phi \rangle &= \langle \Phi, e_4 e_2 \Phi \rangle = \overline{\Phi_1}\Phi_2 - \Phi_1 \overline{\Phi_2} , \\ \langle \Phi, e_1 e_4 \Phi \rangle &= \langle \Phi, e_2 e_3 \Phi \rangle = i(|\Phi_1|^2 - |\Phi_2|^2) . \end{aligned}$$

Thus, using once again (7.6.4), we obtain

$$\begin{aligned} \mathsf{c}(\beta^{\Phi}) = -\frac{1}{2}\big(& (\overline{\Phi_1}\Phi_2 + \Phi_1 \overline{\Phi_2})\sigma_1 + \frac{1}{i}(\overline{\Phi_1}\Phi_2 - \Phi_1 \overline{\Phi_2})\sigma_2 \\ & + (|\Phi_1|^2 - |\Phi_2|^2)\sigma_3 \big) . \end{aligned}$$

On the other hand, decomposing $\mathsf{q}(\Phi)$ defined by (7.6.8) with respect to the basis (7.6.4) yields the same result with the negative sign. ■

Now, the configuration space of the Seiberg–Witten model is defined as

$$\mathscr{C} = \mathscr{A}(P) \times \Gamma^{\infty}(\mathscr{S}^c_+(M)) , \tag{7.6.12}$$

where $\mathscr{A}(P)$ is the affine space of connections on P. Thus, $\mathscr{C}$ is an affine space consisting of pairs (τ, Φ). We stress that the metric g is kept fixed. In a similar way as explained in Sect. 6.1, $\mathscr{C}$ may be treated in a Sobolev space setting, see [487, 553] for details. Clearly, $\mathscr{C}$ is acted upon by the group $\mathscr{G}$ of local gauge transformations. Here, the general transformation laws given by (6.1.2) and (7.1.6) boil down to

$$(\tau, \Phi) \mapsto (\tau + \pi^*(2\rho^{-1}\mathrm{d}\rho), \rho^{-1}\Phi) , \tag{7.6.13}$$

where $\rho : M \to \mathrm{U}(1)$. The action of the Seiberg–Witten model, called the Seiberg–Witten functional, is defined by

$$SW(\tau,\Phi) := \int_M \big(|\Omega_\tau^+|^2 + |\nabla\Phi|^2 + \frac{1}{4}\mathsf{Sc}|\Phi|^2 + \frac{1}{8}|\Phi|^4\big)\mathsf{v_g}\,. \tag{7.6.14}$$

Here, Ω_τ^+ is the self-dual part of the curvature Ω_τ, $\Phi \in \Gamma^\infty(\mathscr{S}_+^c(M))$ and

$$\nabla\Phi = \mathrm{d}\Phi + \frac{1}{2}\sum_{i<j}\omega_{ij}e_ie_j\Phi + \frac{1}{2}\tau\Phi$$

is the covariant derivative defined by the Spin^c-connection ω^τ. In the same way as explained in detail in Sects. 6.2 and 7.2, one derives the Euler–Lagrange equations for the Seiberg–Witten functional (Exercise 7.6.2):

$$\nabla^*\nabla\Phi = -\frac{1}{4}\big(\mathsf{Sc} + |\Phi|^2\big)\Phi\,, \tag{7.6.15}$$

$$\mathrm{d}^*\Omega_\tau^+ = -i\,\mathrm{Im}\big(\langle\nabla\Phi,\Phi\rangle\big)\,. \tag{7.6.16}$$

In our short presentation, we limit our attention to the absolute minima of the Seiberg–Witten functional. They are obtained via the following proposition.

Proposition 7.6.4 *The Seiberg–Witten functional may be rewritten as follows:*

$$SW(\tau,\Phi) = \int_M \big(|\Omega_\tau^+ - \beta^\Phi|^2 + |\mathrm{D}_\tau\Phi|^2\big)\mathsf{v_g}\,.$$

Proof Using (7.6.7), (7.6.11) and (7.6.9), we calculate

$$\begin{aligned}
|\Omega_\tau^+ - \beta^\Phi|^2 &= \frac{1}{2}|\mathsf{c}\big(\Omega_\tau^+\big) + \mathsf{q}(\Phi)|^2\\
&= \frac{1}{2}|\mathsf{c}\big(\Omega_\tau^+\big)|^2 + \frac{1}{2}|\mathsf{q}(\Phi)|^2 + \mathrm{Re}\langle\mathsf{c}\big(\Omega_\tau^+\big),\mathsf{q}(\Phi)\rangle\\
&= |\Omega_\tau^+|^2 + \frac{1}{8}|\Phi|^4 + \frac{1}{2}\langle\mathsf{c}\big(\Omega_\tau^+\big)\Phi,\Phi\rangle\,.
\end{aligned}$$

On the other hand, by Corollary 5.6.6, the Lichnerowicz Formula for D_τ reads

$$\mathrm{D}_\tau^2 = \nabla^*\nabla + \frac{1}{4}\mathsf{Sc} - \frac{1}{2}\mathsf{c}(\Omega_\tau)\,.$$

Thus, since $\Omega_\tau^-(\Phi) = 0$, we obtain

$$|\mathrm{D}_\tau\Phi|^2 = |\nabla\Phi|^2 + \frac{1}{4}\mathsf{Sc}|\Phi|^2 - \frac{1}{2}\langle\mathsf{c}\big(\Omega_\tau^+\big)\Phi,\Phi\rangle\,,$$

and the assertion follows. ■

Proposition 7.6.4 implies the following.

Corollary 7.6.5 *The absolute minima of the Seiberg–Witten functional are determined by the equations*

$$\mathrm{D}_\tau \Phi = 0\,, \quad \Omega_\tau^+ = \beta^\Phi\,. \tag{7.6.17}$$

■

The Eq. (7.6.17) will be referred to as the Seiberg–Witten equations. Equivalently, by (7.6.11), they may be written as

$$\mathrm{D}_\tau \Phi = 0\,, \quad \mathsf{c}(\Omega_\tau^+) = -\mathsf{q}(\Phi)\,. \tag{7.6.18}$$

Remark 7.6.6 (*Gauge transformations*)

1. Consider a gauge transformation (7.6.13) of a solution (τ, Φ) to the Seiberg–Witten equations. From Proposition 6.2.7 we know that (anti-)self-duality of a connection is a property which is invariant under gauge transformations. Here, the situation is even simpler, because in the Abelian case the curvature is gauge invariant. The same is true for $\mathsf{q}(\Phi)$. Moreover, the Dirac operator clearly transforms in the same way as Φ itself,

$$\mathrm{D}_\tau \Phi \mapsto \rho^{-1} \mathrm{D}_\tau \Phi\,.$$

   We conclude that the gauge transformed configuration $(\tau + \pi^*(2\rho^{-1}\mathrm{d}\rho), \rho^{-1}\Phi)$ is a solution of the Seiberg–Witten equations as well.
2. Using elliptic regularity, the following can be shown. If (τ, Φ) is a solution to the Seiberg–Witten equations belonging to an appropriate Sobolev class, then there exists a gauge transformation such that the gauge transformed configuration is smooth and thus, by point 1, a smooth solution, see Theorem 7.11 in [553] for details. ♦

Remark 7.6.7 (*Seiberg–Witten equations and magnetic monopoles*) By the discussion in Sects. 7.4 and 7.5, given a solution (τ, Φ) of (7.6.17) corresponding to a nontrivial first Chern class of P, τ describes a magnetic monopole configuration. Therefore, the Seiberg–Witten equations are also called monopole equations. To make the relation to our previous discussion more transparent, let us consider the Seiberg–Witten equations on Minkowski space, see [215, 467]. In that case, the first of the equations (7.6.17) is the ordinary Dirac equation known from relativistic quantum mechanics for a spin $\frac{1}{2}$ massless particle, coupled to the electromagnetic field, and the second of the equations (7.6.17) puts some conditions on the electromagnetic field strength tensor. It is easy to check (Exercise 7.6.3) that one has the following (static) exact solution of (7.6.17):

$$\mathbb{A}_0 = 0\,, \quad \mathbb{A}_k(x, y, z) = \frac{(-iy, ix, 0)}{2r(r-z)}\,, \tag{7.6.19}$$

$$\Phi(x, y, z) = \frac{1}{\sqrt{2r(r-z)}} \begin{bmatrix} x - iy \\ r - z \end{bmatrix} . \tag{7.6.20}$$

Here, (x, y, z) are the standard coordinates on $\mathbb{R}^3$, $r^2 = x^2 + y^2 + z^2$ and $\mathbb{A}$ is a potential of Ω_τ^+. Clearly, $\mathbb{A}$ describes a magnetic monopole of Dirac type. As expected, calculating the right hand side of the second equation in (7.6.17) for Φ given by (7.6.20) shows that the field strength tensor is of Coulomb type.

Let us add that the Seiberg–Witten equations can be generalized from U(1) to SU(n). Then, one also finds monopole solutions, see [149] for details. ♦

The Lichnerowicz Formula for the Dirac operator implies the following strong a priori estimate for the matter field part of a solution of the Seiberg–Witten equations.

Proposition 7.6.8 *Let (M, g) be an oriented compact Riemannian 4-manifold with scalar curvature Sc, endowed with a Spin^c-structure. If (τ, Φ) is a solution to the Seiberg–Witten equations, then either Φ vanishes identically or, at every point $m \in M$,*

$$|\Phi(m)|^2 \leq -\mathsf{Sc}_{\min} , \tag{7.6.21}$$

where $\mathsf{Sc}_{\min}$ is the minimal value of the scalar curvature on M. In particular, if the scalar curvature is non-negative, then $\Phi = 0$ identically.

Proof By Corollary 5.6.6, the Lichnerowicz Formula for D_τ reads

$$\mathrm{D}_\tau^2 = \nabla^*\nabla + \frac{1}{4}\mathsf{Sc} - \frac{1}{2}\mathsf{c}(\Omega_\tau) . \tag{7.6.22}$$

Thus, for a solution (τ, Φ) of (7.6.18), we have

$$0 = \mathrm{D}_\tau^2 \Phi = \nabla^*\nabla\Phi + \frac{1}{4}\mathsf{Sc}\,\Phi + \frac{1}{4}|\Phi|^2\Phi . \tag{7.6.23}$$

Now, let $m \in M$ be a point where $|\Phi|^2$ takes on a maximum. Then,

$$\mathrm{d}\big(|\Phi|^2\big)(m) = 0 , \quad 0 \leq \big(\Box|\Phi|^2\big)(m) ,$$

where $\Box = \mathrm{d}^*\mathrm{d}$ is the Hodge-Laplace operator of g acting on 0-forms. Using the compatibility of ∇ with the Hermitean fibre metric, together with (2.7.24), (2.7.31) and (7.6.23), we obtain

$$
\begin{aligned}
0 &\leq \frac{1}{2}\mathrm{d}^*\mathrm{d}|\Phi|^2 \\
&= \mathrm{d}^*\Big(\mathrm{Re}(\langle\Phi, \nabla\Phi\rangle)\Big) \\
&= -\sum_j \nabla_{e_j}\big(\mathrm{Re}(\langle\Phi, \nabla\Phi\rangle)\big)(e_j) \\
&= -\sum_j \nabla_{e_j}\big(\mathrm{Re}(\langle\Phi, \nabla_{e_j}\Phi\rangle)\big) + \sum_j \mathrm{Re}(\langle\Phi, \nabla_{\nabla_{e_j}e_j}\Phi\rangle) \\
&= \langle\Phi, \nabla^*\nabla\Phi\rangle - \langle\nabla\Phi, \nabla\Phi\rangle \\
&\leq \langle\Phi, \nabla^*\nabla\Phi\rangle \\
&= -\frac{1}{4}\big(\mathsf{Sc}\,|\Phi|^2 + |\Phi|^4\big)\,,
\end{aligned}
$$

where $\{e_j\}$ is a local orthonormal frame on M. Thus, if $|\Phi|^2_{\max} > 0$, then

$$
0 \leq -\frac{1}{2}\big(\mathsf{Sc} + |\Phi|^2_{\max}\big)\,.
$$

This implies (7.6.21). Finally, if Sc is non-negative, Φ must vanish identically. ∎

Now, recall from Chap. 6 that the study of the moduli space of instantons yields deep insight into the differential topology of 4-manifolds. Here, we deal with a similar situation which, in fact, is much simpler according to the fact that the gauge group is Abelian.[28] Thus, let us consider the moduli space corresponding to the Seiberg–Witten equations. In complete analogy to (6.5.1), we define the moduli space as

$$
\mathfrak{M}_L := \big\{(\tau, \Phi) \in \mathscr{C} : \ \mathrm{D}_\tau\Phi = 0\,,\ \Omega^+_\tau = \beta^\Phi\big\}/\mathscr{G}\,.
$$

As already mentioned at the beginning, as in the Yang–Mills case, all the mappings and spaces involved in the study of $\mathfrak{M}_L$ may be understood within the setting of Sobolev theory. For a presentation including these analytical details, we refer to [553] or [487].

To start with, in sharp contrast to the Yang–Mills case, the following holds.

Theorem 7.6.9 *The Seiberg–Witten moduli space $\mathfrak{M}_L$ is compact.*

For a proof see [393, 553]. The key point is the a priori estimate (7.6.21). Then, by standard bootstrap-type arguments, the assertion follows. We do not work out these details here.

Now, to study $\mathfrak{M}_L$, one can proceed as in the instanton case: one constructs a local model of the moduli space by linearizing the field equations and associates to that linearization an elliptic complex whose index, calculated by the Index Theorem, yields minus the (virtual) dimension of the moduli space.

---

[28] Of course, the Seiberg–Witten equations are nonlinear as well, but the nonlinearity given by the quadratic form $\mathsf{q}(\Phi)$ is much milder than the nonlinearity of the Yang–Mills equation.

Lemma 7.6.10 *The linearized Seiberg–Witten equations at the point* $(\tau, \Phi) \in \mathscr{C}$ *have the following form:*

$$(\mathrm{d}\alpha)^+ = \beta^{\Phi,\phi}\,, \quad \mathrm{D}_\tau \phi + \frac{i}{2}\alpha\Phi = 0\,, \tag{7.6.24}$$

with the indeterminates $\alpha \in \Omega^1(M, i\mathbb{R})$ *and* $\phi \in \Gamma^\infty(\mathscr{S}^c_+(M))$. *Here,* $\beta^{\Phi,\phi} \in \Omega^2_+(M, i\mathbb{R})$ *is given by*

$$\beta^{\Phi,\phi}(X, Y) = \frac{i}{2}\,\mathrm{Im}\,\{\langle\Phi, X\cdot Y\cdot\phi\rangle - \mathsf{g}(X, Y)\langle\Phi, \phi\rangle\}\,. \tag{7.6.25}$$

Proof Consider the 1-parameter families $\tau_t = \tau + t\alpha$ and $\Phi_t = \Phi + t\phi$ generated by (α, ϕ). Then, $\Omega_{\tau_t} = \mathrm{d}\tau + t\mathrm{d}\alpha$ and, thus, $\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\Omega^+_{\tau_t} = (\mathrm{d}\alpha)^+$. We calculate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\beta^{\Phi_t}(X, Y) &= \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\left\{\langle\Phi_t, X\cdot Y\cdot\Phi_t\rangle - \mathsf{g}(X, Y)|\Phi_t|^2\right\}\\
&= \frac{1}{4}\left(\langle\phi, X\cdot Y\cdot\Phi\rangle + \langle\Phi, X\cdot Y\cdot\phi\rangle - \mathsf{g}(X, Y)\big(\langle\Phi, \phi\rangle + \langle\phi, \Phi\rangle\big)\right)\\
&= \frac{i}{2}\mathrm{Im}\,\{\langle\Phi, X\cdot Y\cdot\phi\rangle - \mathsf{g}(X, Y)\langle\Phi, \phi\rangle\}\,.
\end{aligned}$$

This yields the first assertion. To show the second assertion, we note that $\mathrm{D}_{\tau_t}\Phi_t = \mathrm{D}_{\tau_t}\Phi + t\mathrm{D}_{\tau_t}\phi$ and, thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\big(\mathrm{D}_{\tau_t}\Phi_t\big) = \frac{\mathrm{d}}{\mathrm{d}t}{\restriction_0}\big(\mathrm{D}_{\tau_t}\Phi\big) + \mathrm{D}_\tau\phi\,.$$

But,

$$\mathrm{D}_{\tau_t}\Phi = i\sum_i e_i\cdot\left(\nabla_{e_i}\phi + \frac{t}{2}\alpha(e_i)\Phi\right).$$

This yields the second assertion. ∎

We obtain an infinitesimal model for the moduli space by factorizing with respect to the action of $\mathscr{G}$. By (7.6.13), the tangent space to the gauge orbit through (τ, Φ) is

$$\mathrm{T}_{(\tau,\Phi)}\big(\mathscr{G}\cdot(\tau, \Phi)\big) = \left\{(-2\mathrm{d}\xi, \xi\Phi) \in \mathrm{T}_{(\tau,\Phi)}\mathscr{C} : \xi \in \Omega^0(M, i\mathbb{R})\right\}\,. \tag{7.6.26}$$

To summarize, for every solution (τ, Φ) of the Seiberg–Witten equation, we have constructed two natural operators:

$$P^0_{(\tau,\Phi)} : \Omega^0(M, i\mathbb{R}) \to \Omega^1(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_+(M))$$

given by

$$P^0_{(\tau,\Phi)}(\xi) := (-2\mathrm{d}\xi, \xi\Phi)\,,$$

and

$$P^1_{(\tau,\Phi)} : \Omega^1(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_+(M)) \to \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_-(M))$$

defined by

$$P^1_{(\tau,\Phi)}(\alpha, \phi) := \big((\mathrm{d}\alpha)^+ - \beta^{\Phi,\phi}, \mathrm{D}_\tau\phi + \frac{i}{2}\alpha\Phi\big) .$$

Lemma 7.6.11 *For every solution (τ, Φ), the sequence*

$$0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{P^0_{(\tau,\Phi)}} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(\mathscr{S}^c_+(M)) \xrightarrow{P^1_{(\tau,\Phi)}} \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma(\mathscr{S}^c_-(M)) \to 0$$

is an elliptic complex of first order differential operators.

Proof We must show that $P^1_{(\tau,\Phi)} \circ P^0_{(\tau,\Phi)} = 0$. Thus, let $\alpha = -2\mathrm{d}\xi$ and $\phi = \xi\Phi$. Then, by (7.6.25), for any $X, Y \in \mathrm{T}M$,

$$\begin{aligned}\big((\mathrm{d}\alpha)^+ - \beta^{\Phi,\phi}\big)(X, Y) &= -\frac{i}{2}\,\mathrm{Im}\,\big\{\xi\big(\langle\Phi, X\cdot Y\cdot\Phi\rangle - \mathsf{g}(X, Y)|\Phi|^2\big)\big\}\\ &= -2\xi\mathrm{Re}\,\big\{\beta^\Phi(X, Y)\big\}\\ &= 0\,,\end{aligned}$$

because β^Φ and ξ are imaginary-valued. Moreover, using an orthonormal local frame $\{e_i\}$, we compute

$$\begin{aligned}\mathrm{D}_\tau\phi + \frac{i}{2}\alpha\Phi &= \mathrm{D}_\tau(\xi\Phi) - i(\mathrm{d}\xi)\Phi\\ &= i\sum_j e_j\cdot\big\{e_j(\xi)\Phi + \xi\nabla_{e_j}\Phi\big\} - i(\mathrm{d}\xi)\Phi\\ &= \xi\mathrm{D}_\tau\Phi\,.\end{aligned}$$

But $\mathrm{D}_\tau\Phi$ vanishes by (7.6.17). Finally, the complex is elliptic and the operators $P^1_{(\tau,\Phi)}$ and $P^0_{(\tau,\Phi)}$ are Fredholm, because they are built, up to lower-order terms, from the elliptic differential operators discussed in Examples 5.7.22 and 5.7.23. ■

Let us denote the above elliptic complex by $\mathfrak{E}^{SW}$ and call it the Seiberg–Witten complex. In the next step, we have to calculate its index over the reals.

Theorem 7.6.12 *The index of the Seiberg–Witten complex is given by*

$$\mathrm{ind}_{\mathbb{R}}(\mathfrak{E}^{SW}) = -\tfrac{1}{4}\mathfrak{c}_1(L)^2 + \tfrac{1}{4}(2\chi(M) + 3\sigma(M))\,, \tag{7.6.27}$$

where $\mathfrak{c}_1(L)$ is the first Chern index of L and $\chi(M)$ and $\sigma(M)$ are the Euler characteristic and the signature of M, respectively.

Proof Since lower order terms do not contribute, the index of $\mathfrak{E}^{SW}$ is equal to the index of the complex

$$\Omega^0(M, i\mathbb{R}) \xrightarrow{\mathrm{d}\oplus 0} \Omega^1(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_+(M)) \xrightarrow{\mathrm{d}^+\oplus \mathrm{D}_\tau} \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_-(M)),$$

which we denote by $\mathfrak{E}^{SW}_0$. By (5.7.44), in turn, the index of $\mathfrak{E}^{SW}_0$ coincides with minus the index of the assembled complex

$$\Omega^1(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_+(M)) \xrightarrow{(\mathrm{d}^*\oplus \mathrm{d}^+\oplus \mathrm{D}_\tau)} \Omega^0(M, i\mathbb{R}) \oplus \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma^\infty(\mathscr{S}^c_-(M)).$$

Next, using the additivity of the index, we obtain

$$-\operatorname{ind}_{\mathbb{R}}(\mathfrak{E}^{SW}_0) = 2\operatorname{ind}_{\mathbb{C}}(\mathrm{D}_\tau) + \operatorname{ind}_{\mathbb{R}}(\mathrm{d}^+ + \mathrm{d}^*).$$

By (5.8.53), (4.7.15) and (4.7.25), we have

$$\operatorname{ind}_{\mathbb{C}} \mathrm{D}_\tau = \int_M \mathrm{e}^{\frac{1}{2}\mathsf{c}_1(L)}\hat{A}(M) = \int_M \big(1 + \frac{1}{2}\mathsf{c}_1(L) + \frac{1}{8}\mathsf{c}_1(L)^2\big)\big(1 - \frac{1}{24}\mathsf{p}_1(M)\big).$$

Since, by (4.7.11) and the Hirzebruch Theorem 5.9.6, $\sigma(M) = \frac{1}{3}\mathfrak{p}_1(M)$, we obtain

$$\operatorname{ind}_{\mathbb{R}} \mathrm{D}_\tau = \frac{1}{4}\mathsf{c}_1(L)^2 - \frac{1}{4}\sigma(M). \tag{7.6.28}$$

Next, we calculate the index of $T = \mathrm{d}^* + \mathrm{d}^+$. For that purpose, we use the Hodge Theorem 2.7.2 and the remarks thereafter. Let $\alpha \in \Omega^2(M)$. Then, $\alpha \in \ker(T)$ iff $\mathrm{d}^*\alpha = 0$ and $\mathrm{d}^+\alpha = 0$. In this case,

$$(\mathrm{d} + \mathrm{d}^*)(\alpha + *\alpha) = \mathrm{d}\alpha + \mathrm{d} * \alpha = 2\mathrm{d}^+\alpha = 0.$$

Thus, $\mathrm{d}^*\mathrm{d}\alpha = 2\mathrm{d}^*\mathrm{d}^+\alpha = 0$. Taking the L^2-scalar product of this equation with α implies $\mathrm{d}\alpha = 0$. We conclude that the kernel of T coincides with the space of harmonic 1-forms,

$$\ker(T) = \mathscr{H}^1(M) = \ker(\mathrm{d}) \cap \ker(\mathrm{d}^*).$$

Next, we need the adjoint

$$T^* : \Omega^0(M) \oplus \Omega^2_+(M) \to \Omega^1(M), \quad \langle T^*(\xi, \beta), \alpha\rangle = \langle \xi, \mathrm{d}^*\alpha\rangle + \langle \beta, \mathrm{d}^+\alpha\rangle,$$

for any $\alpha \in \Omega^1(M)$. Thus,

$$T^*(\xi, \beta) = \mathrm{d}\xi + \mathrm{d}^*\beta.$$

Now, $(\xi, \beta) \in \ker(T^*)$ iff $\mathrm{d}\xi = 0$ and $\mathrm{d}^*\beta = 0$. But, for a self-dual form $\beta \in \Omega^2_+(M)$ we have $\mathrm{d}^*\beta = 0$ iff $\mathrm{d}\beta = 0$. Thus, we obtain

$$\ker(T^*) = \mathscr{H}^0(M) \oplus \mathscr{H}^2_+(M)\,,$$

where $\mathscr{H}^2_+(M)$ denotes the space of self-dual harmonic 2-forms on M. To summarize, we have

$$\mathrm{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*)) = -b_0 + b_1 - b_2^+\,, \tag{7.6.29}$$

with the b_i denoting the Betti numbers. Now, by definition, $\sigma(M) = b_2^+ - b_2^-$ and, hence, $b_2^+ = \frac{1}{2}(b_2 + \sigma(M))$. Moreover, by Poincaré duality, $\chi(M) = 2(b_0 - b_1) + b_2$. This yields

$$\mathrm{ind}_{\mathbb{R}}(\mathrm{d}^* + \mathrm{d}^+) = -\frac{1}{2}(\chi(M) + \sigma(M))\,. \tag{7.6.30}$$

Adding up (7.6.28) and (7.6.30), we obtain the assertion. ■

Now, $H^1(\mathfrak{E}^{SW})$ serves as an infinitesimal model for the tangent spaces of $\mathfrak{M}_L$. Then, as in the Yang–Mills case, the index of $\mathfrak{E}^{SW}_0$ yields the virtual dimension of $\mathfrak{M}_L$ provided $H^0(\mathfrak{E}^{SW})$ and $H^2(\mathfrak{E}^{SW})$ vanish. First, note that the action of $\mathscr{G}$ is not free when $\Phi = 0$. Such configurations give rise to singular points in the moduli space.

Definition 7.6.13 A solution (τ, Φ) of the Seiberg–Witten equations is called reducible if $\Phi = 0$. Otherwise it is referred to as irreducible.

By (7.6.13), the stabilizer of a reducible configuration is isomorphic to the subgroup $\mathrm{U}(1) \subset \mathscr{G}$ consisting of the constant mappings. Clearly, if (τ, Φ) is irreducible, then $H^0(\mathfrak{E}^{SW})$ vanishes. For later purposes, we also note the following.

Remark 7.6.14 If Φ is a solution to the equation $\mathrm{D}_\tau \Phi = 0$ on a connected manifold, then Φ either vanishes identically, or it is different from zero everywhere on an open dense subset. This is called the Unique Continuation Theorem, see e.g. Theorem E.8 in [553] for a proof. Thus, for an irreducible configuration (τ, Φ), the matter field Φ is nowhere vanishing on an open dense subset of M. ♦

Now, as in the Yang–Mills case, one would like to be able to perturb the system in order to achieve transversality, that is, to achieve the vanishing of $H^0(\mathfrak{E}^{SW})$ and $H^2(\mathfrak{E}^{SW})$. Coming from Yang–Mills theory, it would be desirable to do this by perturbing the metric g and, thus, to obtain a counterpart of the Freed-Uhlenbeck Theorem, see the discussion in Sect. 6.5. Here, the dependence on the metric is, however, more complicated. The system depends on g not only via the Hodge star operator but also via the Spin^c-structure. This leads to a quite complicated variational problem, which to our knowledge has not yet been completely understood in the general case. Sources for this approach are [180, 559]. For a summary of various perturbations used in various special cases and for yet another perturbation approach, we refer to [228]. The most convenient and, probably therefore, the most prominent perturbation is given in terms of a generic self-dual 2-form $\eta \in \Omega^2_+(M, i\mathbb{R})$. From now on, let us limit our attention to that case. Instead of (7.6.17), one considers the perturbed Seiberg–Witten equations

$$\mathrm{D}_\tau \Phi = 0\,, \quad \Omega^+_\tau + \eta = \beta^\Phi\,. \tag{7.6.31}$$

Then, for a reducible solution, we have

$$\Omega^+_\tau + \eta = 0\,. \tag{7.6.32}$$

It turns out that if $b_2^+(M) > 0$, then for a generic choice of η there are no solutions to this equation. In more detail, let $\Omega^{2,+}_c \subset \Omega^2_+(M, i\mathbb{R})$ be the subset of elements η such that there exists a connection $\tau \in \mathscr{A}(P)$ fulfilling (7.6.32).

Lemma 7.6.15 *Assume that $b_2^+(M) > 0$. Then, the set $\Omega^{2,+}_c$ is an affine subspace of $\Omega^2_+(M, i\mathbb{R})$ of codimension $b_2^+(M)$ whose translation vector space is given by the image of* $\mathrm{d}^+ : \Omega^1(M, i\mathbb{R}) \to \Omega^2_+(M, i\mathbb{R})$.

Proof First, we show that $\Omega^{2,+}_c$ is an affine subspace with translation vector space $\mathrm{im}(\mathrm{d}^+)$. For that purpose, let $\eta_0 \in \Omega^{2,+}_c$ and let $\tau_0 \in \mathscr{A}(P)$ be a connection such that $\Omega^+_{\tau_0} + \eta_0 = 0$. On the one hand, for any $\eta \in \Omega^{2,+}_c$ there exists $\tau \in \mathscr{A}(P)$ such that $\Omega^+_\tau + \eta = 0$. Thus, $\eta - \eta_0 = \mathrm{d}^+(\tau_0 - \tau)$. On the other hand, if $\eta = \eta_0 + \mathrm{d}^+\alpha$ for some α, then $\Omega^+_{\tau_0 - \alpha} + \eta = 0$. This implies

$$\Omega^{2,+}_c = \eta_0 + \mathrm{im}(\mathrm{d}^+)\,.$$

It remains to compute the codimension. For that purpose, using Hodge theory, we prove the following direct sum decomposition:

$$\Omega^2_+(M, \mathrm{i}\mathbb{R}) = \mathscr{H}^2_+(M, \mathrm{i}\mathbb{R}) \oplus \mathrm{im}(\mathrm{d}^+)\,. \tag{7.6.33}$$

For any $\eta \in \Omega^2_+(M, i\mathbb{R})$, we have $\eta = \chi + \mathrm{d}\alpha + *\mathrm{d}\beta$, where χ is harmonic and $\alpha, \beta \in \Omega^1(M, i\mathbb{R})$. Thus,

$$\eta = *\eta = *\chi + \mathrm{d}\beta + *\mathrm{d}\alpha\,.$$

This implies $\chi = *\chi$ and $\mathrm{d}\alpha = \mathrm{d}\beta$ and, thus, $\eta = \chi + 2\mathrm{d}^+\alpha$. Since every self-dual harmonic 2-form is orthogonal to the image of d^+, the sum in (7.6.33) is direct. ∎

Passing in Eq. (7.6.32) to the de Rham cohomology classes, we obtain

$$[\eta] = 2\pi \mathrm{i}\, \mathrm{c}_1(L)^+\,. \tag{7.6.34}$$

If this equation holds, then (7.6.32) admits a solution and, in this case, η is said to be bad (with respect to the chosen Spin^c-structure). Otherwise, η is said to be good. By the above discussion, for $b_2^+(M) > 0$, the 2-form η is generically good and, thus, generically every solution of the Seiberg–Witten equations is irreducible. Also note that for $b_2^+(M) = 0$ a reducible solution exists for any metric and for any perturbation.

Now, consider the mapping

$$F : \mathscr{A}(P) \oplus \Gamma(\mathscr{S}^c_+(M) \setminus \{0\}) \oplus \Omega^2_+(M, i\mathbb{R}) \to \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma(\mathscr{S}^c_-(M))$$

given by

$$F(\tau, \Phi, \eta) := (\Omega^+_\tau - \beta^\Phi + \eta, \mathrm{D}_\tau \Phi) .$$

Then, $F^{-1}(\{0\})$ is the set of solutions of the perturbed Seiberg–Witten equations. The tangent mapping

$$P^1_{(\tau,\Phi,\eta)} : \Omega^1(M, i\mathbb{R}) \oplus \Gamma(\mathscr{S}^c_+(M)) \oplus \Omega^2_+(M, i\mathbb{R}) \to \Omega^2_+(M, i\mathbb{R}) \oplus \Gamma(\mathscr{S}^c_-(M))$$

of F is given by

$$P^1_{(\tau,\Phi,\eta)}(\alpha, \phi, \zeta) = \big((\mathrm{d}\alpha)^+ - \beta^{\Phi,\phi} + \zeta, \mathrm{D}_\tau \phi + \frac{i}{2}\alpha\Phi\big) . \tag{7.6.35}$$

The following lemma shows that, for generic η, the second cohomology group of the perturbed Seiberg–Witten complex vanishes.

Lemma 7.6.16 *For a generic perturbation, $P^1_{(\tau,\Phi,\eta)}$ is surjective.*

Proof Let (γ, φ) be in the orthogonal complement of the image of $P^1_{(\tau,\Phi,\eta)}$ in the sense of the L^2-scalar product. Then,

$$0 = \langle(\gamma, \varphi), P^1_{(\tau,\Phi,\eta)}(0, 0, \gamma)\rangle = \parallel \gamma \parallel^2 ,$$

and, thus, $\gamma = 0$. In the same way,

$$0 = \langle(0, \varphi), P^1_{(\tau,\Phi,\eta)}(\alpha, 0, 0)\rangle$$

implies $\langle \frac{i}{2}\alpha\Phi, \varphi\rangle = 0$. But, by assumption, Φ is not vanishing identically and, thus, by Remark 7.6.14, Φ is nowhere vanishing on an open dense subset. It follows that the linear mapping $\alpha \mapsto \alpha\Phi$ is fibrewise injective. This implies $\varphi = 0$. ■

By point 1 of Remark 7.6.6, the zero set of the mapping F agrees with the zero set of the corresponding extended mapping between appropriate Sobolev completions. Thus, we may view F as a mapping between Banach spaces. Then, by the Implicit Function Theorem, $F^{-1}(\{0\})$ is a Banach manifold. Moreover, one can show that the canonical projection

$$\pi : F^{-1}(\{0\}) \to \Omega^2_+(M, i\mathbb{R}) , \quad (\tau, \Phi, \eta) \mapsto \eta ,$$

is a smooth Fredholm mapping.[29] Then, by the Sard–Smale Theorem[30], the set of regular values of π is dense in the target space. Thus, we can choose a regular value η of π and we can build

$$\pi^{-1}(\eta) = F_\eta^{-1}(\{0\})\,.$$

Then, by the Implicit Function Theorem, $F_\eta^{-1}(\{0\})$ is a manifold. Clearly,

$$\mathfrak{M}_{L,\eta} := F_\eta^{-1}(\{0\})/\mathscr{G} \tag{7.6.36}$$

is the moduli space for the perturbed Seiberg–Witten equations with $\mathscr{G}$ acting freely for generic perturbations. Theorem 7.6.12 and Lemmas 7.6.15 and 7.6.16, combined with the above functional analytic arguments, imply the following.

Theorem 7.6.17 *Let $b_2^+(M) > 0$. Then, for generic values of η, the moduli space $\mathfrak{M}_{L,\eta}$ is a smooth manifold whose dimension is given by*

$$\dim \mathfrak{M}_{L,\eta} = \frac{1}{4}\mathfrak{c}_1(L)^2 - \frac{1}{4}(2\chi(M) + 3\sigma(M))\,. \qquad \blacksquare$$

Remark 7.6.18

1. The subset of regular values η is a countable intersection of open and dense sets, see Theorem 7.16 in [553].
2. One can prove that, for generic perturbations, $\mathfrak{M}_{L,\eta}$ is oriented. Let us sketch the idea of the proof. Clearly, a manifold is orientable iff the top exterior power of its tangent bundle is trivial. Then, choosing an orientation at one point yields an orientation everywhere. Thus, here, it is enough to prove that the determinant line bundle $\bigwedge^{\mathrm{top}} \mathrm{T}\,\mathfrak{M}_{L,\eta}$ is trivial. For that purpose, following [159] one embeds $\mathfrak{M}_{L,\eta}$ into $\big(\mathscr{A}(P) \oplus \Gamma^\infty(\mathscr{S}_+^c(M) \setminus \{0\})\big)/\mathscr{G}$. Then, triviality follows from the simply-connectedness of the latter space. Moreover, since the fibres of $\mathrm{T}\,\mathfrak{M}_{L,\eta}$ are given by $H^1(\mathfrak{E}^{SW})$, the bundle $\bigwedge^{\mathrm{top}} \mathrm{T}_{(\tau,\Phi)}\,\mathfrak{M}_{L,\eta}$ coincides with the determinant of the complex $\mathfrak{E}^{SW}$. Analyzing this isomorphism according to Theorem 7.6.12, we obtain a natural bijection between orientations of $\mathfrak{M}_{L,\eta}$ and orientations of the vector space $\mathscr{H}^0(M) \oplus \mathscr{H}^1(M) \oplus \mathscr{H}_+^2(M)\,.$ ♦

In the remainder of this section, we outline that the moduli space gives rise to differential topological invariants, called Seiberg–Witten invariants, which may be used to distinguish between smooth structures on a given topological 4-manifold. By construction, the moduli space depends both on the metric g, the spin structure $\mathfrak{s}$ and the perturbation η. The Seiberg–Witten invariants will be independent of g and η and only dependent on the isomorphism class $[\mathfrak{s}]$ of the Spin^c-structure.[31] Now, let

[29] See e.g. Sect. 3.4 in [459] for an easily readable proof. A smooth Fredholm mapping is a smooth mapping whose tangent mapping is Fredholm.

[30] See e.g. Theorem B.13 in [553].

[31] Two Spin^c-structures $\mathfrak{s}_0$ and $\mathfrak{s}_1$ corresponding to metrics g_0 and g_1 are called equivalent if $(\mathrm{g}_0(X,X))^{-\frac{1}{2}}\mathfrak{s}_0(X) = (\mathrm{g}_1(X,X))^{-\frac{1}{2}}\mathfrak{s}_1(X)$, cf. (5.5.6).

g_0 and g_1 be metrics with equivalent Spin^c-structures $\mathfrak{s}_0$ and $\mathfrak{s}_1$, respectively. Let η_0 and η_1 be regular perturbations corresponding to $(\mathfrak{s}_0, \mathsf{g}_0)$ and $(\mathfrak{s}_1, \mathsf{g}_1)$, respectively. Then, by the same methods as above, one can prove that the corresponding moduli spaces are cobordant. Let us make this statement precise: let $t \mapsto \mathsf{g}_t$ and $t \mapsto \mathfrak{s}_t$ be fixed paths connecting g_0 with g_1 and $\mathfrak{s}_0$ with $\mathfrak{s}_1$, respectively. Consider the space $\mathfrak{Z}$ of all smooth paths $t \mapsto \eta_t$ such that, for every t, η_t is g_t-self-dual. For $\{\eta_t\} \in \mathfrak{Z}$ define

$$\mathfrak{W} := \{(t, \tau, \Phi) : \ t \in [0,1] , \ [(\tau, \Phi)] \in \mathfrak{M}(M, \{\mathfrak{s}_t\}, \{\mathsf{g}_t\}, \{\eta_t\})\} .$$

Now, in general, it will not be possible to find a path $t \to \eta_t$ such that η_t is good for every t. However, if we additionally assume $b_2^+(M) \geq 1$, then there exists a regular[32] subset of $\mathfrak{Z}$ of good paths. For a proof of the following proposition, we refer to Theorem 7.21 of [553].

Proposition 7.6.19 *Let $b_2^+(M) \geq 1$. Then, for every regular path $t \mapsto \eta_t$, $\mathfrak{W}$ is a smooth oriented manifold of dimension*

$$\dim \mathfrak{W} = \frac{1}{4}\mathfrak{c}_1(L)^2 - \frac{1}{4}(2\chi(M) + 3\sigma(M)) + 1 \tag{7.6.37}$$

with boundary

$$\partial\mathfrak{W} = \mathfrak{M}(M, \mathfrak{s}_1, \mathsf{g}_1, \eta_1) - \mathfrak{M}(M, \mathfrak{s}_0, \mathsf{g}_0, \eta_0) .$$

The minus sign accounts for the reversal of the orientation. ∎

Proposition 7.6.19 constitutes the basis for the discussion of invariants. It tells us that, for a chosen equivalence class of Spin^c-structures, different choices of g and η yield cobordant moduli spaces provided $b_2^+(M) \geq 1$. Now, we are prepared to define the Seiberg–Witten invariants. In the remainder, we write $\mathfrak{c}_1$, σ and χ for, respectively, $\mathfrak{c}_1(L)$, $\sigma(M)$ and $\chi(M)$.

First, assume $\dim \mathfrak{M}_{L,\eta} = 0$. Then, by (7.6.37),

$$\frac{1}{4}(\mathfrak{c}_1^2 - \sigma) = \frac{1}{2}(\chi + \sigma) . \tag{7.6.38}$$

Since the left hand side is the real index of the Dirac operator, it is an even number. By (7.6.29), for connected M, the right hand side is equal to $1 - b_1 + b_+^2$ and, thus, $b_+^2 - b_1$ is odd. Moreover, as a zero-dimensional compact manifold, $\mathfrak{M}_{L,\eta}$ consists of a finite number of points for every regular value of η. Its orientation is given as explained under point 2 of Remark 7.6.18. In the case under consideration, $\bigwedge^{\mathrm{top}} \mathrm{T}_{(\tau,\Phi)} \mathfrak{M}_{L,\eta} \cong \mathbb{R}$, see Sect. 7.4 of [553] for details. Thus, the orientation is given by an assignment of ± 1 to each point of $\mathfrak{M}_{L,\eta}$, that is, we assign the number $\nu(\tau, \Phi) = 1$ if the orientation of the determinant line bundle coincides with the natural orientation of $\mathbb{R}$ and -1 otherwise.

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[32] A countable intersection of open and dense sets.

Definition 7.6.20 Let (M, g) be an oriented compact 4-dimensional Riemannian manifold fulfilling $b_2^+ \geq 1$. Let there be chosen a Spin^c-structure $\mathfrak{s}$ of (M, g). If $\dim \mathfrak{M}_{L,\eta} = 0$, where η is a chosen regular self-dual 2-form, then one defines

$$\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta) := \sum \nu(\tau, \Phi)\,, \tag{7.6.39}$$

where the sum runs over the finite set of all equivalence classes $[(\tau, \Phi)] \in \mathfrak{M}_{L,\eta}$.

Then, the following holds.

Theorem 7.6.21 (Seiberg–Witten) *If $b_2^+ > 1$, then the integer $\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta)$ is independent of the choice of g and η. It only depends on the isomorphism class $[\mathfrak{s}]$.* ■

Consequently, the integer $\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta)$ is called the zero-dimensional Seiberg–Witten invariant. Clearly, we can write $\mathrm{sw}(M, \mathfrak{s})$.

Second, assume $\dim \mathfrak{M}_{L,\eta} > 0$. If this dimension is odd, we set

$$\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta) = 0\,.$$

If the dimension is even, $\dim \mathfrak{M}_{L,\eta} = 2d$, we have

$$\frac{1}{4}\left(\mathfrak{c}_1^2 - 2\chi - 3\sigma\right) = 2d\,.$$

This implies that $b_+^2 - b_1$ is again odd. Now, one proceeds as follows. For a chosen point $m_0 \in M$, consider the group of pointed gauge transformations $\mathscr{G}_{m_0} := \{u \in \mathscr{G} : u(m_0) = 1\}$. Then, $\mathscr{C} \to \mathscr{C}/\mathscr{G}_{m_0}$ is a principal U(1)-bundle which we denote by $\mathscr{P}$. Let $\mathsf{c}_1(\mathscr{P})$ be its first Chern class. For any generic perturbation, the moduli space is a compact oriented finite-dimensional submanifold of $\mathscr{C}/\mathscr{G}_{m_0}$. Thus, we can define

$$\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta) := \int_{\mathfrak{M}(M,\mathfrak{s},\mathsf{g},\eta)} \mathsf{c}_1(\mathscr{P})^d\,. \tag{7.6.40}$$

Clearly, $\mathsf{c}_1(\mathscr{P})$ may be viewed as the first Chern class of a finite-dimensional U(1)-bundle obtained by restriction to the submanifold $\mathfrak{M}(M, \mathfrak{s}, \mathsf{g}, \eta)$. Then, we have a counterpart of Theorem 7.6.21.

Theorem 7.6.22 (Seiberg–Witten) *If $b_2^+ > 1$, then the integer $\mathrm{sw}(M, \mathfrak{s}; \mathsf{g}, \eta)$ defined by (7.6.40) is independent of the choice of g and η. It only depends on the isomorphism class $[\mathfrak{s}]$.* ■

Next, we list a few basic properties of the Seiberg–Witten invariants $\mathrm{sw}(M, \mathfrak{s})$, together with consequences following from their non-vanishing. For the proofs we refer to [553].

(a) If $b_2^+ > 1$, then the Seiberg–Witten invariants are zero for all but finitely many Spin^c-structures $\mathfrak{s}$.
(b) If (M, g) has positive scalar curvature and $b_+^2 \geq 2$, then all the Seiberg–Witten invariants vanish.[33] Thus, the non-vanishing of a Seiberg–Witten invariant on a manifold M of the above type means that M does not admit a Riemannian metric with positive scalar curvature. Note that this obstruction depends on the differential (and not merely on the topological) structure of the 4-manifold.
(c) Assume that (M, g) has constant scalar curvature Sc. If $b_+^2 \geq 2$ and $\mathrm{sw}(M, \mathfrak{s}) \neq 0$, then
$$\mathfrak{c}_1^2 \leq \frac{\mathrm{vol}(M)}{32\pi^2} \mathsf{Sc}^2 \,.$$
Equality holds if there exists a pair (τ, Φ) fulfilling
$$|\Omega_\tau^+|^2 = \frac{1}{32} \mathsf{Sc}^2 \,, \quad \Omega_\tau^- = 0 \,, \quad \nabla\Phi = 0 \,, \quad |\Phi|^2 = -\frac{1}{2} \mathsf{Sc} \,.$$
(d) Let (M, g) be an Einstein space. Assume $\mathfrak{c}_1^2 = 2\chi + 3\sigma$, $b_2^+ \geq 2$ and $\mathrm{sw}(M, \mathfrak{s}) \neq 0$, then
$$-2\chi \leq 3\sigma \leq \chi \,.$$
Moreover, $3\sigma = \chi$ iff the universal cover of M is either $\mathbb{R}^4$ or the complex hyperbolic space $\mathrm{SU}(2, 1)/\mathrm{U}(2)$. This result belongs to LeBrun, see [408].

Far beyond the above points, there is a lot of deep applications of Seiberg–Witten theory both in geometry and in differential topology.[34]

(a) First of all, Seiberg–Witten theory yields alternative, much simpler proofs of results obtained via Donaldson theory, see e.g. the proof of the Donaldson Theorem 6.6.3 in [553] or [487]. Nowadays, the Seiberg–Witten invariants belong to the standard tool kit of differential topology of 4-manifolds. In particular, there exists a cut-and-paste technique for the calculation of Seiberg–Witten invariants.
(b) The geometry of embedded algebraic curves in the complex projective 2-space was studied. In this context, the Thom conjecture was proven by Kronheimer and Mrowka, Morgan, Szabo and Taubes and Fintushel and Stern.
(c) Applying Seiberg–Witten theory to symplectic geometry turned out to be especially fruitful. In particular, Taubes identified the Seiberg–Witten invariants of a compact 4-manifold with Gromov invariants. This led to an existence theorem of pseudo-holomorphic curves in such manifolds.

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[33] This is an immediate consequence of the a priori estimate (7.6.21) extended to the perturbed Seiberg–Witten equation.

[34] Many of the results mentioned here were found immediately after the birth of Seiberg–Witten theory. References can be found in the literature cited at the beginning of this section.

Exercises

7.6.1 Prove the statements of Lemma 7.6.2.

7.6.2 Confirm Eqs. (7.6.15) and (7.6.16).

7.6.3 Check that the Eqs. (7.6.19) and (7.6.20) yield a static solution to the Seiberg–Witten equations on Minkowski space.

7.7 The Standard Model of Elementary Particle Physics

From the phenomenological point of view, the electromagnetic, the weak and the strong interactions differ drastically, both in their strength and in their range. Nonetheless, it turns out that the principle of local gauge invariance is applicable to all of them, leading to what nowadays is called the standard model of particle interactions. All the particles described by the standard model are considered to be fundamental, that is, they do not show any internal structure and may be considered as pointlike.[35] The model whose classical field theoretical structure we are going to describe has a long history. First, based on earlier work by Glashow [247] and others, Weinberg [656] and Salam [552] unified the electromagnetic and the weak interactions.[36] One of the basic ingredients was the Higgs mechanism as discussed in Sect. 7.3, see [106, 186, 273, 274, 298–300, 364]. The second piece of the standard model, the theory of strong interactions called Quantum Chromodynamics, was developed at the beginning of the seventies, see [264, 513, 655]. This work was based upon fundamental earlier work by Gell-Mann and collaborators [235, 236]. For an exhaustive presentation of the history of the standard model we refer to [657].

We start with recalling some basics from Chap. 5, see Examples 5.1.21, 5.2.10, 5.3.9 and 5.3.25 where the general structures were illustrated for the case of the Minkowski space. Comparing with these examples, the reader should note some changes in the notation which we invented in order to be as close as possible to the notation in the physics literature.[37]

Consider the Minkowski space (M, g), where $\mathsf{g} = \mathrm{diag}[1, -1 - 1 - 1]$, and its (complexified) Clifford algebra $Cl^c(M, \mathsf{g})$. For its generators $\{\gamma_\mu\}$, $\mu = 0, \dots, 3$, we choose the following representation

$$\gamma_0 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & \tau_k \\ -\tau_k & 0 \end{bmatrix},$$

[35] The Large Hadron Collider (LHC) at CERN allows to study the physics of the standard model down to distances $\Delta x \sim 10^{-18}$cm. The experiments confirm that, down to such distances, the fundamental particles do not show any internal structure, indeed.

[36] For this work, Glashow, Weinberg and Salam received the Nobel prize in 1979.

[37] The conventions used below coincide e.g. with those in [565].

where τ_k , $k = 1, 2, 3$, are the Pauli matrices. Then, the chirality operator[38] is given by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, that is,

$$\gamma_5 = \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} .$$

Thus, $\left(\gamma_5\right)^2 = \mathbb{1}$, that is, γ^5 has eigenvalues ± 1. This yields a direct sum decomposition of the bispinor representation space $\Delta_4 \cong \mathbb{C}^4$ into eigenspaces of γ_5,

$$\Delta_4 = \Delta^+ \oplus \Delta^- .$$

For a bispinor ψ, we denote the elements corresponding to this decomposition by

$$\psi_L := \frac{1}{2}(\mathbb{1} - \gamma^5)\psi , \quad \psi_R := \frac{1}{2}(\mathbb{1} + \gamma^5)\psi , \tag{7.7.1}$$

and call them the left-handed and the right-handed components of ψ, respectively. In the sequel, instead of Δ_4 we will rather write $\mathbb{C}^4$. For building Lagrangians, we will use the standard Hermitean form given by (5.3.55) which, here, will be denoted by $\langle \cdot, \cdot \rangle$. Finally, we should stress that in this section we use the physical representation of gauge potentials, cf. Remark 6.1.1.

Now, we can start building the standard model. It is an $\big(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\big)$-gauge theory, containing three fermionic families, see Table 7.1, a Higgs field and gauge fields mediating the electroweak and the strong interactions. The fermionic families consist of leptons and quarks with equal quantum numbers but different masses. There is no theoretical explanation of this fact. For clearness of presentation, we will limit our attention to the first fermionic family, consisting of the leptons (ν_e, e), where e denotes the electron and ν_e the corresponding neutrino, and the quarks (u, d). The remaining families must be dealt with in essentially the same way. We will comment on that at the end of this section.

Table 7.1 The fermionic families of the standard model. The data are taken from [71]. The quark masses cannot be measured directly, but must be determined indirectly through their influence on hadronic properties

	Particles and their masses in MeV						Charge
Leptons	ν_e	<0.0000006	μ_u	<0.19	ν_τ	<18.2	0
	e	0.510998928 $\pm$0.000000011	μ	105.6583715 $\pm$0.0000035	τ	1776.82 ± 0.16	-1
Quarks	u	$2.3 + 0.7(-0.5)$	c	1275 ± 25	t	173070 ± 890	$+\frac{2}{3}$
	d	$4.8 + 0.5(-0.3)$	s	95 ± 5	b	4650 ± 30	$-\frac{1}{3}$

[38]Note that, in order to apply the general formula (5.3.9) for the chirality element, we must use the presentation (5.1.28) for the generators of Cl_4^c.

We begin with describing the electroweak interaction of the leptons. In the standard notation from particle physics, we associate with e and ν_e a bispinor field on M which we denote by the same letter. It is an experimental fact that in weak interactions parity is not conserved and a right-handed neutrino is not observed. There is no theoretical explanation of this fact within the model. Consequently, we decompose e and ν_e into their left-handed and right-handed parts, according to (7.7.1), and build an SU(2)-doublet from the left-handed parts of e and ν_e and an SU(2)-singlet from the right-handed electron part,

$$L_e = \begin{bmatrix} \nu_{eL} \\ e_L \end{bmatrix}, \quad e_R , \tag{7.7.2}$$

that is, we postulate that L_e transforms under the basic and e_R under the trivial representation of SU(2). From these objects we build

$$\psi_e : M \to \mathbb{C}^4 \otimes \mathbb{C}^3 , \quad \psi_e(\mathbf{x}) := \begin{bmatrix} L_e \\ e_R \end{bmatrix} (\mathbf{x}) . \tag{7.7.3}$$

Here, the bispinor space $\mathbb{C}^4$ carries the representation of the spin group $\mathrm{SL}(2,\mathbb{C})$ of M given by (5.2.15) and $\mathbb{C}^3$ carries the representation of SU(2) just defined,

$$\sigma_L : \mathrm{SU}(2) \times \mathbb{C}^3 \to \mathbb{C}^3 , \quad \sigma_L(a) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ z_3 \end{bmatrix} .$$

In order to accommodate the electromagnetic interaction in this model, we proceed as follows: we introduce a U(1)-symmetry, called weak hypercharge symmetry, acting on $\mathbb{C}^3$ via

$$\sigma_Y : \mathrm{U}(1) \times \mathbb{C}^3 \to \mathbb{C}^3 , \quad \sigma_Y(\exp(i\alpha)) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \exp(iy_L\alpha) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \exp(iy_R\alpha) z_3 \end{bmatrix} ,$$

with $y_L, y_R \in \mathbb{R}$ determined by the following postulate: let τ_a be the Pauli matrices. Consider the bases $\{t_a = \frac{i}{2}\tau_a\}$ and $\{i\}$ of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$, respectively. Denote the generators of the representations σ_L and σ_Y by

$$iT_a := \sigma_L'(t_a) = \left[\begin{array}{c|c} t_a & 0 \\ \hline 0 & 0 \end{array}\right] , \quad iY := \sigma_Y'(i) = i \left[\begin{array}{c|c} y_L \mathbb{1} & 0 \\ \hline 0 & y_R \end{array}\right] \tag{7.7.4}$$

and require that, in any representation, the electric charge generator Q_e be given by[39]

$$Q_e = T_3 + Y . \tag{7.7.5}$$

[39]Note that $[Y, T_3] = 0$.

Applying Q_e to L_e and e_R, from Table 7.1 we read off the eigenvalues $y_L = -\frac{1}{2}$ and $y_R = -1$, respectively.

To summarize, in the terminology of Sect. 7.1, ψ_e is the global representative of a section of type (μ, σ) of the bundle $E = E_s \otimes E_i$ associated with $Q \times_M P$, where

(a) E_s is the spinor bundle with typical fibre $\mathbb{C}^4$ carrying the standard spinor representation μ of $\mathrm{SL}(2, \mathbb{C})$, associated with the (trivial) spin structure bundle $Q(M, \mathrm{SL}(2, \mathbb{C}))$,
(b) E_i is the complex vector bundle with typical fibre $\mathbb{C}^3$ carrying the representation $\sigma = \sigma_L \times \sigma_Y$ of $\mathrm{SU}(2) \times \mathrm{U}(1)$, associated with the (trivial) principal bundle $P(M, \mathrm{SU}(2) \times \mathrm{U}(1))$.

In the next step, we introduce the gauge potential mediating the electroweak interaction. In the geometric terminology, it is described by a connection form on P. Since P is trivial, we can work with a global representative on M. We denote the $\mathfrak{su}(2)$-component of the gauge potential by $\mathbb{W}$ and the $\mathfrak{u}(1)$-component by $\mathbb{B}$, respectively. Since in the analysis below, the coupling constants are relevant, we must use the physical representation, cf. Remark 6.1.1. We denote the coupling constant with respect to the $\mathrm{SU}(2)$-symmetry and the $\mathrm{U}(1)$-symmetry by g and g', respectively, and write $g\mathbb{W}$ and $g'\mathbb{B}$, respectively. By the principle of minimal coupling introduced in Sect. 7.1, the interaction of gauge fields and fermionic matter fields is given via the covariant derivative. According to (7.1.4), we have[40]

$$D\psi_e = \big(\mathrm{d} + g\sigma_L'(\mathbb{W}) + g'\sigma_Y'(\mathbb{B})\big)\psi_e \,. \tag{7.7.6}$$

Now, we are prepared to write down the gauge-invariant Lagrangian describing the $\big(\mathrm{SU}(2) \times \mathrm{U}(1)\big)$-gauge theory of the leptonic family under consideration. According to (6.2.1) and (7.1.9), it reads

$$\mathscr{L}_e = \frac{1}{2}\mathbb{F}_W \dot{\wedge} *\mathbb{F}_W + \frac{1}{2}\mathbb{F}_B \wedge *\mathbb{F}_B + \langle \psi_e, \not{D}\psi_e\rangle \,, \tag{7.7.7}$$

where $\mathbb{F}_W$ and $\mathbb{F}_B$ are the field strength tensors of $\mathbb{W}$ and $\mathbb{B}$, respectively, and $\not{D}$ is the Dirac operator built from (7.7.6), cf. formula (5.5.27). For convenience, in some places below, instead of writing the Lagrangian as a 4-form we will write it as a function on M without further commenting on that.

Remark 7.7.1 In standard coordinates $\{x^\mu\}$ on M and in the Lie algebra bases $\{t_a\}$ of $\mathfrak{su}(2)$ and $\{i\}$ of $\mathfrak{u}(1)$ introduced above, we decompose

$$\mathbb{W} = W^a_\mu\, t_a \otimes \mathrm{d}x^\mu \,, \quad \mathbb{B} = iB_\mu\, \mathrm{d}x^\mu \,. \tag{7.7.8}$$

By (7.7.4) and (7.7.6), we have

$$D_\mu\psi_e = \big(\partial_\mu + igW^a_\mu T_a + ig'B_\mu Y\big)\psi_e \tag{7.7.9}$$

[40] Note that the spin connection $\mathscr{A}_Q$ is trivial here. For simplicity, we omit the factor id_{F_s}.

and the Lagrangian reads

$$\mathscr{L}_e = -\frac{1}{8}\,\mathrm{tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + i\bar{\psi}_e\gamma^\mu D_\mu\psi_e\,, \tag{7.7.10}$$

where

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g[W_\mu, W_\nu]\,, \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

are the representatives of $\mathbb{F}_W$ and $\mathbb{F}_B$, respectively. ♦

We stress that, up until now, all fermions are massless. Naive mass terms of the form $m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ would violate gauge invariance. We will see below that the fermions are endowed with their masses via the Higgs mechanism. This will be our next issue. We add a bosonic scalar field

$$\varphi : M \to \mathbb{C}^2\,, \quad \varphi(\mathbf{x}) := \begin{bmatrix}\varphi^1\\ \varphi^2\end{bmatrix}(\mathbf{x})\,, \tag{7.7.11}$$

carrying the following representation of $\mathrm{SU}(2)\times\mathrm{U}(1)$:

$$\rho_L : \mathrm{SU}(2)\times\mathbb{C}^2 \to \mathbb{C}^2\,, \quad \rho_L(a)\begin{bmatrix}z_1\\ z_2\end{bmatrix} = a\cdot\begin{bmatrix}z_1\\ z_2\end{bmatrix}\,,$$

$$\rho_Y : \mathrm{U}(1)\times\mathbb{C}^2 \to \mathbb{C}^2\,, \quad \rho_Y(\exp(i\alpha))\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \exp(iy_H\alpha)\begin{bmatrix}z_1\\ z_2\end{bmatrix}\,,$$

with $y_H = \frac{1}{2}$.[41] The generators of these representations are given by

$$\rho_L'(t_a) = t_a\,, \quad \rho_Y'(i) = iy_H\mathbb{1}\,. \tag{7.7.12}$$

In the terminology of Sect. 7.1, φ is the global representative of a section of type $(0,\rho)$ of the bundle $E = E_s\otimes E_i$ associated with $Q\times_M P$, where

(a) E_s is the tensor bundle $T^0_0(M)$ associated with the orthonormal frame bundle $Q = O(M)$ carrying the trivial representation of the Lorentz group, that is, φ is a scalar field.
(b) E_i is the complex vector bundle with typical fibre $\mathbb{C}^2$ carrying the representation $\rho = \rho_L\times\rho_Y$ of $\mathrm{SU}(2)\times\mathrm{U}(1)$, associated with the (trivial) principal bundle $P(M, \mathrm{SU}(2)\times\mathrm{U}(1))$.

According to (7.7.12), the covariant derivative of φ reads

$$D_\mu\varphi = \left(\partial_\mu + igW^a_\mu\frac{\tau_a}{2} + i\frac{g'}{2}B_\mu\mathbb{1}\right)\varphi\,.$$

[41] This choice of the eigenvalue y_H is implemented by the postulate of hypercharge conservation in elementary processes, like $e_L \to e_R + \varphi^2$, see [468].

Next, we choose a typical Higgs Lagrangian, see (7.2.1) and (7.2.2),

$$\mathscr{L}_H = \frac{1}{2} D\varphi \stackrel{\cdot}{\wedge} *D\varphi - \lambda \left(\|\varphi\|^2 - \frac{v^2}{2} \right)^2 \mathsf{v}_M \,, \tag{7.7.13}$$

supplemented by a so called Yukawa coupling term, describing the interaction of the leptons with the scalar field,

$$\mathscr{L}_{Yuk} = -c_e \left((\bar{L}_e \varphi) e_R + \bar{e}_R (\varphi^\dagger L_e) \right) \mathsf{v}_M \,. \tag{7.7.14}$$

Here, c_e is a dimensionless coupling constant which can be chosen to be a real non-negative number.

To summarize, the full Lagrangian describing the electroweak interaction of the first lepton family is then given by

$$\mathscr{L} = \mathscr{L}_e + \mathscr{L}_H + \mathscr{L}_{Yuk} \,. \tag{7.7.15}$$

Let us discuss the Higgs mechanism for this model. For that purpose, we observe that $F_{\min}$ coincides with the 2-sphere with radius $\frac{v}{\sqrt{2}}$. We choose[42]

$$\varphi_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v \end{bmatrix} . \tag{7.7.16}$$

Clearly, the stabilizer H of φ_0 under the $\big(\mathrm{SU}(2) \times \mathrm{U}(1)\big)$-action consists of transformations of the form

$$\mathbf{x} \mapsto \exp\left(i\alpha(\mathbf{x}) \left(\frac{\tau_3}{2} + y_H \mathbb{1} \right) \right) ,$$

that is, H is isomorphic to $\mathrm{U}(1)$ and its generator is

$$t_+ := \frac{i}{2}(\tau_3 + \mathbb{1}) = i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

Comparing with (7.7.5), this means that H is generated by Q_e in the representation ρ, that is, H is the electromagnetic subgroup $\mathrm{U}(1)_{\mathrm{em}}$ of $\mathrm{SU}(2) \times \mathrm{U}(1)$. Now, we can apply the general theory of Sect. 7.3. By Proposition 7.3.4, the particle content after symmetry breaking is given by a triple $\big((\hat{\omega}, \tau), \eta\big)$, where $\hat{\omega}$ is the connection form of the residual gauge symmetry H, τ describes the intermediate vector boson and η is the surviving Higgs field. As usual, we denote the Lie algebra of H by $\mathfrak{h}$ and take the orthogonal decomposition

$$\mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{h} \oplus \mathfrak{m} \,.$$

[42] Comparing with the general theory, instead of η_v we simply write v here.

Clearly, $\mathfrak{m}$ is spanned by t_1, t_2 and $t_- := \frac{i}{2}(\tau_3 - \mathbb{1}) = -i\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$. Correspondingly, we decompose

$$igW_\mu^3\frac{\tau_3}{2} + i\frac{g'}{2}B_\mu\mathbb{1} = \frac{1}{2}(gW_\mu^3 + g'B_\mu)t_+ + \frac{1}{2}(gW_\mu^3 - g'B_\mu)t_- . \tag{7.7.17}$$

Then, in the representation ρ', the representative of $\hat{\omega}$ is

$$\mathbb{A} = A_\mu\, t_+ \otimes \mathrm{d}x^\mu\,, \quad A_\mu = \frac{1}{2}(gW_\mu^3 + g'B_\mu)\,, \tag{7.7.18}$$

and that of τ is

$$\mathbb{V} = V_\mu \mathrm{d}x^\mu\,, \quad V_\mu = g\sum_{a=1}^{2} W_\mu^a\, t_a + \frac{1}{2}(gW_\mu^3 - g'B_\mu)t_- . \tag{7.7.19}$$

Remark 7.7.2 The following statements are left to the reader (Exercise 7.7.2). Under a residual local gauge transformation

$$\mathbf{x} \mapsto \begin{bmatrix}\exp(i\alpha(\mathbf{x})) & 0\\ 0 & 1\end{bmatrix},$$

the following transformation laws hold:

$$A_\mu \mapsto A_\mu + \partial_\mu\alpha\,, \quad W_\mu^\pm \mapsto \exp(\pm i\alpha)W_\mu^\pm\,,$$

where

$$W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)\,. \tag{7.7.20}$$

The component $\frac{1}{2}(gW_\mu^3 - g'B_\mu)t_-$ is gauge invariant. Thus, the components $W_\mu^\pm$ constitute a complex (charged) vector field in the fundamental representation of $\mathrm{U}(1)$ and the t_--component is an $\mathbb{R}$-valued (neutral) vector field. ♦

It is now convenient to introduce the Weinberg angle θ_W describing the above mixing via g and g',

$$\tan\theta_W := \frac{g'}{g}\,. \tag{7.7.21}$$

Then, the t_--component in (7.7.19) can be rewritten as $\frac{\sqrt{g'^2+g^2}}{2}Z_\mu$, where

$$Z_\mu := \cos(\theta_W)W_\mu^3 - \sin(\theta_W)B_\mu\,. \tag{7.7.22}$$

In this notation, the mass term (7.3.5) for the intermediate vector boson reads as follows:

$$\frac{g^2 v^2}{4} W^-_\mu W^{+\mu} + \frac{(g^2 + g'^2)v^2}{8} Z_\mu Z^\mu ,$$

that is, the masses of the bosons $W^\pm$ and Z are

$$m_W = \frac{gv}{2}, \quad m_Z = \frac{v\sqrt{g^2 + g'^2}}{2} . \tag{7.7.23}$$

We also see that, via the Yukawa coupling term in (7.7.13), the electron field receives a mass, whereas the neutrino remains massless. Indeed, inserting (7.7.16) into this term, it reduces to

$$-c_e \left\{ (\bar{L}_e \varphi) e_R + \bar{e}_R (\varphi^\dagger L_e) \right\} = -\frac{c_e v}{\sqrt{2}} (\bar{e}_R e_L + \bar{e}_L e_R) ,$$

that is,

$$m_e = \frac{c_e v}{\sqrt{2}} .$$

Finally, for the surviving Higgs field we get the mass

$$m_\eta = 2\lambda v^2 .$$

Now, it remains to identify the electromagnetic gauge potential $\mathbb{A}^{\mathrm{em}}$. It turns out that $\mathbb{A}^{\mathrm{em}}$ does not merely coincide with the full t_+-component $\mathbb{A}$ given by (7.7.17). We will find the correct electromagnetic potential by postulating that after symmetry breaking the minimal coupling term $\frac{1}{2}\langle \psi_e, \not{D}\psi_e \rangle$ in (7.7.7) must produce the correct coupling term $eA^{\mathrm{em}}_\mu j^\mu_{\mathrm{em}}$ with the electromagnetic current[43]

$$j^\mu_{\mathrm{em}} = -(\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R) . \tag{7.7.24}$$

By (7.7.9) and (7.7.10), we get the following interaction term

$$\begin{aligned}
\mathscr{L}^I_e &= -\bar{\psi}_e \gamma^\mu \left(g W^a_\mu T_a + g' B_\mu Y \right) \psi_e \\
&= -\frac{g}{\sqrt{2}} \left(W^+_\mu \bar{\nu}_{eL} \gamma^\mu e_L + W^-_\mu \bar{e}_L \gamma^\mu \nu_{eL} \right) - \frac{\sqrt{g^2 + g'^2}}{2} Z_\mu \bar{\nu}_{eL} \gamma^\mu \nu_{eL} \\
&\quad + A_\mu \bar{e}_L \gamma^\mu e_L + g' B_\mu \bar{e}_R \gamma^\mu e_R .
\end{aligned}$$

We see that the t_+-component $\mathbb{A}$ does not fulfil our postulate, indeed. Now, the following decomposition formulae can be easily checked (Exercise 7.7.1):

[43] We use the sign convention $e > 0$.

$$g'B_\mu = -\frac{g'^2}{\sqrt{g^2+g'^2}} Z_\mu + \frac{g'g}{\sqrt{g^2+g'^2}} A^{\mathrm{em}}_\mu \,, \tag{7.7.25}$$

$$A_\mu = -\frac{g^2-g'^2}{2\sqrt{g^2+g'^2}} Z_\mu + \frac{g'g}{\sqrt{g^2+g'^2}} A^{\mathrm{em}}_\mu \,, \tag{7.7.26}$$

where

$$A^{\mathrm{em}}_\mu := \sin(\theta_W) W^3_\mu + \cos(\theta_W) B_\mu \,. \tag{7.7.27}$$

We denote

$$e := \frac{g'g}{\sqrt{g^2+g'^2}} \,, \tag{7.7.28}$$

define the fermionic currents

$$j^+_\mu := \bar{e}_L \gamma_\mu \nu_{eL} \,, \quad j^-_\mu := \bar{\nu}_{eL} \gamma_\mu e_L \,, \quad j^3_\mu := \frac{1}{2}(\nu_{eL}\gamma^\mu \nu_{eL} - \bar{e}_L \gamma^\mu e_L) \,, \tag{7.7.29}$$

and insert the decompositions (7.7.25) and (7.7.26) into $\mathscr{L}^I_e$. This yields

$$\begin{aligned} \mathscr{L}^I_e &= -eA^{\mathrm{em}}_\mu j^\mu_{\mathrm{em}} - \frac{g}{\sqrt{2}}\big(W^+_\mu j^{-\mu} + W^-_\mu j^{+\mu}\big) \\ &\quad - \sqrt{g^2+g'^2} Z_\mu (j^{3\mu} - \sin^2(\theta_W) j^\mu_{em}) \,. \end{aligned} \tag{7.7.30}$$

From this we see that A^{em}_μ may be interpreted as the electromagnetic potential and e as the electromagnetic coupling constant.

Remark 7.7.3

1. Recall that Z_μ is invariant under local gauge transformations. Thus, it may be viewed as the representative of a horizontal 1-form on the reduced principal H-bundle. Thus, (7.7.26) may be interpreted as a relation between two representatives of connection forms differing by a horizontal 1-form, that is, A^{em}_μ is the representative of a $\mathrm{U}(1)$-connection form on the reduced bundle, indeed.
2. Up until now, the model contains 5 free parameters. They may be chosen as e, $\sin(\theta_W)$, m_e, m_W and m_η. Then,

$$m_Z = \frac{m_W}{\cos(\theta_W)} \,, \quad v = 2m_W \frac{\sin(\theta_W)}{e} \,.$$

   Clearly, the full Lagrangian $\mathscr{L}_e + \mathscr{L}_H$ may be rewritten in terms of the physical fields ψ_e, $W^\pm_\mu$, Z_μ, A^{em}_μ, η and the chosen free parameters above. We omit this lengthy expression here. The model obtained so far is called the Weinberg–Salam model of electroweak interactions. ♦

Table 7.2 Masses and and charges of the gauge bosons and the Higgs boson

	Particle and mass in MeV		Charge
Gauge bosons	γ	$<3 \cdot 10^{-33}$	0
	$\mathrm{W}^{\pm}$	80385 ± 15	± 1
	Z	91187.6 ± 2.1	0
Higgs boson	η	125500 ± 600	0

In Table 7.2, we list the measured values of the masses and the charges of the gauge bosons[44] and of the Higgs boson.[45]
The experimental value of the Weinberg angle was found to be, see [16] for details,

$$\sin^2(\theta_W) = 0.23153 \pm 0.00016 . \tag{7.7.31}$$

Finally, we include the quark family (u, d). Again, we decompose u and d into their left handed and right handed components and build an $\mathrm{SU}(2)$-doublet and two $\mathrm{SU}(2)$-singlets,

$$L_q := \begin{bmatrix} u_L \\ d_L \end{bmatrix} , \quad u_R , \quad d_R . \tag{7.7.32}$$

Now, applying again (7.7.5) and using the fractional electric charges of the quarks provided by the quark model, see Table 7.1, we obtain for Y the eigenvalues $y = \frac{1}{6}$ for L_q, $y = \frac{2}{3}$ for u_R and $y = -\frac{1}{3}$ for d_R. Next, we have to take into account that the quark fields interact also strongly. In the standard model, the corresponding gauge group, called colour group, is chosen to be $\mathrm{SU}(3)$. With respect to this gauge symmetry, the quarks are assumed to build triplets whereas the leptons and the Higgs field are assumed to be singlets. Thus, we introduce the quark matter field

$$\psi_q : M \to \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3 , \quad \psi_q(\mathbf{x}) := \begin{bmatrix} L_q \\ u_R \\ d_R \end{bmatrix} (\mathbf{x}) . \tag{7.7.33}$$

[44] The values are taken from [16].

[45] The existence of the Higgs boson was announced on July 4th 2012 by the ATLAS and CMS Collaborations at CERN and confirmed by later experiments, see [16] for details. The mass value in the table is the one found by ATLAS. The CMS Collaboration found 125.7 ± 0.4 GeV. This discovery confirmed the Higgs sector as a fundamental building block of the standard model experimentally. One year later, the Nobel Prize was awarded jointly to François Englert and Peter W. Higgs.

Here, the first $\mathbb{C}^4$-factor represents the bispinor space carrying the action of the spin group $\mathrm{SL}(2, \mathbb{C})$ of M. The second $\mathbb{C}^4$-factor carries the action of $\mathrm{SU}(2) \times \mathrm{U}(1)$ given by

$$\lambda_L : \mathrm{SU}(2) \times \mathbb{C}^4 \to \mathbb{C}^4 , \quad \lambda_L(a) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} a \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ z_3 \\ z_4 \end{bmatrix} ,$$

$$\lambda_Y : \mathrm{U}(1) \times \mathbb{C}^4 \to \mathbb{C}^4 , \quad \lambda_Y(\exp(i\alpha)) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \exp(iy_L\alpha) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \exp(iy_u\alpha) z_3 \\ \exp(iy_d\alpha) z_3 \end{bmatrix} ,$$

with $y_L = \frac{1}{6}$, $y_u = \frac{2}{3}$ and $y_d = -\frac{1}{3}$. The $\mathbb{C}^3$-factor carries the fundamental representation of $\mathrm{SU}(3)$,

$$\lambda_s : \mathrm{SU}(3) \times \mathbb{C}^3 \to \mathbb{C}^3 , \quad \lambda_s(a) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = a \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} .$$

From these formulae, the reader can easily read off the structure of the associated bundle E of quark matter fields. We postulate that the strong interaction also be mediated by a gauge field. Accordingly, we pass to the (trivial) principal bundle P over M with structure group (the full gauge group of the standard model)

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

and we introduce an additional $\mathrm{SU}(3)$-gauge potential $\mathbb{G}$ mediating the strong interaction. We denote the field strength tensor of $\mathbb{G}$ by $\mathbb{F}_G$. Again, by the principle of minimal coupling introduced in Sect. 7.1, the interaction of gauge fields and quark fields is given via the covariant derivative,

$$D\psi_q = \big(\mathrm{d} + g_s\lambda_s'(\mathbb{G}) + g\lambda_L'(\mathbb{W}) + g'\lambda_Y'(\mathbb{B})\big)\psi_q , \tag{7.7.34}$$

where g_s denotes the strong coupling constant. Now, we can write down the full Lagrangian of the standard model before spontaneous symmetry breaking:

$$\mathscr{L} = \mathscr{L}_g + \mathscr{L}_f + \mathscr{L}_H + \mathscr{L}_{Yuk} , \tag{7.7.35}$$

where

$$\mathscr{L}_g = \frac{1}{2}\mathbb{F}_G \,\dot{\wedge}\, *\mathbb{F}_G + \frac{1}{2}\mathbb{F}_W \,\dot{\wedge}\, *\mathbb{F}_W + \frac{1}{2}\mathbb{F}_B \wedge *\mathbb{F}_B$$

and

$$\mathscr{L}_f = \langle \psi_e, \not{D}\psi_e \rangle + \langle \psi_q, \not{D}\psi_q \rangle .$$

Since the matter field φ is in the trivial representation of $\mathrm{SU}(3)$, the colour symmetry remains unbroken and the Higgs part $\mathscr{L}_H$ is the same as in (7.7.15), that is, the Higgs mechanism described before remains exactly the same. Clearly, the Yukawa coupling term given by (7.7.14) must be modified by adding the corresponding interaction terms of φ with the quark fields,

$$\begin{aligned}\mathscr{L}_{Yuk} = &-c_e\left((\bar{L}_e\varphi)e_R + \bar{e}_R(\varphi^\dagger L_e)\right) - c_u\left((\bar{L}_e\tilde{\varphi})u_R + \bar{u}_R(\tilde{\varphi}^\dagger L_q)\right)\\ &- c_d\left((\bar{L}_e\varphi)d_R + \bar{d}_R(\varphi^\dagger L_q)\right)\,,\end{aligned} \tag{7.7.36}$$

where $\tilde{\varphi} = i\tau_2\varphi^*$.

Remark 7.7.4

1. If one wishes to include the other two fermionic families, then formula (7.7.36) must be modified essentially. Instead of the constants c_e , c_u and c_d, one must allow for complex matrices, called Kobayashi–Maskawa matrices, mixing leptons and quarks of the same charge.[46] Diagonalizing these matrices and passing to fields with a well-defined mass leads to a mixing of the original fields. This change implies that in the charged currents built from the quark fields, mixing matrices show up. The neutral currents are not affected by this change. For a discussion of phenomenological consequences of these facts we refer to [468].
2. It turns out that, on quantum level, the standard model is renormalizable, that is, the renormalized perturbation theory may be applied, see e.g. [656]. The theoretical predictions obtained from this quantum field theory have been very well confirmed by various types of experiments, see [16] for details.
   We stress that the high energy and the low energy properties of the model are quite different. For high energies $E \gg m_Z \sim 100$ GeV, the boson mass corrections of order $\frac{m_z}{E}$ may be neglected. In such an approximation, the full $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$-symmetry is manifest. In contrast, for small energies $E \ll m_Z$, one only sees the broken symmetry $\mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}}$. Schematically, this is often represented as follows:
   $$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \overset{100\,\mathrm{GeV}}{\longrightarrow} \mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}}\,.$$
   In particular, for high energies, one may neglect the electroweak interactions of quarks and one may consider a theory based upon the Lagrangian[47]
   $$\mathscr{L} = \frac{1}{2}\mathbb{F}_G \mathbin{\dot{\wedge}} *\mathbb{F}_G + \langle\psi_q, (\not{D} - m)\psi_q\rangle\,.$$
   This is the Lagrangian of Quantum Chromodynamics (QCD). For large momentum transfers (deep inelastic scattering), the renormalized perturbation theory still

[46]This is due to the postulate of fermion number conservation. Clearly, the matrices must be such that the Yukawa coupling term remains gauge invariant. We refer to Sect. 22.4 of [468] for details.

[47]With m denoting the matrix of quark masses.

may be applied. However, in the low energy sector, perturbative methods do not work appropriately. In particular, it cannot be explained this way why quarks and gluons are not observed. This is the famous quark confinement problem. ♦

To summarize, the full standard model contains the following set of free parameters:

(a) The coupling constants g_s, e, $\sin(\theta_W)$,
(b) the boson masses m_W, m_η,
(c) the lepton masses m_e, m_μ, m_τ,
(d) the quark masses $m_u, m_d, m_c, m_s, m_t, m_b$,
(e) the parameters of the Kobayashi–Maskawa matrix ϑ_1, ϑ_2, ϑ_3, δ.

For a fundamental theory, this number of independent parameters seems to be rather high. Moreover, on the way we have pointed out a number of ad hoc assumptions (taken from the experiment) which could not be explained theoretically. We should add that the standard model predicts massless neutrinos, whereas several experiments require small but non-vanishing neutrino masses. Moreover, the model does not really explain the quantization of electric charge. Thus, the reader may ask himself whether the standard model may be viewed as a truly unified theory.

Consequently, a lot of effort has been put into building further unification schemes. One of the most prominent variants, the so-called grand unification (GUT) was proposed already in 1974 by Georgi and Glashow [240]. The basic idea of grand unification is that, beyond a very high energy scale, elementary particle physics is described by a gauge theory with a simple gauge group G_U, that is, by a theory with a single coupling constant. The Lie group G_U should be large enough so that $G_{SM} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ can be embedded into G_U. At some energy M_U, the symmetry G_U is spontaneously broken to G_{SM}, thus, leading to the standard model. This idea works, indeed [241]: by a renormalization group analysis within the standard model, one shows that the values of the SU(2)- and SU(3)-coupling constants decrease at larger momentum scales, whereas the value of the U(1)-coupling constant increases. The coupling constants approach each other at the energy scale $M_U = 10^{16}$ GeV. This is called the grand unification scale. According to the idea of grand unification, we may now replace the reduction scheme outlined under point 2 of Remark 7.7.4 by

$$G_U \xrightarrow{M_U} G_{SM} \xrightarrow{100\,\mathrm{GeV}} \mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}} \,. \tag{7.7.37}$$

The search for an appropriate simple group G_U was guided by a number of natural requirements: first, as already mentioned, it should be possible to embed G_{SM} into G_U. Thus, G_U must be at least of rank 4 and it should contain SU(3) as a subgroup. Second, it must admit representations allowing for the correct particle spectrum and it should be anomaly free.[48] If one insists to accommodate the fermions in complex representations, then only the simple Lie groups $\mathrm{SU}(n)$, with $n \geq 2$, $\mathrm{SO}(4n+2)$ and the exceptional group E_6 remain as good candidates, see [438]. The requirement that the theory be anomaly free excludes SO(6) and puts limitations on the allowed

[48] See Remark 9.3.8.

representations for the unitary group. For a quite exhaustive study of the underlying group theory as well as of the admissible representation schemes we refer to [238, 597]. In the historical paper of Georgi and Glashow [240], the unifying gauge group SU(5) was proposed. One year later, $G_U = \mathrm{SO}(10)$ was introduced [222, 239].

In the sector of such a unified theory where G_U is unbroken, completely new phenomena occur. Since in this sector the gauge bosons involve both flavor and color, the baryon number is not conserved any more and thus, in most models, proton decay is possible.[49] Another remarkable feature of all realistic grand unifications is the fact that they admit (superheavy) magnetic monopoles.

For an exhaustive review over the first period of the development of GUT's we refer to [400]. For more modern aspects, including supersymmetric GUT's, see [474] and references therein.

In the next section, we are going to present another unification approach which has attracted much attention over the decades.

Exercises

7.7.1 Check the formulae (7.7.25) and (7.7.26).

7.7.2 Prove the statements of Remark 7.7.2.

7.8 Dimensional Reduction. Basics

The idea of dimensional reduction can be traced back to the classical Kaluza-Klein theory invented by Kaluza and Klein [355, 377], Einstein, Bergmann [181, 182] and Weyl [660, 662]. Its application to non-Abelian gauge theories starts with the work of Kerner [363], Forgacs and Manton [207] and Harnad, Shnider and Tafel [284]. Here, we concentrate on dimensional reduction of pure Yang–Mills theories. This variant is often referred to as the CSDR scheme.[50] Our presentation will be along the lines of [394, 546], but we also refer to the review [356]. We will give further references in the text and will comment on other variants of dimensional reduction at the end. Our emphasis is on the method rather than on applications, as dimensional reduction is an important tool for the study of differential equations with symmetries in many branches of physics.

Let us consider a pure Yang–Mills theory on a (pseudo-)Riemannian manifold (M, g) of signature $(-, +, \dots, +)$. In the literature, M is given different names. Often it is called a multidimensional universe, sometimes also a Kaluza-Klein space. We will rather stick to the first term. In short, the idea of dimensional reduction goes as follows: assume we are given a symmetry group K acting on M in such a way that

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[49] But the lifetime of the proton is estimated to be beyond 10^{30} years.

[50] CSDR standing for Coset Space Dimensional Reduction.

the quotient M/K (or some piece of it) may be identified with physical spacetime. Further assume that this symmetry may be lifted to the principal bundle of the gauge theory under consideration. Then, one postulates K-invariance of the gauge field configurations and of the action functional and reduces the latter with respect to this symmetry. This way, interesting unification models may be constructed. In this section, we use the notation and the results of Sect. 1.9.

In more detail, let G be the gauge group and let (P, G, M, Ψ, π) be the gauge principal bundle. We consider a simple group action[51] $\delta : K \times M \to M$ of K on M and assume that it can be lifted to an action $\Delta : K \to \mathrm{Aut}(P)$ such that the induced left action $\rho : (K \times G) \times P \to P$ given by (1.9.1) is also simple. As in Sect. 1.9, we denote the orbit space $P/(K \times G)$ of this action by $\hat{M}$. We limit our presentation to the version described by Remark 1.9.9 and Corollary 1.9.15, that is, given a stabilizer H of δ, we assume that the principal Γ_I/Z-bundle $M_I \to \hat{M}$ is trivial. Note that then also the principal Γ_H-bundle $M_H \to \hat{M}$ is trivial. Here, $\Gamma_H = N_K(H)/H$. Thus, we may choose a global section $s : \hat{M} \to M_H$. As before, we denote $\tilde{M} := s(\hat{M})$. Recall that, in this situation, bundles admitting a lift of the K-action have the following structure, cf. Eq. (1.9.24):

$$P \cong K \times_H \left(G \times_{C_G(\lambda_0(H))} \tilde{P}\right) . \tag{7.8.1}$$

Here, $\lambda_0 : H \to G$ is the Lie group homomorphism given by Eq. (1.9.4), $\tilde{P} \subset P$ is a principal $C_G(\lambda_0(H))$-bundle over $\tilde{M}$ and the right H-action on $K \times \left(G \times_{C_G(\lambda_0(H))} \tilde{P}\right)$ is given by

$$(h, (k, [(g, \tilde{p})])) \mapsto (kh, [(g\lambda_0(h), \tilde{p})]) , \quad h \in H .$$

The diffeomorphism (7.8.1) is given by

$$[(k, [(g, \tilde{p})])] \mapsto \Delta_k \circ \Psi_{g^{-1}}(\tilde{p}) .$$

Now, the setting is given and we may start with the dimensional reduction procedure. In the first step, we must classify the K-invariant configurations (ω, g) entering the action functional. For the gauge field configurations ω, this problem has already been solved: by Corollary 1.9.15, K-invariant connection forms ω on P are in one-to-one correspondence with pairs $(\tilde{\omega}, \tilde{\Phi})$, where

(a) $\tilde{\omega}$ is a connection form on $\tilde{P}$,
(b) $\tilde{\Phi} : \tilde{P} \to L(\mathfrak{m}, \mathfrak{g})^H$ is a $C_G(\lambda_0(H))$-equivariant mapping.

Here, $\mathfrak{m} \subset \mathfrak{k}$ is given by

$$\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p} , \tag{7.8.2}$$

where $\mathfrak{n} = \hat{\mathfrak{n}}_H$ is the Lie algebra of Γ_H. With this notation, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. As usual, let us denote the Ad-invariant scalar products on $\mathfrak{k}$ and $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, respectively.

---

[51] Recall that an action is called simple if it has only one orbit type.

By the discussion in Sect. 1.9, without loss of generality we may assume that the decomposition (1.9.25) is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$. Then, the decomposition (7.8.2) is orthogonal, too. Consequently, from now on we denote

$$\mathfrak{m} \equiv \mathfrak{h}^{\perp}, \quad \mathfrak{p} \equiv \mathfrak{n}^{\perp}. \tag{7.8.3}$$

Next, let us classify the K-invariant metrics on M. For any $y \in \tilde{M}$, we decompose

$$\mathrm{T}_y M = \mathrm{T}_y \tilde{M} \oplus \mathrm{T}_y (K \cdot y), \tag{7.8.4}$$

where $K \cdot y$ is the K-orbit through y. We denote

$$\mathfrak{N}_y := \delta'_y(\mathfrak{n}), \quad \mathfrak{N}_y^{\perp} := \delta'_y(\mathfrak{n}^{\perp}). \tag{7.8.5}$$

Then, $\mathrm{T}_y (K \cdot y) = \mathfrak{N}_y \oplus \mathfrak{N}_y^{\perp}$ and thus

$$\mathrm{T}_y M = \mathrm{T}_y \tilde{M} \oplus \mathfrak{N}_y \oplus \mathfrak{N}_y^{\perp}. \tag{7.8.6}$$

Recall the isotropy representation[52] $\delta'_h : H \to \mathrm{Aut}(\mathrm{T}_y M)$ induced from δ. Since $\mathrm{T}_y (K \cdot y) = \delta'_y(\mathfrak{h}^{\perp})$ and

$$\delta'_h \circ \delta'_y(A) = \delta'_y(\mathrm{Ad}(h)A), \tag{7.8.7}$$

for any $A \in \mathfrak{h}^{\perp}$, the restriction of the isotropy representation to $\mathrm{T}_y (K \cdot y)$ is equivalent to the restriction of the adjoint representation to $\mathfrak{h}^{\perp}$.

In the sequel, in order to exclude pathological situations, we further assume that, for any $y \in \tilde{M}$, none of the components in the decomposition (7.8.6) is tangent to the light cone $\{Y \in \mathrm{T}_y M : \mathsf{g}(Y, Y) = 0\}$.

Lemma 7.8.1 *The subspace $\mathfrak{N}_y^{\perp}$ is g-orthogonal to $\mathrm{T}_y \tilde{M}$ and to $\mathfrak{N}_y$.*

Proof Since g is K-invariant, the isotropy representation $\delta'_h : H \to \mathrm{Aut}(\mathrm{T}_y M)$ is orthogonal. It clearly acts trivially on $\mathrm{T}_y \tilde{M}$. By Remark I/6.2.10, $\mathrm{T}_y (K \cdot y)$ is invariant under the isotropy representation. Thus, the decomposition (7.8.4) is invariant, too.

By reductivity of the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$, the action of H induces an action on $\mathfrak{h}^{\perp}$. Clearly, the maximal subspace of $\mathfrak{h}^{\perp}$ on which $\mathrm{Ad}(H)$ acts trivially is $\mathfrak{n}$. Thus, in the decomposition of $\mathfrak{n}$ and $\mathfrak{n}^{\perp}$ into $\mathrm{Ad}(H)$-irreducible components,

$$\mathfrak{n} = \bigoplus_{i=i}^{m} \mathfrak{n}_i, \quad \mathfrak{n}^{\perp} = \bigoplus_{j=i}^{n} \mathfrak{n}_j^{\perp},$$

the $\mathfrak{n}_i$ are (mutually orthogonal) one-dimensional subspaces. Inserting these decompositions into (7.8.5), we obtain

[52] Cf. point 1 of Remark I/6.2.10.

$$\mathfrak{N}_y = \bigoplus_{i=1}^{m} \delta'_y(\mathfrak{n}_i)\,, \quad \mathfrak{N}_y^\perp = \bigoplus_{j=1}^{n} \delta'_y(\mathfrak{n}_j^\perp)\,.$$

Consider the corresponding components of the metric viewed as mappings

$$\mathsf{g}_{(i,j)} : \ \delta'_y(\mathfrak{n}_i) \to (\delta'_y(\mathfrak{n}_j^\perp))^*\,.$$

Then,

$$\mathsf{g}_{(j,j)}^{-1} \circ \mathsf{g}_{(i,j)} : \delta'_y(\mathfrak{n}_i) \to \delta'_y(\mathfrak{n}_j^\perp)$$

is an operator intertwining the irreducible representations of $\delta'_y(\mathfrak{n}_i)$ and $\delta'_y(\mathfrak{n}_j^\perp)$. This follows from the K-invariance of g. Now, since the representations carried by the $\mathfrak{n}_i$ are trivial and those carried by the $\mathfrak{n}_j^\perp$ are nontrivial, Schur's Lemma implies that this operator must vanish for all pairs (i, j). Consequently, $\mathfrak{N}_y$ and $\mathfrak{N}_y^\perp$ are orthogonal to each other. In the same way, one shows that $\mathrm{T}_y\tilde{M}$ is orthogonal to $\mathfrak{N}_y^\perp$. ■

From Lemma 7.8.1, we immediately get the following.

Corollary 7.8.2 *If Γ_H is discrete, then the decomposition* (7.8.4) *is orthogonal with respect to* g. ■

The following proposition yields the classification of K-invariant metrics.

Proposition 7.8.3 *Let δ and ρ be simple group actions and assume that the bundle $M_H \to \hat{M}$ is trivial. Then, the K-invariant metrics g on M are in one-to-one correspondence with 4-tuples*

$$(\tilde{\mathsf{g}}, \xi, \beta, \beta^\perp)\,, \tag{7.8.8}$$

where $\tilde{\mathsf{g}}$ is a metric on $\tilde{M}$, ξ is a connection form on the principal Γ_H-bundle $M_H \to \hat{M}$ and β and $\beta^\perp$ are functions on $\tilde{M}$ with values in the $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear forms on $\mathfrak{n}$ and $\mathfrak{n}^\perp$, respectively.

Proof By K-invariance, the metric g is completely characterized by its values on $\tilde{M}$. Let $y \in \tilde{M}$ and denote

$$\mathfrak{N}^1 := \mathrm{T}_y\tilde{M}\,, \quad \mathfrak{N}^2 := \mathfrak{N}_y\,, \quad \mathfrak{N}^3 := \mathfrak{N}_y^\perp\,.$$

Let $\mathsf{g}^{(k,l)} : \mathfrak{N}^l \to \left(\mathfrak{N}^k\right)^*$, $k, l = 1, 2, 3$, be the corresponding components of g. By Lemma 7.8.1, we have

$$\mathsf{g}^{(l,3)} = \mathsf{g}^{(3,l)} = 0\,,$$

for $l = 1, 2$. The component $\mathsf{g}^{(1,1)}$ yields $\tilde{\mathsf{g}}$. To define the connection form ξ, consider the right K-action on M defined by $\tilde{\delta}_k := \delta_{k^{-1}}$. Its restriction to Γ_H yields the right principal action on M_H and $x \to \mathfrak{V}_x := \tilde{\delta}'_x(\mathfrak{n})$ is the canonical vertical distribution. As the horizontal distribution $x \to \mathfrak{H}_x$ defining ξ we take the orthogonal complement of $\mathfrak{V}$ with respect to g,

$$\mathrm{T}_x M_H = \mathfrak{V}_x \oplus \mathfrak{H}_x \,.$$

By the K-invariance of g, $\mathfrak{H}$ is Γ_H-invariant and thus a horizontal distribution on the principal bundle $M_H \to \hat{M}$. Finally, the functions β and $\beta^\perp$ are given by

$$\beta_y := \left(\delta'_y\right)^{\mathrm{T}}_{\restriction \mathfrak{n}} \circ \mathsf{g}^{(2,2)}_y \circ \left(\delta'_y\right)_{\restriction \mathfrak{n}} \,, \quad \beta^\perp_y := \left(\delta'_y\right)^{\mathrm{T}}_{\restriction \mathfrak{n}^\perp} \circ \mathsf{g}^{(3,3)}_y \circ \left(\delta'_y\right)_{\restriction \mathfrak{n}^\perp} \,,$$

for any $y \in \tilde{M}$. We check their $\mathrm{Ad}(H)$-invariance. Since $\mathrm{Ad}(H)$ acts trivially on $\mathfrak{n}$, for β the statement is obvious. Using (7.8.7) and the K-invariance of g, for $\beta^\perp_y$ we obtain

$$\begin{aligned}
\beta^\perp_y(\mathrm{Ad}(h)A) &= \left(\delta'_y\right)^{\mathrm{T}}_{\restriction \mathfrak{n}^\perp} \circ \mathsf{g}^{(3,3)}_y \circ \delta'_h \circ \delta'_y(A) \\
&= \left(\delta'_y\right)^{\mathrm{T}}_{\restriction \mathfrak{n}^\perp} \circ \left(\delta'_{h^{-1}}\right)^{\mathrm{T}} \circ \mathsf{g}^{(3,3)}_y \circ \delta'_y(A) \\
&= \mathrm{Ad}^*(h^{-1}) \circ \beta^\perp_y(A) \,.
\end{aligned}$$

The remaining properties are obvious. Finally, for the reconstruction of the K-invariant metric g from the 4-tuple (7.8.8), we need to calculate the connection form ξ. For that purpose, given $y \in \tilde{M}$, we decompose any vector $Y \in \mathrm{T}_y M_H$ with respect to (7.8.6),

$$Y = (\tilde{X}, \tilde{\delta}'_y(A), \tilde{\delta}'_y(B)) \,, \quad \tilde{X} \in \mathrm{T}_y\tilde{M} \,, \, A \in \mathfrak{n} \,, \, B \in \mathfrak{n}^\perp \,,$$

and, using the orthogonality condition, we read off the following vertical part:

$$\mathrm{ver}\,(Y) = \left(0, \tilde{\delta}'_y(A) + \left(\mathsf{g}^{(2,2)}_y\right)^{-1} \circ \mathsf{g}^{(2,1)}_y(\tilde{X}), 0\right) \,.$$

Then, by (1.3.6), we obtain

$$\xi_y(Y) = \left(\left(\tilde{\delta}'_y\right)_{\restriction \mathfrak{n}}\right)^{-1} \circ \left(\mathsf{g}^{(2,2)}_y\right)^{-1} \circ \mathsf{g}^{(2,1)}_y(\tilde{X}) + A \,. \tag{7.8.9}$$

Finally, the values of ξ along the fibre through y are found by transporting ξ_y with $\tilde{\delta}$. Now, it is clear that, given a tuple (7.8.8), g can be reconstructed uniquely. ■

Remark 7.8.4 Proposition 7.8.3 may be taken as a starting point for dimensional reduction of theories including gravity, see [394] for a list of classical references. In particular, we refer to [141] for more details. For an alternative approach based upon reduction theory of the bundle of orthonormal frames, we refer to [538]. In this paper, also the torsion case is included. ♦

Now, we can reduce the action functional

$$S(\omega) = \frac{1}{2} \int_M \Omega \,\dot{\wedge}\, {*\Omega} \,.$$

For clearness of presentation, we limit our attention to the following case. We assume that the metric is of the form

$$\mathsf{g} = \tilde{\mathsf{g}} \oplus \hat{\mathsf{g}}\,, \tag{7.8.10}$$

where $\tilde{\mathsf{g}}$ is obtained from a metric on $\tilde{M}$ by K-invariant extension and $\hat{\mathsf{g}}$ is defined by

$$\hat{\mathsf{g}}(A_*, B_*) := \mathsf{g}_{K/H}(A, B)\,, \quad A, B \in \mathfrak{n}\,.$$

Here, $\mathsf{g}_{K/H}$ is a K-invariant metric on K/H. Then, the decomposition (7.8.4) is orthogonal, that is, together with $\mathfrak{N}_y^{\perp}$, also $\mathfrak{N}_y$ is orthogonal to $\mathrm{T}_y\tilde{M}$ and, consequently, $\xi = 0$ in the classifying 4-tuple (7.8.8). Under this assumption, the canonical volume form on M reads

$$\mathsf{v}_{\mathsf{g}} = \mathsf{v}_{\tilde{\mathsf{g}}} \wedge \mathsf{v}_{\hat{\mathsf{g}}}\,.$$

Using this and (2.7.5), we have[53]

$$S(\omega) = \frac{1}{2}\int_M \mathsf{g}^{-1}(\Omega, \Omega)\, \mathsf{v}_{\tilde{\mathsf{g}}} \wedge \mathsf{v}_{\hat{\mathsf{g}}}\,.$$

Since both the connection and the metric are K-invariant, $\mathsf{g}^{-1}(\Omega, \Omega)$ is K-invariant and, thus, we may integrate over K/H. For that purpose, it is convenient to decompose the scalar product defined by $\beta \oplus \beta^{\perp}$ on each K-orbit with respect to a K-invariant scalar product[54] $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$,

$$\beta(y) = f_0(y)\langle\cdot,\cdot\rangle_{\mathfrak{n}}\,, \quad \beta^{\perp}(y) = f_1(y)\langle\cdot,\cdot\rangle_{\mathfrak{n}^{\perp}}\,.$$

This yields the volume form on the orbit $K \cdot y$ in terms of the canonical volume form $\mathsf{v}_{K/H}$ modified by a function f on $\tilde{M}$:

$$\mathsf{v}_{K\cdot y} = f(y)\, \mathsf{v}_{K/H}\,.$$

Then, integration over the orbits yields:

$$S(\omega) = \frac{1}{2}\mathrm{vol}(K/H)\int_{\tilde{M}} \mathsf{g}^{-1}(\Omega, \Omega) f\, \mathsf{v}_{\tilde{\mathsf{g}}}\,. \tag{7.8.11}$$

Next, we decompose the integrand with respect to the direct product structure (7.8.10) in terms of the classifying objects $(\tilde{\omega}, \tilde{\Phi})$ given by Corollary 1.9.15. We rewrite (7.8.4) as

$$\mathrm{T}_y M = \mathrm{T}_y\tilde{M} \oplus \delta_y'(\mathfrak{h}^{\perp}) \equiv \mathfrak{M}^1 \oplus \mathfrak{M}^2 \tag{7.8.12}$$

---

[53] Note that g^{-1} includes the scalar product in the Lie algebra $\mathfrak{g}$ of the gauge group G. If the manifold M is pseudo-Riemannian with signature $(-, +, \ldots, +)$, then one has to add an overall minus sign.

[54] E.g., minus the Killing form if K is semisimple.

and denote the corresponding components of Ω by $\Omega^{(i,j)}$, $i, j = 1, 2$.

Lemma 7.8.5 *The components of the curvature with respect to the decomposition (7.8.12) are given by*

$$\begin{aligned}
\Omega^{(1,1)} &= \tilde{\Omega} \,, \\
\Omega^{(1,2)} &= \nabla^{\tilde{\omega}} \Phi \,, \\
\Omega^{(2,2)} &= \frac{1}{2}\big([\Phi, \Phi] - \Phi \circ [\cdot, \cdot]_{\mathfrak{h}^\perp} - \lambda_0' \circ [\cdot, \cdot]_{\mathfrak{h}}\big) \,,
\end{aligned}$$

where $\tilde{\Omega}$ is the curvature form of $\tilde{\omega}$ and Φ is the section of the associated bundle $\tilde{P} \times_{C_G(\lambda_0(H))} L(\mathfrak{m}, \mathfrak{g})^H$ corresponding to $\tilde{\Phi}$.

Proof By the Structure Equation, we obviously have $\Omega^{(1,1)} = \tilde{\Omega}$. It remains to calculate $\Omega^{(1,2)}$ and $\Omega^{(2,2)}$. Since ω is K-invariant, it fulfils $\mathscr{L}_{A_*} \omega = 0$, where $(A_*)_p = \Delta_p'(A)$ denotes the Killing vector field of the K-action on P. Thus, for any vector field Y on P we have

$$0 = \big(\mathscr{L}_{A_*} \omega\big)\,(Y) = A_*(\omega(Y)) + \omega([Y, A_*]) \,. \tag{7.8.13}$$

Now, using this and (1.9.48), for any $y \in \tilde{M}$ and $\tilde{p} \in \tilde{P}$ such that $\pi(\tilde{p}) = y$, we calculate[55]

$$\begin{aligned}
\Omega_y^{(1,2)}(\tilde{X}, \delta_y'(A)) &= \iota_{\tilde{p}} \circ \overline{\Omega}_{\tilde{p}}(\tilde{Y}, A_*) \\
&= \iota_{\tilde{p}} \circ \Big(\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]\Big)_{\tilde{p}}(\tilde{Y}, A_*) \\
&= \iota_{\tilde{p}} \circ \Big\{\tilde{Y}_{\tilde{p}}(\omega(A_*)) + [\omega_{\tilde{p}}(\tilde{Y}), \omega_{\tilde{p}}(A_*)]\Big\} \\
&= \iota_{\tilde{p}} \circ \Big(\mathrm{d}\tilde{\Phi}(A) + [\tilde{\omega}, \tilde{\Phi}(A)]\Big)_{\tilde{p}}(\tilde{Y}) \\
&= \Big(\nabla^{\tilde{\omega}} \Phi\Big)_y (A, \tilde{X}) \,,
\end{aligned}$$

where $\tilde{X} \in \mathrm{T}_y\tilde{M}$, $A \in \mathfrak{h}^\perp$ and $\tilde{Y}$ is an arbitrary vector field on $\tilde{P}$ such that $\pi'(\tilde{Y}_{\tilde{p}}) = \tilde{X}$. In the same way, using (7.8.13), we calculate

$$\begin{aligned}
\Omega_y^{(2,2)}(\delta_y'(A), \delta_y'(B)) &= \iota_{\tilde{p}} \circ \overline{\Omega}_{\tilde{p}}(A_*, B_*) \\
&= \iota_{\tilde{p}} \circ \Big(\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]\Big)_{\tilde{p}}(A_*, B_*) \\
&= \iota_{\tilde{p}} \circ \Big\{\omega_{\tilde{p}}([A_*, B_*]) + [\omega_{\tilde{p}}(A_*), \omega_{\tilde{p}}(B_*)]\Big\} \\
&= \iota_{\tilde{p}} \circ \Big\{[\omega_{\tilde{p}}(A_*), \omega_{\tilde{p}}(B_*)] - \omega_{\tilde{p}}([A, B]_*)\Big\}
\end{aligned}$$

---

[55] Here, in order to avoid confusion, the curvature viewed as a horizontal form on P is denoted by $\overline{\Omega}$. For the notation in the last step of the calculation, recall Definition 1.5.2.

$$= \iota_{\tilde{p}} \circ \Big\{ [\tilde{\Phi}(\tilde{p})(A), \tilde{\Phi}(\tilde{p})(B)] - \tilde{\Phi}(\tilde{p})([A,B]_{\mathfrak{h}^\perp}) - \lambda_0'([A,B]_{\mathfrak{h}}) \Big\}$$
$$= [\Phi(y)(A), \Phi(y)(B)] - \Phi(y)([A,B]_{\mathfrak{h}^\perp}) - \lambda_0'([A,B]_{\mathfrak{h}}) .$$

■

We denote the fibre metrics in the spaces of differential forms with values in $\mathfrak{g}$, $(\mathfrak{h}^\perp)^* \otimes \mathfrak{g}$ and $(\bigwedge^2 \mathfrak{h}^\perp)^* \otimes \mathfrak{g}$ by, respectively, $\langle \cdot, \cdot \rangle_{(i)}$, $i = 1, 2, 3$, and write $\Omega^{(2,2)} = \mathscr{P}(\Phi)$, where

$$\mathscr{P}(\Phi) = \frac{1}{2}\big([\Phi, \Phi] - \Phi \circ [\cdot, \cdot]_{\mathfrak{h}^\perp} - \lambda_0' \circ [\cdot, \cdot]_{\mathfrak{h}}\big) . \tag{7.8.14}$$

Then, inserting the decomposition given by Lemma 7.8.5 into (7.8.11), we obtain

$$S(\omega) = \frac{1}{2}\mathrm{vol}(K/H) \int_{\tilde{M}} \big(\langle \tilde{\Omega}, \tilde{\Omega} \rangle_{(1)} + \langle \nabla^{\tilde{\omega}}\Phi, \nabla^{\tilde{\omega}}\Phi \rangle_{(2)} - V(\Phi)\big) f\, \mathsf{v}_{\tilde{\mathrm{g}}} , \tag{7.8.15}$$

where

$$V(\Phi) = -\langle \mathscr{P}(\Phi), \mathscr{P}(\Phi) \rangle_{(3)} . \tag{7.8.16}$$

Remark 7.8.6 For an orthonormal basis $\{\mathbf{e}_k\}$ in $\mathfrak{h}^\perp$, we have

$$V(\Phi) = -\sum_{k,l} \langle F_{kl}, F_{kl} \rangle_{\mathfrak{g}} , \tag{7.8.17}$$

with

$$F_{kl} = [\Phi(\mathbf{e}_k), \Phi(\mathbf{e}_l)] - \Phi\left([\mathbf{e}_k, \mathbf{e}_l]_{\restriction \mathfrak{h}^\perp}\right) - \lambda_0'\left([\mathbf{e}_k, \mathbf{e}_l]_{\restriction \mathfrak{h}}\right) . \tag{7.8.18}$$

♦

To summarize, as a result of dimensional reduction of a pure Yang–Mills theory, we obtain a theory of a Yang–Mills field interacting with a bosonic matter field. The action functional contains a self-interaction term of the matter field which is of fourth order. Thus, it is interesting to ask whether via this method one may construct Higgs potentials. This would lead to a unification scheme for the pure Yang–Mills and the Higgs sector.

This question will be addressed in the next section. For a much deeper discussion of this issue we refer to [394] and a lot of further references therein.

Exercises

7.8.1 Show that ξ defined by (7.8.9) is the connection form of the horizontal distribution $x \to \mathfrak{H}_x$.

7.9 Dimensional Reduction. Model Building

In this section, we show that the dimensional reduction procedure leads to models which are unified in the sense that one obtains constraints between the physical parameters (coupling constants and masses) of the reduced theory. This way, one can obtain predictions for the mass of one or of a number of particles in terms of the remaining parameters.

From now on, we make the following technical assumptions.

(a) Both $\mathfrak{g}$ and $\mathfrak{k}$ are compact simple Lie algebras.
(b) Both $\mathfrak{h}$ and $\lambda_0'(\mathfrak{h})$ are regular[56] Lie subalgebras of $\mathfrak{k}$ and $\mathfrak{g}$, respectively.

From the point of view of unification, the first assumption is certainly natural, because in this case the reduced theory contains the smallest possible number of parameters. In particular, there are unique, up to a constant, Ad-invariant scalar products $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Concerning assumption (b), it is only for regular subalgebras that we have canonical methods for calculating the centralizer and other characteristics, see Appendix C.

Now, if we wish to construct models, we must of course explicitly solve the constraint equation (1.9.47) expressing the H-invariance of $\tilde{\Phi}$,

$$\tilde{\Phi}(\tilde{p}) \circ \mathrm{Ad}(h) = \mathrm{Ad}(\lambda_0(h)) \circ \tilde{\Phi}(\tilde{p}) \,, \quad h \in H \,. \tag{7.9.1}$$

From now on, we will use the following simplified notation:

$$\lambda_0' \equiv \kappa \,, \quad \tilde{\Phi} \equiv \phi \,.$$

Recall that any ϕ fulfilling the above relation may be interpreted as an operator intertwining the representations $\mathrm{Ad}(H)(\mathfrak{h}^\perp)$ and $\mathrm{Ad}(\lambda_0(H))(\mathfrak{g})$. Consequently, to construct ϕ, one has to decompose the representations in (7.9.1) into irreducible components. By Schur's Lemma, ϕ can only intertwine equivalent ones. Technically, it is convenient to pass to the Lie algebraic version of (7.9.1),

$$\phi(\tilde{p}) \circ \mathrm{ad}(B) = \mathrm{ad}(\kappa(B)) \circ \phi(\tilde{p}) \,, \quad B \in \mathfrak{h} \,, \tag{7.9.2}$$

and to work with the complexifications $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Correspondingly, we extend ϕ to the complexified Lie algebras:

$$\phi^{\mathbb{C}}(A_1 + iA_2) := \phi(A_1) + i\phi(A_2) \,, \quad A_1, A_2 \in \mathfrak{h}^\perp \,. \tag{7.9.3}$$

Then,

$$\overline{\phi^{\mathbb{C}}}(A) = \phi^{\mathbb{C}}(\overline{A}) \,, \quad A \in (\mathfrak{h}^\perp)^{\mathbb{C}} \,,$$

with the bar denoting complex conjugation, and we may extend (7.9.2) linearly to

[56] See Appendix C.

$$\phi^{\mathbb{C}} \circ \mathrm{ad}(\mathfrak{h}^{\mathbb{C}}) = \mathrm{ad}(\kappa(\mathfrak{h}^{\mathbb{C}})) \circ \phi^{\mathbb{C}} . \tag{7.9.4}$$

Given a solution $\phi^{\mathbb{C}}$ of this equation, by restriction to $\mathfrak{h}^{\perp} \subset (\mathfrak{h}^{\perp})^{\mathbb{C}}$ one obtains an operator ϕ fulfilling (7.9.2). To summarize, we may first solve (7.9.4) and then obtain the solution by restriction to $\mathfrak{h}^{\perp}$. For the first step, we may use the representation theory of (semi-)simple Lie algebras, see Appendix C or [170, 329] for a detailed exposition.

In model building one is often interested in theories with one irreducible multiplet of scalar fields only. As was shown by Kubyshin and Volobuev [643, 644], for classical Lie groups this is always the case under the following additional assumption.

Proposition 7.9.1 *Let the assumptions* (*a*) *and* (*b*) *be fulfilled, with* K, H *and* G *being classical Lie groups. If, additionally,* K/H *is a simply connected irreducible symmetric space, then the reduced theory contains only one irreducible multiplet of scalar fields.*

Proof The symmetric spaces fulfilling the assumptions are provided by Table 2.1. They read as follows:

$$G_{\mathbb{K}}(m, m+n)\,, \quad \mathrm{Sp}(m)/\mathrm{U}(m)\,, \quad \mathrm{SO}(2m)/\,U(m)\,, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}\,.$$

Now, under the assumptions (a) and (b), $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{h}^{\perp}$.[57] Using this fact, by direct inspection of these spaces in terms of the corresponding root lattices one can show the following: if one passes to the complexification, either $\mathrm{ad}(\mathfrak{h})$ remains irreducible or it yields two inequivalent complex representations which are conjugate to each other. Thus, ϕ either intertwines representations of one type or representations of two types, conjugate to each other. By construction, the centralizer $\mathfrak{c}$ of $\kappa(\mathfrak{h})$ in $\mathfrak{g}$ acts irreducibly on the intertwining operators. Thus, the latter constitute an irreducible multiplet. ■

By Proposition 7.9.1, the above class of symmetric spaces is especially interesting. Thus, we concentrate on this case and, finally, add some comments on more general settings. Let us define the following mappings:

$$f_1 : \bigwedge\nolimits^2 \mathfrak{h}^{\perp} \to \mathfrak{h}\,, \quad f_1(A_1 \wedge A_2) := [A_1, A_2]\,,$$

$$f_2 : \bigwedge\nolimits^2 \mathfrak{h}^{\perp} \to \mathfrak{g}\,, \quad f_2(A_1 \wedge A_2) := [\phi(A_1), \phi(A_2)]\,.$$

By point 1 of Remark 2.5.6, we have

$$[\mathfrak{h}^{\perp}, \mathfrak{h}^{\perp}] = \mathfrak{h}\,. \tag{7.9.5}$$

Thus, the mapping f_1 is surjective. If, additionally, K/H has rank one, then f_1 is an isomorphism of vector spaces. Indeed, in that case, $f_1(A \wedge B) = [A, B] = 0$

[57] Cf. point 2 of Remark 2.5.6.

implies that A and B must be proportional, that is, $A \wedge B = 0$. Moreover, under the assumptions (a) and (b), the representation $\mathrm{ad}(\mathfrak{h})(\mathfrak{h}^\perp)$ is irreducible. This fact has two immediate consequences. First, it implies that the Ad-invariant scalar product on $\mathrm{T}_{[\mathbb{1}]}K/H \cong \mathfrak{h}^\perp$ is unique, up to a constant,[58] and we may write

$$\hat{\mathsf{g}}_{[\mathbb{1}]} = -\frac{1}{m^2}\langle\cdot,\cdot\rangle_{\mathfrak{h}^\perp}\,, \tag{7.9.6}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{h}^\perp}$ denotes the restriction of the canonical scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ to $\mathfrak{h}^\perp$. Second, since ϕ is an intertwiner, the transport of the canonical scalar product on $\mathfrak{g}$ to $\mathfrak{h}^\perp$ is an $\mathrm{Ad}(\mathfrak{h})$-invariant scalar product on $\mathfrak{h}^\perp$ and, thus, by the above assumption, it must be proportional to the canonical scalar product on $\mathfrak{h}^\perp$. Denoting the factor of proportionality by $|\phi|^2$, for any $A_1, A_2 \in \mathfrak{h}^\perp$ we obtain

$$\langle\phi(A_1), \phi(A_2)\rangle_{\mathfrak{g}} = |\phi|^2\langle A_1, A_2\rangle_{\mathfrak{h}^\perp}\,. \tag{7.9.7}$$

Remark 7.9.2 For later purposes, let us calculate $|\phi|^2$ explicitly. Let $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_j\}$ be orthonormal bases in $\mathfrak{h}^\perp$ and $\phi(\mathfrak{h}^\perp) \subset \mathfrak{g}$, respectively. Then, the intertwiner ϕ may be expanded as follows:

$$\phi = \phi^m{}_p(\mathbf{e}^p)^* \otimes \mathbf{E}_m\,.$$

Then, for the left hand side of (7.9.7), we obtain

$$\langle\phi(A_1), \phi(A_2)\rangle_{\mathfrak{g}} = A^i{}_1 A^j{}_2 \phi^m{}_i \phi^n{}_j \delta_{mn}\,,$$

whereas for the right hand side we have

$$|\phi|^2\langle A_1, A_2\rangle_{\mathfrak{h}^\perp} = |\phi|^2 A^i{}_1 A^j{}_2 \delta_{ij}\,.$$

Thus,

$$|\phi|^2 = \frac{1}{\dim(\mathfrak{h}^\perp)}\phi^m{}_i\phi^i{}_m \equiv \frac{1}{\dim(\mathfrak{h}^\perp)}\,\mathrm{tr}(\phi^2)\,. \tag{7.9.8}$$

♦

Let us denote by ε the ratio of indices[59] of $\kappa(\mathfrak{h})$ in $\mathfrak{g}$ and $\mathfrak{h}$ in $\mathfrak{k}$. The following was shown in [547].

Proposition 7.9.3 *Let the assumptions (a) and (b) be fulfilled. Let K/H be a symmetric space of rank* 1*, let $\mathfrak{h}$ be simple and let κ be injective. Then,*

$$V(\phi) = \left(1 - \frac{1}{\varepsilon}|\phi|^2\right)^2 \varepsilon\, m^2\, \widehat{\mathsf{Sc}}\,,$$

[58] Cf. formula (2.5.7).

[59] The index of a simple Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is the factor by which the scalar product on $\mathfrak{h}$ induced from the canonical scalar product of $\mathfrak{g}$ differs from the canonical scalar product of $\mathfrak{h}$.

where $\widehat{\mathsf{Sc}}$ *is the scalar curvature of* $\mathrm{g}_{K/H}$.

Proof Since f_1 is an isomorphism, there exists a mapping

$$f : \mathfrak{h} \to \mathfrak{g}\,, \quad f := f_2 \circ f_1^{-1}\,.$$

Moreover, for any $C \in \mathfrak{h}$, there exists a unique element $\alpha \in \bigwedge^2 \mathfrak{h}^\perp$ such that $f_1(\alpha) = C$. Without loss of generality, we may assume $\alpha = A_1 \wedge A_2$ with $A_1, A_2 \in \mathfrak{h}^\perp$. Then, $C = [A_1, A_2]$ and we have

$$f([A_1, A_2]) = [\phi(A_1), \phi(A_2)]\,. \tag{7.9.9}$$

Using this, together with the Jacobi identity in $\mathfrak{g}$ and the fact that ϕ is an intertwiner, we show that f is an operator intertwining the representations $\mathrm{ad}(\mathfrak{h})$ and $\mathrm{ad}(\kappa(\mathfrak{h}))$: indeed, on the one hand, for any $B \in \mathfrak{h}$ we obtain

$$\begin{aligned}(\mathrm{ad}\kappa(B) \circ f)\,(C) &= [\kappa(B), f([A_1, A_2])] \\ &= [\kappa(B), [\phi(A_1), \phi(A_2)]] \\ &= -[\phi(A_1), [\phi(A_2), \kappa(B)]] - [\phi(A_2), [\kappa(B)\phi(A_1)]] \\ &= [\phi(A_1), \phi([B, A_2])] - [\phi(A_2), \phi([B, A_1])]\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}(f \circ \mathrm{ad}(B))\,(C) &= f_2 \circ f_1^{-1}([B, C]) \\ &= -f_2 \circ f_1^{-1}\,([A_1, [A_2, B]] - [A_2, [B, A_1]]) \\ &= -f_2(A_1 \wedge [A_2, B]) - f_2(A_2 \wedge [B, A_1]]) \\ &= [\phi(A_1), \phi([B, A_2])] - [\phi(A_2), \phi([B, A_1])]\,.\end{aligned}$$

Since, by assumption, κ is an isomorphism onto its image and since $\kappa(\mathfrak{h}) \subset \mathfrak{g}$ is regular, there is only one adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$, namely $\mathrm{ad}(\kappa(\mathfrak{h}))$ acting on $\kappa(\mathfrak{h})$, see [170]. We conclude

$$f(B) = c\kappa(B)\,, \quad B \in \mathfrak{h}\,, \tag{7.9.10}$$

where c is some real constant. Then, for any $A_1, A_2 \in \mathfrak{h}^\perp$ and any $B \in \mathfrak{h}$,

$$\langle f([A_1, A_2]), \kappa(B)\rangle_{\mathfrak{g}} = c\langle \kappa([A_1, A_2]), \kappa(B)\rangle_{\mathfrak{g}} = c\varepsilon\langle [A_1, A_2], B\rangle_{\mathfrak{k}}\,.$$

On the other hand, using the Ad-invariance of the scalar products,

$$\begin{aligned}\langle f([A_1, A_2]), \kappa(B)\rangle_{\mathfrak{g}} &= \langle [\phi(A_1), \phi(A_2)], \kappa(B)\rangle_{\mathfrak{g}} \\ &= \langle \phi(A_2), \phi([B, A_1])\rangle_{\mathfrak{g}} \\ &= |\phi|^2 \langle [A_1, A_2], B\rangle_{\mathfrak{k}} \,.\end{aligned}$$

Thus,

$$c = \frac{1}{\varepsilon}|\phi|^2 \,,$$

and, consequently, by (7.9.9) and (7.9.10),

$$[\phi(A_1), \phi(A_2)] = \frac{1}{\varepsilon}|\phi|^2 \kappa([A_1, A_2]) \,.$$

Next, we observe that for a symmetric space the second term in (7.8.14) vanishes. Thus,

$$\mathscr{P}(\phi) = \frac{1}{2}\left(\frac{1}{\varepsilon}|\phi|^2 - 1\right)\kappa \circ [\cdot, \cdot]_{\mathfrak{h}} \,. \tag{7.9.11}$$

Finally, by (2.5.21), the scalar curvature is given by

$$\widehat{\mathsf{Sc}} = -\sum_{k,l} \hat{\mathsf{g}}([[\mathbf{e}_k, \mathbf{e}_l], \mathbf{e}_l], \mathbf{e}_k) \,, \tag{7.9.12}$$

where $\{\mathbf{e}_k\}$ is an orthonormal basis in $\mathfrak{h}^\perp$ with respect to $\mathsf{g}_{K/H}$. Inserting (7.9.11) into (7.8.17) and using (7.9.6) and (7.9.12), together with the Ad-invariance of the scalar product, we obtain the assertion. ∎

The following example is taken from [547].

Example 7.9.4 (*Georgi–Glashow model*) We consider the case

$$K/H = \mathrm{SO}(l+1)/\mathrm{SO}(l) \,, \quad G = \mathrm{SO}(l+p) \,,$$

with $l = 2n$ and $p = 2k+1$. Then, $\mathfrak{h}$ and $\kappa(\mathfrak{h})$ may be embedded regularly. Let us find the decompositions of the Lie algebras $\mathfrak{k}$ and $\mathfrak{g}$ in terms of the root lattices introduced in Appendix C. The left diagram in Fig. 7.2 shows the decomposition of $\mathfrak{k}$. The roots contained in the triangles with the corners $\alpha_1, \alpha_{n-1}, \alpha(1, n-1)$ and β_1, $\beta_{n-1}, \beta(1, n-1)$ correspond to the Lie subalgebra $\mathfrak{h} = D_n \subset B_n$. To prove this, we observe that the root $\beta_{n-1} = \alpha_{n-1} + 2\alpha_n$, together with the roots $\alpha_1, \dots, \alpha_{n-1}$, may be taken as a system of simple roots of $\mathfrak{h}$. This follows from the fact that $\beta_{n-1} - \alpha_i$ is not a root for $i = 1, \dots, n-1$. On the other hand, $\langle \beta_{n-1}, \alpha_i\rangle_* = 0$ for $i \neq n-2$ and $\langle \beta_{n-1}, \alpha_{n-2}\rangle_* = -1$. Thus, the roots $\alpha_1, \dots, \alpha_{n-1}, \beta_{n-1}$ constitute the Dynkin diagram of D_n, cf. Fig. C.1. The subspace $\mathfrak{h}^\perp$ is spanned by the root vectors of the roots $\alpha(1, n), \dots, \alpha(n, n)$ (filled circles) and the root vectors of the corresponding

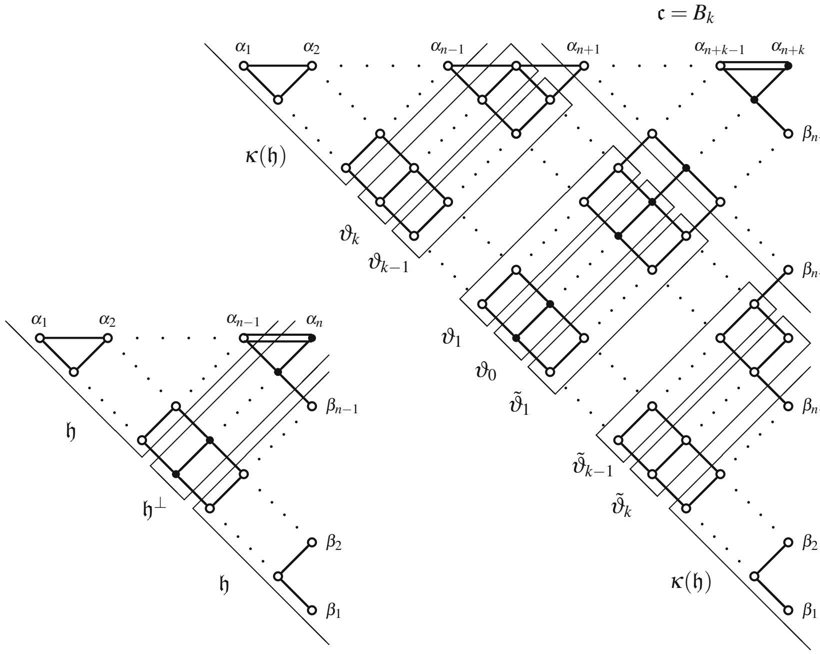

Fig. 7.2 Decomposition of $\mathfrak{k} = B_n$ (*left*) and $\mathfrak{g} = B_{n+k}$ (*right*) in terms of the root diagram

negative roots $-\alpha(1,n),\ldots,-\alpha(n,n)$. Since $\dim\mathfrak{h}^\perp = 2n$, $\mathfrak{h}^\perp$ carries the vector representation of D_n.

Next, let us discuss the right diagram in Fig. 7.2 showing the decomposition of $\mathfrak{g}$. In analogy with the left diagram, the roots in the triangles $(\alpha_1, \alpha_{n-1}, \alpha(1,n-1))$ and $(\beta_1, \beta_{n-1}, \beta(1,n-1))$ build a D_n-subalgebra of B_{n+k}. We denote the root vectors in B_{n+k} by $\mathbf{E}_\alpha$ and choose the homomorphism $\kappa : \mathfrak{h} \to \mathfrak{g}$ as follows:

$$\kappa(\mathbf{e}_{\alpha_i}) := \mathbf{E}_{\alpha_i}\,, \quad \kappa(\mathbf{e}_{\beta_{n-1}}) := \mathbf{E}_{\beta_{n-1}}\,, \quad i = 1,\ldots,n-1\,.$$

Thus, $\varepsilon = 1$, that is,

$$\langle\kappa(B_1),\kappa(B_2)\rangle_{\mathfrak{g}} = \langle B_1, B_2\rangle_{\mathfrak{k}}\,, \quad B_1, B_2 \in \mathfrak{h}\,.$$

Now, the decomposition of the representation $\mathrm{ad}(\mathfrak{g})_{\restriction\kappa(\mathfrak{h})}$ looks as follows: clearly, the triangle $(\alpha_{n+1}, \alpha_{n+k}, \beta_{n+1})$ carries the trivial representation of D_n. Thus, the centralizer of $\kappa(\mathfrak{h})$ in $\mathfrak{g}$ is $\mathfrak{c} = B_k$. Consequently, the structure group $C_G(\lambda(H))$ of the reduced theory is $\mathrm{SO}(2k+1)$. The fact that, in the case under consideration, $\mathfrak{c}$ does not contain any Abelian subalgebra is an immediate consequence of equation (C.1). Next, the two segments $(\alpha(1,n-1+i),\ldots,\alpha(n,n-1+i))$ with $i = 1,\ldots,k+1$ and $(\beta(1,n-1+i),\ldots,\beta(n,n-1+i))$ with $i = 1,\ldots,k$, together with the corresponding negative roots $2k+1$, form vector representations of D_n. More precisely, we have one real representation ϑ_0, and the $2k$ complex representations ϑ_i spanned by the root vectors $(\mathbf{e}_{\alpha(j,n-1+i)}, \dots, \mathbf{e}_{-\beta(j,n-1+i)})$ and $\tilde{\vartheta}_i$ spanned by the root vectors $(\mathbf{e}_{-\alpha(j,n-1+i)}, \dots, \mathbf{e}_{\beta(j,n-1+i)})$, where $j = 1, \dots, n$.

For the sake of completeness, let us write down the scalar field ϕ[60]:

$$\phi = \sum_{k,i} \phi^k (\mathbf{e}^i)^* \otimes \tilde{\mathbf{E}}_{ik} .$$

Here, $\{\mathbf{e}_i\}$ is the basis of root vectors in $\mathfrak{h}^\perp$ and $\{\tilde{\mathbf{E}}_{ik}\}$ are bases in the representations ϑ_0 and ϑ_i, invariant under complex conjugation. Clearly, ϕ carries the vector representation of the centralizer $\mathfrak{c}$. Finally, using (7.9.6) and (7.9.12), one easily calculates the scalar curvature (Exercise 7.9.1),

$$\widehat{\mathsf{Sc}} = m^2 l(l-1) . \tag{7.9.13}$$

This yields

$$V(\phi) = m^4 l(l-1)(1 - |\phi|^2)^2 .$$

If one writes down the scalar products in (7.8.15) explicitly, then the first two terms acquire factors which are functions of the constants m, l and p. Thus, if one wants to bring the reduced action to a canonical form, one must rescale both the gauge potential and the matter field appropriately. After that, the potential takes the following form:

$$V(\phi) = \frac{g^2(l-1)}{4l(p-2)} \left(\frac{m^2 l(p-2)}{g^2} - |\phi|^2 \right)^2 , \tag{7.9.14}$$

where g is the coupling constant of the reduced theory. For $p = 3$ we get the bosonic sector of the Georgi–Glashow model, cf. Example 7.3.7. Using the formulae for the masses of the Higgs boson, the intermediate vector boson W and the monopole as given by 't Hooft [623], we get

$$m_H = m\sqrt{\frac{1}{2}(l-1)} , \quad m_W = \sqrt{l} m , \quad m_{mon} = \frac{4\pi}{g^2} \sqrt{l}\, m\, C\left(\frac{2(l-1)}{l}\right) ,$$

where C is a slowly varying function. Thus,

$$m_H = \sqrt{\frac{2(l-1)}{l}} m_W ,$$

that is, within this unified model one gets a prediction of the Higgs mass in terms of the mass of the W-boson. ♦

[60] By Proposition 7.9.3, in the case of a symmetric space, ϕ need not be calculated explicitly.

In a similar way, one may attempt to construct the bosonic sector of the Weinberg–Salam model, see [423, 644]. In the following example, we present the results of Kubyshin and Volobuev [644]. Since the method is the same as in Example 7.9.4, we omit the calculations.

Example 7.9.5 (Weinberg–Salam model) The results of [644] are summarized in the following table. In each case, one obtains the bosonic sector of the Weinberg–Salam

G	K/H	M_W	M_Z	M_H	$\sin^2\theta_W$
$\mathrm{SO}(l+4)$	$G_{\mathbb{R}}(2,l+2)$	$m\sqrt{l}$	$m\sqrt{l}$	$m\sqrt{2l}$	$\frac{1}{2}$
$\mathrm{SU}(l+2)$	$\mathbb{C}\mathrm{P}^l$	$m\sqrt{l}$	$m\sqrt{2(l+1)}$	$m\sqrt{2(l+1)}$	$\frac{l+2}{2(l+1)}$
$\mathrm{Sp}(l+1)$	$\mathbb{C}\mathrm{P}^l$	$m\sqrt{2l}$	$m\sqrt{2(l+1)}$	$m\sqrt{2(l+1)}$	$\frac{1}{l+1}$

model described in Sect. 7.7. It is interesting to compare the masses and the Weinberg angle obtained via the dimensional reduction method with the experimental data. Instead of the four parameters M_W, M_Z, M_H and $\sin^2\theta_W$ characterizing the bosonic sector, cf. Remark 7.7.3, we have only 3 independent parameters here: the non-Abelian coupling constant g, which is proportional to the coupling constant of the unreduced pure Yang–Mills theory on the multidimensional universe, the reciprocal linear scale m of the internal space, cf. Eq. (7.9.6), and the dimension l of that space. This allows for a prediction of the Higgs mass on tree level. It can be shown that in all the above cases, the correct value of the electric charge e can be obtained by an appropriate choice of the coupling constant g. This should be clear from (7.7.28). Comparing with (7.7.31), we see that the third model of the above table yields the best agreement with the experimental value of the Weinberg angle: for $l = 3$ we obtain $\sin^2\theta_W = 0.25$. Unfortunately, in this case, the predicted value of the Higgs mass is too small. It coincides with the mass of the intermediate vector boson Z. It should be noted that, on the other hand, the relation of the Z- and the W-mass is in quite good agreement with the experimental value. Comparing with the corresponding table in Sect. 7.7, we see that the model in the first row yields a rather nice prediction of the Higgs mass, but $\sin^2\theta_W$ is too large.

Finally, we note that Manton [423] has obtained similar results for models with gauge groups SU(3), SO(5) and G_2 on the six-dimensional universe $M \times \mathrm{S}^2$ with rotational symmetry. ♦

Remark 7.9.6

1. If one departs from the assumption that K/H be symmetric, while keeping κ injective, then new phenomena occur. In this case, $\mathfrak{h}^\perp$ decomposes into $\mathfrak{h}^\perp = \mathfrak{n} \oplus \mathfrak{n}^\perp$, cf. (7.8.2) and (7.8.3). Now, in addition to intertwining non-trivial representations in $\mathfrak{n}^\perp$ with non-trivial representations in $\mathfrak{g}$, in general also trivial representations of $\mathfrak{n}$ and of $\mathfrak{c} \subset \mathfrak{g}$ will be intertwined. The latter phenomenon is new, comparing with the case of a symmetric space. For details, see Proposition 4.2 in [547]. An example of this type is provided by

$$K/H = \mathrm{SU}(5)/\mathrm{SU}(4) \cong \mathrm{S}^9\,, \quad G = \mathrm{SO}(9)\,.$$

If we also drop the assumption that κ be injective, then $\mathfrak{h}$ decomposes into

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2\,, \quad \mathfrak{h}_1 \cong \operatorname{im}\kappa\,, \quad \mathfrak{h}_2 = \ker\kappa\,,$$

and the intertwining condition (7.9.2) reads

$$\phi \circ \operatorname{ad}(B_1) = \operatorname{ad}(\kappa(B_1)) \circ \phi\,, \quad \phi \circ \operatorname{ad}(B_2) = 0\,,$$

for $B_1 \in \mathfrak{h}_1$ and $B_2 \in \mathfrak{h}_2$. The second of these equations says that we get a non-trivial operator ϕ iff there is a trivial representation in $\operatorname{ad}(\mathfrak{k})_{\restriction \mathfrak{h}_2}(\mathfrak{h}^\perp)$. On the other hand, to get a non-trivial self-interaction for ϕ subject to the first of these equations, there must be a non-trivial representation in $\operatorname{ad}(\mathfrak{k})_{\restriction \mathfrak{h}_1}(\mathfrak{h}^\perp)$. An example illustrating this situation is provided by

$$K/H = \mathrm{SO}(9)/(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))\,, \quad G = \mathrm{SU}(n+3)\,.$$

This example is worked out in detail in [547].

2. Finally, we note that a lot of effort has been put into building grand unification models via dimensional reduction, see e.g. [51, 417, 548] and a lot of further references therein. Later on, the method has been extended to include supersymmetric models, see e.g. [337, 422] and further references therein.
3. From the above discussion we see that the dimensional reduction method yields relations between the parameters of the classical theory. It is interesting to ask whether these relations survive in some sense on quantum level. This problem is related to the procedure of reduction of couplings in quantum field theory, cf. [588] and references therein. ♦

Exercises

7.9.1 Prove formula (7.9.13).

7.9.2 Work out the details of the examples provided by point 1 of Remark 7.9.6.

Chapter 8
The Gauge Orbit Space

In the first part of this chapter, we discuss the mathematical structure of the gauge orbit space stratification. In Sect. 8.2, we prove that there is a one-to-one correspondence between orbit types and a certain type of bundle reductions of the principal bundle under consideration. In Sect. 8.3, we study the structure of the gauge orbit stratification in some detail. We prove a Tubular Neighbourhood Theorem and use this to show that the strata are smooth manifolds and that the stratification is regular. In Sect. 8.4, we study the geometry of the strata. We show that every stratum admits a natural Riemannian metric, calculate its volume element and find the corresponding Riemann curvature. We also briefly comment on geodesics.

In the second part of the chapter, we present our results on the enumeration of gauge orbit types in detail. For clearness of presentation, we limit our attention to the case $G = \mathrm{SU}(n)$. The result is given in terms of certain characteristic classes fulfilling a number of algebraic relations. We also show how the natural partial ordering of strata, which contains information on how the strata are linked, can be read off from these relations.

8.1 Introduction

Let us start with a brief introduction to the final two chapters which are closely related. In the present chapter, we study the rich geometric and topological structure of the classical configuration space of gauge theories. In the next chapter, we will discuss some aspects of the significance of this structure for quantum gauge theory.

Roughly speaking, the methods used in quantum field theory may be divided into perturbative and non-perturbative ones. In the case of the standard model, whose classical field theoretic structure was presented in the previous chapter, perturbation theory works well for high energy processes. On the other hand, the low energy hadron physics turns out to be dominated by non-perturbative effects. For the latter there is no

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rigorous theoretical explanation yet. To study them, a variety of different concepts and mathematical methods has been developed. In particular, for some aspects, methods of differential geometry and algebraic topology seem to be unavoidable. This is certainly true if one wants to investigate the influence of the structure of the classical configuration space, the gauge orbit space, on quantum level. Let us discuss some aspects indicating the physical relevance of this structure.

First, studying the geometry and topology of the generic (principal) stratum, one gets an intrinsic topological interpretation of the Gribov ambiguity [258, 591]. We stress that the problem of finding all Gribov copies has been discussed within specific models, see e.g. [401]. For a detailed analysis in the case of 2-dimensional cylindrical spacetime (including the Hamiltonian path integral) we refer to [584]. By investigating the topology of the determinant line bundle over the generic stratum, one gets an understanding of gauge anomalies in terms of the Family Index Theorem [17, 41], see also [114] for the Hamiltonian approach. In particular, one gets anomalies of purely topological type [674], referred to as global anomalies. The latter cannot be seen by perturbative quantum field theory. Moreover, there are partial results and conjectures concerning the relevance of nongeneric strata. First, generally speaking, nongeneric gauge orbits affect the classical motion on the orbit space due to boundary conditions and, in this way, they may produce nontrivial contributions to the path integral. They may lead to localization of certain quantum states, as it was suggested by finite-dimensional examples [185]. Further, the gauge field configurations belonging to nongeneric orbits can possess a magnetic charge, i.e. they can be considered as a kind of magnetic monopole configurations. According to 't Hooft [624], these could be responsible for quark confinement. The role of these configurations was investigated within the framework of Schrödinger quantum mechanics on the gauge orbit space of topological Chern–Simons theory in [25], see also [24] for an approach to 4-dimensional Yang–Mills theories with θ-term. Within 't Hooft's concept, the idea of Abelian projection is of special importance and has been discussed by many authors. For example, this concept was studied within the setting of quantum field theory at finite temperature on the 4-torus [205, 206]. There, a hierarchy of defects, which should be related to the gauge orbit space structure, was discovered. Finally, let us also mention that the existence of additional anomalies corresponding to nongeneric strata was suggested, see [290].

Most of the problems mentioned here are still awaiting a systematic investigation. For that purpose, a deeper insight into the structure of the gauge orbit space is necessary. In a series of papers [296, 297, 543–545] we have made a step in this direction. We have given a complete solution to the problem of determining the strata that are present in the gauge orbit space for gauge theories with the classical gauge groups $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SO}(n)$ in compact Euclidean spacetime of dimension $d = 2, 3, 4$. Our analysis is based on the results of Kondracki and Rogulski [388], who have investigated the general structure of the full gauge orbit space for the first time in detail. In particular, they have shown that the gauge orbit space is a stratified topological space. Moreover, they have described the relation between orbit types and bundle reductions we are using.

Let us mention that there is an approach based upon parameterizing the full gauge orbit space by a so called fundamental domain. The latter is characterized by the property that it is intersected by every gauge orbit exactly once, up to possible identifications on the boundary, see [148, 223, 640, 641, 699] and the review [642] for further references. This concept was developed in order to solve the Gribov problem, see Sect. 9.2 for further details. However, for the study of the stratified structure of the gauge orbit space, this concept seems not to be efficient. Finally, we note that the stratification structure for gauge theories within the Ashtekar approach has also been studied, see [203].

Clearly, as already mentioned above, the main challenge consists in clarifying the possible role of the nongeneric strata on quantum level in a systematic way. For that purpose, one needs a general concept how to implement these strata in a quantum gauge theory. For the case of spaces carrying a Kähler structure, one may use the concept of Hilbert space costratification as proposed by Huebschmann [326]. In [328], these ideas were substantiated for a toy model of Hamiltonian quantum gauge theory on a finite lattice as developed in [368, 369, 386]. Here, the classical phase space may be identified with a product of copies of the complexified structure group, which carries a natural Kähler structure, and the classical stratification is encoded in terms of a costratification of the representation space of the observable algebra. Details will be explained in Sects. 9.6 and 9.7.

8.2 Gauge Orbit Types

We start with recalling some basics from Sect. 6.1. The configuration space $\mathscr{C}$ of a Yang–Mills theory on a principal bundle $P(M, G)$ is the set of connections on P. It carries a natural affine structure with translation vector space

$$\mathscr{T} = \Omega^1(M, \mathrm{Ad}(P)) \cong \Omega^1_{\mathrm{Ad,hor}}(P, \mathfrak{g}) , \tag{8.2.1}$$

and is acted upon by the group of vertical automorphisms $\mathscr{G} = \mathrm{Aut}_M(P)$. By Remark 6.1.2, elements $u \in \mathscr{G}$ may be viewed as sections of the vector bundle $\mathrm{End}(\mathrm{Ad}(P))$. Then, local gauge transformations read

$$\omega^{(u)} = \omega + u^{-1}\nabla^{\omega} u . \tag{8.2.2}$$

As explained in Sect. 6.1, if we assume that G be a compact connected linear Lie group and that M be a compact orientable Riemannian manifold, we can pass to Sobolev completions of $\mathscr{C}$ and $\mathscr{G}$. As before, we denote the Hilbert space of cross sections of Sobolev class k of a vector bundle E by $W^k(E)$. In the sequel, we assume

$$k > \frac{1}{2}\dim(M) + 1 . \tag{8.2.3}$$

Then, by the Sobolev Embedding Theorem 5.7.7, connection forms are of class C^1 and, therefore, have continuous curvature. This theorem also implies that $\mathscr{G}$ is a Hilbert-Lie group with Lie algebra $\mathrm{L}\mathscr{G} = W^{k+1}(\mathrm{Ad}(P))$ and exponential mapping given by (6.1.13) acting smoothly on $\mathscr{C}$. Moreover, by Theorem 6.1.7, the action of $\mathscr{G}$ is proper. Thus, the orbits of the action of $\mathscr{G}$ on $\mathscr{C}$ are closed and the gauge orbit space

$$\mathscr{M} := \mathscr{C}/\mathscr{G} \tag{8.2.4}$$

is Hausdorff.

Remark 8.2.1 Clearly, $\mathscr{M}$ should not depend essentially on the technical parameter k. Thus, let $k' > k$ and let $\mathscr{C}'$, $\mathscr{G}'$ and $\mathscr{M}'$ be the Sobolev completions corresponding to k'. Then, one has natural embeddings $\mathscr{G}' \hookrightarrow \mathscr{G}$ and $\mathscr{C}' \hookrightarrow \mathscr{C}$. As a consequence of the first, the latter projects to a mapping $\varphi : \mathscr{M}' \to \mathscr{M}$. Since the image of $\mathscr{C}'$ in $\mathscr{C}$ is dense, so is $\varphi\left(\mathscr{M}'\right)$ in $\mathscr{M}$. To see that φ is injective, let $\omega_1, \omega_2 \in \mathscr{C}'$ and $u \in \mathscr{G}$ such that $\omega_2 = \omega_1^{(u)}$. Then (8.2.2) implies

$$\mathrm{d}u = u\,\omega_2 - \omega_1\,u\,. \tag{8.2.5}$$

Due to $2k' > 2k > \dim M$, by the multiplication rule for Sobolev functions, the right hand side of (8.2.5) is of class W^{k+1}. Then u is of class W^{k+2}. This can be iterated until the right hand side is of class $W^{k'}$. Hence, $u \in \mathscr{G}'$, so that ω_1 and ω_2 are representatives of the same element of $\mathscr{M}'$. This shows that $\mathscr{M}'$ can be identified with a dense subset of $\mathscr{M}$. Another question is whether the orbit type stratification of $\mathscr{M}$ to be discussed below depends on k. Fortunately, the answer to this question is negative, see Theorem 8.2.8. ♦

As discussed in Chap. 6 of Part I, the orbit space of a proper Lie group action in finite dimensions is a stratified space with the strata being the connected components of the orbit type subsets. Kondracki and Rogulski [388] have shown that in the case of the gauge orbit space, the situation is similar. We will discuss this in Sect. 8.3. For now, let us recall the notion of orbit type and relate orbit types to bundle reductions. For a connection ω and a point p_0, let $\mathscr{H}_{p_0}(\omega)$ and $P_{p_0}(\omega)$ denote the holonomy group and the holonomy bundle of ω based at p_0, respectively. Since, under the assumption (8.2.3), ω is of class C^1, $P_{p_0}(\omega)$ is a bundle reduction of P of class C^2.

The stabilizer of $\omega \in \mathscr{C}$ under the action of $\mathscr{G}$ is given by

$$\mathscr{G}_\omega = \{u \in \mathscr{G} : \omega^{(u)} = \omega\}\,.$$

By the Stabilizer Theorem 6.1.5, this is a compact Lie subgroup of $\mathscr{G}$ with Lie algebra

$$\mathrm{L}\mathscr{G}_\omega = \ker(\nabla^\omega) = \{\xi \in \mathrm{L}\mathscr{G} : \xi_{\restriction P_{p_0}(\omega)} = \mathrm{const}\}\,. \tag{8.2.6}$$

Since

$$\mathscr{G}_{\omega^{(u)}} = u^{-1}\mathscr{G}_\omega u\,,$$

the stabilizers along an orbit form a conjugacy class in $\mathscr{G}$. This class is referred to as the type of that orbit. The set of all orbit types of the action of $\mathscr{G}$ on $\mathscr{C}$ will be denoted by Σ. This set carries a natural partial ordering: for $\tau, \tau' \in \Sigma$, one has $\tau \leq \tau'$ iff there are representatives $\mathscr{G}_\omega$ of τ and $\mathscr{G}_{\omega'}$ of τ' such that $\mathscr{G}_\omega \supset \mathscr{G}_{\omega'}$.[1]

The discussion of the orbit types of the action of $\mathscr{G}$ on $\mathscr{C}$ rests on the fact that they can be expressed in terms of bundle reductions of P. This relation will be analyzed now. Given a subset $A \subset G$, let $\mathrm{C}_G(A)$ denote the centralizer in G. For repeated centralizers, we write $\mathrm{C}^2_G(A) = \mathrm{C}_G\big(\mathrm{C}_G(A)\big)$ etc.
Now, let $p_0 \in P$ be chosen. To every subgroup $S \subset \mathscr{G}$, we assign a subset of P by

$$\Phi_{p_0}(S) = \{p \in P \ : \ u(p) = u(p_0) \text{ for all } u \in S\} \ . \tag{8.2.7}$$

Given Lie subgroups $H \subset K \subset G$ and a reduction Q of P to H, for the induced reduction of P to K we write

$$Q \cdot K = \{p \in P : p = \Psi_k(q) \text{ for some } k \in K \text{ and some } q \in Q\} \,.$$

Lemma 8.2.2

1. *For any $\omega \in \mathscr{C}$,*

$$\Phi_{p_0}\left(\mathscr{G}_\omega\right) = P_{p_0}(\omega) \cdot \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right) \ , \tag{8.2.8}$$

$$\mathscr{G}_\omega = \{u \in \mathscr{G} : u \text{ is constant on } \Phi_{p_0}(\mathscr{G}_\omega)\} \,. \tag{8.2.9}$$

2. *For any $u \in \mathscr{G}$ and any subgroup $S \subset \mathscr{G}$,*

$$\Phi_{p_0}\left(uSu^{-1}\right) = \Psi_{u(p_0)^{-1}} \circ \vartheta_u\big(\Phi_{p_0}(S)\big) \,. \tag{8.2.10}$$

Proof 1. First, we prove (8.2.8). Let $\omega \in \mathscr{C}$ and $u \in \mathscr{G}_\omega$. By Lemma 6.1.4, the restriction of u to $P_{p_0}(\omega)$ is constant. By equivariance, it is also constant on the bundle $P_{p_0}(\omega) \cdot \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right)$: indeed, for $p \in P_{p_0}(\omega)$ and $k \in \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right)$, we have

$$u\left(\Psi_k(p)\right) = k^{-1}u(p)k = k^{-1}u(p_0)k = u(p_0) \,,$$

because $u(p_0) \in \mathrm{C}_G\left(\mathscr{H}_{p_0}(\omega)\right)$ by the Stabilizer Theorem 6.1.5. Thus,

$$P_{p_0}(\omega) \cdot \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right) \subset \Phi_{p_0}\left(\mathscr{G}_\omega\right) \ .$$

Conversely, let $p \in \Phi_{p_0}(\mathscr{G}_\omega)$. Then, $u(p) = u(p_0)$ for all $u \in \mathscr{G}_\omega$. Clearly, there exists $a \in G$ such that $\Psi_a(p) \in P_{p_0}(\omega)$ and, by Lemma 6.1.4,

$$u(p_0) = u\left(\Psi_a(p)\right) = a^{-1}u(p)a = a^{-1}u(p_0)a$$

[1] This choice of partial ordering corresponds to comparing the size of the orbits. It is consistent with [103] but not with [388] and several other authors who choose the inverse partial ordering.

for all $u \in \mathscr{G}_\omega$. Thus, by the Stabilizer Theorem 6.1.5, $a \in \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right)$. Hence,

$$p = \Psi_{a^{-1}}\left(\Psi_a(p)\right) \in P_{p_0}(\omega) \cdot \mathrm{C}^2_G\left(\mathscr{H}_{p_0}(\omega)\right) .$$

Now, consider (8.2.9). Inclusion from left to right holds by definition of $\Phi_{p_0}(\mathscr{G}_\omega)$. Conversely, by (8.2.8), if u is constant on $\Phi_{p_0}(\mathscr{G}_\omega)$, then it is constant on $P_{p_0}(\omega)$. By Lemma 6.1.4, this implies $u \in \mathscr{G}_\omega$.

2. We have $p \in \Phi_{p_0}\left(uSu^{-1}\right)$ iff $u(p)h(p)u(p)^{-1} = u(p_0)h(p_0)u(p_0)^{-1}$ for all $h \in S$. This is equivalent to

$$h(p_0) = u(p_0)^{-1}u(p)h(p)u(p)^{-1}u(p_0) = h\left(\Psi_{u(p_0)}\left(\vartheta_{u^{-1}}(p)\right)\right)$$

for all $h \in S$, that is, it is equivalent to $\Psi_{u(p_0)}\left(\vartheta_{u^{-1}}(p)\right) \in \Phi_{p_0}(S)$. ■

Remark 8.2.3 According to point 1 of Lemma 6.1.4, if the subgroup S is the stabilizer of a connection ω, then $\Phi_{p_0}(S)$ is a bundle reduction of class C^{k+1} of P. In [388], the subbundle $\Phi_{p_0}\left(\mathscr{G}_\omega\right)$ is called the evolution bundle generated by ω. ♦

Definition 8.2.4 (*Howe subgroup*) A subgroup $H \subset G$ is called a Howe subgroup if $H = \mathrm{C}_G(A)$ for some subset $A \subset G$.

Remark 8.2.5

1. Since the centralizer is a closed subgroup, a Howe subgroup is closed and, therefore, a Lie subgroup. Moreover, it is easy to see that $\mathrm{C}^3_G(A) = \mathrm{C}_G(A)$ for any subset $A \subset G$. Thus, if $H = \mathrm{C}_G(A)$, then

   $$\mathrm{C}^2_G(H) = \mathrm{C}^3_G(A) = \mathrm{C}_G(A) = H .$$

   Since $H \subset \mathrm{C}^2_G(H)$, we conclude that $H \subset G$ is Howe iff $H = \mathrm{C}^2_G(H)$.
2. If $H \subset G$ is Howe, then $H' = \mathrm{C}_G(H)$ is Howe, too, and one has

   $$H = \mathrm{C}^2_G(H) = \mathrm{C}_G(H') .$$

   A pair (H, H') of subgroups of G fulfilling $H = \mathrm{C}_G(H')$ and $H' = \mathrm{C}_G(H)$ is referred to as a Howe dual pair in G. To summarize, Howe subgroups are in one-to-one correspondence with Howe dual pairs via $H \to (H, \mathrm{C}_G(H))$.
3. Denote $\mathscr{G}_\omega(p_0) = \{u(p_0) : u \in \mathscr{G}_\omega\}$. By the Stabilizer Theorem 6.1.5, $\mathscr{G}_\omega(p_0) = \mathrm{C}_G(\mathscr{H}_{p_0}(\omega))$. Thus, by point 2, $\left(\mathscr{G}_\omega(p_0), \mathrm{C}^2_G(\mathscr{H}_{p_0}(\omega))\right)$ is a Howe dual pair in G.
4. A Howe dual pair is called reductive iff its members are reductive. Reductive Howe dual pairs play an important role in the representation theory of Lie groups, cf. [319]. There exist, essentially, two methods for the classification theory of reductive Howe dual pairs. One of them applies to the isometry groups of Hermitean spaces and uses the theory of Hermitean forms [456, 524, 561]. The other method applies to complex semisimple Lie algebras and uses root space techniques [537].

♦

Definition 8.2.6 Let $P(M, G)$ be a principal bundle.

1. A bundle reduction of P to a Howe subgroup will be called a Howe subbundle.
2. A bundle reduction Q of P of class C^r is said to be holonomy-induced if there exists a connected bundle reduction $\tilde{Q}$ of P of class C^r to a subgroup $\tilde{H} \subset G$ such that
$$Q = \tilde{Q} \cdot \mathrm{C}^2_G\left(\tilde{H}\right) . \tag{8.2.11}$$

   The set of isomorphism classes of holonomy-induced reductions of P of class C^0, factorized by the action of the structure group G, will be denoted by $\mathrm{Red}_*(P)$.

Remark 8.2.7

1. We equip $\mathrm{Red}_*(P)$ with a partial ordering as follows. For $\eta, \eta' \in \mathrm{Red}_*(P)$ we write $\eta \geq \eta'$ if there exist representatives Q of η and Q' of η' such that $Q \subset Q'$.
2. The reduction Q is the extension of $\tilde{Q}$ to the Howe subgroup of G generated by $\tilde{H}$. In particular, every holonomy-induced reduction is a Howe subbundle.
3. Assume that Q and $\tilde{Q}$ are of class C^0. By Proposition 3.6.2, $\tilde{Q}$ is vertically C^0-isomorphic to a smooth principal $\tilde{H}$-bundle $\tilde{Q}^\infty$. Let Q^∞ and P^∞ denote the extensions of $\tilde{Q}^\infty$ to the structure groups $\mathrm{C}_G(\tilde{H})$ and G, respectively. Every vertical C^0-isomorphism $\tilde{Q} \to \tilde{Q}^\infty$ extends to a vertical C^0-isomorphism $Q \to Q^\infty$ and to a vertical C^0-isomorphism $P \to P^\infty$. Since P and P^∞ are of class C^∞ and vertically C^0-isomorphic, Proposition 3.6.4 implies that they are vertically C^∞-isomorphic. Via such a C^∞-isomorphism, $\tilde{Q}^\infty$ and Q^∞ become beundle reductions of P of class C^∞. This shows that $\mathrm{Red}_*(P)$ coincides with the set of vertical C^∞-isomorphism classes of smooth holonomy-induced bundle reductions. ♦

Theorem 8.2.8 *Let M be a compact connected manifold and assume* $\dim M \geq 2$. *Then, the assignment Φ_{p_0} induces an order-preserving bijection from Σ onto* $\mathrm{Red}_*(P)$.

Proof In the proof, we have to make the Sobolev classes transparent.

Let $\tau \in \Sigma$ and let there be chosen a representative $S \subset \mathscr{G}^{k+1}$. There exists $\omega \in \mathscr{C}^k$ such that $S = \mathscr{G}^{k+1}_\omega$. According to point 1 of Lemma 8.2.2, $\Phi_{p_0}(S)$ is given by the extension of the bundle reduction $P_{p_0}(\omega) \subset P$ to the structure group $\mathrm{C}^2_G(\mathscr{H}_{p_0}(\omega))$. Since $P_{p_0}(\omega)$ is of class C^0, so is $\Phi_{p_0}(S)$. Since $P_{p_0}(\omega)$ is connected, $\Phi_{p_0}(S)$ is holonomy-induced of class C^0. According to point 2 of Lemma 8.2.2, if S is conjugate in $\mathscr{G}^{k+1}$ to some S', $\Phi_{p_0}(S)$ and $\Phi_{p_0}(S')$ are conjugate under the actions of $\mathscr{G}^{k+1}$ and G. Then, since vertical automorphisms from $\mathscr{G}^{k+1}$ are continuous, $\Phi_{p_0}(S)$ and $\Phi_{p_0}(S')$ are C^0-isomorphic. Thus, Φ_{p_0} projects to a mapping from Σ to $\mathrm{Red}_*(P)$.

To check that this mapping is surjective, let an element of $\mathrm{Red}_*(P)$ be given. By Remark 8.2.7/3, we may choose a representative $Q \subset P$ which is smooth and which is generated via (8.2.11) by a smooth connected bundle reduction $\tilde{Q}$. Using the principal action Ψ, we may achieve that $p_0 \in \tilde{Q}$. Since $\dim M \geq 2$, point 5 of Remark 1.7.16 yields that $\tilde{Q}$ carries a smooth connection with holonomy group $\tilde{H}$. This connection extends to a unique smooth connection ω on P obeying $P_{p_0}(\omega) = \tilde{Q}$ and $\mathscr{H}_{p_0}(\omega) = \tilde{H}$. Then, by point 1 of Lemma 8.2.2 and (8.2.11),

$$\Phi_{p_0}(\mathscr{G}_\omega^{k+1}) = \tilde{Q} \cdot \mathrm{C}_G^2(\tilde{H}) = Q\,.$$

This proves surjectivity.

To show that the projected mapping is injective, let $\tau, \tau' \in \Sigma$. Choose representatives S, S' and assume that $\Phi_{p_0}(S')$ and $\Phi_{p_0}(S) \cdot a$ are C^0-isomorphic for some $a \in G$. Since these bundles are of class C^{k+1}, Proposition 3.6.4 implies[2] that there exists a vertical isomorphism of class C^{k+1}. Every such isomorphism extends equivariantly to a vertical C^{k+1}-automorphism ϑ of P. Let $u \in \mathscr{G}^{k+1}$ denote the corresponding equivariant mapping. By construction,

$$\Phi_{p_0}(S') = \vartheta\big(\Psi_a\big(\Phi_{p_0}(S)\big)\big) = \Psi_a\big(\vartheta\big(\Phi_{p_0}(S)\big)\big)\,.$$

By (8.2.10), then

$$\Phi_{p_0}(S') = \Psi_{u(p_0)a}\big(\Phi_{p_0}(uSu^{-1})\big)\,.$$

This implies, in particular, that $\Psi_{u(p_0)a}(p_0)$ is in $\Phi_{p_0}(S')$ again, so that $u(p_0)a$ belongs to the structure group of $\Phi_{p_0}(S')$. Thus, in fact we have

$$\Phi_{p_0}(S') = \Phi_{p_0}(uSu^{-1})\,.$$

Now, the assertion follows from (8.2.9), because S and S' are stabilizers. ■

Remark 8.2.9

1. As a consequence of Theorem 8.2.8, the set Σ does not depend on k.
2. For later use, let us introduce the notation $\mathscr{C}^S$ for the subset of connections with stabilizer S, $\mathscr{C}^\tau$ for the subset of connections of orbit type τ and $\mathscr{M}^\tau$ for the subset of orbits of type τ. Correspondingly, we define

$$\mathscr{C}^{\leq S} := \bigcup_{S' \supseteq S} \mathscr{C}^{S'}\,, \quad \mathscr{C}^{\leq \tau} := \bigcup_{\tau' \leq \tau} \mathscr{C}^{\tau'}\,, \quad \mathscr{M}^{\leq \tau} := \bigcup_{\tau' \leq \tau} \mathscr{M}^{\tau'}\,,$$

   and, by analogy, $\mathscr{C}^{\geq S}$, $\mathscr{C}^{\geq \tau}$, $\mathscr{M}^{\geq \tau}$.
3. The notion of holonomy-induced bundle reduction may be viewed as an abstract version of the notion of evolution subbundle generated by a connection which was introduced in [388]. Correspondingly, Theorem 8.2.8 is an abstract version of Theorem 4.2.1 in [388]. The geometric ideas behind are also contained in [289, Sect. 2]. However, a rigorous proof was not given there.
4. General arguments show that $\mathrm{Red}_*(P)$ is countable, see Theorem 8.3.14 below. Hence, so is Σ. Countability of Σ is a necessary condition for this set to define a stratification. ♦

Theorem 8.2.8 will be used in the study of the gauge orbit stratification in Sect. 8.3 and in the computation of the gauge orbit types in Sects. 8.5 and 8.6.

[2]Clearly, the proposition holds with C^∞ replaced by any differentiability class.

8.3 The Gauge Orbit Stratification

In this section, we discuss the gauge orbit stratification in some detail. Our presentation is along the lines of the work of Kondracki and Rogulski, see [388] for a much more detailed exposition.

To start with, recall from Sect. 6.1 the natural operators

$$\mathrm{d}_\omega\,,\quad \mathrm{d}_\omega^*\,,\quad \Delta_\omega = \mathrm{d}_\omega^* \circ \mathrm{d}_\omega\,,\quad \Box_\omega = \mathrm{d}_\omega \circ \mathrm{d}_\omega^* + \mathrm{d}_\omega^* \circ \mathrm{d}_\omega\,,$$

depending smoothly on ω and sharing the equivariance property (6.1.15). Also recall that for d_ω acting on sections we write ∇^ω. The corresponding operators acting between appropriate Sobolev complections are denoted by the same symbols. By Theorem 6.1.9, these operators give rise to a natural L^2-orthogonal splitting $\mathrm{T}\mathscr{C} = \mathfrak{V} \oplus \mathfrak{H}$, where $\mathfrak{V}_\omega = \mathrm{im}(\nabla^\omega)$ and $\mathfrak{H}_\omega = \ker(\nabla^{\omega *})$. Thus,

$$\mathrm{T}_\omega\mathscr{C} = \mathrm{im}(\nabla^\omega) \oplus \ker(\nabla^{\omega *})\,. \tag{8.3.1}$$

Due to (6.1.15), the distributions $\mathfrak{V}$ and $\mathfrak{H}$ are equivariant,

$$\mathfrak{V}_{\omega^{(u)}} = (\mathfrak{V}_\omega)^{(u)}\,,\quad \mathfrak{H}_{\omega^{(u)}} = (\mathfrak{H}_\omega)^{(u)}\,. \tag{8.3.2}$$

Consequently, $\ker(\nabla^{\omega *})$ may be viewed as a model of the tangent space of $\mathscr{M}$ at $[\omega]$. This will be made precise in the sequel.

The splitting (8.3.1) will be fundamental for all constructions discussed within this and the next two sections. In particular, it guarantees that the gauge orbits are submanifolds, it is basic for the construction of tubes and slices, it provides the fibre bundle structure on each stratum and it induces natural (weak) Riemannian metrics on each stratum of the gauge orbit space via a Kaluza–Klein-type construction.

Theorem 8.3.1 (Orbit Theorem) *For any $\omega \in \mathscr{C}$, the orbit of ω under the action of $\mathscr{G}$ is a smooth embedded submanifold of $\mathscr{C}$, naturally diffeomorphic to $\mathscr{G}/\mathscr{G}_\omega$.*

Proof The orbit mapping $\iota_\omega : \mathscr{G} \to \mathscr{C}$ defined by $\iota_\omega(u) := \omega^{(u)}$ descends to an injective mapping $\tilde{\iota}_\omega : \mathscr{G}/\mathscr{G}_\omega \to \mathscr{C}$. The latter is smooth, because $\mathscr{G} \to \mathscr{G}/\mathscr{G}_\omega$ is a locally trivial principal bundle. It is a homeomorphism onto its image, where the latter is endowed with the relative topology induced from $\mathscr{C}$: this follows from the properness of the action by the same argument as in the finite dimensional case, see Corollary 6.3.5 in Part I. It remains to show that $\tilde{\iota}_\omega$ is an immersion, that is, its tangent mapping at any point is injective and has closed range. Clearly, it suffices to check this at the point $[\mathbb{1}] \in \mathscr{G}/\mathscr{G}_\omega$, the class of the unit element of $\mathscr{G}$. Choose a local section s of $\mathscr{G} \to \mathscr{G}/\mathscr{G}_\omega$ in a neighbourhood of $[\mathbb{1}]$. Then, the image $\mathscr{Y}$ of $\mathrm{T}_{[\mathbb{1}]}\mathscr{G}/\mathscr{G}_\omega$ under the tangent mapping $s'_{[\mathbb{1}]}$ is a closed complement of the subspace $\mathrm{L}\mathscr{G}_\omega$ in $\mathrm{L}\mathscr{G}$. Since

$$(\tilde{\iota}_\omega)'_{[\mathbb{1}]} = (\iota_\omega)'_{\mathbb{1}} \circ s'_{[\mathbb{1}]}\,,$$

it suffices to show that $(\iota_\omega)'_{\mathbb{1}}$ has closed range and that its restriction to $\mathscr{Y}$ is injective. Using (1.8.7), for $\xi \in \mathrm{L}\mathscr{G}$, we compute

$$\iota'_\omega \xi = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \iota_\omega\big(\exp(t\xi)\big) = \frac{\mathrm{d}}{\mathrm{d}t}_{\restriction 0} \omega^{(\exp(t\xi))} = \nabla^\omega \xi \,.$$

Thus, closedness follows from the decomposition (8.3.1) and injectivity follows from the Stabilizer Theorem 6.1.5. ■

Remark 8.3.2 As a consequence of the Orbit Theorem 8.3.1, the vector bundles $\mathrm{T}(\mathscr{G} \cdot \omega)$ and $\mathrm{T}\mathscr{C}_{\restriction \mathscr{G} \cdot \omega}$ are smooth subbundles of $\mathrm{T}\mathscr{C}$. ♦

A second important consequence of the decomposition (8.3.1) is a Tubular Neighbourhood Theorem for the action of $\mathscr{G}$. As we will see, the local slices are simply given by the distribution $\mathfrak{H}$ intersected with local balls in $\mathscr{C}$. The radius of the latter must be defined in accordance with the Sobolev norm. For that purpose, we use the strong Riemannian metric γ^k on $\mathscr{C}$ defined by assigning to $\omega \in \mathscr{C}$ the corresponding Sobolev scalar product given by formula (5.7.8), that is,

$$\gamma^k_\omega(\alpha, \beta) := \int_M \Big\{ \langle \alpha, \beta\rangle + \langle \nabla^{\tilde\omega}\alpha, \nabla^{\tilde\omega}\beta\rangle + \cdots + \langle (\nabla^{\tilde\omega})^k\alpha, (\nabla^{\tilde\omega})^k\beta\rangle \Big\} \, \mathsf{v}_\mathsf{g} \,, \quad (8.3.3)$$

where $\alpha, \beta \in \mathrm{T}_\omega\mathscr{C} = W^k(\mathrm{T}^*M \otimes \mathrm{Ad}(P))$ and $\tilde\omega = \omega^0 + \omega$ with ω^0 denoting the Levi-Civita connection of the metric g on M. Due to $\nabla^{\omega^{(u)}} = \mathrm{Ad}(u^{-1}) \circ \nabla^\omega \circ \mathrm{Ad}(u)$, the metric γ^k is $\mathscr{G}$-invariant,

$$\gamma^k_{\omega^{(u)}}\left(\alpha^{(u)}, \beta^{(u)}\right) = \gamma^k_\omega(\alpha, \beta)\,,$$

where $\alpha^{(u)} = \mathrm{Ad}(u^{-1})\alpha$ according to (8.2.2). Putting $k = 0$ in (8.3.3), we obtain the natural weak Riemannian metric $\gamma^0 \equiv \gamma$ corresponding to the L^2-scalar product.

The following theorem is a generalization of the Tubular Neighbourhood Theorem 6.4.3 of Part I to the infinite-dimensional context under consideration. Except for the use of an invariant metric, the idea of the proof is the same. The notions of tubular neighbourhood and slice, introduced in Definition I/6.4.1, carry over in an obvious way.

Theorem 8.3.3 (Tubular Neighbourhood Theorem) *Every gauge orbit admits a tubular neighbourhood.*

Proof Let $\pi : \mathscr{C} \to \mathscr{M}$ be the canonical projection. By (8.3.1), for any $x \in \mathscr{M}$, the normal bundle of the orbit $\pi^{-1}(x)$ may be identified with

$$\mathrm{N}_x = \mathfrak{H}_{\restriction \pi^{-1}(x)} \,.$$

According to (8.3.2), N_x is equivariant. This, together with the local triviality of the projection $\mathscr{G} \to \mathscr{G}/\mathscr{G}_\omega$, implies that N_x is a smooth locally trivial vector subbundle of $\mathrm{T}\mathscr{C}_{\restriction \pi^{-1}(x)}$. For $\varepsilon > 0$, consider the smooth subbundle

$$\mathrm{N}_{x,\varepsilon} := \left\{ (\omega, \alpha) \in \mathrm{N}_x : \sqrt{\gamma^k_\omega(\alpha, \alpha)} \leq \varepsilon \right\}$$

of N_x.[3] Due to the $\mathscr{G}$-invariance of γ^k, $\mathrm{N}_{x,\varepsilon}$ is equivariant. As $\mathscr{G}$-manifolds, N_x and $\mathrm{N}_{x,\varepsilon}$ are equivariantly diffeomorphic through the rescaling mapping

$$\rho_\varepsilon : \mathrm{N}_x \to \mathrm{N}_{x,\varepsilon} , \quad (\omega, \alpha) \mapsto \left(\omega, \frac{\varepsilon}{\sqrt{\gamma^k_\omega(\alpha, \alpha) + 1}} \alpha \right) \tag{8.3.4}$$

(Exercise 8.3.2). By restriction, the mapping

$$\exp : \mathrm{T}\mathscr{C} \to \mathscr{C} , \quad (\omega, \alpha) \mapsto \omega + \alpha ,$$

which is in fact the exponential mapping with respect to the L^2-metric, defines a smooth $\mathscr{G}$-equivariant mapping $\mathrm{N}_{x,\varepsilon} \to \mathscr{C}$. The image is

$$\mathscr{U}_{x,\varepsilon} = \left\{ \omega + \alpha \in \mathscr{C} : \pi(\omega) = x , (\omega, \alpha) \in \mathrm{N}_{x,\varepsilon} \right\} . \tag{8.3.5}$$

Clearly, $\mathscr{U}_{x,\varepsilon}$ is open in $\mathscr{C}$ and $\mathscr{G}$-invariant. Moreover, by the same argument as in the proof of the Tubular Neighbourhood Theorem 6.4.3 of Part I, one can show that there exists $\varepsilon > 0$ such that the restriction of exp to $\mathrm{N}_{x,\varepsilon} \subset \mathrm{T}\mathscr{C}$ is injective. Consequently, the composition

$$\exp \circ \rho_\varepsilon : \mathrm{N}_x \to \mathscr{C} \tag{8.3.6}$$

is an equivariant diffeomorphism onto $\mathscr{U}_{x,\varepsilon}$, that is, it defines a tubular neighbourhood of the gauge orbit $\pi^{-1}(x)$. ∎

In the following, whenever we write $\mathscr{U}_{x,\varepsilon}$ or $\mathscr{S}_{\omega,\varepsilon}$, it is understood that ε is small enough to make the subset a tubular neighbourhood or a slice, respectively.

Remark 8.3.4 (*Slices*) As a consequence of the Tubular Neighbourhood Theorem 8.3.3, the action of $\mathscr{G}$ on $\mathscr{C}$ admits a slice at every point $\omega \in \mathscr{C}$. According to (8.3.5), this slice is given by the subset

$$\mathscr{S}_{\omega,\varepsilon} := \left\{ \omega + \alpha \in \mathscr{C} : (\omega, \alpha) \in \mathrm{N}_{x,\varepsilon} \right\} \tag{8.3.7}$$

of $\mathscr{U}_{x,\varepsilon}$. By construction, $\mathscr{S}_{\omega,\varepsilon}$ obeys the defining properties of a slice (Exercise 8.3.1):

1. $\mathscr{U}_{x,\varepsilon} = \mathscr{G} \cdot \mathscr{S}_{\omega,\varepsilon}$,
2. $\mathscr{S}_{\omega,\varepsilon}$ is closed in $\mathscr{U}_{x,\varepsilon}$,

[3] Note that $\mathrm{N}_{x,\varepsilon}$ is not just the ε-disk bundle of N_x, because orthogonality and length are taken with respect to different metrics.

3. $\mathscr{S}_{\omega,\varepsilon}$ is invariant under the stabilizer $\mathscr{G}_\omega$,
4. For any $u \in \mathscr{G}$, if $\left(\mathscr{S}_{\omega,\varepsilon}\right)^{(u)} \cap \mathscr{S}_{\omega,\varepsilon} \neq \varnothing$, then $u \in \mathscr{G}_\omega$.

Since $\mathscr{S}_{\omega,\varepsilon}$ is an open subset of the closed affine subspace $\omega + \mathfrak{H}_\omega$ of $\mathscr{C}$, for every $\omega' \in \mathscr{S}_{\omega,\varepsilon}$, the tangent space is given by $\mathrm{T}_{\omega'}\mathscr{S}_{\omega,\varepsilon} = \mathfrak{H}_\omega$. Consequently,

$$\mathrm{T}\mathscr{S}_{\omega,\varepsilon} = \mathscr{S}_{\omega,\varepsilon} \times \mathfrak{H}_\omega \,. \tag{8.3.8}$$

For further use, let us draw the following conclusion from this observation. Since $\mathscr{U}_{\omega,\varepsilon}$ is generated from $\mathscr{S}_{\omega,\varepsilon}$ by the action of $\mathscr{G}$, for every $\omega' \in \mathscr{S}_{\omega,\varepsilon}$ we have $\mathrm{T}_{\omega'}\mathscr{S}_{\omega,\varepsilon} + \mathrm{T}_{\omega'}(\mathscr{G}\cdot\omega') = \mathrm{T}_{\omega'}\mathscr{U}_{\omega,\varepsilon} = \mathscr{T}$, where the sum need not be direct, as $\mathscr{G}_{\omega'}$ may be strictly smaller than $\mathscr{G}_\omega$. Thus, for all $\omega' \in \mathscr{S}_{\omega,\varepsilon}$,

$$\mathfrak{H}_\omega + \mathfrak{V}_{\omega'} = \mathscr{T} \,. \tag{8.3.9}$$

♦

Remark 8.3.5 (*Local Slice Theorem*) The authors of [388] actually prove more: they show that for any $x \in \mathscr{M}$ and any open invariant neighbourhood U of $\pi^{-1}(x)$ in $\mathscr{C}$ there exists $\varepsilon > 0$ such that $\overline{\mathscr{U}_{x,\varepsilon}} \subset U$ and $U \setminus \overline{\mathscr{U}_{x,\varepsilon}} \neq \varnothing$. They call this the Local Slice Theorem. As a consequence, $\mathscr{M}$ is a regular topological space, meaning that whenever one has a closed subset V and a point $x \notin V$, then there exists a neighbourhood of x , whose closure in $\mathscr{M}$ does not intersect V. According to Urysohn's metrization theorem, regularity in combination with second countability (which is due to the separability of $\mathscr{C}$) then implies that $\mathscr{M}$ is a metrizable space. ♦

Theorem 8.3.3 has the following immediate consequence.

Corollary 8.3.6 *For every stabilizer S and every orbit type τ, the following subsets are open:*

$$\mathscr{C}^S \textit{ in } \mathscr{C}^{\leq S}\,, \quad \mathscr{C}^\tau \textit{ in } \mathscr{C}^{\leq\tau}\,, \quad \mathscr{M}^\tau \textit{ in } \mathscr{M}^{\leq\tau}\,.$$

Proof This is a consequence of the fact that property 4 of slice implies that

$$\mathscr{U}_{x,\varepsilon} \subset \mathscr{C}^{\geq\tau}\,, \quad \mathscr{S}_{\omega,\varepsilon} \subset \mathscr{C}^{\geq S} \tag{8.3.10}$$

for any $x \in \mathscr{M}^\tau$ and any $\omega \in \mathscr{C}^S$. Indeed, let $\omega \in \mathscr{C}^S$. Since $\mathscr{U}_{\pi(\omega),\varepsilon}$ is a neighbourhood of ω in $\mathscr{C}$, its intersection with $\mathscr{C}^{\leq S}$ is a neighbourhood of ω in $\mathscr{C}^{\leq S}$. Due to (8.3.10), the intersection is contained in

$$\mathscr{C}^{\geq S} \cap \mathscr{C}^{\leq S} = \mathscr{C}^S \,.$$

The argument applies without change to $\mathscr{C}^\tau$. For $\mathscr{M}^\tau$ it suffices to note that $\mathscr{U}_{x,\varepsilon}$ projects to a neighbourhood of x in $\mathscr{M}$. ■

It is well known that the connections with trivial stabilizer are dense in $\mathscr{C}$, see [591]. More generally, the question arises, whether $\mathscr{C}^\tau$ is dense in $\mathscr{C}^{\leq\tau}$, that is, in other words, whether a connection with nontrivial stabilizer can be approximated by connections with a prescribed, strictly smaller stabilizer. The answer is given by the following result of Kondracki and Rogulski [388].

Theorem 8.3.7 (Approximation Theorem) *Assume* $\dim M \geq 2$. *Let* $\omega \in \mathscr{C}$ *and let* Q *be a connected bundle reduction of* P *to a (not necessarily closed) Lie subgroup. Assume that* Q *contains a holonomy bundle of* ω. *Then, there exists* $\alpha \in \mathscr{T}$ *such that all* $\omega + t\alpha$, $t \in \mathbb{R} \setminus \{0\}$, *have holonomy bundle* Q. ■

Corollary 8.3.8 *For every stabilizer* S *and every orbit type* τ, *the following subsets are dense:*

$$\mathscr{C}^S \text{ in } \mathscr{C}^{\leq S}\,, \quad \mathscr{C}^\tau \text{ in } \mathscr{C}^{\leq\tau}\,, \quad \mathscr{M}^\tau \text{ in } \mathscr{M}^{\leq\tau}\,. \tag{8.3.11}$$

Proof Choose a point $p_0 \in P$. Let $\omega \in \mathscr{C}^{\leq S}$. Then, $S \subset \mathscr{G}_\omega$ and hence $\Phi_{p_0}(\mathscr{G}_\omega) \subset \Phi_{p_0}(S)$. By Lemma 8.2.2, then $\Phi_{p_0}(S)$ contains the holonomy bundle $P_{p_0}(\omega)$. Of course, so does the connected component $(\Phi_{p_0}(S))_{p_0}$ of $\Phi_{p_0}(S)$ containing p_0. Thus, Theorem 8.3.7 yields that ω can be approximated by connections with holonomy bundle $(\Phi_{p_0}(S))_{p_0}$. Since $\Phi_{p_0}(S)$ is holonomy-induced, it is induced via (8.2.11) by $(\Phi_{p_0}(S))_{p_0}$. Hence, all connections with holonomy bundle $(\Phi_{p_0}(S))_{p_0}$ have stabilizer S. This shows that $\mathscr{C}^S$ is dense in $\mathscr{C}^{\leq S}$. Then, denseness of $\mathscr{C}^\tau$ in $\mathscr{C}^{\leq\tau}$ and denseness of $\mathscr{M}^\tau$ in $\mathscr{M}^{\leq\tau}$ follow. ■

Remark 8.3.9

1. Corollaries 8.3.6 and 8.3.8 imply that $\mathscr{C}^S$, $\mathscr{C}^\tau$ and $\mathscr{M}^\tau$ are generic subsets of, respectively, $\mathscr{C}^{\leq S}$, $\mathscr{C}^{\leq\tau}$ and $\mathscr{M}^{\leq\tau}$.
2. For every orbit type τ,

$$\overline{\mathscr{C}^\tau} = \mathscr{C}^{\leq\tau}\,, \quad \overline{\mathscr{M}^\tau} = \mathscr{M}^{\leq\tau}\,. \tag{8.3.12}$$

   The inclusions from right to left are obvious from Corollary 8.3.8. The converse inclusions follow from the Tubular Neighbourhood Theorem: let $\omega \in \overline{\mathscr{C}^\tau}$. Consider $\mathscr{U}_{\omega,\varepsilon} \cap \overline{\mathscr{C}^\tau}$. Since this is a neighbourhood of ω in $\overline{\mathscr{C}^\tau}$, it contains some $\omega' \in \mathscr{C}^\tau$. According to (8.3.10), then τ is greater than or equal to the type of ω. Thus, $\omega \in \mathscr{C}^{\leq\tau}$. The inclusion for $\mathscr{M}^\tau$ then follows by noting that for saturated sets like $\mathscr{C}^\tau$, closure and projection commute.
3. Similarly, for stabilizers S one has

$$\overline{\mathscr{C}^S} = \mathscr{C}^{\leq S}\,. \tag{8.3.13}$$

   While the inclusion from right to left is again due to Corollary 8.3.8, the converse inclusion can be proved without the Tubular Neighbourhood Theorem by the following simple argument. For any $u \in \mathscr{G}$, define a mapping

$$\varphi_u : \mathscr{C} \to \mathscr{T}\,, \quad \varphi_u(\omega) := \omega^{(u)} - \omega\,.$$

Since these mappings are continuous, the subsets $\varphi_u^{-1}(0)$ are closed in $\mathscr{C}$. Then $\mathscr{C}^{\leq S} = \bigcap_{u \in S} \varphi_u^{-1}(0)$ is closed. Hence, $\overline{\mathscr{C}^S} \subset \mathscr{C}^{\leq S}$. ♦

Next, we study the projections

$$\pi^\tau : \mathscr{C}^\tau \to \mathscr{M}^\tau$$

induced from $\pi : \mathscr{C} \to \mathscr{M}$. We will see that they can be equipped with the structure of smooth locally trivial fibre bundles. As a result, in a sense, π fibres over the set of orbit types into such bundles.

We start by showing that the configuration space strata are submanifolds. For any $\omega \in \mathscr{C}^\tau$, define

$$\mathscr{S}^\tau_{\omega,\varepsilon} := \mathscr{S}_{\omega,\varepsilon} \cap \mathscr{C}^\tau \,, \tag{8.3.14}$$

$$\mathfrak{H}^\tau_\omega := \{\alpha \in \mathfrak{H}_\omega : \mathscr{G}_\alpha \supset \mathscr{G}_\omega\} \,, \tag{8.3.15}$$

$$\mathfrak{H}^\tau_{\omega,\varepsilon} := \{\alpha \in \mathfrak{H}^\tau_\omega : \sqrt{\gamma^k(\alpha,\alpha)} < \varepsilon\} \,.$$

Due to (8.3.10), for all $\omega' \in \mathscr{S}^\tau_{\omega,\varepsilon}$, we have

$$\mathscr{G}_{\omega'} = \mathscr{G}_\omega \,. \tag{8.3.16}$$

Proposition 8.3.10 *For every orbit type τ and every stabilizer S, $\mathscr{C}^\tau$ and $\mathscr{C}^S$ are smooth submanifolds of $\mathscr{C}$.*

Proof To prove that $\mathscr{C}^\tau$ is a submanifold of $\mathscr{C}$, it suffices to show that for any $x \in \mathscr{M}^\tau$ the subset

$$\mathscr{U}^\tau_{x,\varepsilon} := \mathscr{U}_{x,\varepsilon} \cap \mathscr{C}^\tau$$

is a submanifold of $\mathscr{U}_{x,\varepsilon}$. By (8.3.16),

$$\mathscr{S}^\tau_{\omega,\varepsilon} = \{\omega + \alpha \in \mathscr{S}_{\omega,\varepsilon} : \mathscr{G}_{\omega+\alpha} = \mathscr{G}_\omega\} \,.$$

Since $\mathscr{G}_{\omega+\alpha} = \mathscr{G}_\omega$ iff $\mathscr{G}_\alpha \supset \mathscr{G}_\omega$, then

$$\mathscr{S}^\tau_{\omega,\varepsilon} = \{\omega + \alpha \in \mathscr{C} : \alpha \in \mathfrak{H}^\tau_{\omega,\varepsilon}\} \tag{8.3.17}$$

and thus

$$\mathscr{U}^\tau_{x,\varepsilon} = \{\omega + \alpha \in \mathscr{C} : \omega \in \pi^{-1}(x) \,,\ \alpha \in \mathfrak{H}^\tau_{\omega,\varepsilon}\} \,.$$

Therefore, the preimage of $\mathscr{U}^\tau_{x,\varepsilon}$ under the equivariant diffeomorphism (8.3.6) is the vector subbundle

$$\mathrm{N}^\tau_x = \bigcup_{\omega \in \pi^{-1}(x)} \mathfrak{H}^\tau_\omega$$

of N_x. Since N_x^τ is equivariant and since its fibres are closed subspaces of $\mathscr{T}$, it is a smooth subbundle of $\mathrm{T}\mathscr{C}_{\restriction \pi^{-1}(x)}$ and hence of N_x. By the Tubular Neighbourhood Theorem, it follows that $\mathscr{U}_{x,\varepsilon}^\tau$ is a smooth submanifold of $\mathscr{U}_{x,\varepsilon}$, for any $x \in \mathscr{M}^\tau$. As a result, $\mathscr{C}^\tau$ is a submanifold of $\mathscr{C}$, as asserted.

To prove that $\mathscr{C}^S$ is a submanifold of $\mathscr{C}$, we observe that $\mathscr{C}^{\leq S}$ is a closed affine subspace of $\mathscr{C}$ with translation vector space given by the closed subspace $\{\alpha \in \mathscr{T} : \mathscr{G}_\alpha \supset S\}$ of $\mathscr{T}$. Thus, the assertion follows from Corollary 8.3.6. ■

Remark 8.3.11

1. The vector subbundle N_x^τ is in fact trivial, with a smooth trivialization given by

$$\mathscr{G}/\mathscr{G}_\omega \times \mathfrak{H}_\omega^\tau \to \mathrm{N}_x^\tau\,, \quad ([u], \alpha) \mapsto (\omega^{(u)}, \alpha^{(u)})\,,$$

for some $\omega \in \pi^{-1}(x)$. Note that this mapping is well defined precisely because $\mathscr{G}_\alpha \supset \mathscr{G}_\omega$. It follows that $\mathscr{U}_{x,\varepsilon}^\tau$ also has a direct product structure. This can be made explicit by introducing mappings

$$\chi_{\omega,\varepsilon}^\tau : \mathscr{S}_{\omega,\varepsilon}^\tau \times \mathscr{G}/\mathscr{G}_\omega \to \mathscr{U}_{\pi(\omega),\varepsilon}^\tau\,, \quad (\omega', [u]) \mapsto \omega'^{(u)}\,, \tag{8.3.18}$$

which are easily seen to be diffeomorphisms. Note that, for obvious reasons, the roles of fibre and base are interchanged here.

2. Clearly, $\mathfrak{H}_\omega^\tau$ is a closed subspace of $\mathscr{C}$ for every $\omega \in \mathscr{C}^\tau$. Hence, by (8.3.17), the partial slice $\mathscr{S}_{\omega,\varepsilon}^\tau$ is an open subset of the closed affine subspace $\omega + \mathfrak{H}_\omega^\tau$ of $\mathscr{C}$. By analogy with the case of the full slice $\mathscr{S}_{\omega,\varepsilon}$ in Remark 8.3.4, this implies

$$\mathrm{T}\mathscr{S}_{\omega,\varepsilon}^\tau = \mathscr{S}_{\omega,\varepsilon}^\tau \times \mathfrak{H}_\omega^\tau\,. \tag{8.3.19}$$

♦

Proposition 8.3.12 *For every orbit type τ, the following hold true.*

1. *$\mathscr{M}^\tau$ is a smooth manifold.*
2. *$\mathscr{C}^\tau$ has the structure of a locally trivial fibre bundle over $\mathscr{M}^\tau$ with typical fibre $\mathscr{G}/\mathscr{G}_\omega$, where $\omega \in \mathscr{C}^\tau$.*

Proof We shall construct an atlas of the stratum $\mathscr{M}^\tau$ using the partial slices $\mathscr{S}_{\omega,\varepsilon}^\tau$, $\omega \in \mathscr{C}^\tau$. For any $x \in \mathscr{M}^\tau$, define

$$V_{x,\varepsilon}^\tau := \pi(\mathscr{U}_{x,\varepsilon}^\tau)\,.$$

This is is an open subset of $\mathscr{M}^\tau$, because $V_{x,\varepsilon}^\tau = \mathscr{M}^\tau \cap \pi(\mathscr{U}_{x,\varepsilon})$ and $\pi(\mathscr{U}_{x,\varepsilon})$ is open in $\mathscr{M}$. By restriction in domain and range, for any $\omega \in \pi^{-1}(x)$, π defines a mapping

$$\pi_{\omega,\varepsilon}^\tau : \mathscr{S}_{\omega,\varepsilon}^\tau \to V_{x,\varepsilon}^\tau\,. \tag{8.3.20}$$

By (8.3.16), we have that $\pi^\tau_{\omega,\varepsilon}$ is bijective. We show that it is open and hence a homeomorphism onto $V^\tau_{x,\varepsilon}$. Let $U \subset \mathscr{S}^\tau_{\omega,\varepsilon}$ be open. Then, $U = \mathscr{S}^\tau_{\omega,\varepsilon} \cap U'$, where $U' \subset \mathscr{S}_{\omega,\varepsilon}$ is open. Using a local trivialization of the normal bundle N_x, one can check that the saturation $\tilde{U}' = U'^{(\mathscr{G})}$ is open in $\mathscr{U}_{x,\varepsilon}$ and hence in $\mathscr{C}$. Therefore, $\pi(\tilde{U}')$ is open in $\mathscr{M}$. Using (8.3.10) and the fact that $\tilde{U}'$ is saturated, we obtain

$$\pi(U) = \pi(\mathscr{S}^\tau_{\omega,\varepsilon} \cap \tilde{U}') = \pi(\mathscr{S}^\tau_{\omega,\varepsilon}) \cap \pi(\tilde{U}') = V^\tau_{x,\varepsilon} \cap \pi(\tilde{U}')\,,$$

which shows that $\pi(U)$ is open in $V^\tau_{x,\varepsilon}$. Hence, (8.3.20) is a homeomorphism, indeed. Combining this with the observation of Remark 8.3.11/2 that the partial slices $\mathscr{S}^\tau_{\omega,\varepsilon}$ are open subsets of closed affine subspaces of $\mathscr{C}$, we conclude that the family $(V^\tau_{\pi(\omega),\varepsilon}, (\pi^\tau_{\omega,\varepsilon})^{-1})$, $\omega \in \mathscr{C}^\tau$, provides a covering of $\mathscr{M}^\tau$ by local charts (one can make this more explicit by further mapping $\mathscr{S}^\tau_{\omega,\varepsilon} \to \mathfrak{H}^\tau_{\omega,\varepsilon}$).

It remains to check that the transition mappings between these charts are smooth. Due to (8.3.18), for any $\omega_1, \omega_2 \in \mathscr{C}^\tau$, we have a diffeomorphism

$$\left(\chi^\tau_{\omega_2,\varepsilon_2}\right)^{-1} \circ \chi^\tau_{\omega_1,\varepsilon_1} : \mathscr{S}^\tau_{\omega_1,\varepsilon_1} \cap \mathscr{U}^\tau_{\pi(\omega_2),\varepsilon_2} \times \mathscr{G}/\mathscr{G}_{\omega_1} \longrightarrow \mathscr{S}^\tau_{\omega_2,\varepsilon_2} \cap \mathscr{U}^\tau_{\pi(\omega_1),\varepsilon_1} \times \mathscr{G}/\mathscr{G}_{\omega_2}\,.$$

Now, the transition mapping $\left(\pi^\tau_{\omega_2,\varepsilon_2}\right)^{-1} \circ \pi^\tau_{\omega_1,\varepsilon_1}$ is given by the composition of the embedding $\omega' \mapsto (\omega', [\mathbb{1}])$, the above diffeomorphism, and projection to the first component. Hence, it is smooth. Thus, the atlas we have constructed equips $\mathscr{M}^\tau$ with the structure of a smooth Hilbert manifold. This proves the first assertion.

To prove the second assertion, we observe that the local diffeomorphisms $\chi^\tau_{\omega,\varepsilon}$ define local diffeomorphisms

$$V^\tau_{\pi(\omega),\varepsilon} \times \mathscr{G}/\mathscr{G}_\omega \xrightarrow{(\pi^\tau_{\omega,\varepsilon})^{-1} \times \mathrm{id}} \mathscr{S}^\tau_{\pi(\omega),\varepsilon} \times \mathscr{G}/\mathscr{G}_\omega \xrightarrow{\chi^\tau_{\omega,\varepsilon}} \mathscr{U}^\tau_{\pi(\omega),\varepsilon}\,.$$

These mappings provide a covering of $\mathscr{C}^\tau$ by local trivializations of the projection $\pi^\tau : \mathscr{C}^\tau \to \mathscr{M}^\tau$. Thus, the latter is a smooth locally trivial fibre bundle with typical fibre $\mathscr{G}/\mathscr{G}_\omega$. ∎

Remark 8.3.13 Let us consider, in particular, the principal orbit type $\tau = \tau_\mathrm{p}$, which is the conjugacy class consisting of the subgroup $\tilde{\mathrm{Z}}(G)$ of constant functions $P \to \mathrm{Z}(G)$, where $\mathrm{Z}(G)$ denotes the center of G. Since $\tilde{\mathrm{Z}}(G)$ is normal in $\mathscr{G}$, the smooth locally trivial fibre bundle

$$\pi^\mathrm{p} : \mathscr{C}^\mathrm{p} \to \mathscr{M}^\mathrm{p}$$

is in fact principal, with structure group $\widetilde{\mathscr{G}} := \mathscr{G}/\tilde{\mathrm{Z}}(G)$. This bundle has been studied intensively [454, 455, 476, 591]. An important aspect is that the nontriviality of this bundle is an obstruction to the existence of smooth (or even continuous) gauges [591]. This explains the Gribov ambiguity [258] in geometric terms, see Sect. 9.2 for a detailed discussion.

For the other orbit types, representatives S are not normal in $\mathscr{G}$. In order to have a similar picture as in the case of the principal stratum, one would have to take the submanifold $\mathscr{C}^S$ of connections with stabilizer S. $\mathscr{C}^S$ is acted upon freely by N/S, where N denotes the normalizer of S in $\mathscr{G}$. Provided one could show that N is a Lie subgroup of $\mathscr{G}$ (a problem which, to our knowledge, has not been solved yet) the projection $\pi^S : \mathscr{C}^S \to \mathscr{M}^\tau$ would be a smooth locally trivial principal fibre bundle and $\pi^\tau : \mathscr{C}^\tau \to \mathscr{M}^\tau$ would be associated with this bundle via the action of N/S on $\mathscr{G}/S$. ♦

In the remainder, we discuss the properties of the decomposition of $\mathscr{M}$ into the orbit type subsets $\mathscr{M}^\tau$. Following the terminology of Kondracki and Rogulski [388],[4] a stratification of a topological space X is a countable disjoint decomposition into smooth manifolds X_i, called the strata, such that for all i, i' the frontier condition holds:

$$X_i \cap \overline{X_{i'}} \neq \varnothing \quad \Rightarrow \quad X_i \subset \overline{X_{i'}} .$$

As this notion is rather weak, one usually adds additional assumptions about the linking between the strata, thus arriving at special types of stratification. According to [388], the type of stratification appropriate here is defined by the additional property

$$X_i \cap \overline{X_{i'}} \neq \varnothing \quad \Rightarrow \quad X_i \text{ closed in } X_i \cup X_{i'} .$$

Such a stratification is called regular.

The following result belongs to Kondracki and Rogulski [388].

Theorem 8.3.14 (Stratification Theorem) *The decomposition of $\mathscr{M}$ by orbit types is a regular stratification.*

Proof We first check countability of orbit types. By Theorem 8.2.8, orbit types are in one-to-one correspondence with certain reductions of P to Howe subgroups, modulo isomorphy of the reductions and modulo conjugacy of the subgroups. In view of this, the following facts ensure countability:

(a) Howe subgroups are closed.
(b) There are at most countably many conjugacy classes of closed subgroups in a compact Lie group [383].[5]
(c) There are at most countably many isomorphism classes of principal bundles with a given compact structure group G over a given manifold M: by Theorem 3.4.23, these classes are in one-to-one correspondence with homotopy classes of mappings from M to the classifying space BG. Since both M and BG can be given a CW-complex structure and since that of M is finite, the Cellular Approximation Theorem implies that there are at most countably many such classes.

---

[4] See the remarks on the notion of stratification in Sect. 6.6 of Part I.

[5] Let us note that the number of Howe subgroups in a compact Lie group is actually finite. This follows from the fact that any centralizer in a compact Lie group is generated by finitely many elements [92, Chap. 9] and that a compact group action on a compact manifold has a finite number of orbit types [103].

It remains to prove the frontier and regularity conditions. Let τ, τ' be orbit types such that $\overline{\mathscr{M}^\tau} \cap \mathscr{M}^{\tau'} \neq \varnothing$. According to the closure formula (8.3.12), $\overline{\mathscr{M}^\tau}$ is a union of strata. If $\mathscr{M}^{\tau'}$ intersects this union, then it must in fact coincide with one of these strata. This implies $\mathscr{M}^{\tau'} \subset \overline{\mathscr{M}^\tau}$. Thus, the frontier condition is fulfilled.

On the other hand, by Corollary 8.3.6, $\mathscr{M}^\tau$ is open in $\mathscr{M}^{\leq\tau}$ and hence in $\overline{\mathscr{M}^\tau}$. Then $\mathscr{M}^\tau$ is open in $\mathscr{M}^\tau \cup \mathscr{M}^{\tau'}$, because the latter is a subset of $\overline{\mathscr{M}^\tau}$ due to the frontier condition. Then $\mathscr{M}^{\tau'}$, being the complement, is closed. Hence, the decomposition by orbit types is a regular stratification. ■

Remark 8.3.15

1. Define a relation $\geq$ on the set of strata by

$$\mathscr{M}^\tau \geq \mathscr{M}^{\tau'} \quad \Leftrightarrow \quad \overline{\mathscr{M}^\tau} \cap \mathscr{M}^{\tau'} \neq \varnothing\,.$$

   Clearly, this relation is reflexive. By the frontier condition, it is transitive and thus a quasi-ordering (the natural quasi-ordering of the stratification). By the regularity condition, it is also anti-symmetric and hence a partial ordering. By construction, $\mathscr{M}^\tau \geq \mathscr{M}^{\tau'}$ iff $\tau \geq \tau'$. That is, the natural partial ordering of strata is compatible with the natural partial ordering of orbit types.
2. Instead of using Sobolev techniques one can also stick to smooth connection forms and gauge transformations. Then one obtains essentially analogous results about the stratification of the corresponding gauge orbit space where, roughly speaking, one has to replace 'Hilbert manifold' and 'Hilbert Lie group' by 'tame Fréchet manifold' and 'tame Fréchet Lie group', see [1, 2]. ♦

Exercises

8.3.1 Prove the properties 1–4 of slices listed in Remark 8.3.4.

8.3.2 Prove that the mapping defined by (8.3.4) is a diffeomorphism.

8.4 Geometry of Strata

In this section, we will show that the weak Riemannian metric γ^0 on $\mathscr{C}$ induces a weak Riemannian metric on each stratum $\mathscr{M}^\tau$. This was discussed for the case of the principal stratum in [47, 592] and for the general case in [75]. The basic idea consists in restricting the tangent bundle splitting $\mathrm{T}\mathscr{C} = \mathfrak{V} \oplus \mathfrak{H}$ given by Theorem 6.1.9 to the strata. This yields a natural smooth connection on each bundle which allows to lift tangent vectors, thus projecting γ^0 to a metric on each stratum. By restriction, the distribution $\mathfrak{V}$ made up by the tangent spaces of the orbits induces a distribution $\mathfrak{V}^\tau$ on $\mathscr{C}^\tau$. Contrary to $\mathfrak{V}$, $\mathfrak{V}^\tau$ is smooth and locally trivial, because $\mathfrak{V}^\tau = \ker((\pi^\tau)')$ and π^τ is a smooth submersion. Let $\mathfrak{H}^\tau$ denote the distribution orthogonal to $\mathfrak{V}^\tau$ with respect to γ^0. By construction,

$$\mathfrak{H}^\tau = \mathfrak{H} \cap \mathrm{T}\mathscr{C}^\tau \,,$$

and, thus, we have the L^2-orthogonal splitting

$$\mathrm{T}\mathscr{C}^\tau = \mathfrak{V}^\tau \oplus \mathfrak{H}^\tau \,. \tag{8.4.1}$$

Moreover, by (8.3.2), $\mathfrak{H}^\tau$ is $\mathscr{G}$-equivariant,

$$\mathfrak{H}^\tau_{\omega^{(u)}} = \left(\mathfrak{H}^\tau_\omega\right)^{(u)} \,, \quad u \in \mathscr{G} \,.$$

We draw the attention of the reader to the fact that we had already introduced the notation $\mathfrak{H}^\tau_\omega$ for the subspace of $\mathfrak{H}_\omega$ consisting of elements invariant under $\mathscr{G}_\omega$, see (8.3.15). This notation suggests that $\mathfrak{H}^\tau_\omega$ is in fact the fibre at ω of the distribution $\mathfrak{H}^\tau$. To see that this holds indeed, recall that $\mathfrak{H}_\omega = \mathrm{T}_\omega \mathscr{S}_{\omega,\varepsilon}$. Hence, the fibre of $\mathfrak{H}^\tau$ is

$$\mathrm{T}_\omega \mathscr{S}_{\omega,\varepsilon} \cap \mathrm{T}_\omega \mathscr{C}^\tau = \mathrm{T}_\omega \mathscr{S}^\tau_{\omega,\varepsilon} \,.$$

According to (8.3.19), the right hand side is given by $\mathfrak{H}^\tau_\omega$.

Proposition 8.4.1 *The distribution $\mathfrak{H}^\tau$ is smooth.*

Proof Recall from Theorem 6.1.9 that the orthogonal projectors onto $\mathfrak{V}_\omega$ and $\mathfrak{H}_\omega$ are given by

$$\mathbf{v}_\omega = \nabla^\omega \mathrm{G}_\omega \nabla^{\omega *} \,, \quad \mathbf{h}_\omega = \mathrm{id} - \mathbf{v}_\omega \,,$$

respectively, where G_ω is the Green's operator of Δ_ω. To prove that $\mathfrak{H}^\tau$ is smooth it is enough to show that the restriction of $\mathbf{v}$ to $\mathrm{T}\mathscr{C}^\tau \subset \mathrm{T}\mathscr{C}$ is smooth. By restriction, $\mathbf{v}$ induces a mapping (denoted by the same symbol)

$$\mathbf{v} : \mathscr{C}^\tau \to \mathrm{B}(\mathscr{T}) \,, \quad \omega \mapsto \mathbf{v}_\omega := \nabla^\omega \mathrm{G}_\omega \nabla^{\omega *} \,,$$

where $\mathrm{B}(\mathscr{T})$ denotes the Banach space of bounded operators on $\mathscr{T}$. Since the diagonal embedding $\mathscr{C}^\tau \to \mathscr{C}^\tau \times \mathscr{C}^\tau \times \mathscr{C}^\tau$ is smooth and since ∇^ω and $\nabla^{\omega *}$ depend smoothly on ω, it suffices to prove the smoothness of the mapping

$$\mathscr{C}^\tau \to \mathrm{B}(W^{k-1}(\mathrm{Ad}(P)), W^{k+1}(\mathrm{Ad}(P))) \,, \quad \omega \mapsto \mathrm{G}_\omega \,. \tag{8.4.2}$$

Pulling the latter back with a local trivialization $\chi^\tau_{\omega_0,\varepsilon}$, $\omega_0 \in \mathscr{C}^\tau$, see (8.3.18), we obtain a mapping

$$\mathscr{S}^\tau_{\omega_0,\varepsilon} \times \mathscr{G}/\mathscr{G}_{\omega_0} \to \mathrm{B}(W^{k-1}(\mathrm{Ad}(P)), W^{k+1}(\mathrm{Ad}(P))) \,, \quad (\omega, [u]) \mapsto \mathrm{G}_{\omega^{(u)}} \,, \tag{8.4.3}$$

which is well defined, because $\mathscr{G}_\omega = \mathscr{G}_{\omega_0}$ for all $\omega \in \mathscr{S}^\tau_{\omega_0,\varepsilon}$. Due to (6.1.29), this mapping is smooth along $\mathscr{G}/\mathscr{G}_{\omega_0}$. Thus, what we actually have to show is that the restrictions of the mapping (8.4.2) to the partial slices $\mathscr{S}^\tau_{\omega_0,\varepsilon}$, $\omega_0 \in \mathscr{C}^\tau$, are smooth.

For that purpose, recall from (5.7.34) that G_ω is constructed from the (bounded) inverse of the operator
$$\varDelta_\omega : \ker(\varDelta_\omega)^\perp \to \mathrm{im}(\varDelta_\omega)\,. \tag{8.4.4}$$

Due to $\mathscr{G}_\omega = \mathscr{G}_{\omega_0}$, the Stabilizer Theorem 6.1.5, Eq. (6.1.23) and the Hodge Decomposition Theorem 5.7.18, we have

$$\ker(\varDelta_\omega) = \ker(\varDelta_{\omega_0})\,, \quad \mathrm{im}(\varDelta_\omega) = \mathrm{im}(\varDelta_{\omega_0})\,. \tag{8.4.5}$$

Hence, (8.4.4) reads
$$\varDelta_\omega : \ker(\varDelta_{\omega_0})^\perp \to \mathrm{im}(\varDelta_{\omega_0})\,,$$

for all $\omega \in \mathscr{S}^\tau_{\omega_0,\varepsilon}$. Thus, the mapping under consideration decomposes into

$$\mathscr{S}^\tau_{\omega_0,\varepsilon} \xrightarrow{\varDelta} \mathrm{inv}\left(\ker(\varDelta_{\omega_0})^\perp, \mathrm{im}(\varDelta_{\omega_0})\right) \xrightarrow{\mathrm{inv}} \mathrm{inv}\left(\mathrm{im}(\varDelta_{\omega_0}), \ker(\varDelta_{\omega_0})^\perp\right),$$

followed by prolongation to a bounded operator $W^{k-1}(\mathrm{Ad}(P)) \to W^{k+1}(\mathrm{Ad}(P))$. Here $\mathrm{inv}(\cdot,\cdot) \subset \mathrm{B}(\cdot,\cdot)$ denotes the open subset of invertible bounded operators and inv stands for the inversion mapping. Since the first step factorizes into continuous linear mappings and composition of bounded operators and since the inversion mapping of linear operators is smooth, we conclude that the restrictions of the mapping (8.4.2) to the partial slices $\mathscr{S}^\tau_{\omega_0,\varepsilon}$ are smooth, indeed. ■

Next, we show that the distribution $\mathfrak{H}^\tau$ on $\mathscr{C}^\tau$ is locally trivial. For that purpose, recall that, due to (8.3.10), $\mathscr{S}_{\omega_0,\varepsilon}$ is transversal to any orbit in $\mathscr{C}^\tau$ it meets. Hence,

$$\mathfrak{H}_{\omega_0} \cap \mathfrak{V}_\omega = \ker(\nabla^{\omega_0 *}) \cap \mathrm{im}(\nabla^\omega) = \{0\} \tag{8.4.6}$$

and thus (8.3.9) implies that we have a direct sum decomposition

$$\mathscr{T} = \mathfrak{H}_{\omega_0} \oplus \mathfrak{V}_\omega = \ker(\nabla^{\omega_0 *}) \oplus \mathrm{im}(\nabla^\omega) \tag{8.4.7}$$

for all $\omega \in \mathscr{S}^\tau_{\omega_0,\varepsilon}$. Consider the mapping

$$\varDelta_{\omega_0\omega} :\; W^{k+1}(\mathrm{Ad}(P)) \to W^{k-1}(\mathrm{Ad}(P))\,, \quad \varDelta_{\omega_0\omega} := \nabla^{\omega_0 *}\nabla^\omega\,. \tag{8.4.8}$$

It will be referred to as the Faddeev–Popov operator. Let us construct the corresponding Green's operator. By (8.4.6), we have $\ker(\varDelta_{\omega_0\omega}) = \ker(\nabla^\omega)$. Since $\mathscr{G}_\omega = \mathscr{G}_{\omega_0}$, the Stabilizer Theorem 6.1.5 implies $\ker(\nabla^\omega) = \ker(\nabla^{\omega_0}) = \ker(\varDelta_{\omega_0})$. Thus,

$$\ker(\varDelta_{\omega_0\omega}) = \ker(\varDelta_{\omega_0})\,.$$

On the other hand, by (8.4.7),

$$\mathrm{im}(\varDelta_{\omega_0\omega}) = \mathrm{im}(\nabla^{\omega_0 *}) = \mathrm{im}(\varDelta_{\omega_0})\,.$$

As a consequence, by restriction, $\varDelta_{\omega_0\omega}$ induces an isomorphism (denoted by the same symbol)

$$\varDelta_{\omega_0\omega} : \ker(\varDelta_{\omega_0})^{\perp} \to \operatorname{im}(\varDelta_{\omega_0})\,.$$

Consequently, we can define a Green's operator

$$\mathrm{G}_{\omega_0\omega} : W^{k-1}(\mathrm{Ad}(P)) \to W^{k+1}(\mathrm{Ad}(P))$$

similar to G_{ω_0} and G_{ω}. Then,

$$\mathrm{G}_{\omega_0\omega}\varDelta_{\omega_0\omega} = \mathrm{G}_{\omega_0}\varDelta_{\omega_0} = \mathrm{G}_{\omega}\varDelta_{\omega}\,,$$

because $\mathrm{G}_{\omega_0\omega}\varDelta_{\omega_0\omega}$ is the identical mapping on $\ker(\varDelta_{\omega_0})^{\perp}$ and trivial on $\ker(\varDelta_{\omega_0})$, and $\ker(\varDelta_{\omega_0}) = \ker(\varDelta_{\omega})$. Similarly,

$$\varDelta_{\omega_0\omega}\mathrm{G}_{\omega_0\omega} = \varDelta_{\omega_0}\mathrm{G}_{\omega_0}\,,$$

because $\varDelta_{\omega_0\omega}\mathrm{G}_{\omega_0\omega}$ is the identical mapping on $\operatorname{im}(\varDelta_{\omega_0})$ and trivial on $\operatorname{im}(\varDelta_{\omega_0})^{\perp}$.

Lemma 8.4.2 *Let $\omega \in \mathscr{S}^{\tau}_{\omega_0,\varepsilon}$. Then,*

1. *the Faddeev-Popov operator $\varDelta_{\omega_0\omega}$ is formally self-adjoint,*
2. *the operator*

$$\mathbf{v}_{\omega_0\omega} := \nabla^{\omega}\mathrm{G}_{\omega_0\omega}\nabla^{\omega_0 *} \tag{8.4.9}$$

is the projector onto the subspace $\mathfrak{V}_{\omega} = \operatorname{im}(\nabla^{\omega})$ in the decomposition (8.4.7).

Proof Using the Ad-invariance of the L^2-scalar product, for any $\xi \in W^{k+1}(\mathrm{Ad}(P))$ and $\eta \in W^{k-1}(\mathrm{Ad}(P))$, we calculate

$$\begin{aligned}
\left\langle \eta, \left(\nabla^{\omega *}\nabla^{\omega_0} - \nabla^{\omega_0 *}\nabla^{\omega}\right)\xi \right\rangle &= \left\langle \nabla^{\omega}\eta, \nabla^{\omega_0}\xi \right\rangle - \left\langle \nabla^{\omega_0}\eta, \nabla^{\omega}\xi \right\rangle \\
&= \left\langle \nabla^{\omega}\eta - \nabla^{\omega_0}\eta, \nabla^{\omega_0}\xi \right\rangle - \left\langle \nabla^{\omega_0}\eta, \nabla^{\omega}\xi - \nabla^{\omega_0}\xi \right\rangle \\
&= \left\langle [\omega - \omega_0, \eta], \nabla^{\omega_0}\xi \right\rangle - \left\langle \nabla^{\omega_0}\eta, [\omega - \omega_0, \xi] \right\rangle \\
&= \left\langle \omega - \omega_0, [\eta, \nabla^{\omega_0}\xi] - [\xi, \nabla^{\omega_0}\eta] \right\rangle \\
&= \left\langle \nabla^{\omega_0 *}(\omega - \omega_0), [\eta, \xi] \right\rangle .
\end{aligned}$$

Since $\omega \in \mathscr{S}^{\tau}_{\omega_0,\varepsilon}$, the right hand side vanishes. To prove the second assertion, we use that $\mathrm{G}_{\omega_0\omega}\varDelta_{\omega_0\omega}$ is the identical mapping on $\ker(\varDelta_{\omega_0})^{\perp}$. Since $\ker(\varDelta_{\omega_0})^{\perp} = \operatorname{im}(\mathrm{G}_{\omega_0\omega})$, this implies

$$\mathbf{v}^2_{\omega_0\omega} = \nabla^{\omega}\mathrm{G}_{\omega_0\omega}\varDelta_{\omega_0\omega}\mathrm{G}_{\omega_0\omega}\nabla^{\omega_0 *} = \mathbf{v}_{\omega_0\omega}\,,$$

showing that $\mathbf{v}_{\omega_0\omega}$ is a projector. Since $\ker(\varDelta_{\omega_0}) = \ker(\nabla^{\omega})$, this furthermore implies

$$\mathbf{v}_{\omega_0\omega}(\nabla^{\omega}\xi) = \nabla^{\omega}\mathrm{G}_{\omega_0\omega}\varDelta_{\omega_0\omega}\xi = \nabla^{\omega}\xi\,,$$

showing that $\operatorname{im}(\mathbf{v}_{\omega_0\omega}) = \operatorname{im}(\nabla^{\omega})$. Since $\mathbf{v}_{\omega_0\omega}$ acts trivially on $\ker(\nabla^{\omega_0 *})$, it is the projector onto the subspace $\operatorname{im}(\nabla^{\omega})$ in the decomposition (8.4.7), indeed. ■

Remark 8.4.3 Correspondingly, $\mathbf{h}_{\omega_0\omega} := \mathbb{1} - \mathbf{v}_{\omega_0\omega}$ is the projector onto the subspace $\mathfrak{H}_{\omega_0} = \ker(\nabla^{\omega_0 *})$ in the decomposition (8.4.7). Associated with $\mathbf{v}_{\omega_0\omega}$ and $\mathbf{h}_{\omega_0\omega}$, we have their adjoints,

$$\mathbf{v}^*_{\omega_0\omega} = \nabla^{\omega_0} \mathrm{G}_{\omega_0\omega} \nabla^{\omega *}\,, \quad \mathbf{h}^*_{\omega_0\omega} = \mathbb{1} - \mathbf{v}^*_{\omega_0\omega} = \mathbb{1} - \nabla^{\omega_0} \mathrm{G}_{\omega_0\omega} \nabla^{\omega *}\,,$$

which are the projectors onto $\mathfrak{V}_{\omega_0}$ and $\mathfrak{H}_{\omega}$, respectively. By (6.1.26),

$$\mathbf{h}_{\omega_0\omega}\mathbf{h}_{\omega} = \mathbf{h}_{\omega_0}\mathbf{h}_{\omega_0\omega} = \mathbf{h}_{\omega_0\omega}\,, \quad \mathbf{h}_{\omega_0\omega}\mathbf{h}_{\omega_0} = \mathbf{h}_{\omega_0}\,, \quad \mathbf{h}_{\omega}\mathbf{h}_{\omega_0\omega} = \mathbf{h}_{\omega}\,, \tag{8.4.10}$$

$$\mathbf{h}_{\omega}\mathbf{h}^*_{\omega_0\omega} = \mathbf{h}^*_{\omega_0\omega}\mathbf{h}_{\omega_0} = \mathbf{h}^*_{\omega_0\omega}\,, \quad \mathbf{h}_{\omega_0}\mathbf{h}^*_{\omega_0\omega} = \mathbf{h}_{\omega_0}\,, \quad \mathbf{h}^*_{\omega_0\omega}\mathbf{h}_{\omega} = \mathbf{h}_{\omega} \tag{8.4.11}$$

for all $\omega \in \mathscr{S}^{\tau}_{\omega_0,\varepsilon}$. Similar formulae hold for $\mathbf{v}$. ♦

Now, for $\omega \in \mathscr{S}^{\tau}_{\omega_0,\varepsilon}$, consider the induced action of $\mathscr{G}_{\omega} = \mathscr{G}_{\omega_0}$ on $\mathrm{T}_{\omega}\mathscr{C}^{\tau}$. It leaves the decomposition

$$\mathrm{T}_{\omega}\mathscr{C}^{\tau} = \mathfrak{V}_{\omega} \oplus \mathfrak{H}^{\tau}_{\omega}$$

invariant. Moroever, it leaves $\mathfrak{H}^{\tau}_{\omega}$ invariant pointwise. Hence, denoting the subspace of the $\mathscr{G}_{\omega}$-invariant elements of $\mathfrak{V}_{\omega}$ by $\hat{\mathfrak{V}}_{\omega}$, we have the decomposition

$$\left(\mathrm{T}_{\omega}\mathscr{C}^{\tau}\right)^{\mathscr{G}_{\omega_0}} = \hat{\mathfrak{V}}_{\omega} \oplus \mathfrak{H}^{\tau}_{\omega}\,.$$

In particular, this decomposition holds for $\omega = \omega_0$. By point 2 of Lemma 8.4.2 and the equivariance property (6.1.30), $\mathbf{v}_{\omega_0\omega}$ induces an isomorphism

$$\hat{\mathbf{v}}_{\omega_0\omega} : \hat{\mathfrak{V}}_{\omega_0} \to \hat{\mathfrak{V}}_{\omega}\,. \tag{8.4.12}$$

Correspondingly, $\mathbf{h}_{\omega_0\omega}$ induces an isomorphism

$$\hat{\mathbf{h}}_{\omega_0\omega} : \mathfrak{H}^{\tau}_{\omega} \to \mathfrak{H}^{\tau}_{\omega_0}\,. \tag{8.4.13}$$

Clearly, the projectors $\mathbf{v}_{\omega_0\omega}$ and $\mathbf{h}_{\omega_0\omega}$ define a splitting of the restriction of the tangent bundle of the stratum to the slice $\mathscr{S}^{\tau}_{\omega_0,\varepsilon}$. We will now see that these splittings yield a system of local trivializations of the vector bundle $\mathfrak{H}^{\tau}$.

Proposition 8.4.4 *The distribution $\mathfrak{H}^{\tau}$ is a locally trivial subbundle of* $\mathrm{T}\mathscr{C}^{\tau}$.

Proof To construct a local trivialization of $\mathfrak{H}^{\tau}$, choose $\omega_0 \in \mathscr{C}^{\tau}$ and consider the distribution $\mathfrak{D}^{\tau}_{\omega_0,\varepsilon}$ on $\mathscr{S}^{\tau}_{\omega_0,\varepsilon} \times \mathscr{G}/\mathscr{G}_{\omega_0}$, made up by the subspaces tangent to $\mathscr{S}^{\tau}_{\omega_0,\varepsilon}$. Due to (8.3.19), it is trivial. We claim that the mapping

$$\mathfrak{D}^{\tau}_{\omega_0,\varepsilon} \to \mathrm{T}(\mathscr{S}^{\tau}_{\omega_0,\varepsilon} \times \mathscr{G}/\mathscr{G}_{\omega_0}) \xrightarrow{(\chi^{\tau}_{\omega_0,\varepsilon})'} \mathrm{T}\mathscr{U}^{\tau}_{\pi(\omega_0),\varepsilon} \xrightarrow{\mathbf{h}} \mathfrak{H}^{\tau}_{\restriction \mathscr{U}^{\tau}_{\omega_0,\varepsilon}} \tag{8.4.14}$$

is a smooth vector bundle isomorphism and, thus, provides a local trivialization of $\mathfrak{H}^\tau$. By equivariance of $(\chi^\tau_{\omega_0,\varepsilon})'$ and $\mathbf{h}$, it suffices to show that the mapping

$$\left(\mathfrak{D}^\tau_{\omega_0,\varepsilon}\right)_{\restriction \mathscr{S}^\tau_{\omega_0,\varepsilon}} = \mathscr{S}^\tau_{\omega_0,\varepsilon} \times \mathfrak{H}^\tau_{\omega_0} \to \mathfrak{H}^\tau_{\restriction \mathscr{S}^\tau_{\omega_0,\varepsilon}}, \quad (\omega, \alpha) \mapsto (\omega, \mathbf{h}_\omega \alpha), \tag{8.4.15}$$

is a smooth vector bundle isomorphism. By the same argument as in the proof of Proposition 8.4.1, one can show that the mapping

$$\mathscr{S}^\tau_{\omega_0,\varepsilon} \to \mathrm{B}(\mathscr{T}), \quad \omega \mapsto \mathbf{h}_{\omega_0\omega},$$

is smooth. Moreover, by (8.4.10), for $\alpha \in \mathfrak{H}^\tau_{\omega_0}$, one finds $\mathbf{h}_{\omega_0\omega}\mathbf{h}_\omega\alpha = \mathbf{h}_{\omega_0\omega}\alpha = \alpha$. Hence, the mapping

$$\mathfrak{H}^\tau_{\restriction \mathscr{S}^\tau_{\omega_0,\varepsilon}} \to \mathscr{S}^\tau_{\omega_0,\varepsilon} \times \mathfrak{H}^\tau_{\omega_0}, \quad (\omega, \alpha) \mapsto \left(\omega, \mathbf{h}_{\omega_0\omega}\alpha\right),$$

provides a smooth inverse of (8.4.15). ■

Remark 8.4.5 Associated with the distribution $\mathfrak{H}$ there is an equivariant differential form Z on $\mathscr{C}$ with values in $\mathrm{L}\mathscr{G}$ given by

$$\mathrm{Z}(\omega, \alpha) := \mathrm{G}_\omega \nabla^{\omega *}\alpha, \quad (\omega, \alpha) \in \mathscr{C} \times \mathscr{T} = \mathrm{T}\mathscr{C}. \tag{8.4.16}$$

By definition, Z_ω annihilates the elements of $\mathfrak{H}_\omega$. If ω belongs to the principal stratum $\mathscr{C}^{\mathrm{p}}$, we have

$$\mathrm{Z}(\omega, \nabla^\omega \xi) = \xi \tag{8.4.17}$$

for all $\xi \in \mathrm{L}\mathscr{G}$, showing that Z restricts to an ordinary connection form on the principal bundle $\mathscr{C}^{\mathrm{p}} \to \mathscr{M}^{\mathrm{p}}$. For ω in another stratum, however, Z_ω maps the value at ω of the Killing field generated by ξ to the projection of ξ onto the L^2-orthogonal complement of $\mathrm{L}\mathscr{G}_\omega$ in $\mathrm{L}\mathscr{G}$. We will comment on that below. ♦

The natural connection $\mathfrak{H}^\tau$ and the (weak) Riemannian metric $\gamma = \gamma^0$ induce a Riemannian metric γ^τ on $\mathscr{M}^\tau$ as follows. Due to the Open Mapping Theorem, the restriction of $(\pi^\tau)'$ to a fibre $\mathfrak{H}^\tau_\omega$, $\omega \in \mathscr{C}^\tau$, induces a Banach space isomorphism onto $\mathrm{T}_{\pi(\omega)}\mathscr{M}^\tau$. This allows to lift tangent vectors at $x \in \mathscr{M}^\tau$ to horizontal tangent vectors at $\omega \in \pi^{-1}(x)$ and to evaluate their scalar product with respect to γ. Due to equivariance of $\mathfrak{H}^\tau$ and invariance of γ, the result does not depend on the choice of the representative ω. Due to smoothness of $\mathfrak{H}^\tau$, the Riemannian metric γ^τ on $\mathscr{M}^\tau$ so constructed is smooth.

Let us determine the local representatives of γ^τ in the charts provided by the slices $\mathscr{S}^\tau_{\omega_0,\varepsilon}$, cf. (8.3.20). Let $\omega \in \mathscr{S}^\tau_{\omega_0,\varepsilon}$. For $(\omega, \alpha_i) \in \mathrm{T}_\omega\mathscr{S}^\tau_{\omega_0,\varepsilon} = \mathscr{S}^\tau_{\omega_0,\varepsilon} \times \mathfrak{H}^\tau_{\omega_0}$, we have

$$\left((\pi^\tau_{\omega_0,\varepsilon})^*\gamma^\tau\right)\left((\omega, \alpha_1), (\omega, \alpha_2)\right) = \gamma^\tau\left((\pi^\tau_{\omega_0,\varepsilon})'(\omega, \alpha_1), (\pi^\tau_{\omega_0,\varepsilon})'(\omega, \alpha_2)\right).$$

The Z-horizontal lifts of $(\pi^\tau_{\omega_0,\varepsilon})'(\omega, \alpha_i)$ to ω are given by $(\omega, \mathbf{h}_\omega\alpha_i)$. Hence,

$$\big((\pi^\tau_{\omega_0,\varepsilon})^*\gamma^\tau\big)\big((\omega,\alpha_1),(\omega,\alpha_2)\big) = \langle\alpha_1, \mathbf{h}_\omega\alpha_2\rangle_{L^2}\,. \tag{8.4.18}$$

By (8.4.10),

$$\langle\alpha_1, \mathbf{h}_\omega\alpha_2\rangle_{L^2} = \langle\alpha_1, \mathbf{h}_{\omega_0}\mathbf{h}_\omega\mathbf{h}_{\omega_0}\alpha_2\rangle_{L^2}\,.$$

Since, by restriction, $\mathbf{h}_\omega$ defines an isomorphisms

$$\hat{\mathbf{h}}_\omega : \mathfrak{H}^\tau_{\omega_0} \to \mathfrak{H}^\tau_\omega\,, \tag{8.4.19}$$

in the chart provided by the slice $\mathscr{S}^\tau_{\omega_0,\varepsilon}$, the metric is given by the smooth mapping

$$\mathscr{S}^\tau_{\omega_0,\varepsilon} \to \mathrm{B}(\mathfrak{H}^\tau_{\omega_0})\,,\quad \omega \mapsto \gamma^\tau_\omega := \hat{\mathbf{h}}_{\omega_0}\hat{\mathbf{h}}_\omega\hat{\mathbf{h}}_{\omega_0}\,. \tag{8.4.20}$$

By (8.4.10) and (8.4.11), the inverse of the metric is given by

$$\big(\gamma^\tau_\omega\big)^{-1} = \hat{\mathbf{h}}_{\omega_0\omega}\hat{\mathbf{h}}^*_{\omega_0\omega}\,. \tag{8.4.21}$$

Thus, in particular, γ^τ_ω is a Banach space isomorphism.

Remark 8.4.6 (*Kaluza–Klein-type structure*) For every orbit type τ, the restriction of the $\mathscr{G}$-invariant L^2-metric γ to $\mathscr{C}^\tau$ is uniquely characterized by the triple

$$\big(\gamma^\tau, \mathrm{Z}, \langle\cdot,\cdot\rangle_{\mathrm{L}\mathscr{G}}\big)\,,$$

where $\langle\cdot,\cdot\rangle_{\mathrm{L}\mathscr{G}}$ denotes the L^2-scalar product on $\mathrm{L}\mathscr{G}$. This is a structure similar to that in Kaluza-Klein theory, cf. Proposition 7.8.3, where K-invariant metrics g on a K-bundle Q with fibre K/H over the manifold M are in one-to-one correspondence with triples $(\mathsf{g}_M, \xi, \langle\cdot,\cdot\rangle)$. Here g_M is a metric on M , ξ is a connection form on the principal bundle P with structure group N/H associated with Q and $\langle\cdot,\cdot\rangle$ is an $\mathrm{Ad}(K)$-invariant scalar product on the Lie algebra of K. Moreover, N denotes the normalizer of H in K . According to Remark 8.3.13, in the case under consideration, it is unclear whether the normalizer of a given stabilizer $\mathscr{G}_\omega$ in $\mathscr{G}$ is a Lie subgroup. Thus, we cannot construct the above associated principal bundle and give an interpretation of Z as a connection form on this bundle. Such an interpretation is only possible on the principal stratum. ♦

Let us write down the formal volume element of the metric γ^τ on $\mathscr{M}^\tau$ in the local charts provided by the slices $\mathscr{S}^\tau_{\omega_0,\varepsilon}$. This generalizes a result for the generic stratum due to Babelon and Viallet [46]. Recall that Δ_{ω_0} maps the orthogonal complement of $\ker(\Delta_{\omega_0})$ in $W^{k+1}(\mathrm{Ad}(P))$ isomorphically onto $\mathrm{im}(\Delta_{\omega_0})$, which by (6.1.21) coincides with the L^2-orthogonal complement of $\ker(\Delta_{\omega_0})$ in $W^{k-1}(\mathrm{Ad}(P))$. Thus, we may view Δ_{ω_0} as an operator on the closed subspace $\mathrm{im}(\Delta_{\omega_0})$ of $W^{k-1}(\mathrm{Ad}(P))$ which is densely defined and which has an inverse, given by the Green's operator G_{ω_0}. Since $\ker(\Delta_\omega) = \ker(\Delta_{\omega_0\omega}) = \ker(\Delta_{\omega_0})$, this applies also to Δ_ω and the Faddeev–Popov operator $\Delta_{\omega_0\omega}$, as well as the corresponding Green's operators. By equivariance, all these operators restrict to operators on the subspace $\mathrm{im}(\Delta_{\omega_0})^{\mathscr{G}_{\omega_0}}$ of $\mathscr{G}_{\omega_0}$-invariants. Let us denote the restricted operators by $\hat{\Delta}_{\omega_0}$, $\hat{\Delta}_{\omega}$, $\hat{\Delta}_{\omega_0\omega}$, and $\hat{G}_{\omega_0}$, $\hat{G}_{\omega}$, $\hat{G}_{\omega_0\omega}$. The following expression is formal in the sense that the determinants involved have to be regularized. For the regularization of determinants, we refer to Appendix D.

Proposition 8.4.7 *In the local chart defined by a slice $\mathscr{S}^{\tau}_{\omega_0,\varepsilon}$, the formal volume element at $\omega \in \mathscr{S}^{\tau}_{\omega_0,\varepsilon}$ of the metric γ^{τ} on $\mathscr{M}^{\tau}$ is given by*

$$\det\left(\gamma^{\tau}_{\omega}\right)^{1/2} = \frac{\det(\hat{\Delta}_{\omega_0\omega})}{\det(\hat{\Delta}_{\omega_0})^{1/2}\det(\hat{\Delta}_{\omega})^{1/2}} . \tag{8.4.22}$$

Proof Define a mapping $\chi : \mathscr{T} \to \mathscr{T}$ by $\chi := \mathbf{h}_{\omega_0}\mathbf{v}_{\omega}$. Then, $\chi^* = \mathbf{v}_{\omega}\mathbf{h}_{\omega_0}$ and hence

$$(1 - \chi\chi^*)_{\restriction \mathfrak{H}^{\tau}_{\omega_0}} = (1 - \mathbf{h}_{\omega_0}\mathbf{v}_{\omega}\mathbf{h}_{\omega_0})_{\restriction \mathfrak{H}^{\tau}_{\omega_0}} = (\mathbf{h}_{\omega_0}\mathbf{h}_{\omega}\mathbf{h}_{\omega_0})_{\restriction \mathfrak{H}^{\tau}_{\omega_0}} = \hat{\mathbf{h}}_{\omega_0}\hat{\mathbf{h}}_{\omega}\hat{\mathbf{h}}_{\omega_0} .$$

Therefore,

$$\gamma^{\tau}_{\omega} = (1 - \chi\chi^*)_{\restriction \mathfrak{H}^{\tau}_{\omega_0}} . \tag{8.4.23}$$

On the other hand, consider the isomorphism

$$\phi : \mathrm{im}(\Delta_{\omega_0})^{\mathscr{G}_{\omega_0}} \to \mathrm{im}(\Delta_{\omega_0})^{\mathscr{G}_{\omega_0}} , \quad \phi := \hat{\mathrm{G}}_{\omega}\hat{\Delta}^*_{\omega_0\omega}\hat{\mathrm{G}}_{\omega_0}\hat{\Delta}_{\omega_0\omega} .$$

Using that the Faddeev–Popov operator is self-adjoint, we obtain

$$\det(\phi) = \frac{\det(\hat{\Delta}_{\omega_0\omega})^2}{\det(\hat{\Delta}_{\omega_0})\det(\hat{\Delta}_{\omega})} . \tag{8.4.24}$$

Next, by (8.4.16),

$$Z_{\omega}\nabla^{\omega} = \left(\mathrm{G}_{\omega}\nabla^{\omega *}\nabla^{\omega}\right)_{\restriction \mathrm{im}(\Delta_{\omega_0})} = \mathrm{id}_{\mathrm{im}(\Delta_{\omega_0})} ,$$

and

$$\nabla^{\omega}Z_{\omega} = (\nabla^{\omega}\mathrm{G}_{\omega}\nabla^{\omega *})_{\restriction \mathfrak{V}_{\omega}} = \mathbf{v}_{\omega \restriction \mathfrak{V}_{\omega}} = \mathrm{id}_{\mathfrak{V}_{\omega}} .$$

We conclude that $Z_{\omega} : \mathfrak{V}_{\omega} \to \mathrm{im}(\Delta_{\omega_0})$ is an isomorphism inverse to ∇^{ω}. By equivariance, Z_{ω} and ∇^{ω} induce mutually inverse isomorphisms

$$\hat{Z}_{\omega} : \hat{\mathfrak{V}}_{\omega} \to \mathrm{im}(\Delta_{\omega_0})^{\mathscr{G}_{\omega_0}} , \quad \hat{\nabla}^{\omega} : \mathrm{im}(\Delta_{\omega_0})^{\mathscr{G}_{\omega_0}} \to \hat{\mathfrak{V}}_{\omega} .$$

Thus,

$$\tilde{\phi} := \hat{\nabla}^{\omega}\phi\hat{Z}_{\omega} : \hat{\mathfrak{V}}_{\omega} \to \hat{\mathfrak{V}}_{\omega}$$

is an isomorphism. We compute

$$(1-\chi^*\chi)_{\restriction\hat{\mathfrak{V}}_\omega} = (1-\mathbf{v}_\omega\mathbf{h}_{\omega_0}\mathbf{v}_\omega)_{\restriction\hat{\mathfrak{V}}_\omega} = (\mathbf{v}_\omega\mathbf{v}_{\omega_0}\mathbf{v}_\omega)_{\restriction\hat{\mathfrak{V}}_\omega} = \hat{\nabla}^\omega\phi\hat{Z}_\omega\,.$$

Thus,

$$\tilde{\phi} = (1-\chi^*\chi)_{\restriction\hat{\mathfrak{V}}_\omega}\,.$$

This implies

$$\det(\tilde{\phi}) = \det(\phi) = \det\left((\mathbb{1}-\chi^*\chi)_{\restriction\hat{\mathfrak{V}}_\omega}\right)\,. \tag{8.4.25}$$

Now, assume that $u\in\hat{\mathfrak{V}}_\omega$ is an eigenvector of $\chi^*\chi$ with a nonzero eigenvalue λ. Then, $\chi\chi^*(\chi u) = \lambda(\chi u)$ and $\chi u \neq 0$. Thus, χu is an eigenvector of $\chi\chi^*$ with the same eigenvalue λ. Moreover, $\chi u \in \mathfrak{H}^\tau_{\omega_0}$. Consequently, χ defines isomorphisms between the eigenspaces of $(\chi^*\chi)_{\restriction\hat{\mathfrak{V}}_\omega}$ and $(\chi\chi^*)_{\restriction\mathfrak{H}^\tau_{\omega_0}}$ corresponding to nonzero eigenvalues. Thus, by (8.4.23) and (8.4.25),

$$\det(\gamma^\tau_\omega) = \det\left((\mathbb{1}-\chi\chi^*)_{\restriction\mathfrak{H}^\tau_{\omega_0}}\right) = \det\left((\mathbb{1}-\chi^*\chi)_{\restriction\hat{\mathfrak{V}}_\omega}\right) = \det(\phi)\,.$$

This yields the assertion. ■

Remark 8.4.8

1. In the case of the principal stratum $\mathscr{M}^{\mathrm{p}}$, we have $\ker(\varDelta_{\omega_0}) = 0$ and $\mathrm{L}\mathscr{G}_{\omega_0} = 0$. Thus, by the Hodge Theorem, $\mathrm{im}(\varDelta_{\omega_0})^{\mathscr{G}_{\omega_0}} = W^{k+1}(\mathrm{Ad}(P))$ and (8.4.22) reproduces the formula given in [46]:

$$\det\left(\gamma^{\mathrm{p}}_\omega\right)^{1/2} = \frac{\det(\varDelta_{\omega_0\omega})}{\det(\varDelta_{\omega_0})^{1/2}\det(\varDelta_\omega)^{1/2}}\,. \tag{8.4.26}$$

2. Since the Faddeev–Popov operator is not elliptic, the standard ζ-function regularization procedure for determinants of elliptic operators on compact manifolds as summarized in Appendix D does not apply directly. However, it can be shown [506] that this procedure may be extended to a larger class of operators including the Faddeev–Popov operator. ♦

Next, we compute the Riemann curvature tensor of γ^τ. This is again a generalization of a result of Babelon and Viallet for the principal stratum, see [47]. Our proof will be along the lines of Groisser and Parker [262] who use the O'Neill Formula for a Riemannian submersion [495]. Indeed, by construction of γ^τ, the canonical projection $\pi : \mathscr{C}^\tau \to \mathscr{M}^\tau$ is a Riemannian submersion, that is, it has maximal rank and it preserves the length of horizontal vectors. For $\alpha\in\mathscr{T}$, define an operator

$$\mathrm{C}_\alpha : \varOmega^p(M,\mathrm{Ad}(P)) \to \varOmega^{p+1}(M,\mathrm{Ad}(P))\,,\quad \mathrm{C}_\alpha(\beta) := [\alpha,\beta] \tag{8.4.27}$$

and let C^*_α denote the adjoint with respect to the L^2-scalar product. Then,

$$\mathrm{d}_\omega = \mathrm{d}_{\omega_0} + \mathrm{C}_\alpha\,,\quad \alpha = \omega-\omega_0\,. \tag{8.4.28}$$

Let $\nabla^{\mathscr{C}}$ and $\overline{\nabla}^\tau$ denote the Levi-Civita connections of the Riemannian metrics γ on $\mathscr{C}$ and γ^τ on $\mathscr{M}^\tau$, respectively, and let $\mathsf{R}^{\mathscr{C}}$ and $\overline{\mathsf{R}}^\tau$ denote the corresponding Riemann curvature tensors. According to [495, Lemma 1], for given vector fields $\overline{\alpha}, \overline{\beta}$ on $\mathscr{M}^\tau$ and their horizontal lifts α, β to $\mathscr{C}^\tau$, the covariant derivatives $\nabla^{\mathscr{C}}_\alpha \beta$ and $\overline{\nabla}^\tau_{\overline{\alpha}}\overline{\beta}$ are π-related.

Proposition 8.4.9 *The Riemann curvature of γ^τ is given by*

$$\left\langle \overline{\mathsf{R}}^\tau_{[\omega]}(\overline{\alpha}, \overline{\beta})\overline{\rho}, \overline{\zeta}\right\rangle = \left\langle \mathrm{C}^*_\alpha \zeta, \mathrm{G}_\omega \mathrm{C}^*_\beta \rho\right\rangle - \left\langle \mathrm{C}^*_\beta \zeta, \mathrm{G}_\omega \mathrm{C}^*_\alpha \rho\right\rangle + 2\left\langle \mathrm{C}^*_\zeta \rho, \mathrm{G}_\omega \mathrm{C}^*_\alpha \beta\right\rangle,$$

where α, β, ρ and ζ are the horizontal lifts to $\mathfrak{H}^\tau_\omega$ of $\overline{\alpha}$, $\overline{\beta}$, $\overline{\rho}$ and $\overline{\zeta} \in \mathrm{T}_{[\omega]}\mathscr{M}^\tau$.

Proof As already noted, $\pi : \mathscr{C}^\tau \to \mathscr{M}^\tau$ is a Riemannian submersion. Thus, by formula {4} in Theorem 2 of [495], for $\overline{\alpha}, \overline{\beta} \in \mathrm{T}_{[\omega]}\mathscr{M}^\tau$ we have

$$\left\langle \overline{\mathsf{R}}^\tau_{[\omega]}(\overline{\alpha}, \overline{\beta})\overline{\beta}, \overline{\alpha}\right\rangle = \left\langle \mathsf{R}^{\mathscr{C}}_\omega(\alpha, \beta)\beta, \alpha\right\rangle + \tfrac{3}{4}\left\|\mathbf{v}_\omega[\tilde{\alpha}, \tilde{\beta}]\right\|^2, \tag{8.4.29}$$

where the commutator is that of vector fields on $\mathscr{C}^\tau$ and where $\tilde{\alpha}$ and $\tilde{\beta}$ are arbitrary extensions of α and β, respectively, to horizontal vector fields on $\mathscr{C}^\tau$. We choose $\tilde{\alpha}$ and $\tilde{\beta}$ so that

$$\tilde{\alpha}_{\omega'} = \mathbf{h}_{\omega'}\alpha\,, \quad \tilde{\beta}_{\omega'} = \mathbf{h}_{\omega'}\beta$$

for all $\omega' \in \mathscr{C}^\tau$. Since $\mathscr{C}^\tau$ is open in $\mathscr{C}^{\leq\tau}$, the curve $t \mapsto \omega' + t\tilde{\alpha}_{\omega'}$ is contained in $\mathscr{C}^\tau$ for small t and has tangent vector $\tilde{\alpha}_{\omega'}$ at $t = 0$. Hence, using (6.1.26) and (8.4.28), we may compute

$$\begin{aligned}
\left(\nabla^{\mathscr{C}}_{\tilde{\alpha}}\tilde{\beta}\right)_{\omega'} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\restriction_0 \tilde{\beta}_{\omega'+t\tilde{\alpha}_{\omega'}} \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big\restriction_0 \left(\nabla^{\omega'+t\tilde{\alpha}_{\omega'}}\mathrm{G}_{\omega'+t\tilde{\alpha}_{\omega'}}\left(\nabla^{\omega'+t\tilde{\alpha}_{\omega'}}\right)^*\beta\right) \\
&= -\mathrm{C}_{\tilde{\alpha}_{\omega'}} G_{\omega'}\nabla^{\omega'*}\beta - \nabla^{\omega'}\mathrm{G}_{\omega'}\mathrm{C}^*_{\tilde{\alpha}_{\omega'}}\beta \\
&\qquad + \nabla^{\omega'}G_{\omega'}\left(\mathrm{C}^*_{\tilde{\alpha}_{\omega'}}\nabla^{\omega'} + \nabla^{\omega'*}\mathrm{C}_{\tilde{\alpha}_{\omega'}}\right)G_{\omega'}\nabla^{\omega'*}\beta\,.
\end{aligned} \tag{8.4.30}$$

Using this, as well as $\nabla^{\omega*}\beta = 0$ and $\mathrm{C}^*_\alpha\beta = -\mathrm{C}^*_\beta\alpha$, by a tedious but straightforward calculation (Exercise 8.4.1) one finds

$$\begin{aligned}
\left([\nabla^{\mathscr{C}}_{\tilde{\alpha}}, \nabla^{\mathscr{C}}_{\tilde{\beta}}]\tilde{\beta}\right)_\omega &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\restriction_0 \frac{\mathrm{d}}{\mathrm{d}s}\Big\restriction_0 \tilde{\beta}_{\omega+t\alpha+s\tilde{\beta}_{\omega+t\alpha}} - \frac{\mathrm{d}}{\mathrm{d}t}\Big\restriction_0 \frac{\mathrm{d}}{\mathrm{d}s}\Big\restriction_0 \tilde{\beta}_{\omega+t\beta+s\tilde{\alpha}_{\omega+t\beta}} \\
&= 2\left(\nabla^\omega G_\omega \mathrm{C}^*_\beta\right)^2\alpha\,.
\end{aligned} \tag{8.4.31}$$

Moreover, (8.4.30) yields

$$\left[\tilde{\alpha}, \tilde{\beta}\right]_\omega = \left(\nabla^{\mathscr{C}}_{\tilde{\alpha}}\tilde{\beta} - \nabla^{\mathscr{C}}_{\tilde{\beta}}\tilde{\alpha}\right)_\omega = 2\nabla^\omega \mathrm{G}_\omega \mathrm{C}^*_\beta \alpha\,. \tag{8.4.32}$$

On the one hand, plugging in $[\tilde{\alpha}, \tilde{\beta}]$ for $\tilde{\alpha}$ and ω for ω' in (8.4.30), from (8.4.32) we obtain

$$\left(\nabla^{\mathscr{C}}_{[\tilde{\alpha},\tilde{\beta}]}\tilde{\beta}\right)_{\omega} = 2\nabla^{\omega}G_{\omega}\mathrm{C}^*_{\beta}\big[\tilde{\alpha}, \tilde{\beta}\big]_{\omega} = 2\big(\nabla^{\omega}G_{\omega}\mathrm{C}^*_{\beta}\big)^2\alpha$$

and thus

$$\mathsf{R}^{\mathscr{C}}_{\omega}\big(\tilde{\alpha}, \tilde{\beta}\big)\tilde{\beta} = 0\,.$$

On the other hand, (8.4.32) yields

$$\big\|\mathbf{v}_{\omega}\big[\tilde{\alpha}, \tilde{\beta}\big]\big\|^2 = 4\big\|\nabla^{\omega}\mathrm{G}_{\omega}\mathrm{C}^*_{\alpha}\beta\big\|^2 = 4\big\langle \mathrm{C}^*_{\alpha}\beta, \mathrm{G}_{\omega}\mathrm{C}^*_{\alpha}\beta\big\rangle$$

and thus

$$\big\langle\overline{\mathsf{R}}^{\tau}_{[\omega]}(\overline{\alpha}, \overline{\beta})\overline{\beta}, \overline{\alpha}\big\rangle = 3\big\langle \mathrm{C}^*_{\alpha}\beta, \mathrm{G}_{\omega}\mathrm{C}^*_{\alpha}\beta\big\rangle\,. \tag{8.4.33}$$

The assertion now follows by using the symmetry

$$\big\langle\overline{\mathsf{R}}^{\tau}_{[\omega]}(\overline{\alpha}, \overline{\beta})\overline{\rho}, \overline{\zeta}\big\rangle = \big\langle\overline{\mathsf{R}}^{\tau}_{[\omega]}(\overline{\rho}, \overline{\zeta})\overline{\alpha}, \overline{\beta}\big\rangle\,, \tag{8.4.34}$$

and the multilinearization formula given in the proof of Proposition 2.4.2 (Exercise 8.4.2). ■

Remark 8.4.10 From Proposition 8.4.9, or directly from (8.4.33), we read off the sectional curvature K of a 2-plane $\mathfrak{P} \subset \mathrm{T}_{[\omega]}\mathscr{M}^{\tau}$,

$$\mathsf{K}_{\omega}(\mathfrak{P}) = 3\langle \mathrm{C}^*_{\alpha}\beta, \mathrm{G}_{\omega}\mathrm{C}^*_{\alpha}\beta\rangle\,,$$

where $\alpha, \beta \in \mathfrak{H}^{\tau}_{\omega}$ are the horizontal lifts of two orthonormal vectors spanning $\mathfrak{P}$. We claim that the sectional curvature is non-negative, as in the case of the principal stratum [47, 592]. To see this, denote $\xi = \mathrm{C}^*_{\alpha}\beta$. Using that

$$\mathrm{im}(G_{\omega}) = \ker(\Delta_{\omega})^{\perp} \subset \mathrm{im}(\Delta_{\omega})$$

and that, according to (6.1.22), $\Delta_{\omega}G_{\omega}$ is the L^2-orthogonal projector onto $\mathrm{im}(\Delta_{\omega})$, we find

$$\langle \xi, \mathrm{G}_{\omega}\xi\rangle = \langle \Delta_{\omega}\mathrm{G}_{\omega}\xi, \mathrm{G}_{\omega}\xi\rangle = \|\nabla_{\omega}\mathrm{G}_{\omega}\xi\|^2\,.$$

For an analysis of the scalar curvature we refer to [591]. ♦

We conclude this section with a brief discussion of geodesics. In [75], a proof of the following proposition was outlined.

Proposition 8.4.11 *Let $\omega \in \mathscr{C}^{\tau}$ and $\alpha \in \mathfrak{H}^{\tau}_{\omega}$. Let I denote the connected component of 0 in $\{t \in \mathbb{R} : \omega + t\alpha \in \mathscr{C}^{\tau}\}$. Then I is nonempty, open, and*

$$I \to \mathscr{M}^{\tau}\,, \quad t \mapsto \pi^{\tau}(\omega + t\alpha)\,,$$

is a geodesic in $\mathscr{M}^\tau$. Conversely, any geodesic in $\mathscr{M}^\tau$ is of this form.

Proof Clearly, the curve is contained in $\mathscr{C}^\tau$ and is a geodesic in $\mathscr{C}$. Hence, it is a geodesic in $\mathscr{C}^\tau$. Since it is perpendicular to the $\mathscr{G}$-orbit through ω, Corollary 26.12 in [447] yields that its projection to $\mathscr{M}^\tau$ is a geodesic in $\mathscr{M}^\tau$.

Conversely, let $\overline{\gamma}$ be a geodesic in $\mathscr{M}^\tau$ and let γ be the horizontal lift of $\overline{\gamma}$ starting at some representative ω of $\overline{\gamma}(0)$. By Lemma 26.11 in [447], γ is a geodesic in $\mathscr{C}^\tau$. Since the segment containing ω of the straight line $t \mapsto \omega + t\dot{\gamma}(0)$ in $\mathscr{C}^\tau$ is a geodesic with the same initial conditions as γ, the latter coincides with that segment. ■

Remark 8.4.12 In the proof we have used that the straight line $\omega + t\alpha$ is perpendicular to the orbit through ω. Since $\mathrm{C}_\alpha^* \alpha = 0$ (Exercise 8.4.3), we have

$$\nabla^{\omega + t\alpha\, *} \alpha = \nabla^{\omega *} \alpha = 0\,, \tag{8.4.35}$$

that is, this straight line is perpendicular to any orbit it meets. This is consistent with the general situation, where one can prove that if a geodesic in a Riemannian submersion is perpendicular to one fibre, then it is perpendicular to all fibres it meets, cf. Corollary 26.12 in [447].

Thus, Proposition 8.4.11 states that the geodesics in $\mathscr{M}^\tau$ are given by projections of segments of straight lines inside $\mathscr{C}^\tau$ which are perpendicular to orbits. In particular, the charts defined by the slices $\mathscr{S}^\tau_{\omega_0,\varepsilon}$ provide normal coordinates. ♦

In [75], the above characterization of geodesics is used to prove that the principal stratum need not be geodesically complete. In fact, the argument given there proves the following.

Proposition 8.4.13 *$\mathscr{M}^\tau$ is geodesically complete iff there is no τ' with $\tau' < \tau$.*

Proof Indeed, for $\omega \in \mathscr{C}^\tau$ and $\alpha \in \mathfrak{H}^\tau_\omega$, we have $\mathscr{G}_{\omega + t\alpha} \supset \mathscr{G}_\omega \cap \mathscr{G}_\alpha = \mathscr{G}_\omega$. Therefore,

$$\omega + t\alpha \in \mathscr{C}^{\leq \tau}\,, \tag{8.4.36}$$

for all $t \in \mathbb{R}$. In particular, if there is no τ' with $\tau' < \tau$, the geodesic associated to ω and α is defined for all values $t \in \mathbb{R}$.

Now assume that $\tau' < \tau$ for some τ'. Choose $x' \in \mathscr{M}^{\tau'}$ and a tube $\mathscr{U}_{x',\varepsilon}$ about the orbit $\pi^{-1}(x')$. Since $\mathscr{U}_{x',\varepsilon}$ is a neighbourhood of $\pi^{-1}(x')$ in $\mathscr{C}$, the denseness properties (8.3.11) imply $\mathscr{U}_{x',\varepsilon} \cap \mathscr{C}^\tau \neq \varnothing$. Hence, we find $\omega' \in \pi^{-1}(x')$ and $\omega \in \mathscr{C}^\tau$ such that $\omega \in \mathscr{S}_{\omega',\varepsilon}$. Let $\alpha \in \mathscr{T}$ such that $\omega' = \omega + \alpha$. Then, $\alpha \in \mathfrak{H}_{\omega'}$ and hence $\nabla^{\omega' *}\alpha = 0$. Since $\mathrm{C}_\alpha^*\alpha = 0$, this implies $\nabla^{\omega *}\alpha = 0$ and hence $\alpha \in \mathfrak{H}_\omega$. Since ω and ω' are invariant under $\mathscr{G}_\omega$, so is α. Thus, $\alpha \in \mathfrak{H}^\tau_\omega$. By Proposition 8.4.11, then a segment of the straight line $t \mapsto \omega + t\alpha$ projects to a geodesic in $\mathscr{M}^\tau$. Clearly, this geodesic cannot be prolonged to $t = 1$. ■

Proposition 8.4.13 implies, in particular, that the principal stratum is geodesically complete iff there are no secondary strata.

Proposition 8.4.14 *Let $\omega \in \mathscr{C}^\tau$ and $\alpha \in \mathfrak{H}^\tau_\omega$. The set of values $t \in \mathbb{R}$ for which $\omega + t\alpha \notin \mathscr{C}^\tau$ is discrete.*

Proof Consider the continuous mapping $\eta : \mathbb{R} \to \mathscr{C}$ defined by $\eta(t) := \omega + t\alpha$. According to (8.4.36), the preimage $\eta^{-1}(\mathscr{C}^\tau)$ of $\mathscr{C}^\tau$ is open in $\mathbb{R}$, because $\mathscr{C}^\tau$ is open in $\mathscr{C}^{\leq\tau}$. Hence, $\mathbb{R} \setminus \eta^{-1}(\mathscr{C}^\tau)$ is closed in $\mathbb{R}$.

Let $t_0 \in \mathbb{R} \setminus \eta^{-1}(\mathscr{C}^\tau)$. By Remark 8.4.12, $\alpha \in \ker(\nabla^*_{\eta(t_0)})$, so that the Tubular Neighbourhood Theorem implies that $\eta(t) = \eta(t_0) + (t - t_0)\alpha \in \mathscr{S}_{\eta(t_0),\varepsilon}$ for t close to t_0. If t_0 were an accumulation point of $\mathbb{R} \setminus \eta^{-1}(\mathscr{C}^\tau)$, there would exist $t_1 \neq t_0$ such that $\eta(t_1) \in \mathscr{S}_{\eta(t_0),\varepsilon} \cap \mathscr{C}^{\tau'}$ for some $\tau' < \tau$. By the slice properties, $\mathscr{G}_{\eta(t_1)} \subset \mathscr{G}_{\eta(t_0)}$. Since $\eta(t_1) = \eta(t_0) + (t_1 - t_0)\alpha$, then $\mathscr{G}_\alpha \supset \mathscr{G}_{\eta(t_1)}$. Writing $\omega = \eta(t_1) - t_1\alpha$ one sees that then $\mathscr{G}_{\eta(t_1)} \subset \mathscr{G}_\omega$ (contradiction). Hence, $\mathbb{R} \setminus \eta^{-1}(\mathscr{C}^\tau)$ consists of isolated points. Due to closedness, it is then discrete. ■

Exercises

8.4.1 Prove formula (8.4.31).

8.4.2 Derive the formula for the Riemann curvature given in Proposition 8.4.9 from (8.4.33), using (8.4.34) and the multilinearization formula given in the proof of Proposition 2.4.2.

8.4.3 Show that $\mathrm{C}^*_\alpha \alpha = 0$ for all $\alpha \in \mathscr{T}$.

8.5 Classification of Howe Subgroups

According to Theorem 8.2.8, to determine the gauge orbit types of a gauge theory defined on a principal G-bundle $P(M, G)$, one has to classify the holonomy-induced bundle reductions up to isomorphy and conjugacy under the principal action. Thus, one has to work through the following programme.

1. Classify the Howe subgroups of G up to conjugacy.
2. Classify the Howe subbundles of P up to isomorphy.
3. Extract the Howe subbundles which are holonomy-induced.
4. Factorize by the principal action.
5. Determine the natural partial ordering.

In a series of papers, we have accomplished this programme for M having dimension 2, 3 or 4 and G being $\mathrm{SU}(n)$ [541, 543, 544] or another classical compact Lie group [296, 297]. Here, we will discuss the case $G = \mathrm{SU}(n)$ in detail. In the present section, we determine the Howe subgroups, thus accomplishing the first step of the programme.

Recall that, by Definition 8.2.4, a subgroup H of a Lie group G is called Howe if there exists a subset $A \subset G$ such that $H = \mathrm{C}_G(A)$. The basic properties of Howe subgroups have been listed in Remark 8.2.5. In order to determine the set of conjugacy classes of Howe subgroups of $\mathrm{SU}(n)$, we consider $\mathrm{SU}(n)$ as a subset of $\mathrm{M}_n(\mathbb{C})$, the associative algebra of complex $(n \times n)$-matrices. By Remark 8.2.5, any Howe subgroup H may be represented by its associated Howe dual pair $(H, \mathrm{C}_G(H))$. A Howe dual pair is called reductive iff its members are reductive. In our case this condition is automatically satisfied, because $\mathrm{SU}(n)$ is compact and Howe subgroups are always closed.

Let $\mathrm{K}(n)$ denote the collection of pairs

$$J = (\mathbf{k}, \mathbf{m}) = ((k_1, \ldots, k_r), (m_1, \ldots, m_r)) \ , \quad r = 1, 2, 3, \ldots, n \, ,$$

of sequences of equal length consisting of positive integers which obey

$$\mathbf{k} \cdot \mathbf{m} = \sum_{i=1}^{r} k_i m_i = n \, . \tag{8.5.1}$$

For any permutation σ of r elements, define $\sigma J = (\sigma\mathbf{k}, \sigma\mathbf{m})$. Every $J \in \mathrm{K}(n)$ generates a decomposition

$$\mathbb{C}^n = \left(\mathbb{C}^{k_1} \otimes \mathbb{C}^{m_1}\right) \oplus \cdots \oplus \left(\mathbb{C}^{k_r} \otimes \mathbb{C}^{m_r}\right) \tag{8.5.2}$$

and an associated injective homomorphism

$$\prod_{i=1}^{r} \mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) \, , \quad (D_1, \ldots, D_r) \mapsto \bigoplus_{i=1}^{r} \left(D_i \otimes \mathbb{1}_{m_i}\right) \, . \tag{8.5.3}$$

We denote the image of this homomorphism by $\mathrm{M}_J(\mathbb{C})$ and define

$$\mathrm{U}J := \mathrm{M}_J(\mathbb{C}) \cap \mathrm{U}(n) \, , \quad \mathrm{SU}J := \mathrm{M}_J(\mathbb{C}) \cap \mathrm{SU}(n) \, .$$

Clearly, $\mathrm{U}J$ is the image of the subset $\prod_{i=1}^r \mathrm{U}(k_i) \subset \prod_{i=1}^r \mathrm{M}_{k_i}(\mathbb{C})$ under (8.5.3).

Lemma 8.5.1 *A subgroup of* $\mathrm{SU}(n)$ *is Howe iff it is conjugate to* $\mathrm{SU}J$ *for some* $J \in \mathrm{K}(n)$.

Proof One can check that the Howe subgroups of $\mathrm{SU}(n)$ are obtained from the Howe subgroups of $\mathrm{U}(n)$ by intersection with $\mathrm{SU}(n)$ and that, for the latter, conjugacy under $\mathrm{U}(n)$ boils down to conjugacy under $\mathrm{SU}(n)$ (Exercise 8.5.1). Hence, it suffices to prove that a subgroup of $\mathrm{U}(n)$ is Howe iff it is conjugate to $\mathrm{U}J$ for some $J \in \mathrm{K}(n)$.

First, let H be a Howe subgroup of $\mathrm{U}(n)$. Let $H' = \mathrm{C}_{\mathrm{U}(n)}(H)$ and let K denote the subgroup generated by H and H'. The vector space $\mathbb{C}^n$ decomposes into an orthogonal direct sum of K-irreducible subspaces,

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r \, . \tag{8.5.4}$$

Each V_i is invariant under H and thus decomposes orthogonally into H-irreducible subspaces,

$$V_i = W_{i,1} \oplus \cdots \oplus W_{i,m_i} .$$

Since V_i is K-irreducible, the subgroup H' acts by intertwining all of these representations with one another. Thus, by Schur's Lemma, all $W_{i,j}$ are isomorphic to $W_{i,1}$. Choosing an orthonormal basis in $W_{i,1}$ and denoting $k_i := \dim(W_i)$, we obtain $V_i \cong \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i}$. Since H and H' are mutual centralizers, under this isomorphism,

$$H_{\restriction V_i} = \{a_i \otimes \mathbb{1}_{m_i} : a_i \in \mathrm{U}(k_i)\}\,, \quad H'_{\restriction V_i} = \{\mathbb{1}_{k_i} \otimes b_i : b_i \in \mathrm{U}(m_i)\}\,.$$

As a result, in an appropriate orthonormal basis in $\mathbb{C}^n$, the elements of H can be written in the form

$$a_1 \otimes \mathbb{1}_{m_1} \oplus \cdots \oplus a_r \otimes \mathbb{1}_{m_r}\,, \quad a_i \in \mathrm{U}(k_i)\,.$$

Thus, H is conjugate under $\mathrm{U}(n)$ to the subgroup $\mathrm{U}J$ with $J = (\mathbf{k}, \mathbf{m})$.

Conversely, let $J \in \mathrm{K}(n)$. It suffices to show that $\mathrm{U}J$ is Howe. Consider the centralizer $M' := \mathrm{C}_{\mathrm{M}_n(\mathbb{C})}(\mathrm{M}_J(\mathbb{C}))$. Since $\mathrm{M}_J(\mathbb{C})$ is a unital $*$-subalgebra, so is M'. In particular, M' is spanned by the subset $\tilde{M}' = M' \cap \mathrm{U}(n)$. Moreover, the Double Commutant Theorem yields $\mathrm{C}_{\mathrm{M}_n(\mathbb{C})}(M') = \mathrm{M}_J(\mathbb{C})$. Thus, we obtain

$$\mathrm{C}_{\mathrm{U}(n)}(\tilde{M}') = \mathrm{C}_{\mathrm{M}_n(\mathbb{C})}(\tilde{M}') \cap \mathrm{U}(n) = \mathrm{C}_{\mathrm{M}_n(\mathbb{C})}(M') \cap \mathrm{U}(n) = \mathrm{M}_J(\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{U}J\,.$$

This shows that $\mathrm{U}J$ is Howe. ■

Lemma 8.5.2 *For $J, J' \in \mathrm{K}(n)$, the Howe subgroups $\mathrm{SU}J$ and $\mathrm{SU}J'$ of $\mathrm{SU}(n)$ are conjugate iff there exists a permutation σ such that $J' = \sigma J$.*

Proof It suffices to check that the subalgebras $\mathrm{M}_J(\mathbb{C})$ and $\mathrm{M}_{J'}(\mathbb{C})$ of $\mathrm{M}_n(\mathbb{C})$ are conjugate under $\mathrm{SU}(n)$ iff $J' = \sigma J$ for some permutation σ. If σ exists, one can construct a matrix $T \in \mathrm{SU}(n)$ mapping the factors $\mathbb{C}^{k'_i} \otimes \mathbb{C}^{l'_i}$ of the decomposition (8.5.2) defined by J' identically onto the factors $\mathbb{C}^{k_{\sigma(i)}} \otimes \mathbb{C}^{m_{\sigma(i)}}$ of the decomposition defined by J. Then, $\mathrm{M}_{J'}(\mathbb{C}) = T^{-1}\mathrm{M}_J(\mathbb{C})T$. Conversely, if $\mathrm{M}_{J'}(\mathbb{C}) = T^{-1}\mathrm{M}_J(\mathbb{C})T$ for some $T \in \mathrm{SU}(n)$, then $\mathrm{M}_J(\mathbb{C})$ and $\mathrm{M}_{J'}(\mathbb{C})$ are isomorphic. Hence, $\mathbf{k}' = \sigma\mathbf{k}$ for some permutation σ. Since T is an isomorphism of the representations $\mathrm{M}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{k_r}(\mathbb{C}) \xrightarrow{J} \mathrm{M}_n(\mathbb{C})$ and

$$\mathrm{M}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{k_r}(\mathbb{C}) \xrightarrow{\sigma} \mathrm{M}_{k'_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{k'_r}(\mathbb{C}) \xrightarrow{J'} \mathrm{M}_n(\mathbb{C})\,,$$

where J, J' indicate the respective embeddings (8.5.3), it does not change the multiplicities of the irreducible factors. Thus, $\mathbf{m}' = \sigma\mathbf{m}$. It follows $J' = \sigma J$. ■

As a consequence of Lemma 8.5.2, we can introduce an equivalence relation on the set $\mathrm{K}(n)$ by putting $J \sim J'$ iff $J' = \sigma J$ for some permutation σ. Let $\hat{\mathrm{K}}(n)$ denote the set of equivalence classes. Lemmas 8.5.1 and 8.5.2 yield the following.

Theorem 8.5.3 *The assignment $J \mapsto \mathrm{SU}J$ induces a bijection from $\hat{\mathrm{K}}(n)$ onto the set of conjugacy classes of Howe subgroups of* $\mathrm{SU}(n)$. ■

This concludes the classification of Howe subgroups of $\mathrm{SU}(n)$.

In the remainder, we calculate the homotopy groups of $\mathrm{SU}J$. This will be needed for the discussion of the Howe subbundles in the next section. For a given positive integer l, define the homomorphisms

$$p_l : \mathrm{U}(1) \to \mathrm{U}(1)\,, \quad p_l(z) := z^l\,, \tag{8.5.5}$$

$$j_l : \mathbb{Z}_l \to \mathrm{U}(1)\,, \qquad j_l(k) := \mathrm{e}^{\mathrm{i}2\pi k/l}\,. \tag{8.5.6}$$

Moreover, let

$$j_J : \mathrm{SU}J \to \mathrm{U}J\,, \quad i_J : \mathrm{U}J \to \mathrm{U}(n)\,, \quad \mathrm{pr}_i^{\mathrm{U}J} : \mathrm{U}J \to \mathrm{U}(k_i)$$

denote the natural inclusion mappings and the natural projections to the factors. Finally, for a given element $J = (\mathbf{k}, \mathbf{m})$ of $\mathrm{K}(n)$, let g denote the greatest common divisor of the members of $\mathbf{m}$ and define $\tilde{\mathbf{m}} = (\tilde{m}_1, \dots, \tilde{m}_r)$ by $g\tilde{m}_i = m_i$ for all i. For $D \in \mathrm{U}J$, we compute

$$\det\nolimits_{\mathrm{U(n)}} \circ\, i_J(D) = \prod_{i=1}^{r} p_{m_i} \circ \det\nolimits_{\mathrm{U}(k_i)} \circ\, \mathrm{pr}_i^{\mathrm{U}J}(D) = p_g\left(\prod_{i=1}^{r} p_{\tilde{m}_i} \circ \det\nolimits_{\mathrm{U}(k_i)} \circ\, \mathrm{pr}_i^{\mathrm{U}J}(D)\right).$$

Thus, if we define a group homomorphism $\lambda_J : \mathrm{U}J \to \mathrm{U}(1)$ by

$$\lambda_J(D) := \prod_{i=1}^{r} p_{\tilde{m}_i} \circ \det\nolimits_{\mathrm{U}(k_i)} \circ\; \mathrm{pr}_i^{\mathrm{U}J}(D)\,, \tag{8.5.7}$$

then

$$\det\nolimits_{\mathrm{U(n)}} \circ\, i_J = p_g \circ \lambda_J\,. \tag{8.5.8}$$

As a consequence, the restriction of λ_J to the subgroup $\mathrm{SU}J$ takes values in $\ker p_g = j_g(\mathbb{Z}_g)$. Hence, we obtain an induced homomorphism

$$\lambda_J^{\mathrm{S}} : \mathrm{SU}J \to \mathbb{Z}_g$$

satisfying

$$\lambda_J \circ j_J = j_g \circ \lambda_J^{\mathrm{S}}\,. \tag{8.5.9}$$

The situation can be summarized in the commutative diagram

$$\begin{array}{ccccc} \mathrm{SU}J & \xrightarrow{j_J} & \mathrm{U}J & \xrightarrow{i_J} & \mathrm{U}(n) \\ \lambda^{\mathrm{S}}_J \downarrow & & \downarrow \lambda_J & & \downarrow \det_{\mathrm{U(n)}} \\ \mathbb{Z}_g & \xrightarrow{j_g} & \mathrm{U}(1) & \xrightarrow{p_g} & \mathrm{U}(1) \end{array} \tag{8.5.10}$$

Lemma 8.5.4 *The homomorphism λ^{S}_J projects to an isomorphism from the group of connected components $\mathrm{SU}J/\mathrm{SU}J_0$ onto $\mathbb{Z}_g$.*

Proof Since $\mathbb{Z}_g$ is discrete, λ^{S}_J must be constant on connected components. Hence $\mathrm{SU}J_0 \subset \ker \lambda^{\mathrm{S}}_J$ and λ^{S}_J projects to a homomorphism $\mathrm{SU}J/\mathrm{SU}J_0 \to \mathbb{Z}_g$. The latter is surjective, because so is λ^{S}_J. To prove injectivity, we show $\ker \lambda^{\mathrm{S}}_J \subset \mathrm{SU}J_0$. Let $D \in \ker \lambda^{\mathrm{S}}_J$ and denote $D_i = \mathrm{pr}^{\mathrm{U}J}_i \circ j_J(D)$. Define the homomorphism

$$\varphi : \mathrm{U}(1)^r \to \mathrm{U}(1)\,, \quad (z_1, \dots, z_r) \mapsto z_1^{\tilde{m}_1} \cdots z_r^{\tilde{m}_r}\,.$$

Then,

$$\lambda^{\mathrm{S}}_J(D) = \varphi\left(\det_{\mathrm{U}(k_1)} D_1, \dots, \det_{\mathrm{U}(k_r)} D_r\right)\,.$$

By assumption, $(\det_{\mathrm{U}(k_1)} D_1, \dots, \det_{\mathrm{U}(k_r)} D_r) \in \ker \varphi$. Since the exponents defining φ have greatest common divisor 1, $\ker \varphi$ is connected. Thus, there exists a path $t \mapsto (\gamma_1(t), \dots, \gamma_r(t))$ in $\ker \varphi$ running from $(\det_{\mathrm{U}(k_1)} D_1, \dots, \det_{\mathrm{U}(k_r)} D_r)$ to $(1, \dots, 1)$. For each $i = 1, \dots, r$, define a path $t \mapsto G_i(t)$ in $\mathrm{U}(k_i)$ as follows: first, go from D_i to $(\det_{\mathrm{U}(k_i)} D_i) \oplus \mathbb{1}_{k_i - 1}$, keeping the determinant constant, thus using connectedness of $\mathrm{SU}(k_i)$. Then, use the path $t \mapsto \gamma_i(t) \oplus \mathbb{1}_{k_i-1}$ to get to $\mathbb{1}_{k_i}$. By construction, the image of $(G_1(t), \dots, G_r(t))$ under the embedding (8.5.3) is a path in $\mathrm{SU}J$ connecting D with $\mathbb{1}_n$. This proves $\ker \lambda^{\mathrm{S}}_J \subset \mathrm{SU}J_0$. ∎

Theorem 8.5.5 *The homotopy groups of* $\mathrm{SU}J$ *are*

$$\pi_i\big(\mathrm{SU}J\big) = \begin{cases} \mathbb{Z}_g & i = 0\,, \\ \mathbb{Z}^{\oplus(r-1)} & i = 1\,, \\ \pi_i\big(\mathrm{U}(k_1)\big) \oplus \cdots \oplus \pi_i\big(\mathrm{U}(k_r)\big) & i > 1\,. \end{cases}$$

In particular, $\pi_1(\mathrm{SU}J)$ and $\pi_3(\mathrm{SU}J)$ are torsion-free.

Proof For $i = 0$, this follows from Lemma 8.5.4. For $i > 1$, the assertion follows from the exact homotopy sequence induced by the principal $\mathrm{SU}J$-bundle $\mathrm{U}J \to \mathrm{U}(1)$ with projection $q = \det_{\mathrm{U}(n)} \circ i_J$. For $i = 1$, consider the following portion of this sequence:

$$\begin{array}{ccccccccccc} \pi_2(\mathrm{U}(1)) & \to & \pi_1(\mathrm{SU}J) & \to & \pi_1(\mathrm{U}J) & \xrightarrow{q_*} & \pi_1(\mathrm{U}(1)) & \to & \pi_0(\mathrm{SU}J) & \to & \pi_0(\mathrm{U}J)\,. \\ \| & & & & \| & & \| & & \| & & \| \\ 0 & & & & \mathbb{Z}^{\oplus r} & & \mathbb{Z} & & \mathbb{Z}_g & & 0 \end{array}$$

One has $\mathbb{Z}^{\oplus r}/\ker(q_*) \cong \mathrm{im}(q_*)$ and exactness implies

$$\ker(q_*) \cong \pi_1(\mathrm{SU}J)\,, \quad \mathrm{im}(q_*) = g\mathbb{Z} \cong \mathbb{Z}\,.$$

It follows that $\pi_1(\mathrm{SU}J) \cong \mathbb{Z}^{\oplus(r-1)}$, as asserted. ■

Exercises

8.5.1 Show that the Howe subgroups of $\mathrm{SU}(n)$ are obtained from the Howe subgroups of $\mathrm{U}(n)$ by intersection with $\mathrm{SU}(n)$.

8.6 Classification of Howe Subbundles

In this section, we are going to derive the Howe subbundles of principal $\mathrm{SU}(n)$-bundles up to vertical isomorphisms. By the results of the previous section, we can restrict attention to the structure groups $\mathrm{SU}J$, $J \in \mathrm{K}(n)$. Thus, let $J \in \mathrm{K}(n)$ be fixed.

We shall first derive a description of principal $\mathrm{SU}J$-bundles in terms of suitable characteristic classes and then characterize those which are redutions of a given principal $\mathrm{SU}J$-bundle P. We start from the general classification result of Chap. 3 stating that there exists a bijective correspondence between the set of vertical isomorphism classes of $\mathrm{SU}J$-bundles over M and the set $[M, \mathrm{BSU}J]$ of homotopy classes of continuous mappings from M to the classifying space $\mathrm{BSU}J$, given by assigning to $f : M \to \mathrm{BSU}J$ the pullback under f of the universal $\mathrm{SU}J$-bundle $\mathrm{ESU}J$. Recall that $\mathrm{BSU}J$ can be realized as a CW-complex, cf. Remark 3.4.19. In general, $[M, \mathrm{BSU}J]$ is hard to handle and it cannot be expected to be classified by characteristic classes. However, Theorem 4.8.7 allows us to successively construct the Postnikov tower of $\mathrm{BSU}J$ up to the fifth stage, thus obtaining a 5-equivalent approximation $(\mathrm{BSU}J)_5$. Thus, if we assume that $\dim(M) \leq 4$, then $[M, \mathrm{BSU}J] = [M, (\mathrm{BSU}J)_5]$ and the explicit form of $(\mathrm{BSU}J)_5$ allows for finding the kind of characteristic classes which are necessary to classify principal $\mathrm{SU}J$-bundles. Finally, we shall construct these classes explicitly. The procedure described is common if one deals with bundle classification problems, see for example [43] or [677].

Now, let us construct $(\mathrm{BSU}J)_5$. We use the results and the notation of Sect. 4.8. Recall, in particular, that for a given Abelian group A and a given integer $l \geq 0$, the Eilenberg–MacLane space $K(A, l)$ is defined up to homotopy equivalence by having the homotopy group A in dimension l and trivial homotopy groups in all other dimensions, cf. Appendix G. Let r^* denote the number of indices i for which $k_i > 1$.

Theorem 8.6.1 *The fifth stage of the Postnikov tower of* $\mathrm{BSU}J$ *is given by*

$$(\mathrm{BSU}J)_5 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2) \times \prod_{j=1}^{r^*} K(\mathbb{Z}, 4)\,. \tag{8.6.1}$$

Proof In the proof, we denote $B \equiv \mathrm{BSU}J$. First, we check that B is simple, that is, that the natural action of $\pi_1(B)$ on $\pi_i(B)$ is trivial for all $i \geq 1$. According to Proposition 3.2.9, it suffices to check that the natural action of $\pi_0(\mathrm{SU}J)$ on $\pi_{i-1}(\mathrm{SU}J)$, induced by inner automorphisms, is trivial. This follows by observing that any inner automorphism of $\mathrm{SU}J$ is generated by an element of $(\mathrm{SU}J)_0$ and hence is homotopic to the identity automorphism. Thus, having realized B as a CW-complex, we can apply Theorem 4.8.7 to construct the Postnikov tower. According to Theorem 8.5.5, the relevant homotopy groups are

$$\pi_1(B) = \mathbb{Z}_g\,, \quad \pi_2(B) = \mathbb{Z}^{\oplus(r-1)}\,, \quad \pi_3(B) = 0\,, \quad \pi_4(B) = \mathbb{Z}^{\oplus r^*}\,. \tag{8.6.2}$$

Moreover, we shall need that $H^*_{\mathbb{Z}}(K(\mathbb{Z}, 2))$ is torsion-free and that

$$H^{2i+1}_{\mathbb{Z}}(K(\mathbb{Z}, 2)) = 0\,, \quad H^{2i+1}_{\mathbb{Z}}(K(\mathbb{Z}_g, 1)) = 0\,, \quad i = 0, 1, 2, \dots\,, \tag{8.6.3}$$

see Appendix G.

Stage 1. B_1 is contractible and may therefore be replaced by $B_1 = *$.

Stage 2. B_2 is weakly homotopy equivalent to the total space of the pullback of the path-loop fibration over $K(\pi_1(B), 2)$ under a mapping $\theta_1 : B_1 \to K(\pi_1(B), 2)$. Since $B_1 = *$, the total space coincides with the fibre $K(\pi_1(B), 1)$. Thus, B_2 is weakly homotopy equivalent to $K(\mathbb{Z}_g, 1)$. Realizing the latter as a CW-complex, we can conclude that B_2 is in fact homotopy equivalent to $K(\mathbb{Z}_g, 1)$ and, therefore, can be replaced by the latter:

$$B_2 = K(\mathbb{Z}_g, 1)\,. \tag{8.6.4}$$

Stage 3. B_3 is weakly homotopy equivalent to the total space of the path-loop fibration over $K(\pi_2(B), 3)$ by some mapping $\theta_2 : B_2 \to K(\pi_2(B), 3)$. In view of (8.6.4) and (8.6.2), θ_2 is a mapping $K(\mathbb{Z}_g, 1) \to K(\mathbb{Z}^{\oplus(r-1)}, 3)$. Using

$$K(A_1 \oplus A_2, l) = K(A_1, l) \times K(A_2, l)$$

and (G.1), we find

$$\left[K(\mathbb{Z}_g, 1), K\left(\mathbb{Z}^{\oplus(r-1)}, 3\right)\right] = \prod_{i=1}^{r-1} \left[K(\mathbb{Z}_g, 1), K(\mathbb{Z}, 3)\right] = \prod_{i=1}^{r-1} H^3_{\mathbb{Z}}(K(\mathbb{Z}_g, 1))\,.$$

By (8.6.3), the right hand side vanishes. Hence, θ_2 is homotopic to a constant mapping. It follows that B_3 is weakly homotopy equivalent to, and thus may be replaced by,

$$B_3 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2)\,. \tag{8.6.5}$$

Stage 4. B_4 is weakly homotopy equivalent to the total space of the pullback of the path-loop fibration over $K(\pi_3(B), 4)$ under a mapping $\theta_3 : B_3 \to K(\pi_3(B), 4)$. In view of (8.6.2), the total space coincides with the base space and we obtain $B_4 = B_3$.

Stage 5. B_5 is weakly homotopy equivalent to the total space of the pullback of the path-loop fibration over $K(\pi_4(B), 5)$ under a mapping $\theta_4 : B_4 = B_3 \to K(\pi_4(B), 5)$. By analogy with stage 3,

$$\left[B_3, K\left(\mathbb{Z}^{\oplus r^*}, 5\right)\right] = \prod_{i=1}^{r^*} H^5_{\mathbb{Z}}(B_3) . \tag{8.6.6}$$

According to (8.6.5), since $H^*_{\mathbb{Z}}(K(\mathbb{Z}, 2))$ is torsion-free, we can apply the Künneth Theorem for cohomology [598, Thm. 5.5.11] to write $H^5_{\mathbb{Z}}(B_3)$ as a sum over tensor products

$$H^j_{\mathbb{Z}}(K(\mathbb{Z}_g, 1)) \otimes H^{j_1}_{\mathbb{Z}}(K(\mathbb{Z}, 2)) \otimes \cdots \otimes H^{j_{r-1}}_{\mathbb{Z}}(K(\mathbb{Z}, 2)) ,$$

where $j + j_1 + \cdots + j_{r-1} = 5$. Due to this constraint, each summand contains a tensor factor of odd degree and hence is trivial by (8.6.3). Thus, θ_4 is homotopic to a constant mapping. It follows that B_5 may be replaced by the direct product of B_3 with the fibre $K(\mathbb{Z}^{\oplus r^*}, 4) = \prod_{i=1}^{r^*} K(\mathbb{Z}, 4)$. This proves the theorem. ■

Corollary 8.6.2 *Let $J \in \mathrm{K}(n)$ and let P and P' be principal* SUJ*-bundles over M,* $\dim M \leq 4$. *If $\alpha(P) = \alpha(P')$ for every characteristic class α defined by an element of $H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$, $H^2_{\mathbb{Z}}(\mathrm{BSU}J)$ or $H^4_{\mathbb{Z}}(\mathrm{BSU}J)$, then P and P' are isomorphic.*

Proof As before, we denote $B = \mathrm{BSU}J$. Let $\mathrm{pr}_1, \mathrm{pr}_{21}, \dots, \mathrm{pr}_{2r-1}$, and $\mathrm{pr}_{41}, \dots, \mathrm{pr}_{4r^*}$ denote the natural projections of the direct product (8.6.1) onto its factors. Let γ_1, γ_2 and γ_4 be characteristic elements of, respectively, $H^1_{\mathbb{Z}_g}(K(\mathbb{Z}_g, 1))$, $H^2_{\mathbb{Z}}(K(\mathbb{Z}, 2))$ and $H^4_{\mathbb{Z}}(K(\mathbb{Z}, 4))$. Let $y_5 : B \to B_5$ be the 5-equivalence provided by Theorem 4.8.5. Composition with y_5 defines a bijection $[M, B] \to [M, B_5]$, cf. Corollary VII.11.13 in [104]. Hence, Theorem 8.6.1 and equation (G.1) imply that the mapping

$$\varphi : [M, B] \to H^1_{\mathbb{Z}_g}(M) \times \prod_{i=1}^{r-1} H^2_{\mathbb{Z}}(M) \times \prod_{i=1}^{r^*} H^4_{\mathbb{Z}}(M) ,$$

defined by

$$\varphi(f) := \Big(f^*(\mathrm{pr}_1 \circ y_5)^*\gamma_1 \ , \ \big(f^*(\mathrm{pr}_{2i} \circ y_5)^*\gamma_2\big)_{i=1}^{r-1} \ , \ \big(f^*(\mathrm{pr}_{4i} \circ y_5)^*\gamma_4\big)_{i=1}^{r^*}\Big)$$

is a bijection. Here, for all i,

$$(\mathrm{pr}_1 \circ y_5)^*\gamma_1 \in H^1_{\mathbb{Z}_g}(B) , \quad (\mathrm{pr}_{2i} \circ y_5)^*\gamma_2 \in H^2_{\mathbb{Z}}(B) , \quad (\mathrm{pr}_{4i} \circ y_5)^*\gamma_4 \in H^4_{\mathbb{Z}}(B) .$$

As a consequence, given classifying mappings $f, f' : M \to B$ for P and P', respectively, the assumption implies $\varphi(f) = \varphi(f')$. Hence, f and f' are homotopic. ■

From the proof of Corollary 8.6.2 we read off that the cohomology elements $\left(\mathrm{pr}_1 \circ y_5\right)^* \gamma_1$, $\left(\mathrm{pr}_{2i} \circ y_5\right)^* \gamma_2$, $i = 1, \dots, r-1$, and $\left(\mathrm{pr}_{4i} \circ y_5\right)^* \gamma_4$, $i = 1, \dots, r^*$, of $\mathrm{BSU}J$ define a set of characteristic classes which classifies $\mathrm{SU}J$-bundles over manifolds of dimension ≤ 4. These classes are independent and surjective. However, they are hard to handle, because we do not know the homomorphism y_5^* explicitly. Therefore, we prefer to work with characteristic classes defined by some natural generators of the cohomology groups in question. The price we have to pay is that the corresponding classes are subject to a relation and that we have to determine their image explicitly. Thus, our next aim is to construct generators of $H^2_{\mathbb{Z}}(\mathrm{BSU}J)$, $H^4_{\mathbb{Z}}(\mathrm{BSU}J)$ and $H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$.

First, let us discuss the integral cohomology groups. Generators for $H^*_{\mathbb{Z}}(\mathrm{BSU}J)$ can be obtained as follows. Consider the classifying mappings

$$\mathrm{BSU}J \xrightarrow{\mathrm{B}j_J} \mathrm{BU}J \xrightarrow{\mathrm{B\,pr}_i^{\mathrm{U}J}} \mathrm{BU}(k_i)\,,$$

cf. Definition 3.7.1. By Theorem 4.2.1, $H^*_{\mathbb{Z}}(\mathrm{BU}(k_i))$ is the polynomial ring over $\mathbb{Z}$ in the universal Chern classes $\mathsf{c}_j^{\mathrm{U}(k_i)} \in H^{2j}_{\mathbb{Z}}(\mathrm{BU}(k_i))$, $j = 1, \dots, k_i$. Define

$$\mathsf{c}_j^{\mathrm{U}J,i} := \left(\mathrm{B\,pr}_i^{\mathrm{U}J}\right)^* \mathsf{c}_j^{\mathrm{U}(k_i)} \in H^{2j}_{\mathbb{Z}}(\mathrm{BU}J)\,, \tag{8.6.7}$$

$$\mathsf{c}_j^{\mathrm{SU}J,i} := (\mathrm{B}j_J)^* \, \mathsf{c}_j^{\mathrm{U}J,i} \in H^{2j}_{\mathbb{Z}}(\mathrm{BSU}J) \tag{8.6.8}$$

and write

$$\mathsf{c}^{\mathrm{U}J,i} = 1 + \mathsf{c}_1^{\mathrm{U}J,i} + \cdots + \mathsf{c}_{k_i}^{\mathrm{U}J,i}\,, \quad \mathsf{c}^{\mathrm{SU}J,i} = 1 + \mathsf{c}_1^{\mathrm{SU}J,i} + \cdots + \mathsf{c}_{k_i}^{\mathrm{SU}J,i}\,, \quad i = 1, \dots, r\,,$$

as well as $\mathbf{c}^{\mathrm{U}J} = \left(\mathsf{c}^{\mathrm{U}J,1}, \dots, \mathsf{c}^{\mathrm{U}J,r}\right)$ and $\mathbf{c}^{\mathrm{SU}J} = \left(\mathsf{c}^{\mathrm{SU}J,1}, \dots, \mathsf{c}^{\mathrm{SU}J,r}\right)$.

Lemma 8.6.3 *$H^*_{\mathbb{Z}}(\mathrm{BU}J)$ is the polynomial ring over $\mathbb{Z}$ in the generators $\mathsf{c}_j^{\mathrm{U}J,i}$, $j = 1, \dots, k_i$, $i = 1, \dots, r$.*

Proof As a consequence of Theorem 4.2.1 and the Künneth Theorem for cohomology, $H^*_{\mathbb{Z}}(\prod_i \mathrm{BU}(k_i))$ is the polynomial ring over $\mathbb{Z}$ in the generators

$$1_{\mathrm{BU}(k_1)} \times \cdots \times 1_{\mathrm{BU}(k_{i-1})} \times \mathsf{c}_j^{\mathrm{U}(k_i)} \times 1_{\mathrm{BU}(k_{i+1})} \times \cdots \times 1_{\mathrm{BU}(k_r)}\,,$$

where $j = 1, \dots, k_i$, $i = 1, \dots, r$ and $\times$ denotes the cohomology cross product. By means of the isomorphism

$$\mathrm{U}J \xrightarrow{\Delta_r} \prod_{i=1}^r \mathrm{U}J \xrightarrow{\prod_{i=1}^r \mathrm{pr}_i^{\mathrm{U}J}} \prod_{i=1}^r \mathrm{U}(k_i)\,,$$

where Δ_r denotes r-fold diagonal embedding, this yields the assertion. ∎

Lemma 8.6.4 *The homomorphism* $(\mathrm{B}j_J)^* : H^*_{\mathbb{Z}}(\mathrm{BU}J) \to H^*_{\mathbb{Z}}(\mathrm{BSU}J)$ *is surjective.*

Proof According to Proposition 3.7.8/2, the mapping $\mathrm{B}j_J : \mathrm{BSU}J \to \mathrm{BU}J$ is the projection in a principal bundle with structure group $\mathrm{U}J/\mathrm{SU}J \cong \mathrm{U}(1)$. Denote this bundle by Q. Being orientable, Q has a Gysin sequence, cf. Theorem 4.1.10,

$$\cdots \to H^l_{\mathbb{Z}}(\mathrm{BU}J) \xrightarrow{(\mathrm{B}j_J)^*} H^l_{\mathbb{Z}}(\mathrm{BSU}J) \xrightarrow{\varphi} H^{l-1}_{\mathbb{Z}}(\mathrm{BU}J) \xrightarrow{\cup \mathsf{c}_1(Q)} H^{l+1}_{\mathbb{Z}}(\mathrm{BU}J) \to \cdots$$

If Q were trivial, we would have $\pi_1(\mathrm{BSU}J) \cong \pi_1(\mathrm{BU}J \times \mathrm{U}(1)) \cong \mathbb{Z}$, which would contradict Theorem 8.5.5. Hence, Q is nontrivial. According to Theorem 4.8.1, then $\mathsf{c}_1(Q) \neq 0$. Due to Lemma 8.6.3, $H^*_{\mathbb{Z}}(\mathrm{BU}J)$ does not have zero divisors. It follows that multiplication by $\mathsf{c}_1(Q)$ is an injective operation on $H^*_{\mathbb{Z}}(\mathrm{BU}J)$. Then, exactness of the Gysin sequence implies that the connecting homomorphism φ is trivial and, therefore, $(\mathrm{B}j_J)^*$ is surjective. ■

Lemmas 8.6.3 and 8.6.4 yield the following.

Corollary 8.6.5 (Integral cohomology of BSUJ) *The ring $H^*_{\mathbb{Z}}(\mathrm{BSU}J)$ is generated by $\mathsf{c}_j^{\mathrm{SU}J,i}$, $j = 1, \dots, k_i$, $i = 1, \dots, r$.* ■

The generators $\mathsf{c}_j^{\mathrm{SU}J,i}$ are subject to a relation. Since this relation turns out to be a consequence of a more fundamental relation which will be derived below, it does not play a role in the sequel.

Next, we construct generators for $H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$. For that purpose, we use the homomorphism $\lambda^{\mathrm{S}}_J : \mathrm{SU}J \to \mathbb{Z}_g$ defined by (8.5.9).

Lemma 8.6.6 *The mapping* $\left(\mathrm{B}\lambda^{\mathrm{S}}_J\right)^* : H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g) \to H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$ *is an isomorphism.*

Proof By Lemma 8.5.4, the induced homomorphism $\lambda^{\mathrm{S}}_{J*} : \pi_0(\mathrm{SU}J) \to \pi_0(\mathbb{Z}_g)$ is an isomorphism. Hence, so is $\left(\mathrm{B}\lambda^{\mathrm{S}}_J\right)_* : \pi_1(\mathrm{BSU}J) \to \pi_1(\mathrm{B}\mathbb{Z}_g)$. Now, the assertion follows by the Hurewicz Theorem and the Universal Coefficient Theorem. ■

We conclude that generators of $H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$ can be obtained as the images of generators of $H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g)$ under $\left(\mathrm{B}\lambda^{\mathrm{S}}_J\right)^*$. Since according to the discussion prior to Theorem 4.8.3, $\mathrm{B}\mathbb{Z}_g$ is an Eilenberg–MacLane space of type $K(\mathbb{Z}_g, 1)$, we have

$$H^1_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g) \cong \mathrm{Hom}(\mathbb{Z}_g, \mathbb{Z}_g) \cong \mathbb{Z}_g \,.$$

To choose a generator, we use the homomorphism $j_g : \mathbb{Z}_g \to \mathrm{U}(1)$ defined by (8.5.6) and the short exact sequence of coefficient groups

$$0 \to \mathbb{Z} \xrightarrow{\mu_g} \mathbb{Z} \xrightarrow{\rho_g} \mathbb{Z}_g \to 0\,, \tag{8.6.9}$$

where μ_g denotes multiplication by g and ρ_g reduction modulo g. Recall that this sequence induces a long exact sequence of coefficient homomorphisms [104, Sect. IV.5],

$$\cdots \to H^i_{\mathbb{Z}}(\cdot) \xrightarrow{\mu_g} H^i_{\mathbb{Z}}(\cdot) \xrightarrow{\rho_g} H^i_{\mathbb{Z}_g}(\cdot) \xrightarrow{\beta_g} H^{i+1}_{\mathbb{Z}}(\cdot) \to \cdots\,, \tag{8.6.10}$$

where β_g is the Bockstein homomorphism.

Lemma 8.6.7 *There exists a unique element $\delta_g \in H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g)$ such that*

$$\beta_g(\delta_g) = \big(\mathrm{B}j_g\big)^* \, \mathsf{c}_1^{\mathrm{U}(1)} , \tag{8.6.11}$$

and this element is a generator of $H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g)$.

Proof Clearly, both $\beta_g(\delta_g)$ and $\big(\mathrm{B}j_g\big)^* \, \mathsf{c}_1^{\mathrm{U}(1)}$ are elements of $H^2_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g)$ so that equation (8.6.11) makes sense.

Since $\mathrm{B}\mathbb{Z}_g$ is an Eilenberg–MacLane space of type $K(\mathbb{Z}_g, 1)$, we can read off $H^*_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g)$ from (G.4) to obtain the following portion of the exact sequence (8.6.10):

$$\begin{array}{ccccccc}
H^1_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g) & \xrightarrow{\rho_g} & H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g) & \xrightarrow{\beta_g} & H^2_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g) & \xrightarrow{\mu_g} & H^2_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g) \\
\| & & \| & & \| & & \| \\
0 & & \mathbb{Z}_g & & \mathbb{Z}_g & & \mathbb{Z}_g
\end{array}$$

We conclude that $\ker(\beta_g) = 0$ and that μ_g is trivial. Thus, β_g is an isomorphism. This proves existence and uniqueness of δ_g.

To check that δ_g is a generator, consider the pair $J^\circ = \big((1), (g)\big) \in \mathrm{K}(g)$. Observe that $\mathbb{Z}_g \cong \mathrm{SU}J^\circ$, $\mathrm{U}(1) \cong \mathrm{U}J^\circ$, and that j_g corresponds to $j_{J^\circ} : \mathrm{SU}J^\circ \to \mathrm{U}J^\circ$. Then, Lemma 8.6.4 implies that $\big(\mathrm{B}j_g\big)^*$ is surjective. Thus, $H^2_{\mathbb{Z}}(\mathrm{B}\mathbb{Z}_g)$ is generated by $\big(\mathrm{B}j_g\big)^* \, \mathsf{c}_1^{\mathrm{U}(1)}$ and, therefore, $H^1_{\mathbb{Z}_g}(\mathrm{B}\mathbb{Z}_g)$ is generated by δ_g. ∎

We define

$$\delta_J := \big(\mathrm{B}\lambda^{\mathrm{S}}_J\big)^* \, \delta_g \,.$$

As a consequence of Lemmas 8.6.6 and 8.6.7, we obtain the following.

Corollary 8.6.8 *The cohomology group $H^1_{\mathbb{Z}_g}(\mathrm{BSU}J)$ is generated by δ_J.* ∎

By naturality of the Bockstein homomorphism, the relation (8.6.11) entails

$$\beta_g(\delta_J) = \big(\mathrm{B}\lambda^{\mathrm{S}}_J\big)^* \, \big(\mathrm{B}j_g\big)^* \, \mathsf{c}_1^{\mathrm{U}(1)} . \tag{8.6.12}$$

This relation leads to a relation between the generators δ_J and $\mathsf{c}_j^{\mathrm{SU}J,i}$ as follows. Given a topological space X and a finite sequence of non-negative integers $\mathbf{a} = (a_1, \dots, a_s)$, define a mapping

$$E_{\mathbf{a}} : \prod_{i=1}^{s} H^*_{\mathbb{Z}}(X) \to H^*_{\mathbb{Z}}(X) \,, \quad (\alpha_1, \dots, \alpha_s) \mapsto \alpha_1^{a_1} \cup \dots \cup \alpha_s^{a_s} \,, \tag{8.6.13}$$

where powers are taken with respect to the cup product. Let $E_{\mathbf{a},j}$ denote the composition with the projection to $H^{2j}_{\mathbb{Z}}(X)$. One can check (Exercise 8.6.1) that for elements of the form $\alpha_i = 1 + \alpha_{i,1} + \alpha_{i,2} + \cdots$ with $\alpha_{i,j} \in H^{2j}_{\mathbb{Z}}(X)$, the components in degree 2 and 4 are given by

$$E_{\mathbf{a},1}(\alpha_1, \dots, \alpha_s) = \sum_{i=1}^{s} a_i \, \alpha_{i,1} \,, \tag{8.6.14}$$

$$E_{\mathbf{a},2}(\alpha_1, \dots, \alpha_s) = \sum_{i=1}^{s} a_i \alpha_{i,2} + \sum_{i=1}^{s} \frac{a_i(a_i-1)}{2} \, \alpha_{i,1}^2 + \sum_{i<j} a_i a_j \, \alpha_{i,1} \cup \alpha_{j,1} \,, \tag{8.6.15}$$

respectively. For such elements, (8.6.14) implies that for every $l \in \mathbb{Z}$,

$$E_{l\mathbf{a},1}(\alpha_1, \dots, \alpha_s) = l E_{\mathbf{a},1}(\alpha_1, \dots, \alpha_s) \,. \tag{8.6.16}$$

Recall that $\tilde{\mathbf{m}} = (\tilde{m}_1, \dots, \tilde{m}_r)$ is defined by $g\tilde{m}_i = m_i$ for all i.

Lemma 8.6.9 *We have*

$$(\mathrm{B} i_J)^* \, \mathsf{c}^{\mathrm{U}(n)} = E_{\mathbf{m}} \left(\mathbf{c}^{\mathrm{U}J} \right) \,, \tag{8.6.17}$$

$$(\mathrm{B}\lambda_J)^* \, \mathsf{c}_1^{\mathrm{U}(1)} = E_{\tilde{\mathbf{m}},1} \left(\mathbf{c}^{\mathrm{U}J} \right) \,. \tag{8.6.18}$$

Proof To prove (8.6.17), we decompose i_J into

$$\mathrm{U}J \xrightarrow{\Delta_r} \prod_i \mathrm{U}J \xrightarrow{\prod_i \mathrm{pr}_i^{\mathrm{U}J}} \prod_i \mathrm{U}(k_i) \xrightarrow{\prod_i \Delta_{m_i}} \prod_i \left(\mathrm{U}(k_i) \times \overset{m_i}{\cdots} \times \mathrm{U}(k_i) \right) \xrightarrow{j} \mathrm{U}(n) \,.$$

Here j stands for the natural blockwise embedding. By Theorem 4.3.1,

$$(\mathrm{B}j)^* \, \mathsf{c}^{\mathrm{U}(n)} = (\mathsf{c}^{\mathrm{U}(k_1)} \times \overset{m_1}{\cdots} \times \mathsf{c}^{\mathrm{U}(k_1)}) \times \cdots \times (\mathsf{c}^{\mathrm{U}(k_r)} \times \overset{m_r}{\cdots} \times \mathsf{c}^{\mathrm{U}(k_r)}) \,.$$

Using this, we compute

$$\begin{aligned}
(\mathrm{B}\, i_J)^* \, \mathsf{c}^{\mathrm{U}(n)} &= \Delta_r^* \circ \left(\prod_i \mathrm{B}\, \mathrm{pr}_i^{\mathrm{U}J} \right)^* \circ \left(\prod_i \Delta_{m_i} \right)^* \circ \left(\mathrm{B}j \right)^* \mathsf{c}^{\mathrm{U}(n)} \\
&= \Delta_r^* \circ \left(\prod_i \mathrm{B}\, \mathrm{pr}_i^{\mathrm{U}J} \right)^* \left((\mathsf{c}^{\mathrm{U}(k_1)})^{m_1} \times \cdots \times (\mathsf{c}^{\mathrm{U}(k_r)})^{m_r} \right) \\
&= \Delta_r^* \left(\left(\mathsf{c}^{\mathrm{U}J,1} \right)^{m_1} \times \cdots \times \left(\mathsf{c}^{\mathrm{U}J,r} \right)^{m_r} \right) \\
&= \left(\mathsf{c}^{\mathrm{U}J,1} \right)^{m_1} \cup \ldots \cup \left(\mathsf{c}^{\mathrm{U}J,r} \right)^{m_r} .
\end{aligned}$$

This yields (8.6.17). To prove (8.6.18), we observe that (8.5.10) implies

$$(\mathrm{B}\lambda_J)^* \left(\mathrm{B}p_g \right)^* \mathsf{c}_1^{\mathrm{U}(1)} = (\mathrm{B}\, i_J)^* \left(\mathrm{B}\, \det\nolimits_{\mathrm{U}(n)} \right)^* \mathsf{c}_1^{\mathrm{U}(1)} \tag{8.6.19}$$

and compute $(\mathrm{B}p_g)^* \mathsf{c}_1^{\mathrm{U}(1)} = g \, \mathsf{c}_1^{\mathrm{U}(1)}$ and $(\mathrm{B}\det_{\mathrm{U}(n)})^* \mathsf{c}_1^{\mathrm{U}(1)} = \mathsf{c}_1^{\mathrm{U}(n)}$. Plugging this into (8.6.19) and using (8.6.17) and (8.6.16), we obtain

$$g (\mathrm{B}\lambda_J)^* \mathsf{c}_1^{\mathrm{U}(1)} = E_{\mathbf{m},1}(\mathbf{c}^{\mathrm{U}J}) = g \, E_{\tilde{\mathbf{m}},1}(\mathbf{c}^{\mathrm{U}J}) \,.$$

Since this holds in $H^2_{\mathbb{Z}}(\mathrm{BU}J)$, which is free Abelian, (8.6.18) follows. ■

Theorem 8.6.10 *The generators δ_J and $\mathsf{c}_j^{\mathrm{SU}J,i}$ satisfy the relation*

$$\beta_g(\delta_J) = E_{\tilde{\mathbf{m}},1}(\mathbf{c}^{\mathrm{SU}J}) . \tag{8.6.20}$$

Proof Using (8.6.12), (8.5.9) and (8.6.18), we compute

$$\beta_g(\delta_J) = \left(\mathrm{B}\lambda_J^{\mathrm{S}}\right)^* \left(\mathrm{B}j_g\right)^* \mathsf{c}_1^{\mathrm{U}(1)} = (\mathrm{B}j_J)^* \, (\mathrm{B}\lambda_J)^* \, \mathsf{c}_1^{\mathrm{U}(1)} = (\mathrm{B}j_J)^* \, E_{\tilde{\mathbf{m}},1} \left(\mathbf{c}^{\mathrm{U}J}\right) .$$

By definition of $\mathbf{c}^{\mathrm{SU}J}$, this yields the assertion. ■

Remark 8.6.11 To summarize, we can replace the $1 + (r-1) + r^*$ independent generators

$$\left(\mathrm{pr}_1 \circ y_5\right)^* \gamma_1 , \quad \left(\mathrm{pr}_{2i} \circ y_5\right)^* \gamma_2 , \ i = 1, \dots, r-1 , \quad \left(\mathrm{pr}_{4j} \circ y_5\right)^* \gamma_4 , \ j = 1, \dots, r^* ,$$

which arise from the construction of the Postnikov tower and are hardly manageable, by the $1 + r + r^*$ natural generators

$$\delta_J , \quad \mathsf{c}_j^{\mathrm{SU}J,i} , \ i = 1, \dots, r, j = 1, \dots, r^* ,$$

fulfilling the relation (8.6.20). This relation is, in effect, a consequence of (8.5.9). ♦

Now, we discuss the characteristic classes for principal SUJ-bundles Q defined by the cohomology elements $\mathsf{c}_j^{\mathrm{SU}J,i}$ and δ_J. We denote them by, respectively, $\mathsf{c}_j^i(Q)$ and $\delta_J(Q)$. Let

$$\mathsf{c}^i(Q) = 1 + \mathsf{c}_1^i(Q) + \cdots + \mathsf{c}_{2k_i}^i(Q) , \quad \mathbf{c}(Q) = \left(\mathsf{c}^1(Q), \dots, \mathsf{c}^r(Q)\right) .$$

Then,

$$\mathsf{c}_j^i(Q) = f^* \mathsf{c}_j^{\mathrm{SU}J,i} , \quad \mathsf{c}^i(Q) = f^* \mathsf{c}^{\mathrm{SU}J,i} , \quad \mathbf{c}(Q) = f^* \mathbf{c}^{\mathrm{SU}J} , \tag{8.6.21}$$

for any classifying mapping $f : M \to \mathrm{BSU}J$ for Q. Theorem 8.6.10 entails that the characteristic classes c^i and δ_J satisfy the relation

$$\beta_g\left(\delta_J(Q)\right) = E_{\tilde{\mathbf{m}},1}\left(\mathbf{c}(Q)\right) \tag{8.6.22}$$

for all principal SUJ-bundles Q. As a consequence of Corollary 4.1.4, they can furthermore be expressed in terms of the ordinary characteristic classes of associated principal $\mathrm{U}(k_i)$-bundles and $\mathbb{Z}_g$-bundles (Exercise 8.6.2):

$$\mathsf{c}^i \, (Q) = \mathsf{c}\left(Q^{[\mathrm{pr}_i^{\mathrm{U}J} \circ j_J]}\right) , \tag{8.6.23}$$

$$\delta_J \, (Q) = \delta_g\left(Q^{[\lambda_J^{\mathrm{S}}]}\right) . \tag{8.6.24}$$

The characteristic classes c^i and δ_J allow for classifying principal $\mathrm{SU}J$-bundles. To state the result, let $H^J_{\mathbb{Z}}(M)$ denote the subset of $\prod_{i=1}^r H^*_{\mathbb{Z}}(M)$ consisting of the sequences $\alpha = (\alpha_1, \dots, \alpha_r)$ whose members are of the form $\alpha_i = 1 + \alpha_{i,1} + \cdots + \alpha_{i,k_i}$ with $\alpha_{i,j} \in H^{2j}_{\mathbb{Z}}(M)$ and define

$$\mathrm{K}(M, J) := \left\{(\alpha, \xi) \in H^J_{\mathbb{Z}}(M) \times H^1_{\mathbb{Z}_g}(M) \; : \; E_{\tilde{\mathbf{m}},1}(\alpha) = \beta_g(\xi)\right\}.$$

Theorem 8.6.12 (Classification of principal $\mathrm{SU}J$-bundles) *Let M be a manifold of dimension ≤ 4 and let $J \in \mathrm{K}(n)$. Then, the characteristic classes c^i and δ_J define a bijection from the set of vertical isomorphism classes of principal* $\mathrm{SU}J$*-bundles over M onto* $\mathrm{K}(M, J)$.

Proof By Corollary 8.6.2, it remains to prove that for every $(\alpha, \xi) \in \mathrm{K}(M, J)$, there exists a principal $\mathrm{SU}J$-bundle Q over M such that $\mathbf{c}(Q) = \alpha$ and $\delta_J(Q) = \xi$. Since $\dim M \leq 4$, by Theorem 4.8.8, there exist principal $\mathrm{U}(k_i)$-bundles over M such that $\mathsf{c}(Q_i) = \alpha_i$, $i = 1, \dots, r$. Consider the fibre product $\tilde{Q} = Q_1 \times_M \cdots \times_M Q_r$, which has structure group $\prod_{i=1}^r \mathrm{U}(k_i)$ and may thus be interpreted as a $\mathrm{U}J$-bundle. Then,

$$\tilde{Q}^{[\,\mathrm{pr}_i^{\mathrm{U}J}\,]} \cong Q_i\,, \quad i = 1, \dots, r\,. \tag{8.6.25}$$

The desired $\mathrm{SU}J$-bundle Q will arise as a reduction of $\tilde{Q}$. To find it, consider the associated principal $\mathrm{U}(1)$-bundle $\tilde{Q}^{[\lambda_J]}$.

We claim that $\tilde{Q}^{[\lambda_J]}$ admits a reduction to the subgroup $\mathbb{Z}_g \subset \mathrm{U}(1)$. By Theorem 4.8.3, there exists a principal $\mathbb{Z}_g$-bundle R over M such that $\delta_g(R) = \xi$. Consider the principal $\mathrm{U}(1)$-bundle $R^{[j_g]}$ obtained by extension with the homomorphism j_g defined by (8.5.6). On the one hand, using Corollary 4.1.4, Lemma 8.6.7 and naturality of the Bockstein homomorphism β_g, we find $\mathsf{c}_1\left(R^{[j_g]}\right) = \beta_g(\xi)$. On the other hand, a similar calculation using (8.6.18) yields $\mathsf{c}_1\left(\tilde{Q}^{[\lambda_J]}\right) = E_{\tilde{\mathbf{m}},1}(\alpha)$. Since $(\alpha, \xi) \in \mathrm{K}(M, J)$, these classes coincide. As a consequence, Theorem 4.8.1 implies that $\tilde{Q}^{[\lambda_J]}$ and $R^{[j_g]}$ are vertically isomorphic. Hence, R is a reduction of $\tilde{Q}^{[\lambda_J]}$.

Now, we can define Q to be the preimage of the reduction R of $\tilde{Q}^{[\lambda_J]}$ under the natural bundle morphism $\tilde{Q} \to \tilde{Q}^{[\lambda_J]}$. By construction, Q is a reduction of $\tilde{Q}$ to the subgroup $\lambda_J^{-1}(\mathbb{Z}_g) = \mathrm{SU}J \subset \mathrm{U}J$.

It remains to compute $\mathsf{c}^i(Q)$ and $\delta_J(Q)$. Since $Q^{[j_J]} = \tilde{Q}$, using (8.6.23) and (8.6.25), we find

$$\mathsf{c}^i(Q) = \mathsf{c}\Big(Q^{[\mathrm{pr}_i^{\mathrm{U}J} \circ j_J]}\Big) = \mathsf{c}\Big(\Big(Q^{[j_J]}\Big)^{[\mathrm{pr}_i^{\mathrm{U}J}]}\Big) = \mathsf{c}\Big(\tilde{Q}^{[\mathrm{pr}_i^{\mathrm{U}J}]}\Big) = \mathsf{c}\,(Q_i) = \alpha_i\,.$$

Finally, since $Q^{[\lambda_J^{\mathrm{S}}]} = R$, the relation (8.6.24) implies $\delta_J\,(Q) = \delta_g\,(R) = \xi$. ∎

To classify the Howe subbundles of a given principal $\mathrm{SU}(n)$-bundle P up to vertical isomorphy, it remains to characterize the reductions of P to the subgroups $\mathrm{SU}J$ in terms of the characteristic classes $\mathbf{c}$ and δ_J. Let $i_J^{\mathrm{S}} : \mathrm{SU}J \to \mathrm{SU}(n)$ denote the natural inclusion mapping.

Lemma 8.6.13 *For every principal* SUJ*-bundle* Q*, we have* $\mathsf{c}\big(Q^{[i_J^{\mathrm{S}}]}\big) = E_{\mathbf{m}}\big(\mathbf{c}(Q)\big)$.

Proof Denoting the natural inclusion mapping $\mathrm{SU}(n) \to \mathrm{U}(n)$ by j, we find

$$\mathsf{c}\big(Q^{[i_J^{\mathrm{S}}]}\big) = \mathsf{c}\big(Q^{[j \circ i_J^{\mathrm{S}}]}\big) = \mathsf{c}\big(Q^{[i_J \circ j_J]}\big)\,.$$

Using Corollary 4.1.4 and equation (8.6.17), one can check that

$$\mathsf{c}\big(Q^{[i_J \circ j_J]}\big) = E_{\mathbf{m}}\Big(\mathsf{c}\big(Q^{[\mathrm{pr}_1^{\mathrm{U}J} \circ j_J]}\big), \dots, \mathsf{c}\big(Q^{[\mathrm{pr}_r^{\mathrm{U}J} \circ j_J]}\big)\Big) \tag{8.6.26}$$

(Exercise 8.6.3). Then, (8.6.23) yields the assertion. ■

Define

$$\mathrm{K}(P, J) = \big\{(\alpha, \xi) \in \mathrm{K}(M, J) : E_{\mathbf{m}}(\alpha) = \mathsf{c}(P)\big\}\,.$$

Theorem 8.6.14 (Classification of Howe subbundles) *Let* P *be a principal* $\mathrm{SU}(n)$*-bundle over a manifold* M *of dimension* ≤ 4 *and let* $J \in \mathrm{K}(n)$*. Then, the characteristic classes* c^i *and* δ_J *define a bijection from the set of vertical isomorphism classes of reductions of* P *to the subgroup* SUJ *onto* $\mathrm{K}(P, J)$.

Proof Let $Q \subset P$ be a principal SUJ-bundle over M. By Theorem 8.6.12, the pair $\big(\mathbf{c}(Q), \delta_J(Q)\big)$ belongs to $\mathrm{K}(M, J)$. Lemma 8.6.13 implies that it belongs to the subset $\mathrm{K}(P, J)$ iff $\mathsf{c}\big(Q^{[i_J^{\mathrm{S}}]}\big) = \mathsf{c}(P)$. Since $\dim M \leq 4$, by Theorem 4.8.8, the latter is equivalent to $Q^{[i_J^{\mathrm{S}}]} \cong P$, that is, to the condition that Q be a reduction of P. ■

Remark 8.6.15 The equation $E_{\mathbf{m}}(\alpha) = \mathsf{c}(P)$ actually contains the two equations $E_{\mathbf{m},1}(\alpha) = 0$ and $E_{\mathbf{m},2}(\alpha) = \mathsf{c}_2(P)$. However, under the assumption that (α, ξ) belongs to $\mathrm{K}(M, J)$, the first one is redundant, because in this case, due to (8.6.16), one has $E_{\mathbf{m},1}(\alpha) = g\, E_{\tilde{\mathbf{m}},1}(\alpha) = g\, \beta_g(\xi) = 0$. Thus, the relevant equations are

$$E_{\tilde{\mathbf{m}},1}(\alpha) = \beta_g(\xi)\,, \tag{8.6.27}$$

$$E_{\mathbf{m},2}(\alpha) = \mathsf{c}_2(P)\,, \tag{8.6.28}$$

where $\alpha \in H^J_{\mathbb{Z}}(M)$ and $\xi \in H^1_{\mathbb{Z}_g}(M)$. The set of solutions of equation (8.6.27) yields $\mathrm{K}(M, J)$ and hence the principal SUJ-bundles over M. The set of solutions of both Eqs. (8.6.27) and (8.6.28) yields $\mathrm{K}(P, J)$ and, therefore, the reductions of P to the subgroup SUJ. ♦

This concludes the classification of Howe subbundles of P, that is, Step 2 of our programme.

Example 8.6.16 Let us discuss some examples of $J \in \mathrm{K}(n)$, including the two trivial ones, corresponding to the center and the whole group. For brevity, we shall write J in the form $J = (k_1, \dots, k_r | m_1, \dots, m_r)$.

1. $J = (1|n)$. We have $\mathrm{SU}J = \mathbb{Z}_n$, the center of $\mathrm{SU}(n)$, and hence $g = n$. Variables are $\xi \in H^1_{\mathbb{Z}_n}(M)$ and $\alpha = 1 + \alpha_1$, with $\alpha_1 \in H^2_{\mathbb{Z}}(M)$. According to (8.6.14) and (8.6.15), Eqs. (8.6.27) and (8.6.28) read

$$\alpha_1 = \beta_n(\xi)\,, \quad \frac{n(n-1)}{2}\alpha_1^2 = \mathsf{c}_2(P)\,.$$

   Since the first equation yields $n\alpha_1 = 0$, the second one requires $\mathsf{c}_2(P) = 0$. It follows that $\mathrm{K}(P, J)$ is nonempty iff P is trivial. In that case, the first equation implies that $\mathrm{K}(P, J)$ is parameterized by ξ. This coincides with what is known about $\mathbb{Z}_n$-reductions of $\mathrm{SU}(n)$-bundles.
2. $J = (n|1)$. We have $\mathrm{SU}J = \mathrm{SU}(n)$ and hence $g = 1$. Accordingly, the variable is $\alpha = 1 + \alpha_1 + \alpha_2$. Equations (8.6.27) and (8.6.28) read

$$\alpha_1 = 0\,, \quad \alpha_2 = \mathsf{c}_2(P)\,.$$

   Thus, as expected, $\mathrm{K}(P, J)$ consists of P itself.
3. $J = (1, 1|2, 2) \in \mathrm{K}(4)$. One can check that $\mathrm{SU}J$ has the connected components

$$\{\mathrm{diag}(z, z, z^{-1}, z^{-1}) \,:\, z \in \mathrm{U}(1)\}\,, \quad \{\mathrm{diag}(z, z, -z^{-1}, -z^{-1}) : z \in \mathrm{U}(1)\}\,.$$

   It is therefore isomorphic to $\mathrm{U}(1) \times \mathbb{Z}_2$. Variables are $\xi \in H^1_{\mathbb{Z}_2}(M)$ and $\alpha_i = 1 + \alpha_{i,1}$, $i = 1, 2$. Equations (8.6.27) and (8.6.28) read

$$\alpha_{1,1} + \alpha_{2,1} = \beta_2(\xi)\,, \quad \alpha_{1,1}^2 + \alpha_{2,1}^2 + 4\alpha_{1,1} \cup \alpha_{2,1} = \mathsf{c}_2(P)\,.$$

   Since products including $\beta_2(\xi)$ vanish, by eliminating $\alpha_{2,1}$ we obtain

$$-2\alpha_{1,1}^2 = \mathsf{c}_2(P)\,. \tag{8.6.29}$$

4. $J = (2, 3|1, 1) \in \mathrm{K}(5)$. We have $\mathrm{SU}J = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ which is isomorphic to the symmetry group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ of the standard model. In the grand unified $\mathrm{SU}(5)$-model this is the subgroup to which $\mathrm{SU}(5)$ is broken by the heavy Higgs field. Variables are $\alpha_i = 1 + \alpha_{i,1} + \alpha_{i,2}$. Equations (8.6.27) and (8.6.28) read

$$\alpha_{1,1} + \alpha_{2,1} = 0\,, \quad \alpha_{1,2} + \alpha_{2,2} + \alpha_{1,1} \cup \alpha_{2,1} = \mathsf{c}_2(P)\,.$$

   Eliminating $\alpha_{2,1} = -\alpha_{1,1}$ and $\alpha_{2,2} = \mathsf{c}_2(P) - \alpha_{1,2} + \alpha_{1,1}^2$, we see that $\mathrm{K}(P, J)$ can be parameterized by α_1 (or α_2), that is, by the Chern class of one of the factors $\mathrm{U}(2)$ or $\mathrm{U}(3)$. Due to the important role $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is playing in elementary particle physics, this has been known for a long time [338]. ♦

Remark 8.6.17 As an illustration, let us discuss Eq. (8.6.29) explicitly for the base manifolds $M = \mathrm{S}^4$, $\mathrm{S}^2 \times \mathrm{S}^2$ and $\mathrm{L}^3_p \times \mathrm{S}^1$, where L^3_p denotes the 3-dimensional lens space of order p. Since M is compact and orientable, we have $H^4_{\mathbb{Z}}(M) = \mathbb{Z}$.

1. $M = \mathrm{S}^4$. Since $H^2_{\mathbb{Z}}(M) = 0$, $\mathrm{K}(P, J)$ is nonempty iff $\mathsf{c}_2(P) = 0$. In that case, it consists of the trivial $\mathrm{U}(1) \times \mathbb{Z}_2$-bundle only.
2. For $M = \mathrm{S}^2 \times \mathrm{S}^2$, we choose a generator (orientation) γ_2^{S} of $H^2_{\mathbb{Z}}(M)$ to expand

$$\alpha_{1,1} = a\,\gamma_2^{\mathrm{S}} \times 1 + b\,1 \times \gamma_2^{\mathrm{S}}\,, \quad \mathsf{c}_2(P) = c\gamma_2^{\mathrm{S}} \times \gamma_2^{\mathrm{S}}$$

with $a, b, c \in \mathbb{Z}$. Then, Eq. (8.6.29) becomes

$$-4ab = c\,. \tag{8.6.30}$$

If $c = 0$, there are two obvious series of solutions. In particular, $\mathrm{K}(P, J)$ is infinite here. If $c = 4l$ for some $l \neq 0$, then a runs through the positive and negative divisors of l and $b = -l/a$. If c is not divisible by 4, then $\mathrm{K}(P, J)$ is empty.
3. $M = \mathrm{L}^3_p \times \mathrm{S}^1$. The relevant cohomology groups of L^p_3 are

$$H^1_{\mathbb{Z}}(\mathrm{L}^3_p) = 0\,, \quad H^2_{\mathbb{Z}}(\mathrm{L}^3_p) = \mathbb{Z}_p\,, \quad H^1_{\mathbb{Z}_g}(\mathrm{L}^3_p) = \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_g) = \mathbb{Z}_{\langle p,g\rangle}\,,$$

where $\langle p, g\rangle$ denotes the greatest common divisor of p and g. Hence, by the Künneth Theorem for cohomology,

$$H^1_{\mathbb{Z}_g}(M) = \mathbb{Z}_{\langle p,g\rangle} \oplus \mathbb{Z}_g\,, \quad H^2_{\mathbb{Z}}(M) = \mathbb{Z}_p\,.$$

Since $H^2_{\mathbb{Z}}(M)$ is torsion, $\mathrm{K}(P, J)$ is nonempty iff $\mathsf{c}_2(P) = 0$. In that case, it is parameterized independently by $\xi \in \mathbb{Z}_{\langle 2,p\rangle} \oplus \mathbb{Z}_2$ and $\alpha_{1,1} \in \mathbb{Z}_p$.

The case of base manifold $\mathrm{S}^2 \times \mathrm{S}^2$ illustrates that equation (8.6.28) generally leads to a Diophantine equation. Here, this equation is bilinear. For bilinear Diophantine equations, there exists an algorithm to parameterize the set of solutions [596]. The situation is different, for example, for the base manifold $M = \mathbb{C}\mathrm{P}^2$. Here the equation obtained from (8.6.28) is quadratic and, therefore, substantially harder to discuss. ♦

Exercises

8.6.1 Confirm Eqs. (8.6.14) and (8.6.15).

8.6.2 Use Corollary 4.1.4 to verify the relations (8.6.23) and (8.6.24).

8.6.3 Use Corollary 4.1.4 and Eq. (8.6.17) to prove (8.6.26).

8.6.4 Analyze Eqs. (8.6.27) and (8.6.28) for $J = (1, 1|2, 3) \in \mathrm{K}(5)$ and $J = (2|2) \in \mathrm{K}(4)$, cf. Example 8.6.16.

8.7 Enumeration of Gauge Orbit Types

In this section, we complete the enumeration of gauge orbit types. First, we accomplish step 3 of our programme, that is, we determine which Howe subbundles of a given principal $\mathrm{SU}(n)$-bundle P are holonomy-induced.

Lemma 8.7.1 *Let H and H' be Howe subgroups of* $\mathrm{SU}(n)$ *such that $H \subset H'$. If* $\dim H = \dim H'$*, then $H = H'$.*

Proof There exist $J, J' \in \mathrm{K}(n)$ such that H and H' are conjugate to $\mathrm{SU}J$ and $\mathrm{SU}J'$, respectively. Consider $\mathrm{U}J$ and $\mathrm{U}J'$. Since $H \subset H'$, we can find $D \in \mathrm{SU}(n)$ such that $D^{-1}\mathrm{U}JD \subset \mathrm{U}J'$. By assumption,

$$\dim(\mathrm{U}J') = \dim(H) + 1 = \dim(H') + 1 = \dim(\mathrm{U}J)\,.$$

Since $\mathrm{U}J'$ is connected and $D^{-1}\mathrm{U}JD$ is closed in $\mathrm{U}J'$, equality of dimension implies $D^{-1}\mathrm{U}JD = \mathrm{U}J'$. Then, $D^{-1}\mathrm{SU}JD = \mathrm{SU}J'$ and hence $H = H'$. ■

Theorem 8.7.2 *Any Howe subbundle of a principal* $\mathrm{SU}(n)$*-bundle is holonomy-induced.*

Proof Let P be a principal $\mathrm{SU}(n)$-bundle and let Q be a Howe subbundle of P with structure group H. Choose a connected component $\tilde{Q}$ of Q and let $\tilde{H}$ denote the corresponding structure group. Since H is Howe, $\mathrm{C}^2_{\mathrm{SU}(n)}(\tilde{H}) \subset \mathrm{C}^2_{\mathrm{SU}(n)}(H) = H$. Since $\dim \tilde{H} = \dim H$, the subgroups $\mathrm{C}^2_{\mathrm{SU}(n)}(\tilde{H})$ and H have the same dimension. Then, Lemma 8.7.1 implies $\mathrm{C}^2_{\mathrm{SU}(n)}(\tilde{H}) = H$ and, hence, the assertion. ■

One may wonder whether there exist Howe subbundles which are not holonomy-induced. Let us give an example. Consider the subgroup $H = \{\mathbb{1}_3, \mathrm{diag}(-1, -1, 1)\}$ of $\mathrm{SO}(3)$. One can check that H is Howe. Thus, the reduction $Q = M \times H$ of the trivial bundle $M \times \mathrm{SO}(3)$ is a Howe subbundle. Any connected reduction $\tilde{Q}$ of Q has the center $Z = \{\mathbb{1}_3\}$ as its structure group. Since the center is Howe itself, we find $\tilde{Q} \cdot \mathrm{C}^2_G(Z) = \tilde{Q} \neq Q$. Thus, Q is not holonomy-induced.

Now, we turn to step 4 of our programme, that is, we determine which of the isomorphism classes of Howe subbundles get identified under the principal $\mathrm{SU}(n)$-action on P. Since this action conjugates the structure groups, it suffices to restrict attention to the reductions to the subgroups $\mathrm{SU}J$ with $J \in \mathrm{K}(n)$. Define

$$\mathrm{K}(P) = \bigsqcup_{J \in \mathrm{K}(n)} \mathrm{K}(P, J)\,. \tag{8.7.1}$$

We shall denote the elements of $\mathrm{K}(P)$ by L and write them in the form $L = (J; \alpha, \xi)$, where $J \in \mathrm{K}(n)$ and $(\alpha, \xi) \in \mathrm{K}(P, J)$. By a Howe subbundle of P of type $L = (J; \alpha, \xi)$ we mean a bundle reduction Q of P to the subgroup $\mathrm{SU}J$ with the characteristic classes $\mathbf{c}(Q_L) = \alpha$ and $\delta_J(Q_L) = \xi$. On the set $\mathrm{K}(P)$, we introduce the following equivalence relation: $(J;\alpha,\xi)\sim(J';\alpha',\xi')$ iff there exists a permutation σ such that $J'=\sigma J$ and $\alpha'=\sigma\alpha$. Clearly, in that case, the sequences constituting J and J' must have the same length r. Let us furthermore introduce the following notation. For every combination of elements $J,J'\in \mathrm{K}(n)$, we put

$$\mathrm{N}(J,J') := \{D\in \mathrm{SU}(n)\ :\ D^{-1}\mathrm{SU}J D\subset \mathrm{SU}J'\}\,.$$

This is a subset of $\mathrm{SU}(n)$. Every element $D\in \mathrm{N}(J,J')$ defines an algebra embedding

$$h_D^{\mathrm{M}}:\mathrm{M}(J)\to\mathrm{M}(J')\,,\quad C\mapsto D^{-1}CD\,,$$

and, by restriction, Lie subgroup embeddings $h_D^{\mathrm{U}}:\mathrm{U}J\to\mathrm{U}J'$ and $h_D^{\mathrm{S}}:\mathrm{SU}J\to\mathrm{SU}J'$.

Lemma 8.7.3 *Let $L,L'\in\mathrm{K}(P)$ and let Q and Q' be Howe subbundles of P of type L and L', respectively. Then, Q' is vertically isomorphic to $\Psi_D(Q)$ for some $D\in\mathrm{SU}(n)$ iff $L'\sim L$.*

Proof Let $L=(J;\alpha,\xi)$ and $L'=(J';\alpha',\xi')$. One can check that

$$\Psi_D(Q)\cong Q^{[h_D^{\mathrm{S}}]}\,.$$

Accordingly, by Proposition 3.7.2/1, if Q has classifying mapping f, then $\Psi_D(Q)$ has classifying mapping $\mathrm{B}h_D^{\mathrm{S}}\circ f$. Due to $\lambda_{J'}^{\mathrm{S}}\circ h_D^{\mathrm{S}}=\lambda_J^{\mathrm{S}}$, this implies, in particular,

$$\delta_{J'}\big(\Psi_D(Q)\big)=\delta_J(Q)\,. \tag{8.7.2}$$

First, assume that Q' is vertically isomorphic to $\Psi_D(Q)$ for some $D\in\mathrm{SU}(n)$. Then,

$$\mathbf{c}\big(\Psi_D(Q)\big)=\alpha'\,,\quad \delta_{J'}\big(\Psi_D(Q)\big)=\xi'\,. \tag{8.7.3}$$

In view of (8.7.2), the second equation implies $\xi'=\xi$. Moreover, $D\in\mathrm{N}(J,J')$ and h_D^{U} and h_D^{S} are isomorphisms. Consequently, there exists a permutation σ such that h_D^{U} maps the $\sigma(i)$-th factor of $\mathrm{U}J$ isomorphically onto the i-th factor of $\mathrm{U}J'$. Then, in particular, $J'=\sigma J$. It remains to show that $\alpha'=\sigma\alpha$. For that purpose, we bring D to a normal form as follows. Given σ, we can find $D_\sigma\in\mathrm{N}(J,J')$ such that $\mathrm{pr}_i^{\mathrm{U}J'}\circ h_{D_\sigma}^{\mathrm{U}}=\mathrm{pr}_{\sigma(i)}^{\mathrm{U}J}$ for all i. Then,

$$\mathrm{pr}_i^{\mathrm{U}J'}\circ j_{J'}\circ h_{D_\sigma}^{\mathrm{S}}=\mathrm{pr}_{\sigma(i)}^{\mathrm{U}J}\circ j_J\,. \tag{8.7.4}$$

Moreover, $C=DD_\sigma^{-1}\in\mathrm{N}(J,J)$ and h_C^{U} is an automorphism of $\mathrm{U}J$ which leaves each factor invariant separately. One can check that h_C^{U}, and hence h_C^{S}, is inner. Since any inner automorphism of $\mathrm{SU}J$ can be generated by an element of the connected component of the identity, we conclude $\mathrm{B}h_C^{\mathrm{S}}=\mathrm{id}_{\mathrm{BSU}J}$ and thus $\mathrm{B}h_D^{\mathrm{S}}=\mathrm{B}h_{D_\sigma}^{\mathrm{S}}$. As a consequence, $\mathrm{B}h_{D_\sigma}^{\mathrm{S}}\circ f$ is a classifying mapping for $\Psi_D(Q)$. Using this and Corollary

4.1.4, from (8.7.4) we derive

$$\mathbf{c}\big(\Psi_D(Q)\big) = \sigma\big(\mathbf{c}(Q)\big)\,. \tag{8.7.5}$$

In view of (8.7.3), this implies $\alpha' = \sigma\alpha$ and hence, finally, $L' \sim L$.

Conversely, assume that $\xi' = \xi$ and $\alpha' = \sigma\alpha$, $J' = \sigma J$ for some permutation σ. Then, in particular, there exists $D = D_\sigma \in \mathrm{N}(J, J')$ satisfying (8.7.4) and hence (8.7.5). It follows that $\mathbf{c}\big(\Psi_D(Q)\big) = \sigma\alpha = \alpha' = \mathbf{c}(Q')$. Similarly, (8.7.2) yields $\delta_{J'}\big(\Psi_D(Q)\big) = \delta_{J'}(Q')$. As a consequence, Theorem 8.6.12 implies that $\Psi_D(Q)$ and Q' are vertically isomorphic. ■

Let $\hat{\mathrm{K}}(P)$ denote the set of equivalence classes in $\mathrm{K}(P)$. Combining Lemma 8.7.3 with Theorem 8.6.14, we finally arrive at the following result. Recall that $\mathrm{Red}_*(P)$ denotes the set of holonomy-induced bundle reductions of P modulo vertical isomorphisms and conjugacy under the principal action on P.

Theorem 8.7.4 (Classification of holonomy-induced bundle reductions) *Let P be a principal $\mathrm{SU}(n)$-bundle over a manifold M of dimension ≤ 4. The assignment to $L \in \mathrm{K}(P)$ of a bundle reduction Q of P of type L induces a bijection from $\hat{\mathrm{K}}(P)$ onto $\mathrm{Red}_*(P)$.* ■

With Theorem 8.7.4 we have accomplished the enumeration of gauge orbit types. As a result, these orbit types are in bijective correspondence with the elements of $\hat{\mathrm{K}}(P)$. Let us summarize.

Corollary 8.7.5 (Enumeration of gauge orbit types) *For $G = \mathrm{SU}(n)$ and $\dim M = 2, 3, 4$, gauge orbit types are in one-to-one correspondence with symbols $[(J; \alpha, \xi)]$, where*

1. *$J = ((k_1, \dots, k_r), (m_1, \dots, m_r))$ is a pair of sequences of positive integers obeying*
$$\sum_{i=1}^{r} k_i m_i = n\,,$$
2. *$\alpha = (\alpha_1, \dots, \alpha_r)$ is a sequence of elements $\alpha_i \in H^*_{\mathbb{Z}}(M)$ representing admissible values of the Chern classes of $\mathrm{U}(k_i)$-bundles over M,*
3. *$\xi \in H^1_{\mathbb{Z}_g}(M)$, where g is the greatest common divisor of $(m_1, \dots, m_r)$.*

The cohomology elements α_i and ξ are subject to the relations

$$\sum_{i=1}^{r} \frac{m_i}{g}\alpha_{i,1} = \beta_g(\xi)\,, \quad \alpha_1^{m_1} \cup \dots \cup \alpha_r^{m_r} = \mathsf{c}(P)\,,$$

where $\beta_g : H^1_{\mathbb{Z}_g}(M) \to H^2_{\mathbb{Z}}(M)$ is the Bockstein homomorphism associated with the short exact sequence of coefficient groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_g \to 0$. For any permutation σ of $\{1, \dots, r\}$, the symbols $[(J; \alpha, \xi)]$ and $[(\sigma J; \sigma\alpha, \xi)]$ have to be identified. ■

8.8 Partial Ordering

In this section we are going to characterize the natural partial ordering of gauge orbit types in terms of the classifying set $\hat{\mathrm{K}}(P)$. For the technical details, we refer to [544].

According to Theorem 8.2.8, the partial ordering of gauge orbit types corresponds to the partial ordering on $\hat{\mathrm{K}}(P)$ which is induced from the inclusion relation between bundle reductions. Thus, we put $[L] \leq [L']$ if there exist bundle reductions Q of type L and Q' of type L' such that $\Psi_D(Q) \subset Q'$ for some $D \in \mathrm{SU}(n)$. Let $L = (J; \alpha, \xi)$ with $J = (\mathbf{k}, \mathbf{m}) = ((k_1, \ldots, k_r), (m_1, \ldots, m_r))$ and $L' = (J'; \alpha', \xi')$ with $J' = (\mathbf{k}', \mathbf{m}') = ((k'_1, \ldots, k'_r), (m'_1, \ldots, m'_r))$ be given.

First, we observe that $\Psi_D(Q) \subset Q'$ implies $D \in \mathrm{N}(J, J')$. Since $\mathrm{M}_J(\mathbb{C})$ and $\mathrm{M}_{J'}(\mathbb{C})$ are finite-dimensional unital C^*-algebras, the embedding h^{M}_D defined by $C \mapsto D^{-1}CD$ is characterized by a so-called inclusion matrix Δ. This is an $(r' \times r)$-matrix whose entries $\Delta_{i'i}$ are given by the numbers of basic representations contained in the representations

$$\mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_J(\mathbb{C}) \xrightarrow{h^{\mathrm{M}}_D} \mathrm{M}_{J'}(\mathbb{C}) \to \mathrm{M}_{k'_{i'}}(\mathbb{C}) \,,$$

where the first arrow is the canonical embedding to the ith factor of $\mathrm{M}_J(\mathbb{C})$ and the third arrow is the natural projection to the i'th factor of $\mathrm{M}_{J'}(\mathbb{C})$. Since the embedding h^{M}_D is unital, $\sum_i \Delta_{i'i} k_i = k'_{i'}$ for all i'. Since conjugation of $\mathrm{M}_J(\mathbb{C})$ by D^{-1} preserves the total number of basic representations of the factor $\mathrm{M}_{k_i}(\mathbb{C})$ in $\mathrm{M}_n(\mathbb{C})$, we have $\sum_{i'} \Delta_{i'i} m'_{i'} = m_i$ for all i. Thus, Δ solves the system of equations

$$\Delta\,\mathbf{k} = \mathbf{k}' \,, \quad \mathbf{m} = \Delta^{\mathrm{T}} \mathbf{m}' \,. \tag{8.8.1}$$

Conversely, assume that a solution Δ of (8.8.1) is given. Then, the decompositions (8.5.2) associated with J and J' admit subdecompositions

$$\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \left(\bigoplus_{i'=1}^{r'} \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}} \right), \quad \mathbb{C}^n = \bigoplus_{i'=1}^{r'} \left(\bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \right) \otimes \mathbb{C}^{m'_{i'}} \,,$$

respectively, which differ by a permutation of the factors $\mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}}$. From this permutation, a matrix $D \in \mathrm{N}(J, J')$ with inclusion matrix Δ can be constructed. It follows that $\mathrm{SU}J \subset \mathrm{SU}J'$ up to conjugacy iff the system of equations (8.8.1) has a solution Δ.

Second, we observe that the extension of $\Psi_D(Q)$ to the structure group $\mathrm{SU}J'$ is vertically isomorphic to $Q^{[h^{\mathrm{S}}_D]}$. Hence, $\Psi_D(Q) \subset Q'$ iff

$$\mathbf{c}\big(Q^{[h^{\mathrm{S}}_D]}\big) = \alpha' \,, \quad \delta_J\big(Q^{[h^{\mathrm{S}}_D]}\big) = \xi' \,. \tag{8.8.2}$$

By (8.6.23) and $j_{J'} \circ h^{\mathrm{S}}_D = h^{\mathrm{U}}_D \circ j_J$,

$$\mathsf{c}^{i'}\big(Q^{[h^{\mathrm{S}}_D]}\big) = \mathsf{c}\big(Q^{[\mathrm{pr}^{\mathrm{U}J'}_{i'} \circ h^{\mathrm{U}}_D \circ j_J]}\big) \,.$$

A computation analogous to the proof of formula (8.6.17) then yields

$$\mathsf{c}^{i'}\big(Q^{[h_D^{\mathrm{S}}]}\big) = \big(\mathsf{c}^1(Q)\big)^{\Delta_{i'1}} \cdots \big(\mathsf{c}^r(Q)\big)^{\Delta_{i'r}} .$$

Thus, using the notation

$$E_\Delta(\alpha) := \left(\alpha_1^{\Delta_{11}} \cdots \alpha_r^{\Delta_{1r}}, \ldots, \alpha_1^{\Delta_{r'1}} \cdots \alpha_r^{\Delta_{r'r}}\right) ,$$

which is a generalization of (8.6.13), we obtain

$$\mathbf{c}\big(Q^{[h_D^{\mathrm{S}}]}\big) = E_\Delta\big(\mathbf{c}(Q)\big) . \tag{8.8.3}$$

By (8.6.24) and $\lambda_{J'}^{\mathrm{S}} \circ h_D^{\mathrm{S}} = \rho_{g'} \circ \lambda_J^{\mathrm{S}}$, we have

$$\delta_J\big(Q^{[h_D^{\mathrm{S}}]}\big) = \delta_{g'}\big(Q^{[\rho_{g'} \circ \lambda_D^{\mathrm{S}}]}\big) .$$

Here, reduction mod g' is well defined on $\mathbb{Z}_g$-valued cohomology, because the second equation in (8.8.1) implies that g' divides g. Using that the characteristic class of the mod g'-reduction of a $\mathbb{Z}_g$-bundle is given by the mod g'-reduction of the characteristic class of this bundle (Exercise 8.8.3), we obtain

$$\delta_J\big(Q^{[h_D^{\mathrm{S}}]}\big) = \rho_{g'}\big(\delta_J(Q)\big) . \tag{8.8.4}$$

From (8.8.2), (8.8.3) and (8.8.4) we conclude that $\Psi_D(Q) \subset Q'$ iff

$$E_\Delta(\alpha) = \alpha' , \tag{8.8.5}$$

$$\rho_{g'}(\xi) = \xi' . \tag{8.8.6}$$

Let us introduce the following notation. If (8.8.6) holds, let $\mathrm{N}(L, L')$ be the set of solutions of the system of equations (8.8.1) and (8.8.5). If (8.8.6) does not hold, let $\mathrm{N}(L, L') = \varnothing$. To summarize, we have shown the following.

Theorem 8.8.1 *Let* $L, L' \in \mathrm{K}(P)$*. Then* $[L] \leq [L']$ *if and only if* $\mathrm{N}(L, L') \neq \varnothing$*.* ■

Example 8.8.2 Consider the trivial bundle $P = M \times \mathrm{SU}(4)$. Let $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi') \in \mathrm{K}(P)$ with $J = (1, 1|2, 2)$ and $J' = (2, 2|1, 1)$. Then, $\mathrm{SU}J \cong \mathrm{U}(1) \times \mathbb{Z}_2$. The subgroup $\mathrm{SU}J'$ can be parameterized by

$$\mathrm{SU}J' = \left\{ \begin{pmatrix} zA & 0 \\ 0 & z^{-1}B \end{pmatrix} : \, z \in \mathrm{U}(1), \;\; A, B \in \mathrm{SU}(2) \right\} .$$

It is therefore isomorphic to $\big(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)\big)/\mathbb{Z}_2$. To determine $\mathrm{N}(L, L')$, we first consider the system of equations (8.8.1):

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} , \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \Delta_{11} & \Delta_{21} \\ \Delta_{12} & \Delta_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

The solutions are

$$\Delta^a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \quad \Delta^b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} , \quad \Delta^c = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} .$$

For $\alpha = (\alpha_1, \alpha_2)$, they yield

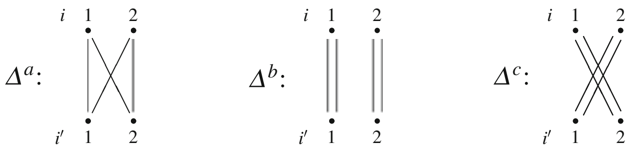
$$E_{\Delta^a}(\alpha) = (\alpha_1\alpha_2, \alpha_1\alpha_2) , \quad E_{\Delta^b}(\alpha) = (\alpha_1^2, \alpha_2^2) , \quad E_{\Delta^c}(\alpha) = (\alpha_2^2, \alpha_1^2) .$$

Condition (8.8.6) is trivially satisfied due to $g' = 1$. Thus, $\mathrm{N}(L, L') \neq \varnothing$ precisely in one of the following cases:

$$(a)\ \alpha_1' = \alpha_2' = \alpha_1\alpha_2 , \quad (b)\ \alpha_1' = \alpha_1^2,\ \alpha_2' = \alpha_2^2 , \quad (c)\ \alpha_1' = \alpha_2^2,\ \alpha_2' = \alpha_1^2 .$$

♦

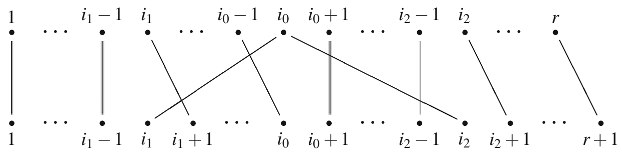
Remark 8.8.3 Any inclusion matrix Δ can be visualized by a diagram consisting of a series of upper vertices, labelled by $i = 1, \ldots, r$, and a series of lower vertices, labelled by $i' = 1, \ldots, r'$. For each combination of i and i', the corresponding vertices are connected by $\Delta_{i'i}$ edges. For example, the matrices Δ^a, Δ^b and Δ^c in the above example give rise to the diagrams



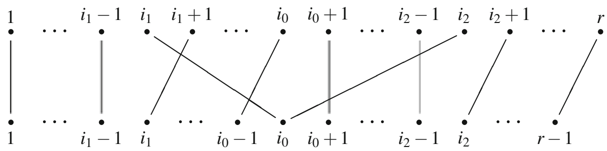
The diagrams assigned in this way to the elements of $\mathrm{N}(J, J')$ are special cases of so-called Bratteli diagrams [101]. The latter have, in general, several stages picturing the subsequent inclusion matrices associated to an ascending sequence of finite dimensional von-Neumann algebras $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \mathbf{A}_3 \subset \cdots$. For this reason, we refer to the diagram associated to $\Delta \in \mathrm{N}(J, J')$ as the Bratteli diagram of Δ. We remark that, due to the first equation in (8.8.1), Δ cannot have a zero row. By the second equation, it cannot have a zero column either. Accordingly, each vertex of the Bratteli diagram of Δ is met by at least one edge. ♦

In what follows, we give a brief survey about the characterization and generation of direct successors and direct predecessors. Proofs can be found in [544].

Theorem 8.8.4 (Characterization of direct successors and predecessors) *Let* $L = (J; \alpha, \xi)$, $L' = (J'; \alpha', \xi') \in \mathrm{K}(P)$. *Then,* $[L']$ *is a direct successor of* $[L]$ *if and only if* $\mathrm{N}(L, L')$ *contains an element with Bratteli diagram*



or



for some i_0 *and* $i_1 < i_2$.

To generate direct successors and predecessors, let $L = (J; \alpha, \xi) \in \mathrm{K}(P)$ with $J = (\mathbf{k}, \mathbf{m})$. Consider the following operations applied to L. We leave it to the reader to check that all the tuples $L' = (J'; \alpha', \xi')$ produced by these operations belong to $\mathrm{K}(P)$ (Exercise 8.8.2).

1. Splitting. Choose i_0 such that $m_{i_0} \neq 1$ and decompose $m_{i_0} = m_{i_0,1} + m_{i_0,2}$ with strictly positive integers $m_{i_0,1}, m_{i_0,2}$. Define $\mathbf{k}'$ and α' by doubling the entries k_{i_0} and α_{i_0}, respectively, and $\mathbf{m}'$ by replacing the single entry m_{i_0} by the two entries $m_{i_0,1}, m_{i_0,2}$. Then, by construction, the greatest common divisor g' of $\mathbf{m}'$ divides g and we can put $\xi' = \rho_{g'}(\xi)$.
2. Merging. Choose $i_1 < i_2$ such that $m_{i_1} = m_{i_2}$. Define $\mathbf{k}'$, $\mathbf{m}'$ and α' by deleting the i_2-th entry and replacing the entry k_{i_1} of $\mathbf{k}$ by $k_{i_1} + k_{i_2}$ and the entry α_{i_1} of α by $\alpha_{i_1} \cup \alpha_{i_2}$. Then, $g' = g$ and we can put $\xi' = \xi$.
3. Inverse splitting. Choose $i_1 < i_2$ such that $k_{i_1} = k_{i_2}$ and $\alpha_{i_1} = \alpha_{i_2}$. Define $\mathbf{k}'$, $\mathbf{m}'$ and α' by deleting the i_2-th entry and replacing the entry m_{i_1} of $\mathbf{m}$ by $m_{i_1} + m_{i_2}$. Then, g divides g'. Choose $\xi' \in H^1_{\mathbb{Z}_{g'}}(M)$ such that $\xi = \rho_g(\xi')$ and $\beta_{g'}(\xi') = E_{\tilde{\mathbf{m}}',1}(\alpha')$.
4. Inverse Merging. Choose i_0 such that $k_{i_0} \neq 1$ and decompose $k_{i_0} = k_{i_0,1} + k_{i_0,2}$ with strictly positive integers $k_{i_0,1}, k_{i_0,2}$. For $l = 1, 2$, choose cohomology elements $\alpha_{i_0,l} = 1 + \alpha_{i_0,l,1} + \cdots + \alpha_{i_0,l,k_{i_0,l}}$ with $\alpha_{i_0,l,j} \in H^{2j}_{\mathbb{Z}}(M)$ such that $\alpha_{i_0,1} \cup \alpha_{i_0,2} = \alpha_{i_0}$. Define $\mathbf{k}'$ and α' by replacing the corresponding i_0-th entry by the two entries $k_{i_0,1}, k_{i_0,2}$ and $\alpha_{i_0,1}, \alpha_{i_0,2}$, respectively, and define $\mathbf{m}'$ by doubling the i_0-the entry. Then, $g' = g$ and we can put $\xi' = \xi$.

Theorem 8.8.5 (Generation of direct successors and predecessors) *Let* $[L] \in \hat{\mathrm{K}}(P)$ *and let L be a representative. The direct successors (predecessors) of* $[L]$ *are obtained by applying all possible splittings and mergings (inverse splittings and inverse mergings) to L and passing to equivalence classes.*

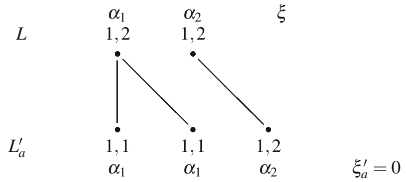
Remark 8.8.6

1. It may happen that for certain elements of $\mathrm{K}(P)$ no splittings or no mergings can be applied. Among such elements are, for example, those with $m_i = 1$ for all i (no splitting) and those having pairwise distinct m_i (no merging). An analogous statement is true for inverse splitting and inverse merging.
2. A direct inspection shows that for every $L \in \mathrm{K}(P)$, the number of splitting or merging operations which can be applied to L is finite. It follows that an element of $\hat{\mathrm{K}}(P)$ can have at most finitely many direct successors and hence at most finitely many successors. ♦

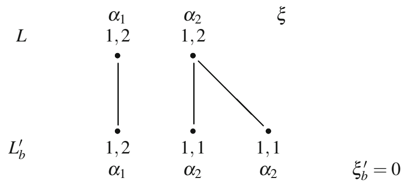
In the remainder of this section, we discuss two examples.

Example 8.8.7 Let P be a principal $\mathrm{SU}(4)$-bundle and consider $L = (J; \alpha, \xi)$ with $J = (1, 1|2, 2)$. Here, $\alpha_i = 1 + \alpha_{i,1}$, $i = 1, 2$, and $\xi \in H^1_{\mathbb{Z}_2}(M)$.

First, there are two splitting operations which can be applied to L. One is given by $i_0 = 1$ and the decomposition $m_1 = 2 = 1 + 1$. It yields $L'_a = (J'_a; \alpha'_a, \xi'_a)$, where $J'_a = (1, 1, 1|1, 1, 2)$, $\alpha'_a = (\alpha_1, \alpha_1, \alpha_2)$, and $\xi'_a = 0$. The passage from L to L'_a can be conveniently visualized by a Bratteli diagram whose vertices are labelled by the respective quantities k_i, m_i and α_i:

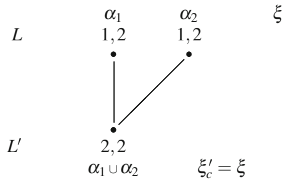


The other splitting operation is given by $i_0 = 2$ and $m_2 = 2 = 1 + 1$. It yields L'_b represented by the labelled Bratteli diagram



The equivalence classes of L'_a and L'_b coincide iff $\alpha_1 = \alpha_2$. In order to see for which bundles P this can happen, consider the Eqs. (8.6.27) and (8.6.28). The first one requires $\alpha_{1,1} = \alpha_{2,1}$ to be a torsion element. Then, due to $\alpha_{1,2} = \alpha_{2,2} = 0$, the second one implies $\mathsf{c}_2(P) = 0$. Thus, L'_a and L'_b can be equivalent only if P is trivial.

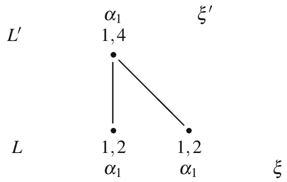
Next, there is a single merging operation, given by $i_1 = 1$, $i_2 = 2$. It yields L' represented by



As a result, generically, $[L]$ has three direct successors, represented by L'_a, L'_b and L'_c. Now, we turn to the generation of direct predecessors of $[L]$. Inverse splittings can be applied only if $\alpha_1 = \alpha_2$. In this case, $J' = (1|4)$ and $\alpha' = (\alpha_1)$. Every solution $\xi \in H^1_{\mathbb{Z}_4}(M)$ of the system of equations

$$\xi' \bmod 2 = \xi\,, \quad \beta_4(\xi') = \alpha_{1,1}\,, \tag{8.8.7}$$

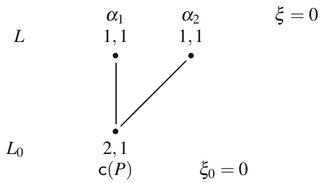
complements J' and α' to an element L' of $\mathrm{K}(P)$. The passage from L to L' can be summarized in the labelled Bratteli diagram



which has to be read upwards. Since the L' differ in the class ξ', they generate a separate equivalence class each. Finally, since $k_1 = k_2 = 1$, inverse mergings cannot be applied to L. Thus, in the case $\alpha_1 = \alpha_2$, the direct predecessors of the equivalence class of L are labelled by the solutions of equations (8.8.7), whereas in the case $\alpha_1 \neq \alpha_2$ direct predecessors do not exist. Recall that the first case can only occur if P is trivial. ♦

Example 8.8.8 Let P be a principal $\mathrm{SU}(2)$-bundle. We shall construct the partially ordered set $\hat{\mathrm{K}}(P)$, starting from its maximal element.

Let L_0 denote the unique representative of the maximal element of $\hat{\mathrm{K}}(P)$. Since the latter corresponds to P itself, L_0 is given by $J_0 = (2|1)$, $\alpha_0 = \mathsf{c}(P)$ and $\xi_0 = 0$. Inverse mergings yield the following elements L:



Here, $\alpha_i = 1 + \alpha_{i,1}$ such that $\alpha_1\alpha_2 = \mathsf{c}(P)$. Sorting by degree yields the equations $\alpha_{1,1} + \alpha_{2,1} = 0$ and $\alpha_{1,1}\alpha_{2,1} = \mathsf{c}_2(P)$. The first one implies $\alpha_{2,1} = -\alpha_{1,1}$ and the second one then reads

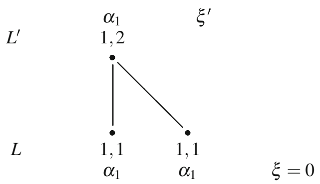
$$-\,\alpha_{1,1}^2 = \mathsf{c}_2(P)\,. \tag{8.8.8}$$

The solutions $\alpha_{1,1}$ and $-\alpha_{1,1}$ yield equivalent direct predecessors.[6]

Next, we determine the direct predecessors of the classes generated by L. Inverse mergings cannot be applied. Inverse splittings can be applied provided $\alpha_1 = \alpha_2$. In this case, every solution $\xi' \in H^1_{\mathbb{Z}_2}(M)$ of the equation

$$\beta_2\left(\xi'\right) = \alpha_{1,1}\,, \tag{8.8.9}$$

yields an element L' by



and each of these elements generates a separate equivalence class. Clearly, $J' = (1|2)$ labels the center $\mathbb{Z}_2$ of $\mathrm{SU}(2)$ and ξ' is the natural characteristic class provided by Theorem 4.8.3 of the corresponding reduction. In particular, L' does not have predecessors and we are done.

Let us present the Hasse diagram of the partially ordered set $\hat{\mathrm{K}}(P)$ for the base manifolds $M = \mathrm{S}^4$, $\mathrm{S}^2 \times \mathrm{S}^2$ and $\mathrm{L}^3_{2p} \times \mathrm{S}^1$. In a Hasse diagram, vertices stand for the elements of the partially ordered set and edges indicate the relation 'left vertex $\leq$ right vertex'. When viewing the elements of $\hat{\mathrm{K}}(P)$ as Howe subbundles, the vertex on the right hand side represents the class corresponding to P itself, whereas the vertices in the middle and on the left hand side represent reductions of P to the Howe subgroups $\mathrm{U}(1)$ and $\mathbb{Z}_2$, respectively. When viewing the elements of $\hat{\mathrm{K}}(P)$ as orbit types, or strata of the gauge orbit space, the vertex on the right hand side represents the generic stratum, whereas the vertices in the middle and on the left hand side represent the secondary strata.

1. $M = \mathrm{S}^4$. If $\mathsf{c}_2(P) = 0$, Eq. (8.8.8) is trivially satisfied by $\alpha_{1,1} = 0$. Then, Eq. (8.8.9) is trivially satisfied by $\xi' = 0$. Since $H^1_{\mathbb{Z}_2}(M) = 0$ and $H^2_{\mathbb{Z}}(M) = 0$, there are no further solutions for either one. Thus, in the case where P is trivial, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is

$$\bullet\!-\!\!-\!\!-\!\bullet\!-\!\!-\!\!-\!\bullet$$

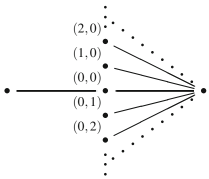
[6]We note that the Howe subgroup labelled by $J = (1, 1|1, 1)$ is the toral subgroup $\mathrm{U}(1)$ of $\mathrm{SU}(2)$ and that $\alpha_{1,1}$ is just the first Chern class of the corresponding reduction of P. By virtue of this transliteration, Eq. (8.8.8) is consistent with the literature [338].

If P is nontrivial, $\hat{\mathrm{K}}(P)$ consists only of the class corresponding to P itself. On the level of strata, this result means that in the sector of vanishing topological charge the gauge orbit space decomposes into the generic stratum, a $\mathrm{U}(1)$-stratum, and an $\mathrm{SU}(2)$-stratum. If, on the other hand, a topological charge is present, then only the generic stratum survives.

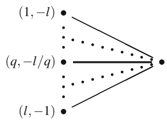
2. $M = \mathrm{S}^2 \times \mathrm{S}^2$. Choosing a generator of $H^2_{\mathbb{Z}}(M)$ and expanding $\alpha_{1,1}$ and $\mathsf{c}_2(P)$ as in Remark 8.6.17/2, Eq. (8.8.8) yields

$$-4ab = c\,,$$

cf. (8.6.30). Since $H^1_{\mathbb{Z}_2}(M) = 0$, only the solution $a = b = 0$ has a direct predecessor. Thus, if $\mathsf{c}_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



The vertices in the middle are labelled by the corresponding values of (a, b). Note that passage to equivalence classes requires that solutions (a, b) and $(-a, -b)$ are identified. If $c = 2l$, the Hasse diagram is



where, according to the identification $(a, b) \sim (-a, -b)$, q runs through the positive divisors of l only. Finally, if c is odd, $\hat{\mathrm{K}}(P)$ has one element, corresponding to P itself.

3. $M = \mathrm{L}^3_{2p} \times \mathrm{S}^1$. The relevant cohomology groups of L^3_{2p} are given in Remark 8.6.17/3. Let γ^{L}_1 and γ^{L}_2 be generators of $H^1_{\mathbb{Z}_g}(\mathrm{L}^3_{2p})$ and $H^2_{\mathbb{Z}}(\mathrm{L}^3_{2p})$, respectively. In addition, choose a generator γ^{S}_1 of $H^1_{\mathbb{Z}}(\mathrm{S}^1)$. Then, $H^1_{\mathbb{Z}_g}(M) = \mathbb{Z}_{\langle 2p,g\rangle} \oplus \mathbb{Z}_{2p}$ is generated by $\gamma^{\mathrm{L}}_1 \times 1$ and $1 \times \rho_{2p}(\gamma^{\mathrm{S}}_1)$ and $H^2_{\mathbb{Z}}(M) = \mathbb{Z}_{2p}$ is generated by γ^{L}_2. One can check that γ^{L}_2 can be chosen so that the Bockstein homomorphism β_g is given by

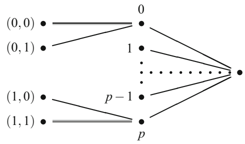
$$\beta_g\big(\gamma^{\mathrm{L}}_1 \times 1\big) = \frac{2p}{\langle 2p, g\rangle}\, \gamma^{\mathrm{L}}_2 \times 1\,, \quad \beta_g\big(1 \times \rho_{2p}(\gamma^{\mathrm{S}}_1)\big) = 0\,, \tag{8.8.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the greatest common divisor. We expand

$$\alpha_{1,1} = a\ \gamma^{\mathrm{L}}_2 \times 1\,, \quad \xi' = \xi'_{\mathrm{L}}\ \gamma^{\mathrm{L}}_1 \times 1 + \xi'_{\mathrm{S}}\ 1 \times \rho_g(\gamma^{\mathrm{S}}_1)\,.$$

First, consider Eq. (8.8.8). Since $H^2_{\mathbb{Z}_{2p}}(\mathrm{L}^3_{2p})$ is torsion, we have $\alpha_\mathrm{I}^2 = 0$. Hence, (8.8.8) admits a solution iff $\mathsf{c}_2(P) = 0$. In this case, the solutions are given by $a \in \mathbb{Z}_{2p}$. Since when passing to equivalence classes we have to identify a with $-a$, the direct predecessors are labelled by $a = 0, \ldots, p$.

Now, consider Eq. (8.8.9). According to (8.8.10), in the present situation it reads $p\,\xi'_\mathrm{L} = a$. Thus, only the elements labelled by $a = 0$ and $a = p$ have direct predecessors. These are given by the values $\xi'_\mathrm{L} = 0$, $\xi'_\mathrm{S} = 0, 1$ and $\xi'_\mathrm{L} = 1$, $\xi'_\mathrm{S} = 0, 1$, respectively. As a result, if $\mathsf{c}_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



Here the vertices on the left hand side are labelled by $(\xi'_\mathrm{L}, \xi'_\mathrm{S})$, whereas those in the middle are labelled by a. If $\mathsf{c}_2(P) \neq 0$, then $\hat{\mathrm{K}}(P)$ is trivial. ♦

Exercises

8.8.1 Determine the direct successors and the direct predecessors for $J = (2|2)$.

8.8.2 Verify that the tuples L' obtained by splitting, merging, inverse splitting and inverse merging belong to $\mathrm{K}(P)$.

8.8.3 Let g and g' be positive integers such that g' divides g and let Q be a principal $\mathbb{Z}_g$-bundle. Show that $\delta_{g'}\big(Q^{[\rho_{g'}]}\big) = \rho_{g'}\big(\delta_g(Q)\big)$.

Chapter 9
Elements of Quantum Gauge Theory

In this chapter, we discuss some elements of quantum gauge theory with the main emphasis on those aspects which are related to the structure of the classical gauge orbit space in one or the other way. In Sects. 9.1 and 9.2, we present the classical Faddeev–Popov path integral quantization procedure, address the famous Gribov problem and formulate the latter in the language of differential geometry. In this formulation, the problem boils down to the study of the obstruction against the existence of a global section (a global gauge) of the generic stratum of the gauge orbit space. Following Singer, we prove that for some model classes, there does not exist any global gauge at all. Next, in Sect. 9.3, we discuss another general aspect of quantum gauge theories. It turns out that a symmetry of the classical Lagrangian of a gauge model is not necessarily maintained on quantum level. If this happens, one speaks of an anomaly. We discuss this phenomenon for models of gauge fields coupled to fermionic matter. We address Abelian and gauge anomalies in detail and comment on global anomalies at the end. The discussion is based on the path integral formulation and heavily uses the Atiyah–Singer Index Theorem.

In the second part of this chapter, we present some of our results on non-perturbative quantum gauge theory for (finite) lattice models in the Hamiltonian framework. In Sect. 9.4, we construct the quantum model via canonical quantization and in Sect. 9.5 we derive the field algebra and the observable algebra of the system. We show that, for the finite lattice model, these algebras are uniquely defined up to equivalence. We discuss the Gauß law, indicate how to classify irreducible representations of the observable algebra in terms of global colour charge and, finally, also comment on recent results for the infinite lattice model. In Sect. 9.6, we explain how to include the nongeneric gauge orbit strata on quantum level. This presentation is based upon the concept of a Hilbert space costratification in the sense of Huebschmann and uses the generalized Segal–Bargmann transform of Hall. Finally, in Sect. 9.7, we discuss the costratification for a toy model.

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9.1 Path Integral Quantization

In this section, we limit our attention to the principal orbit type $\tau = \tau_{\mathrm{p}}$, which is the conjugacy class consisting of the subgroup $\tilde{\mathrm{Z}}(G)$ of constant functions $P \to \mathrm{Z}(G)$, where $\mathrm{Z}(G)$ denotes the center of G. As already noted, since $\tilde{\mathrm{Z}}(G)$ is normal in $\mathscr{G}$, the smooth locally trivial fibre bundle

$$\pi^{\mathrm{p}} : \mathscr{C}^{\mathrm{p}} \to \mathscr{M}^{\mathrm{p}} \tag{9.1.1}$$

is in fact principal with structure group

$$\tilde{\mathscr{G}} := \mathscr{G}/\tilde{\mathrm{Z}}(G)\,.$$

This bundle has been studied intensively [454, 455, 476, 591]. For convenience, we assume that $\tilde{\mathrm{Z}}(G)$ is discrete. Thus, $\mathrm{L}\mathscr{G} = \mathrm{L}\tilde{\mathscr{G}}$.

Below, we will describe a procedure for quantizing a gauge theory within the functional integral approach which was proposed in 1967 by Faddeev and Popov [188], building on earlier work by Feynman [194, 195] and De Witt [151]. Basically, the functional integral obtained in this way[1] serves as a tool for perturbation theory, see e.g. [340]. In this approach, effects which potentially may come from the possible nontriviality of the bundle P over spacetime M where the gauge connections live on are not taken into account. Thus, we will represent the gauge connections ω by their local representatives $\mathbb{A}$ on M. As before, the local representative of the field strength will be denoted by $\mathbb{F}$. Moreover, we pass from spacetime to Euclidean space, also denoted by M, and consider the functional integral there. This step is achieved by replacing real time t by imaginary time it.

Thus, the starting point is the Euclidean Yang–Mills action

$$S_{\mathrm{YM}}(\mathbb{A}) = \frac{1}{2}\int_M |\mathbb{F}|^2 \mathsf{v}_{\mathsf{g}}\,, \tag{9.1.2}$$

see (6.2.2), together with the naive generating functional[2]

$$Z(J) = \int [\mathrm{d}\mathbb{A}]\mathrm{e}^{-S_{\mathrm{YM}}(\mathbb{A})+\int_M \mathrm{d}\mathbf{x} J(\mathbf{x})\cdot\mathbb{A}(\mathbf{x})}\,. \tag{9.1.3}$$

Here, $[\mathrm{d}\mathbb{A}] := \prod \mathrm{d}\mathbb{A}(\mathbf{x})$ is the formal measure on $\mathscr{C}^{\mathrm{p}}$ and $\int_M J\cdot\mathbb{A}$ is called the source term. For the time being, let us drop it.[3]

The Faddeev–Popov procedure may be written down for the theory on physical spacetime as well. Anyway, the above functional integral is not defined rigorously.

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[1] Combined with the machinery of renormalization, see [532, 556, 656] and references therein.

[2] For the time being, we neglect matter fields. They will be included in Sect. 9.3. For convenience, the Planck constant $\hbar$ is set equal to 1.

[3] Note that this term is not gauge-invariant.

Passing to the Euclidean space is the first step in the constructive programme of quantum field theory. Before we continue, let us briefly outline the main steps of this programme.

Remark 9.1.1 (*Non-perturbative quantum gauge theory*) In the programme of constructive quantum field theory, one proceeds as follows.

1. Approximate the underlying classical field theory on a finite lattice in Euclidean space.
2. Quantize this system via the functional integral approach. This way, one obtains a rigorously defined finite-dimensional quantum statistical model.[4]
3. Construct the continuum limit of this theory. This includes both passing to an infinite lattice (thermodynamical limit) and passing with the lattice spacing to 0 (ultraviolet limit). In particular, this way, one constructs the measure in the functional integral and, consequently, the Euclidean Green's functions (Schwinger functions) rigorously.
4. Use Osterwalder–Schrader type arguments [497, 498] to pass to the model on Minkowski space, the ultimate goal being the construction of Wightman functions fulfilling the Wightman axioms [275, 605].

For some types of models, this programme has been fully accomplished, see e.g. [192, 586]. However, for gauge theories on 4-dimensional spacetime this is still an (extremely hard) open problem. As a matter of fact, it is one of the famous Millennium problems formulated by the Clay Mathematics Institute, see [160, 347] for details and references to the main results obtained in this field. In this context, we also refer to the textbooks [248, 532].

Alternatively, one may try to develop a rigorous approach within the Hamiltonian framework. Here, one starts with an infinite-dimensional Hamiltonian system with a symmetry (the gauge symmetry) and one may try to develop a rigorous quantum theory by possibly extending methods working for finite-dimensional systems to the infinite-dimensional context. Again, lattice approximation may be helpful as an intermediate step. In Sect. 9.4, we will explain the finite lattice version of gauge theory in some detail. Clearly, the above problem does not become simpler by just passing to the Hamiltonian framework. But, as a matter of fact, different methods of functional analysis, in particular, spectral theory and operator algebras, play a role here. In this context, we refer to a series of deep papers by Bach, Fröhlich and Sigal,[5] see the review [49] and further references therein. We also refer to [48, 50] for further developments. ♦

Now, disregarding the hard problems discussed in the above remark, let us explain the Faddeev–Popov procedure in some detail. To start with, let us assume that the principal bundle (9.1.1) is trivial. Obstructions against this property will be discussed

[4] Combined with appropriate computer methods, like Monte-Carlo simulation, this also serves as a tool for non-perturbative calculations in elementary particle physics, see [143, 233, 536].

[5] These authors have studied the theory of non-relativistic electrons bound to static nuclei and interacting with the quantized radiation field in the Hamiltonian approach on a rigorous level.

later. In this case, we can choose a global section $s : \mathscr{M}^{\mathrm{p}} \to \mathscr{C}^{\mathrm{p}}$, called a gauge. Clearly, s can be defined by a local gauge fixing condition

$$f(\mathbb{A}) = 0\,,$$

where $f : \mathscr{C}^{\mathrm{p}} \to \mathrm{L}\mathscr{G}$ is a smooth mapping whose restriction to

$$\mathrm{im}(s) = \left\{\mathbb{A} \in \mathscr{C}^{\mathrm{p}} : f(\mathbb{A}) = 0\right\}$$

is of maximal rank.[6] Then, by an infinite-dimensional version of the Level Set Theorem, f determines s uniquely, indeed.

Remark 9.1.2 We have met already a number of gauge fixing conditions in the previous chapters. In view of the natural splitting (8.3.1), the covariant Lorenz gauge defined by

$$f(\mathbb{A}) = \nabla^{\overline{\mathbb{A}}*}(\mathbb{A} - \overline{\mathbb{A}}) \tag{9.1.4}$$

is somewhat distinguished. Here, $\overline{\mathbb{A}} \in \mathscr{C}^{\mathrm{p}}$ is referred to as the background gauge potential. Another gauge popular in the context of functional integrals is the axial gauge. It is defined by

$$f(\mathbb{A}) = \mathsf{n} \cdot \mathbb{A}\,, \tag{9.1.5}$$

where $\mathsf{n} \in M$ is a fixed vector. We will further comment on axial-like gauges below. ♦

Now, Faddeev and Popov proposed to implement the gauge fixing defined by s in the functional integral and, thus, to remove the unphysical gauge freedom from the naive functional integral (9.1.3) as follows. By the assumption on f, the restriction to $\mathrm{im}(s)$ of the derivative of f in the vertical direction,

$$f'_{[\mathbb{A}]} : \mathfrak{V}_{s([\mathbb{A}])} \to \mathrm{L}\mathscr{G}\,,$$

is an isomorphism for every $[\mathbb{A}] \in \mathscr{M}^{\mathrm{p}}$. Recall from Sect. 6.1 that the distribution $\mathfrak{V}$ is spanned by the Killing vector fields of the $\mathscr{G}$-action,

$$\mathfrak{V}_{\mathbb{A}} = \nabla^{\mathbb{A}}(\mathrm{L}\mathscr{G})\,,$$

and, thus, $\mathfrak{V}_{s([\mathbb{A}])}$ may be identified with $\mathrm{L}\mathscr{G}$. Thus, for every $[\mathbb{A}] \in \mathscr{M}^{\mathrm{p}}$, we have an isomorphism $f'_{[\mathbb{A}]} \circ \nabla^{s([\mathbb{A}])} : \mathrm{L}\mathscr{G} \to \mathrm{L}\mathscr{G}$. Clearly, this is the derivative of the mapping

$$\Phi_{[\mathbb{A}]} : \tilde{\mathscr{G}} \to \mathrm{L}\mathscr{G}\,, \quad \Phi_{[\mathbb{A}]}(u) = f\left(\mathbb{A}^{(u)}\right),$$

[6] By Proposition 1.1.6, s uniquely determines an equivariant mapping $\kappa : \mathscr{C} \to \overline{\mathscr{G}}$. Given κ, one can take $f := \kappa'$. But, clearly, s does not determine f uniquely.

where $\mathbb{A} = s([\mathbb{A}])$ and $\mathbb{A}^{(u)}$ is the local gauge transformation of $\mathbb{A}$ generated by $u \in \mathscr{G}$, see (6.1.3). That is,

$$\Phi'_{[\mathbb{A}]} = f'_{[\mathbb{A}]} \circ \nabla^{s([\mathbb{A}])} . \tag{9.1.6}$$

Note that for the covariant Lorenz gauge (9.1.4), $\Phi'_{[\mathbb{A}]}$ coincides with the Faddeev–Popov operator as given by (8.4.8). Therefore, we call $\Phi'_{[\mathbb{A}]}$ the (generalized) Faddeev–Popov operator. As a consequence, formally, we now can generalize the standard formula

$$1 = \int \mathrm{d}^n \mathbf{x} \left|\det\left(\varphi'\right)\right|_{\restriction \varphi(\mathbf{x})=0} \delta\left(\varphi(\mathbf{x})\right)$$

for a bijective smooth mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ to the case under consideration:

$$1 = \int [\mathrm{d}\rho] \left|\det\left(\Phi'_{[\mathbb{A}]}\right)\right|_{\restriction f(\mathbb{A})=0} \delta\left(\Phi_{[\mathbb{A}]}(\rho)\right) , \tag{9.1.7}$$

where $[\mathrm{d}\rho] := \prod \mathrm{d}\rho(\mathbf{x})$ is the formal Haar measure on $\tilde{\mathscr{G}}$. We denote

$$\Delta_f(\mathbb{A}) := \left|\det\left(\Phi'_{[\mathbb{A}]}\right)\right|_{\restriction f(\mathbb{A})=0} \tag{9.1.8}$$

and call it the Faddeev–Popov determinant. Inserting the identity (9.1.7) into the generating function (9.1.3) with $J = 0$ and using the gauge invariance of $[\mathrm{d}\mathbb{A}]$, $\Delta_f(\mathbb{A})$ and $S_{\mathrm{YM}}(\mathbb{A})$, we obtain

$$Z(0) = \int [\mathrm{d}\rho] \int [\mathrm{d}\mathbb{A}] \Delta_f(\mathbb{A}) \delta\left(f\left(\mathbb{A}\right)\right) \mathrm{e}^{-S_{\mathrm{YM}}(\mathbb{A})} .$$

The volume $\int [\mathrm{d}\rho]$ of $\mathscr{G}$ is an infinite constant factor which may be dropped. Thus, also adding the source term again, we finally get

$$Z(J) = \int [\mathrm{d}\mathbb{A}] \Delta_f(\mathbb{A}) \delta\left(f\left(\mathbb{A}\right)\right) \mathrm{e}^{-S_{\mathrm{YM}}(\mathbb{A}) + \int_M \mathrm{d}\mathbf{x} J(\mathbf{x}) \cdot \mathbb{A}(\mathbf{x})} . \tag{9.1.9}$$

This is an integral over the gauge fixing submanifold $\mathrm{im}(s)$.

Remark 9.1.3

1. In the covariant Lorenz gauge, using (8.4.26), one can rewrite (9.1.9) as an integral over the gauge orbit space with the volume form induced from the natural weak Riemannian metric on $\mathscr{M}^{\mathrm{p}}$. We refer to [46, 349] for further details, see also [234] for a rigorous study on the lattice.
2. If we choose a system of local coordinates $\{x^i\}$ on M, a basis $\{\mathbf{e}_a\}$ in the Lie algebra of G and a local frame $\{\xi^a\}$ in $\mathrm{L}\mathscr{G} \cong W^{k+1}(\mathrm{Ad}(P))$, then $\Phi'_{[\mathbb{A}]}$ may be represented by a matrix-valued distribution as follows.[7]

[7] We use the notation of functional derivative as common in physics.

$$Q^a{}_b(\mathbf{x}, \mathbf{y}) \equiv \left(\Phi'_{[\mathbb{A}]}\right)^a{}_b(\mathbf{x}, \mathbf{y}) = \left[\frac{\delta f^a\left(\mathbb{A}^{(\rho)}(\mathbf{x})\right)}{\delta \xi^b(\mathbf{y})}\right]_{\restriction \xi=0\,,\, f(\mathbb{A})=0} ,$$

where $\rho = \exp(\xi^a \mathbf{e}_a)$. Using (6.1.8), we calculate

$$\begin{aligned} \frac{\delta f^a\left(\mathbb{A}^{(\rho)}(\mathbf{x})\right)}{\delta \xi^b(\mathbf{y})} &= \int \mathrm{d}\mathbf{z} \frac{\delta f^a\left(\mathbb{A}(\mathbf{x})\right)}{\delta \mathbb{A}^c_\mu(\mathbf{z})} \frac{\delta\left(\mathbb{A}^{(\rho)}\right)^c_\mu(\mathbf{z})}{\delta \xi^b(\mathbf{y})} \\ &= \int \mathrm{d}\mathbf{z} \frac{\delta f^a\left(\mathbb{A}(\mathbf{x})\right)}{\delta \mathbb{A}^c_\mu(\mathbf{z})} \mathrm{D}^c_{\mu b} \delta(\mathbf{y} - \mathbf{z}) , \end{aligned}$$

where $\mathrm{D}^c_{\mu b} = \delta^c{}_b \partial_\mu + \mathrm{ad}(\mathbb{A}_\mu)^c{}_b$. Thus,

$$Q^a{}_b(\mathbf{x}, \mathbf{y}) = \left[\int \mathrm{d}\mathbf{z} \frac{\delta f^a\left(\mathbb{A}(\mathbf{x})\right)}{\delta \mathbb{A}^c_\mu(\mathbf{z})} \mathrm{D}^c_{\mu b} \delta(\mathbf{y} - \mathbf{z})\right]_{\restriction f(\mathbb{A})=0} . \tag{9.1.10}$$

Clearly, this is the local form of (9.1.6). For the gauges given in Remark 9.1.2, this matrix-valued distribution may be calculated easily. For the Lorenz gauge, putting for simplicity $\overline{\mathbb{A}} = 0$, we get

$$Q^a{}_b(\mathbf{x}, \mathbf{y}) = \left(\delta^a{}_b \Box + \mathrm{ad}(\mathbb{A}_\mu)^a{}_b \partial^\mu\right) \delta(\mathbf{x} - \mathbf{y}) , \tag{9.1.11}$$

whereas for the axial gauge we obtain

$$Q^a{}_b(\mathbf{x}, \mathbf{y}) = \delta^a{}_b \mathsf{n}^\mu \partial_\mu \delta(\mathbf{x} - \mathbf{y}) . \tag{9.1.12}$$

♦

To make formula (9.1.9) tractable for perturbative calculations, one represents the Faddeev–Popov determinant in terms of a Berezin integral [66],[8]

$$\Delta_f(\mathbb{A}) = \int [\mathrm{d}c][\mathrm{d}\overline{c}] \mathrm{e}^{\int \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \overline{c}_a(\mathbf{x}) Q^a{}_b(\mathbf{x},\mathbf{y}) c^b(\mathbf{y})} , \tag{9.1.13}$$

where c and $\overline{c}$ are Graßmann-valued Lorentz scalars carrying the adjoint representation of the Lie algebra of G. They are called Faddeev–Popov ghosts and anti-ghosts, respectively.[9] Finally, one usually gets rid of the δ-distribution by averaging over an arbitrary auxiliary field with a Gaussian weight. This way the δ-distribution gets replaced by a factor

[8] Here, we temporarily assume that the determinant is positive. Consequently we neglect the absolute value.

[9] This naming goes back to Feynman. It is due to the fact that c and $\overline{c}$ do not contribute to the spectrum of observables of the quantum theory. In the language of perturbation theory, these quantities cannot occur in external lines of Feynman diagrams.

$$\mathrm{e}^{-\frac{1}{2\alpha}\int_M \mathrm{d}\mathbf{x} f(\mathbb{A}(\mathbf{x}))^2},$$

where α is the width of the Gaussian weight. As a result, the generating functional now reads as follows:

$$Z(J) = \int [\mathrm{d}\mathbb{A}][\mathrm{d}c][\mathrm{d}\bar{c}]\mathrm{e}^{-\left(S_{\mathrm{YM}}(\mathbb{A})+S_{gf}(\mathbb{A},c,\bar{c})\right)+\int_M \mathrm{d}\mathbf{x} J(\mathbf{x})\cdot\mathbb{A}(\mathbf{x})}, \tag{9.1.14}$$

with the gauge fixing term given by

$$S_{gf}(\mathbb{A}, c, \bar{c}) = -\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\bar{c}_a(\mathbf{x}) Q^a{}_b(\mathbf{x},\mathbf{y}) c^b(\mathbf{y}) + \frac{1}{2\alpha}\int_M \mathrm{d}\mathbf{x}\, f(\mathbb{A}(\mathbf{x}))^2 . \tag{9.1.15}$$

Now, the Euclidean quantum expectation value of an observable $\mathscr{O}$ is defined by

$$\langle\mathscr{O}\rangle = \frac{1}{Z(0)}\int [\mathrm{d}\mathbb{A}][\mathrm{d}c][\mathrm{d}\bar{c}]\,\mathscr{O}[\mathbb{A}]\,\mathrm{e}^{-\left(S_{\mathrm{YM}}(\mathbb{A})+S_{gf}(\mathbb{A},c,\bar{c})\right)} . \tag{9.1.16}$$

Correspondingly, using $Z(J)$, one defines the Euclidean n-point Green's functions (Schwinger functions).

Remark 9.1.4 Clearly, there are now various gauges corresponding to various choices of α. In particular, the case $\alpha = 1$ is usually referred to as the Feynman gauge. The choice $\alpha = 0$ is called the Landau gauge. Note that in this case the width of the Gaussian weight vanishes and so we are actually back to the Lorenz gauge $\mathrm{d}^*\mathbb{A} = 0$.[10] ♦

By the above gauge fixing procedure, the local gauge symmetry has been broken. However, a new symmetry occurs. To exhibit it, we further rewrite the functional integral (9.1.14) as

$$Z(J) = \int [\mathrm{d}\mathbb{A}][\mathrm{d}B][\mathrm{d}c][\mathrm{d}\bar{c}]\, e^{-\left(S_{\mathrm{YM}}(\mathbb{A})+\tilde{S}_{gf}(\mathbb{A},B,c,\bar{c})\right)+\int_M \mathrm{d}\mathbf{x} J(\mathbf{x})\cdot\mathbb{A}(\mathbf{x})}, \tag{9.1.17}$$

where B is a bosonic scalar field in the adjoint representation, called the Nakanishi–Lautrup field, and

$$\tilde{S}_{gf}(\mathbb{A}, B, c, \bar{c}) = -\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,\bar{c}_a(\mathbf{x}) Q^a{}_b(\mathbf{x},\mathbf{y}) c^b(\mathbf{y}) - \int_M \mathrm{d}\mathbf{x}\big(B(\mathbf{x})\cdot f(\mathbb{A}(\mathbf{x})) + \frac{\alpha}{2}B(\mathbf{x})^2\big) .$$

[10] Some authors, however, reserve the term Lorenz gauge for the more general condition $\mathrm{d}^*\mathbb{A} = B$, where B is an arbitrary scalar field in the adjoint representation.

The functional integral (9.1.14) is reobtained by integrating out the B-field. Consider the following transformation of $\phi = (\mathbb{A}, B, c, \overline{c})$:

$$\delta_\lambda \mathbb{A}^a_\mu = \lambda \mathrm{D}^a_{\mu b} c^b \,, \quad \delta_\lambda B^a = 0 \,, \quad \delta_\lambda \overline{c}^a = -\lambda B^a \,, \quad \delta_\lambda c^a = -\frac{1}{2}\lambda f^a{}_{bd} c^b c^d \,,$$

where λ is a constant parameter which anti-commutes with the ghost fields (and with all fermionic matter fields of the theory) and f^{abd} are the structure constants of the Lie algebra of G. For any functional F of ϕ, one defines the Slavnov variation sF by

$$\delta_\lambda F(\phi) := \lambda\big(sF(\phi)\big) \,.$$

By definition, s is an odd derivation. One can prove that it is nilpotent and, using this fact, one shows that the effective action $S_{YM} + \tilde{S}_{gf}$ is s-invariant.[11] The symmetry obtained this way was found independently by Becchi, Rouet and Stora [62] and by Tyutin [635] and it is, therefore, referred to as the BRST symmetry. It constitutes the basic technical tool both for the proof of the renormalizability and of the unitarity of Yang–Mills theory in the perturbative approach. For these topics we refer to the standard literature, see [656].

9.2 The Gribov Problem

Unfortunately, in general, the procedure explained in the previous section does not work globally. That is, there are obstructions against the existence of a global gauge section $s : \mathscr{M}^{\mathrm{p}} \to \mathscr{C}^{\mathrm{p}}$. This observation was first made by Gribov [258] in 1978 in the context of the Lorenz gauge $\mathrm{d}^* \mathbb{A} = B$, see Remark 9.1.4. He showed that a gauge orbit can intersect a Lorenz gauge section more than once.[12] To understand the geometry of this phenomenon, we proceed in two steps:

(a) We reformulate the arguments of Gribov in the geometric language.
(b) Following Singer [591], we show that, in general, there does not exist any global gauge fixing at all.

To discuss point (a), we denote

$$\mathscr{S}_\omega := \{\omega + \alpha : \ \alpha \in \mathfrak{H}_\omega\} \tag{9.2.1}$$

[11] For a detailed proof of these facts we refer to Volume II of [656].

[12] As a matter of fact, he also mentioned the possibility that some orbits may not intersect a chosen gauge at all, but he seemingly was not aware of any example. This can happen, indeed, e.g. in the axial gauge with periodic boundary conditions, see [684].

for any $\omega \in \mathscr{C}^{\mathrm{p}}$. We keep on assuming that the metric g on spacetime M has Euclidean signature, but now we additionally assume that M is compact. Below, standard examples will be $M = \mathrm{S}^4$ or S^3 which may be viewed as being obtained from Euclidean space via a one-point compactification.[13]

Proposition 9.2.1 (Singer) *Let $\omega_0 \in \mathscr{C}^{\mathrm{p}}$. Then, for every line $\omega_0 + t\alpha \in \mathscr{S}_{\omega_0}$, there exists a vector $\tau \in \mathfrak{H}_{\omega_0}$ which is tangent to the orbit at $\omega_0 + t_0\alpha$ for some $t_0 \in \mathbb{R}$.*

Proof The tangent space to the orbit at $\omega_0 + t\alpha$ is spanned by elements of the form $\nabla^{\omega_0+t\alpha}\xi = \big(\nabla^{\omega_0} + t\mathrm{C}_\alpha\big)\xi$ with $\xi \in \mathrm{L}\mathscr{G}$ and C_α given by $\mathrm{C}_\alpha\xi = [\alpha, \xi]$. Thus, a vector $\tau \in \mathfrak{H}_{\omega_0}$ is tangent to the orbit at $\omega_0 + t\alpha$ iff there exists $\xi \in \mathrm{L}\mathscr{G}$ such that

$$\tau = \big(\nabla^{\omega_0} + t\mathrm{C}_\alpha\big)\xi \,. \tag{9.2.2}$$

Together with $\nabla^{\omega_0 *}\tau = 0$, this implies

$$\big(\nabla^{\omega_0 *}\nabla^{\omega_0} + t\nabla^{\omega_0 *} \circ \mathrm{C}_\alpha\big)\xi = 0 \,.$$

Now, since g is positive definite, $\nabla^{\omega_0 *}\nabla^{\omega_0}$ is a self-adjoint positive operator. Moreover, since the symbol of the self-adjoint operator $\nabla^{\omega_0 *} \circ \mathrm{C}_\alpha$ is not non-negative, this operator is not non-negative. Thus, there exists a smallest finite value $t_0 \in \mathbb{R}$ such that the operator

$$\mathrm{P}^{\omega_0}(t_0) := \nabla^{\omega_0 *}\nabla^{\omega_0} + t_0\nabla^{\omega_0 *} \circ \mathrm{C}_\alpha$$

has a nontrivial kernel. Any element ξ belonging to that kernel yields via (9.2.2) an element τ which is tangent to the orbit at $\omega_0 + t_0\alpha$. ∎

Remark 9.2.2 (*Gribov ambiguity*) The connection $\omega_0 + t_0\alpha$ is said to be on the Grivov horizon around ω_0 in the direction α. At every point of this horizon, there exists a vector from $\mathfrak{H}_{\omega_0}$ which is tangent to the orbit through that point. Moreover, the operator $\mathrm{P}^{\omega_0}(t_0)$ coincides with the Faddeev–Popov operator $\Delta_{\omega_0\omega}$, where $\omega = \omega_0 + t_0\alpha$. Thus, extended to the Grivov horizon, the Faddeev–Popov operator has zero modes and, consequently, the Faddeev–Popov determinant vanishes on the horizon. This means that this determinant can switch sign and, thus, the Faddeev–Popov procedure fails. In the language of geodesics, the exponential mapping around ω_0 becomes singular at the horizon. ♦

Now, let us turn to point (b). In [591], Singer has shown that for some spacetime manifolds the bundle (9.1.1) is nontrivial and thus there does not exist any global gauge fixing at all. The idea of the proof goes as follows. First, show that the homotopy

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[13] Which may be implied by the requirement of considering finite energy field configurations only, cf. Chap. 6. More generally speaking, passing to a compact manifold may be viewed as the introduction of an infrared cutoff needed as an intermediate step for a non-perturbative understanding of Yang–Mills theory. In this spirit, as already mentioned by Gribov himself, the Gribov problem is likely to be related to non-perturbative problems like the quark confinement problem.

groups of the principal stratum $\mathscr{C}^{\rm p}$ vanish. Assume that the bundle (9.1.1) were trivial. Then,

$$\mathscr{C}^{\rm p} \cong \mathscr{M}^{\rm p} \times \widetilde{\mathscr{G}} \,.$$

Since $\pi_i(\mathscr{C}^{\rm p}) = 0$, we could conclude that the homotopy groups $\pi_i(\widetilde{\mathscr{G}})$ vanish for $i \geq 1$. Since in many cases this is not true, it follows that in these cases (9.1.1) is nontrivial. Below, we present Singer's arguments in some detail.

Proposition 9.2.3 *The homotopy groups of the principal stratum $\mathscr{C}^{p}$ vanish.*

Proof Let $\omega \in \mathscr{B} := \mathscr{C} \backslash \mathscr{C}^{\rm p}$. Then, by Remark 8.8.6/2, the orbit type τ of ω has finitely many successors $\tau_1, \ldots, \tau_r \neq$ p with respect to the partial ordering.[14] By the Tubular Neighbourhood Theorem and formula (8.3.10), there exists a neighbourhood $\mathscr{U}$ of ω such that $\mathscr{U} \subset \mathscr{C}^{\geq \tau}$. It follows that

$$\mathscr{U} \backslash \mathscr{B} = \mathscr{U} \backslash \left(\mathscr{C}^{\leq \tau_1} \cup \cdots \cup \mathscr{C}^{\leq \tau_r} \right) \,.$$

The subsets $\mathscr{C}^{\leq \tau_i}$ are affine subspaces. Since they have infinite codimension in $\mathscr{C}$, there exists an infinite dimensional affine subspace which is orthogonal to all $\mathscr{C}^{\leq \tau_i}$. By a standard deformation argument, it follows that $\pi_j(\mathscr{U} \backslash \mathscr{B}) = 0$ for all j. Now, let $f : \partial \Delta^{l+1} \to \mathscr{C}^{\rm p}$ be a continuous mapping representing an element of $\pi_l(\mathscr{C}^{\rm p})$. Since $\mathscr{C}$ is affine, f can be extended to a continuous mapping $\tilde{f} : \Delta^{l+1} \to \mathscr{C}$. By the Simplicial Approximation Theorem, there exists a subdivision of Δ^{l+1} and a homotopic mapping $\tilde{g} : \Delta^{l+1} \to \mathscr{C}$ such that $\tilde{g}$ maps each subsimplex to either $\mathscr{C}^{\rm p}$ or to some $\mathscr{U}$. Since $\pi_j(\mathscr{U} \backslash \mathscr{B}) = 0$ for all these $\mathscr{U}$, by induction on the dimension of the skeleta of the subdivision, we can deform $\tilde{g}$ homotopically in such a way that it takes values in $\mathscr{C}^{\rm p}$. The deformed mapping induces a homotopy from f to a constant mapping $\partial \Delta^{l+1} \to \mathscr{C}^{p}$. ■

From now on, we limit our attention to $G = \mathrm{SU}(n)$. For some chosen point $m \in M$, consider the pointed gauge group[15]

$$\mathscr{G}_m := \{u \in \mathscr{G} : u(m) = \mathbb{1}\} \,.$$

Lemma 9.2.4 *For $M = \mathrm{S}^r$, the pointed gauge group $\mathscr{G}_m$ is weakly homotopy equivalent to the space of continuous mappings $(\mathrm{S}^r, m) \to (\mathrm{SU}(n), \mathbb{1})$ endowed with the compact-open topology.*[16] *Moreover, for all j,*

$$\pi_j(\mathscr{G}_m) \cong \pi_{j+r}(\mathrm{SU}(n)) \,. \tag{9.2.3}$$

[14] See Sect. 8.8.

[15] Here, we view $u \in \mathscr{G}$ as a section of the associated bundle $P \times_G G$, cf. Remark 6.1.2.

[16] See Sect. 3.1.

Proof Let $m = \mathbf{e}_0$ be the north pole of S^r and let S^r_+ and S^r_- denote the upper and the lower hemisphere, respectively. Elements u of $\mathscr{G}_m$ correspond to pairs of W^{k+1}-mappings $u_\pm : \mathrm{S}^r_\pm \to \mathrm{SU}(n)$ fulfilling

$$u_+(\mathbf{e}_0) = \mathbb{1}\,, \quad u_-(\mathbf{x}) = \rho(\mathbf{x}) \cdot u_+(\mathbf{x}) \cdot \rho(\mathbf{x})^{-1} \tag{9.2.4}$$

for all $\mathbf{x}$ on the equator S^{r-1}, where ρ denotes the transition mapping of a chosen pair of local trivializations. Consider the homomorphism

$$\varphi : \mathscr{G}_m \to W^{k+1}\big((\mathrm{S}^r_+, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big)\,, \quad \varphi(u) := u_+\,.$$

Its kernel is $\ker(\varphi) = \{u \in \mathscr{G}_m : u_+ = \mathbb{1}\}$. By (9.2.4), the assignment $u \mapsto u_-$ defines a mapping

$$\ker(\varphi) \to W^{k+1}\big((\mathrm{S}^r_-, \mathrm{S}^{r-1}), (\mathrm{SU}(n), \mathbb{1})\big)\,, \tag{9.2.5}$$

which clearly is an isomorphism. Since the group $W^{k+1}\big((\mathrm{S}^r_+, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big)$ is contractible, the natural inclusion mapping $\ker(\varphi) \to \mathscr{G}_m$ is a weak homotopy equivalence. Composing this with the isomorphism (9.2.5) and identifying

$$W^{k+1}\big((\mathrm{S}^r_-, \mathrm{S}^{r-1}), (\mathrm{SU}(n), \mathbb{1})\big) \cong W^{k+1}\big((\mathrm{S}^r, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big)\,,$$

we obtain a weak homotopy equivalence

$$\mathscr{G}_m \sim W^{k+1}\big((\mathrm{S}^r, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big)\,.$$

Finally, using the Smoothing Homotopy Theorem, one can check that the natural inclusion mapping $W^{k+1}\big((\mathrm{S}^r, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big) \to C\big((\mathrm{S}^r, \mathbf{e}_0), (\mathrm{SU}(n), \mathbb{1})\big)$ is a weak homotopy equivalence, too. This yields the first assertion. The second assertion follows by iterated application of Theorem 3.1.5/2. ■

Remark 9.2.5 The first assertion of Lemma 9.2.4 carries over to arbitrary compact manifolds of dimension $r \leq 4$. Indeed, for $r < 4$, P is trivial, which means that $\mathscr{G}_m = W^{k+1}\big((M, m), (\mathrm{SU}(n), \mathbb{1})\big)$. For $r = 4$, $\mathrm{SU}(n)$-bundles P are classified by the second Chern class $\mathsf{c}_2(P)$. Hence, one may apply the argument for S^4 to all elements of a set of generators of $H^4_{\mathbb{Z}}(M)$. ♦

The following propositions are simple generalizations of Theorem 3 in [591].

Proposition 9.2.6 *Let $M = \mathrm{S}^r$ with $r \geq 2$ and assume $n > r/2$. Then, $\pi_1(\tilde{\mathscr{G}}) \neq 0$.*

Proof The exact homotopy sequences of the principal bundles $\mathbb{Z}_n \to \mathscr{G} \to \tilde{\mathscr{G}}$ and $\mathscr{G}_m \to \mathscr{G} \to \mathrm{SU}(n)$ are given by

$$\cdots \longrightarrow \pi_k(\mathbb{Z}_n) \longrightarrow \pi_k(\mathscr{G}) \longrightarrow \pi_k(\tilde{\mathscr{G}}) \longrightarrow \pi_{k-1}(\mathbb{Z}_n) \longrightarrow \cdots \tag{9.2.6}$$

$$\cdots \longrightarrow \pi_k(\mathscr{G}_m) \longrightarrow \pi_k(\mathscr{G}) \longrightarrow \pi_k\big(\mathrm{SU}(n)\big) \longrightarrow \pi_{k-1}(\mathscr{G}_m) \longrightarrow \cdots \tag{9.2.7}$$

First, consider the piece

$$\pi_1(\tilde{\mathscr{G}}) \longrightarrow \pi_0(\mathbb{Z}_n) = \mathbb{Z}_n \longrightarrow \pi_0(\mathscr{G}) \tag{9.2.8}$$

of (9.2.6). Since $\pi_0(\mathrm{SU}(n)) = \pi_1(\mathrm{SU}(n)) = 0$, exactness of (9.2.7) and Lemma 9.2.4 imply that for $M = \mathrm{S}^r$ we have

$$\pi_0(\mathscr{G}) = \pi_0(\mathscr{G}_m) = \pi_r(\mathrm{SU}(n))\,.$$

For $n > r/2$ and $r \geq 2$, $\pi_r(\mathrm{SU}(n)) = 0$ or $\mathbb{Z}$, see e.g. [104], Example VII.8.5. In the first case, exactness of (9.2.8) implies that $\pi_1(\tilde{\mathscr{G}}) \neq 0$. Since the only homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is the trivial one, $\pi_1(\tilde{\mathscr{G}}) \neq 0$ must hold in the second case, too. ■

Proposition 9.2.6 covers the cases S^2 and S^3 for any n and S^4 for $n > 2$. For S^4 and $n = 2$, we need another argument.

Proposition 9.2.7 *Let $M = \mathrm{S}^4$ and $n = 2$. Then, $\pi_3(\tilde{\mathscr{G}}) \neq 0$.*

Proof Consider the piece

$$\pi_3(\mathscr{G}) \longrightarrow \pi_3\big(\mathrm{SU}(2)\big) = \mathbb{Z} \longrightarrow \pi_2(\mathscr{G}_m)$$

of (9.2.7). By exactness of (9.2.6), $\pi_3(\mathscr{G}) = \pi_3(\tilde{\mathscr{G}})$ and by Lemma 9.2.4,

$$\pi_2(\mathscr{G}_m) = \pi_6(\mathrm{SU}(2)) = \mathbb{Z}_{12}\,.$$

Since there is no injective homomorphism $\mathbb{Z} \to \mathbb{Z}_{12}$, we conclude that $\pi_3(\tilde{\mathscr{G}}) \neq 0$. ■

Remark 9.2.8

1. By Propositions 9.2.6 and 9.2.7, the Gribov problem occurs on S^2, S^3 and S^4 for every unitary group $\mathrm{SU}(n)$.
2. Using general results on the structure of the mapping space $C(M, G)$ for M being a product of spheres, Killingback [376] has shown that the Gribov ambiguity is also present in $\mathrm{SU}(n)$-gauge theory on the 4-torus and in $\mathrm{SU}(2)$-gauge theory on $\mathrm{S}^2 \times \mathrm{S}^2$. ♦

We emphasize once again that the whole discussion above is limited to the principal stratum.

Finally, we comment on attempts to overcome the Gribov ambiguity.

(a) One approach consists in trying to reformulate the functional integral explicitly in terms of local gauge invariant quantities, see e.g. [366, 372] and references therein. In the process of constructing local gauge invariants, topologically non-trivial configurations show up in a natural way. Typically, they are of magnetic monopole or magnetic vortex type, leading to a hydrodynamical picture of matter.[17] There were many speculations on the usefulness of such a formulation for the proof of quark confinement, see the papers of Mandelstam [421] and 't Hooft [624–626]. According to these authors, a non-Abelian gauge model with matter fields may exhibit various phases:

- a Georgi–Glashow phase containing photons, charged particles and magnetic monopoles,
- a superconductivity phase containing magnetic monopoles which are confined by magnetic vortices.

In [540], these phases have been analyzed for a model with gauge group $\mathrm{SU}(3)$. Together with the classical paper of Montonen and Olive [457], the above papers of 't Hooft and Mandelstam may be viewed as precursors of modern charge-monopole duality, see also the discussion and the references at the beginning of Sect. 7.6. Finally, we also refer to the papers of Asorey and collaborators, see [24, 25] and further references therein, which are close in spirit.

(b) Another approach, already suggested by Gribov in his classical paper [258], was developed by Zwanziger [696–699]. Consider the covariant Lorenz gauge (9.1.4) and define the Gribov region[18]

$$\Omega = \left\{ \mathbb{A} \in \mathscr{C}^{\mathrm{p}} : \; \nabla^{\overline{\mathbb{A}}*}(\mathbb{A} - \overline{\mathbb{A}}) = 0 \,, \; \Phi'_{[\mathbb{A}]} > 0 \right\} . \tag{9.2.9}$$

Equivalently, Ω may be viewed as the set of relative minima of the family of Morse functionals $\mu_{\mathbb{A}}$ defined by

$$\mu_{\mathbb{A}}(\rho) = \|\mathbb{A}^{(\rho)} - \overline{\mathbb{A}}\|^2 \,.$$

Indeed,

$$\left[\frac{\delta \mu_{\mathbb{A}}}{\delta \rho}\right]_{\restriction \rho = \mathbb{1}} = -2\nabla^{\overline{\mathbb{A}}*}(\mathbb{A} - \overline{\mathbb{A}}) \,, \quad \left[\frac{\delta^2 \mu_{\mathbb{A}}}{\delta \rho^2}\right]_{\restriction \rho = \mathbb{1}} = -2\nabla^{\overline{\mathbb{A}}*}\nabla^{\mathbb{A}} \,,$$

showing that the Hessian of μ coincides with the Faddeev–Popov operator. Using this, it was shown that every gauge orbit intersects with the Gribov region. Moreover, it was proven that Ω is a convex set bounded in every direction. Unfortunately, Ω still contains Gribov copies [581]. To improve the situation, one passes to the subset $\hat{\Omega} \subset \Omega$, called the fundamental modular domain, consisting

[17] For electrodynamics interacting with matter fields, such a hydrodynamical description was found already in the nineteen fifties, see the classical paper of Takabayashi [607]. We also refer to [367, 370, 371] and further references therein.

[18] One can consider different gauges. In particular, an axial-like gauge on the torus has been analyzed in detail, see [401]. In this case, the fundamental modular domain was found to be an orbifold, obtained by factorizing the Gribov region with respect to an infinite discrete group.

of the absolute minima of the Morse functionals $\mu_{\mathbb{A}}$. That is, on every gauge orbit one selects the gauge configuration closest to the origin,

$$\hat{\Omega} = \left\{ \mathbb{A} \in \mathscr{C}^{\mathrm{p}} : \; \nabla^{\overline{\mathbb{A}}*}(\mathbb{A} - \overline{\mathbb{A}}) = 0 \,, \; \mu_{\mathbb{A}}(\rho) \geq \mu_{\mathbb{A}}(\mathbb{1}) \text{ for all } \rho \in \mathscr{G} \right\} . \tag{9.2.10}$$

Again, $\hat{\Omega}$ is convex and bounded in every direction and all gauge orbits intersect with $\hat{\Omega}$. Moreover, the interior of $\hat{\Omega}$ contains at most one representative of each gauge orbit. However, on the boundary $\partial\hat{\Omega}$ Gribov copies still can and do occur [640, 641]. The general idea now consists in restricting the functional integral to $\hat{\Omega}$ and arguing that the contributions from the boundary should be neglectable. In this context, a lot of work has been done including case studies, numerical simulations and, in particular, calculations within the lattice approximation. For further reading we refer to the review [642].

9.3 Anomalies

In this section, we will meet another peculiar property of gauge theories. It turns out that a symmetry of the classical Lagrangian is not necessarily maintained on quantum level. If this happens, one speaks of an anomaly. We discuss this issue for models of gauge fields coupled to fermionic matter. As before, we assume that spacetime M is a compact four-dimensional manifold with Euclidean signature. Since we are going to deal with spin structures, we assume moreover that the first two Stiefel-Whitney classes of M vanish. As explained in Sect. 7.1, fermionic matter fields are classically described in terms of sections of the canonical spinor bundle $\mathscr{S}(M)$ twisted with a vector bundle E carrying a representation of the gauge group G and, possibly, some further flavour-type representation. In Chap. 7, we have seen a number of relevant examples of that type.

To pass to quantum theory, we use the concept of the functional integral as explained in the first section. In this approach, fermions are represented by anti-commuting Graßmann-valued variables ψ and $\overline{\psi}$ taking values in sections of E. As already mentioned in the first section, the functional integration for Graßmann-valued fields has been developed by Berezin [66, 67]. Using this concept, the (naive) functional integral of a theory of gauge fields interacting with fermionic matter fields reads

$$Z(0) = \int [\mathrm{d}\mathbb{A}][\mathrm{d}\psi][\mathrm{d}\overline{\psi}]\mathrm{e}^{-S_{\mathrm{YM}}(\mathbb{A}) - S_{mat}(\psi,\overline{\psi},\mathbb{A})} \,, \tag{9.3.1}$$

where $S_{\mathrm{YM}}(\mathbb{A})$ is given by (6.2.2). As before, we keep on representing the gauge connections ω by their local representatives $\mathbb{A}$, that is, we assume that the principal gauge bundle P is trivial.[19] If we assume that the matter fields are massless, the matter field action is of the form

[19] We will comment on the nontrivial bundle case on the way.

$$S_{mat}(\psi, \overline{\psi}, \mathbb{A}) = \int_M \mathrm{d}\mathbf{x} \langle \psi(\mathbf{x}), \not{D}_{\mathbb{A}} \psi(\mathbf{x}) \rangle \,, \tag{9.3.2}$$

see (7.1.9), where $\not{D}_{\mathbb{A}}$ is the Dirac operator of the twisted Dirac bundle $\mathscr{E} = \mathscr{S}(M) \otimes E$. Beware that, while the canonical Hermitean scalar product for fermions on Minkowski space is given by (5.3.55), for the Euclidean signature we have

$$\langle \psi, \phi \rangle = \psi^\dagger \phi \,.$$

We keep on using the notation $\overline{\psi}$, but here $\overline{\psi} = \psi^\dagger$. Carrying out the fermionic integration in (9.3.1) in the sense of Berezin yields a fermionic determinant. The latter turns out to be the crucial object for the study of the question whether an anomaly occurs with respect to a given classical symmetry. Below, we discuss two types of anomalies in some detail: Abelian[20] anomalies and gauge anomalies. Finally, we add some remarks on global anomalies. For an exhaustive treatment of the subject, including also gravitational anomalies, we refer to [74, 530]. We stress that anomlies may also be discussed within the Hamiltonian approach, see [114, 187, 189, 448].

(a) Abelian Anomalies

We use the approach developed by Fujikawa [224–226] and combine it with the Index Theorem. Consider the Dirac operator $\not{D}_{\mathbb{A}}$ of a twisted Dirac bundle $\mathscr{E} = \mathscr{S}(M) \otimes E$, locally given by

$$\not{D}_{\mathbb{A}} \psi = i \sum_\mu \gamma^\mu \nabla_\mu \psi \,, \quad \nabla_\mu \psi = \left(\partial_\mu + \Gamma_\mu + \mathbb{A}_\mu \right) \psi \,, \tag{9.3.3}$$

with Γ_μ representing the spin connection. For the Euclidean signature, the following choice of γ-matrices is convenient:

$$\gamma^0 := \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \gamma^k := \begin{bmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}, \quad k = 2, 3, 4 \,, \tag{9.3.4}$$

cf. (5.1.28). Then, the chirality operator is given by

$$\gamma^5 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{9.3.5}$$

Now, consider the chiral transformations

$$\psi \mapsto \mathrm{e}^{i\alpha\gamma^5} \psi \,, \quad \overline{\psi} \mapsto \overline{\psi} \mathrm{e}^{i\alpha\gamma^5} \,, \quad \alpha \in \mathbb{R} \,. \tag{9.3.6}$$

Since $\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0$, they leave the fermionic action invariant. For local chiral transformations with functions $\mathbf{x} \to \alpha(\mathbf{x})$, the fermionic action transforms as (Exercise 9.3.1)

[20] Also referred to as axial anomalies or as Adler–Bell–Jackiw anomalies [10, 65].

$$S_{mat}(\psi, \overline{\psi}, \mathbb{A}) \to S_{mat}(\psi, \overline{\psi}, \mathbb{A}) + \int_M \mathrm{d}\mathbf{x}\, \alpha(\mathbf{x}) \partial^\mu j^5_\mu(\mathbf{x}) \,, \tag{9.3.7}$$

where

$$j^5_\mu(\mathbf{x}) = \overline{\psi}(\mathbf{x}) \gamma^\mu \gamma^5 \psi(\mathbf{x}) \tag{9.3.8}$$

is referred to as the axial current. Thus, the chiral transformations constitute a classical symmetry with the Noether current j^5_μ.

Now, let us study the behaviour of the fermionic functional integral

$$\int [\mathrm{d}\psi][\mathrm{d}\overline{\psi}] \mathrm{e}^{-S_{mat}(\psi, \overline{\psi}, \mathbb{A})} = \det(\not{D}_{\mathbb{A}}) \tag{9.3.9}$$

under chiral transformations.[21] To find the transformation of the measure, we first perform a formal calculation and then we introduce a gauge invariant regularization making the calculation meaningful. By Propositions 5.7.4 and 5.7.11, $\not{D}_{\mathbb{A}}$ is a self-adjoint elliptic operator admitting a complete orthonormal basis $\psi_1, \psi_2, \ldots$ of $L^2(\mathscr{E})$ consisting of eigenvectors, that is, $\not{D}_{\mathbb{A}} \psi_n = \lambda_n \psi_n$. Moreover, the eigenspaces are all finite-dimensional and $\lim_{n \to \infty} |\lambda_n| = \infty$. Thus, we can expand

$$\psi = \sum a_i \psi_i \,, \quad \overline{\psi} = \sum \overline{b}_i \psi_i^\dagger \,.$$

The coefficients are Graßmann variables fulfilling

$$[a_i, a_j]_+ = 0 \,, \quad [\overline{b}_i, \overline{b}_j]_+ = 0 \,, \quad [a_i, \overline{b}_j]_+ = 0 \,.$$

By the orthonormality of the basis $\{\psi_i\}$,

$$[\mathrm{d}\psi][\mathrm{d}\overline{\psi}] = \prod_k \mathrm{d}a_k \mathrm{d}\overline{b}_k \,, \quad \int_M \mathrm{d}\mathbf{x} \langle \overline{\psi}(\mathbf{x}), \not{D}_{\mathbb{A}} \psi(\mathbf{x}) \rangle = \sum_k \lambda_k \overline{b}_k a_k \,.$$

Thus, the fermionic functional integral (9.3.9) takes the form

$$\det(\not{D}_{\mathbb{A}}) = \int \prod_k \mathrm{d}a_k \mathrm{d}\overline{b}_k \mathrm{e}^{-\sum_k \lambda_k \overline{b}_k a_k} = \prod_k \lambda_k \,, \tag{9.3.10}$$

justifying the notation in (9.3.9). Now, consider an infinitesimal local chiral transformation $(\psi, \overline{\psi}) \to (\psi', \overline{\psi}')$ induced by a function $\mathbf{x} \to \alpha(\mathbf{x})$. Then, the corresponding transformation of the coefficients a_k and $\overline{b}_k$ reads (Exercise 9.3.2)

[21] By Appendix F, $\det(\not{D}_{\mathbb{A}})$ must be viewed as a section of the determinant bundle $\mathrm{Det}(\not{D}_{\mathbb{A}})$ over the gauge orbit space, as will be explained later. However, it turns out that, for the study of the Abelian anomaly, it is enough to consider $\det(\not{D}_{\mathbb{A}})$ for a fixed background field $\mathbb{A}$.

$$a_k \to a'_k = \sum_j C_{kj} a_j\,, \quad \overline{b}_k \to \overline{b}'_k = \sum_j C_{jk} \overline{b}_j\,,$$

where

$$C_{kj} = \delta_{kj} + i \int_M \mathrm{d}\mathbf{x}\, \alpha(\mathbf{x}) \psi_k^\dagger(\mathbf{x}) \gamma^5 \psi_j(\mathbf{x})\,, \tag{9.3.11}$$

and, according to the Berezin calculus,

$$\prod_k \mathrm{d}a'_k = \big(\det(C)\big)^{-1} \prod_k \mathrm{d}a_k\,, \quad \prod_k \mathrm{d}\overline{b}'_k = \big(\det(C)\big)^{-1} \prod_k \mathrm{d}\overline{b}_k\,. \tag{9.3.12}$$

Next, using $\det(C) = \exp\big(\mathrm{tr}(\ln C)\big)$ and expanding the logarithm up to first order, we obtain

$$\prod_k \mathrm{d}a'_k \mathrm{d}\overline{b}'_k = \prod_k \mathrm{d}a_k \mathrm{d}\overline{b}_k\, \mathrm{e}^{-2i \int_M \mathrm{d}\mathbf{x} \alpha(\mathbf{x}) \mathfrak{A}(\mathbf{x})}\,, \tag{9.3.13}$$

where

$$\mathfrak{A}(\mathbf{x}) = \sum_k \psi_k^\dagger(\mathbf{x}) \gamma^5 \psi_k(\mathbf{x})\,. \tag{9.3.14}$$

This shows that the measure is not invariant under chiral transformations.

Clearly, $\mathfrak{A}$ is not well defined. Following Fujikawa, we regularize it by damping the contributions coming from the large eigenvalues of $\not{D}_{\mathbb{A}}$,

$$\mathfrak{A}(\mathbf{x}) \to \mathfrak{A}_\Lambda(\mathbf{x}) = \sum_k \psi_k^\dagger(\mathbf{x}) \gamma^5 \mathrm{e}^{-\frac{\not{D}_{\mathbb{A}}^2}{\Lambda^2}} \psi_k(\mathbf{x})\,. \tag{9.3.15}$$

Clearly, in the end, one has to take the limit $\Lambda \to \infty$. Now, consider

$$\int_M \mathrm{d}\mathbf{x}\, \mathfrak{A}_\Lambda(\mathbf{x}) = \sum_k \int_M \mathrm{d}\mathbf{x}\, \psi_k^\dagger(\mathbf{x}) \gamma^5 \mathrm{e}^{-\frac{\not{D}_{\mathbb{A}}^2}{\Lambda^2}} \psi_k(\mathbf{x})\,. \tag{9.3.16}$$

Since $\not{D}_{\mathbb{A}}$ anti-commutes with γ^5, the spinor field $\gamma^5 \psi_k$ is an eigenvector with eigenvalue $-\lambda_k$. Thus, by the orthogonality of the basis $\{\psi_k\}$, all contributions in (9.3.16) coming from non-vanishing eigenvalues cancel and we obtain a reduction to the sum over zero-modes. Next, since $[\gamma^5, \not{D}_{\mathbb{A}}] = 2\gamma^5 \not{D}_{\mathbb{A}}$, restricted to the eigenspace of zero modes, γ^5 and $\not{D}_{\mathbb{A}}$ commute. Thus, this space decomposes into subspaces with fixed chirality,

$$\gamma^5 \psi_{k,\pm}^{(0)} = \pm \psi_{k,\pm}^{(0)}\,.$$

This yields

$$\int_M \mathrm{d}\mathbf{x}\, \mathfrak{A}_\Lambda(\mathbf{x}) = \sum_k \int_M \mathrm{d}\mathbf{x}\, \psi_{k,+}^{(0)\dagger}(\mathbf{x})\psi_{k,+}^{(0)}(\mathbf{x}) - \sum_k \int_M \mathrm{d}\mathbf{x}\, \psi_{k,-}^{(0)\dagger}(\mathbf{x})\psi_{k,-}^{(0)}(\mathbf{x})\,. \tag{9.3.17}$$

Since the eigenfunctions are normalized, the right hand side coincides with the difference of the numbers of zero modes with positive and negative chirality, respectively, that is,[22]

$$\int_M \mathrm{d}\mathbf{x}\, \mathfrak{A}(\mathbf{x}) = \mathrm{ind}(\not{D}_{\mathbb{A}})\,, \tag{9.3.18}$$

with $\mathrm{ind}(\not{D}_{\mathbb{A}})$ given by (5.8.16). Here,

$$\not{D}_{\mathbb{A}}^{+} = \not{D}_{\mathbb{A}}\frac{1}{2}\big(1+\gamma^5\big)\,, \quad \not{D}_{\mathbb{A}}^{-} = \not{D}_{\mathbb{A}}\frac{1}{2}\big(1-\gamma^5\big)\,.$$

Now, by the Atiyah–Singer Index Theorem 5.8.14,

$$\int_M \mathrm{d}\mathbf{x}\, \mathfrak{A}(\mathbf{x}) = \int_M \hat{A}(M) \wedge \mathrm{ch}(\mathscr{E}|\mathscr{S})\,, \tag{9.3.19}$$

where $\hat{A}(M)$ is the $\hat{A}$-genus form of M and $\mathrm{ch}(\mathscr{E}|\mathscr{S})$ is the relative Chern character form of $\mathscr{E}$. Here, $\mathrm{ch}(\mathscr{E}|\mathscr{S}) = \mathrm{ch}(E)$. Moreover, by (9.3.15),

$$\mathfrak{A} = \mathrm{tr}\left(\gamma^5 \mathrm{e}^{-\frac{\not{D}_{\mathbb{A}}^2}{\Lambda^2}}\right)\,.$$

Thus, by the heat kernel analysis in the proof of Theorem 5.8.14 leading to the Local Index Theorem,[23] viewing $\mathfrak{A}$ as a differential form, we obtain

$$\mathfrak{A} = \hat{A}(M) \wedge \mathrm{ch}(E)\,. \tag{9.3.20}$$

In particular, for $M = \mathrm{S}^4$, we have $\hat{A}(M) = 1$ and the axial anomaly is given by the second Chern class of E. Thus, for $G = \mathrm{SU}(n)$, we obtain

$$\mathfrak{A} = \frac{1}{8\pi^2}\,\mathrm{tr}(\mathbb{F}\wedge\mathbb{F}) = \frac{1}{32\pi^2}\varepsilon^{\mu\nu\kappa\lambda}\,\mathrm{tr}(\mathbb{F}_{\mu\nu}\mathbb{F}_{\kappa\lambda})\,\mathsf{v}_{\mathrm{S}^4}\,.$$

Now, let us calculate the Euclidean vacuum expectation value of $\partial^\mu j^5_\mu$, treating $\mathbb{A}$ as a classical background field:

[22] Note that in the course of this calculation, the regularization term is automatically gone.

[23] Cf. Remark 5.8.15.

$$\begin{aligned}\langle \partial^\mu j^5_\mu(\mathbf{x})\rangle &= \frac{1}{Z(0)}\int [\mathrm{d}\psi][\mathrm{d}\overline{\psi}]\left(\partial^\mu j^5_\mu(\mathbf{x})\right)\,\mathrm{e}^{-S_{mat}(\psi,\overline{\psi},\mathbb{A})}\\ &= \frac{1}{Z(0)}\frac{\delta}{\delta\alpha(\mathbf{x})}_{\restriction\alpha=0}\int [\mathrm{d}\psi][\mathrm{d}\overline{\psi}]\mathrm{e}^{-\int_M \mathrm{d}\mathbf{y}\left(\overline{\psi}\,\not{D}_{\mathbb{A}}\,\psi-\alpha\partial^\mu j^5_\mu\right)}\,.\end{aligned}$$

By (9.3.7) and (9.3.13), a chiral transformation of ψ and $\overline{\psi}$ in this integral yields

$$\langle \partial^\mu j^5_\mu(\mathbf{x})\rangle = \frac{1}{Z(0)}\frac{\delta}{\delta\alpha(\mathbf{x})}_{\restriction\alpha=0}\int [\mathrm{d}\psi][\mathrm{d}\overline{\psi}]\mathrm{e}^{-\int_M \mathrm{d}\mathbf{y}\left(\overline{\psi}\,\not{D}_{\mathbb{A}}\,\psi+2i\alpha\mathfrak{A}\right)}\,,$$

that is, $\langle \partial^\mu j^5_\mu(\mathbf{x})\rangle = -2i\,\mathfrak{A}(\mathbf{x})$. For $M = \mathrm{S}^4$ and $G = \mathrm{SU}(n)$, we obtain

$$\langle \partial^\mu j^5_\mu\rangle = -\frac{i}{16\pi^2}\varepsilon^{\mu\nu\kappa\lambda}\,\mathrm{tr}(\mathbb{F}_{\mu\nu}\mathbb{F}_{\kappa\lambda})\,. \tag{9.3.21}$$

This is the classical result of Adler, Bell and Jackiw [10, 65],[24]

Remark 9.3.1

1. The above result does not depend on the concrete choice of the regularization as given by (9.3.15). The factor $\mathrm{e}^{-\Lambda^{-1}\not{D}^2_{\mathbb{A}}}$ may be replaced by $f\left(\Lambda^{-1}\not{D}^2_{\mathbb{A}}\right)$, where f is any smooth function decreasing rapidly at infinity. It is easy to see that this choice yields the same anomaly [74, 224].
2. In perturbation theory, the above anomaly is found by a one-loop calculation (axial-vector triangle diagram). It turns out that radiative corrections do not provide additional contributions to the anomaly. They merely result in a renormalization of fields and charges. This deep result is due to Adler and Bardeen [11] who carried out the analysis for spinor electrodynamics and for a σ-model. Later, this result has been generalized to arbitrary gauge theories with fermionic matter fields by various authors using various techniques, see e.g. [412, 429, 687]. So, it is the Adler–Bardeen Theorem which guarantees that the above functional integral calculation, with the gauge potential treated as a classical background field, yields the correct anomaly. ♦

(b) Gauge Anomalies

Now, we consider invariance under local gauge transformations. In the same spirit as before, if local gauge invariance cannot be maintained on quantum level, then we speak of a gauge anomaly. A gauge anomalous theory should be discarded. We refer to the classical papers [54, 85, 263, 603, 693, 695].

Locally, gauge transformations are given by

$$\mathbb{A} \mapsto \mathbb{A}^{(\rho)} = \rho^{-1}\mathbb{A}\rho + \rho^{-1}\mathrm{d}\rho\,,\quad \psi \mapsto \psi^{(\rho)} = \rho^{-1}\psi\,,$$

[24] When passing to Minkowski space, the $-i$ in the formula below must be replaced by 1, for the convention $\varepsilon^{0123} = 1$.

cf. (6.1.3) and (7.1.8). Let $\not{D}_{\mathbb{A}} : \mathscr{E} \to \mathscr{E}$ be the Dirac operator of a twisted Dirac bundle $\mathscr{E} = \mathscr{S}(M) \otimes E$. We have to study the behaviour of the fermionic determinant of $\not{D}_{\mathbb{A}}$ under local gauge transformations. To start with, we note that

$$\not{D}_{\mathbb{A}^{(\rho)}} = \rho^{-1} \not{D}_{\mathbb{A}} \rho \,.$$

This implies that $\not{D}_{\mathbb{A}^{(\rho)}}$ and $\not{D}_{\mathbb{A}}$ have identical spectra and that, in particular,

$$\ker\big(\not{D}_{\mathbb{A}^{(\rho)}}\big) = \rho\big(\ker(\not{D}_{\mathbb{A}})\big) \,.$$

That is, the index of $\not{D}_{\mathbb{A}}$ viewed as an element of $K(\mathscr{C})$-theory, see Appendix E, is equivariant under the action of $\mathscr{G}$ on $\mathscr{C}$. Thus, it descends to an element of $K(\mathscr{M})$ where $\mathscr{M} = \mathscr{C}/\mathscr{G}$ is the gauge orbit space. As in the previous section, we limit our attention to the principal stratum $\mathscr{M}^{\mathrm{p}}$. By Appendix F, the Quillen determinant $\det(\not{D}_{\mathbb{A}})$ must be viewed as a section of the determinant bundle $\mathrm{Det}(\not{D}_{\mathbb{A}})$ over $\mathscr{M}^{\mathrm{p}}$. If this bundle is trivial, then $\det(\not{D}_{\mathbb{A}})$ can be globally represented by a $\mathbb{C}$-valued function and, then, no anomaly can occur.

First, consider the fermionic action $S_{mat}(\psi, \overline{\psi}, \mathbb{A}) = \int_M \mathrm{d}\mathbf{x} \langle \psi, \not{D}_{\mathbb{A}} \psi \rangle$. In the physics literature, this case is referred to as the vector coupling. By Theorem 5.7.17, $\not{D}_{\mathbb{A}}$ is a Fredholm operator with index zero, that is, the index bundle of $\not{D}_{\mathbb{A}}$ is zero-dimensional. Thus, by Appendix F, the determinant bundle of $\not{D}_{\mathbb{A}}$ is also zero-dimensional and, consequently, no anomaly can occur.

In the remainder, let us consider the case where $\mathscr{E}$ has a natural $\mathbb{Z}_2$-grading induced by the chirality operator γ^5. Accordingly, the Dirac operator decomposes into its chirality components,

$$\not{D}_{\mathbb{A}} = \not{D}^+_{\mathbb{A}} + \not{D}^-_{\mathbb{A}} \,.$$

In physical models such as the standard model,[25] we have parity violating fermionic actions,

$$S_{mat}(\psi, \overline{\psi}, \mathbb{A}) = \int_M \mathrm{d}\mathbf{x} \, \langle \psi, \not{D}^+_{\mathbb{A}} \psi \rangle \,,$$

where $\not{D}^+_{\mathbb{A}} : \Gamma^\infty(\mathscr{S}^+(M) \otimes E) \to \Gamma^\infty(\mathscr{S}^-(M) \otimes E)$. Locally, $\not{D}^+_{\mathbb{A}}$ is given by

$$\not{D}^+_{\mathbb{A}} = i \sum_\mu \gamma^\mu \big(\partial_\mu + \Gamma_\mu + \mathbb{A}_\mu\big) \frac{1}{2} \big(1 + \gamma^5\big) \,.$$

The corresponding axial current is given by

$$j^\mu_a = i \overline{\psi} \, \gamma^\mu \sigma(t_a) \frac{1}{2} \big(1 + \gamma^5\big) \psi \,, \tag{9.3.22}$$

[25] See Sect. 7.7.

where σ is a representation of G and $\{t_a\}$ is a basis of the Lie algebra of G. On classical level we have the conservation law

$$\nabla_\mu j^\mu = 0\,.$$

In the sequel, for simplicity, we suppress the chirality index and write $\not{D}_{\mathbb{A}}$ instead of $\not{D}^+_{\mathbb{A}}$. We proceed along the lines of Atiyah and Singer [41]:

1. We show that the determinant of $\not{D}_{\mathbb{A}}$ gives rise to an element $[\mu]$ of the first de Rham cohomology group of $\tilde{\mathscr{G}}$. This element will be identified with the gauge anomaly. We prove that $[\mu]$ is the transgression of the first Chern class c_1 of the determinant line bundle.
2. Using the Atiyah–Singer Family Index Theorem, we express c_1 in terms of the characteristic classes of a universal principal bundle over $M \times \mathscr{M}^{\mathrm{p}}$ and calculate its transgression explicitly via secondary cohomology classes.

To accomplish point 1, choose a reference connection $\mathbb{A}_0$ such that $\not{D}_{\mathbb{A}_0}$ has index zero and consider the operator

$$P_{\mathbb{A}} := \not{D}^\dagger_{\mathbb{A}_0}\not{D}_{\mathbb{A}} : \Gamma^\infty(\mathscr{S}^+(M)\otimes E) \to \Gamma^\infty(\mathscr{S}^+(M)\otimes E)\,,$$

for any $\mathbb{A} \in \mathscr{C}^{\mathrm{p}}$.

Remark 9.3.2 Assume $M = \mathrm{S}^4$ and $G = \mathrm{SU}(n)$. Then, by Theorem 4.8.8, principal G-bundles P over M are classified by their second Chern class. But, by the Atiyah–Singer Index Theorem, vanishing of the index of $\not{D}_{\mathbb{A}_0}$ implies vanishing of the second Chern class. We conclude that, in this case, the above assumption implies that P is trivial. ♦

Since $\mathrm{ind}\big(\not{D}_{\mathbb{A}_0}\big) = 0$, by the deformation invariance of the index, we can pass to a gauge potential $\mathbb{A}_0$ fulfilling $\ker\big(\not{D}_{\mathbb{A}_0}\big) = 0$ without violating the condition that the index be zero. But, then, also the kernel of $\not{D}^\dagger_{\mathbb{A}_0}$ is empty. Thus, the determinant line bundle of the family $\{P_{\mathbb{A}}\}$ may be identified with $\mathrm{Det}\big(\not{D}_{\mathbb{A}}\big)$. Under this identification, $\det(P_{\mathbb{A}})$ gets identified with the Quillen determinant $\det(\not{D}_{\mathbb{A}})$. Thus, instead of studying the determinant of the family $\{\not{D}_{\mathbb{A}}\}$, we can study the section $\det(P_{\mathbb{A}})$ of the determinant bundle of $\{\not{D}_{\mathbb{A}}\}$.[26] Note that, for every $\mathbb{A} \in \mathscr{C}$, the operator $P_{\mathbb{A}}$ is elliptic with symbol $\xi \mapsto |\xi|^2$. Thus, $P_{\mathbb{A}}$ can only have a finite number of zero and negative eigenvalues, that is, we are in the situation described in Appendix D, see formula (D.4), and we can apply ζ-function regularization for $\det(P_{\mathbb{A}})$.[27] This way, we obtain a section[28]

[26] In the language of physics, the above transformation results in a constant factor in front of the functional integral, see [430] for further details.

[27] As mentioned in Appendix D, this regularization procedure may be extended to the case where zero eigenvalues occur.

[28] Note that $P_{\mathbb{A}}$ does not transform equivariantly under gauge transformations. Thus, the regularized determinant will not be gauge invariant, that is, it does not descend to a function on $\mathscr{M}^{\mathrm{p}}$.

$$\Phi : \mathscr{M}^{\mathrm{p}} \to \mathrm{Det}(\not{D}_{\mathbb{A}})\,, \quad \Phi([\mathbb{A}]) := \det_\zeta(P_{\mathbb{A}})\,. \tag{9.3.23}$$

On connected components where the index of $\not{D}_{\mathbb{A}}$ is nonzero, this section vanishes. Let $\hat{\mathscr{C}}$ be the open subbundle of $\mathscr{C}$ where $\Phi([\mathbb{A}]) \neq 0$. For any $\mathbb{A} \in \hat{\mathscr{C}}$, consider the function

$$f_{\mathbb{A}} : \tilde{\mathscr{G}} \to \mathbb{C}\,, \quad f_{\mathbb{A}}(\rho) := \det_\zeta(P_{\mathbb{A}^{(\rho)}})\,. \tag{9.3.24}$$

By construction, it is smooth and nowhere vanishing. Thus, denoting the exterior differential on $\tilde{\mathscr{G}}$ by $\hat{\delta}$, for any $\mathbb{A} \in \hat{\mathscr{C}}$,

$$\mu_{\mathbb{A}} := \frac{1}{2\pi i}\,\frac{\hat{\delta} f_{\mathbb{A}}}{f_{\mathbb{A}}} \tag{9.3.25}$$

is a closed 1-form on $\tilde{\mathscr{G}}$ and, therefore, it defines an element $[\mu_{\mathbb{A}}] \in H^1_{\mathrm{dR}}(\tilde{\mathscr{G}})$. This quantity is referred to as the gauge anomaly.

Now, as an immediate consequence of the exact homotopy sequence of the principal $\tilde{\mathscr{G}}$-bundle $\mathscr{C}^{\mathrm{p}} \to \mathscr{M}^{\mathrm{p}}$ and Proposition 9.2.3, we have $\pi_i(\mathscr{M}^{\mathrm{p}}) \cong \pi_{i-1}(\tilde{\mathscr{G}})$. In particular,

$$\pi_1(\tilde{\mathscr{G}}) \cong \pi_2(\mathscr{M}^{\mathrm{p}})\,, \tag{9.3.26}$$

where the isomorphism is given by the connecting homomorphism of this sequence. Explicitly, this isomorphism is realized as follows: any 2-sphere Σ in $\mathscr{M}^{\mathrm{p}}$ may be viewed as being obtained from projecting a disc D in $\mathscr{C}^{\mathrm{p}}$ whose boundary ∂D lies completely in the gauge orbit of some reference point $\mathbb{A}$. On the other hand, via $\mathbb{A}$ the boundary ∂D defines a loop γ in $\tilde{\mathscr{G}}$. The assignment $\Sigma \mapsto \gamma$ descends to a mapping $\pi_2(\mathscr{M}^{\mathrm{p}}) \to \pi_1(\tilde{\mathscr{G}})$ yielding the above isomorphism. Next, since $\Sigma\setminus\pi(\gamma)$ is diffeomorphic to the interior of D, for the first Chern class of the determinant line bundle we obtain

$$\int_\Sigma \mathsf{c}_1 = \int_{\Sigma\setminus\pi(\gamma)} \mathsf{c}_1 = \int_D \pi^*\mathsf{c}_1\,. \tag{9.3.27}$$

Since $\mathscr{C}^{\mathrm{p}}$ is weakly contractible, the 2-form $\pi^*\mathsf{c}_1$ on $\mathscr{C}^{\mathrm{p}}$ is exact, that is, there exists a 1-form β_1 such that $\pi^*\mathsf{c}_1 = \mathrm{d}\beta_1$. Thus, by Stokes' Theorem,

$$\int_D \pi^*\mathsf{c}_1 = \int_D \mathrm{d}\beta_1 = \int_\gamma \beta_1\,. \tag{9.3.28}$$

The restriction of β_1 to the orbit through $\mathbb{A}$ is a closed 1-form $t(\mathsf{c}_1)$ on $\tilde{\mathscr{G}}$ which is referred to as the transgression of c_1.

The following proof is along the lines of [430].

Proposition 9.3.3 *The anomaly form $\mu_{\mathbb{A}}$ is cohomologous to $t(\mathsf{c}_1)$.*

Proof As above, let γ be a loop in the fibre through $\mathbb{A}$. By the isomorphism (9.3.26), there exists a disc D with $\partial D = \gamma$. Let Σ be the corresponding 2-sphere in $\mathscr{M}^{\mathrm{p}}$

obtained by projecting D. Consider any loop $\tilde{\gamma}$ on Σ. Then, $\pi^{-1}(\tilde{\gamma})$ is a loop in D homotopic to γ. Thus, the winding number of the S^1-valued function $|f_{[\mathbb{A}]}|^{-1} f_{[\mathbb{A}]}$ on γ coincides with the winding number of the S^1-valued function $|\Phi_{[\mathbb{A}]}|^{-1}\Phi_{[\mathbb{A}]}$ on $\tilde{\gamma}$. On the other hand, by standard arguments, for any loop γ in $\tilde{\mathscr{G}}$,

$$\deg\left(|f_{[\mathbb{A}]}|^{-1} f_{[\mathbb{A}]}\right) = \int_\gamma \mu_{\mathbb{A}}\,, \quad \deg\left(|\Phi_{[\mathbb{A}]}|^{-1}\Phi_{[\mathbb{A}]}\right) = \int_\Sigma \mathsf{c}_1\,.$$

We conclude

$$\int_\gamma \mu_{\mathbb{A}} = \int_\Sigma \mathsf{c}_1\,. \tag{9.3.29}$$

Combining (9.3.29) with (9.3.27) and (9.3.28), we obtain

$$\int_\gamma \mu_{\mathbb{A}} = \int_\gamma t(\mathsf{c}_1)\,, \tag{9.3.30}$$

for any loop γ in the fibre over $[\mathbb{A}]$. This shows that the 1-form $\mu_{\mathbb{A}}$ is cohomologous to $t(\mathsf{c}_1)$. ■

Remark 9.3.4 As announced in [41], one can give an analytic proof of Proposition 9.3.3 as well. For that purpose, view the restriction of the index bundle to any 2-sphere $\Sigma \subset \mathscr{M}^{\mathrm{p}}$ as being associated with the corresponding restriction of the principal $\tilde{\mathscr{G}}$-bundle $\mathscr{C}^{\mathrm{p}} \to \mathscr{M}^{\mathrm{p}}$, take the connection induced from the natural connection Z given by (8.4.16) and calculate c_1 via its curvature. Formally, this quantity is given by

$$\delta\,\mathrm{tr}\left(\not{D}_{\mathbb{A}}^{-1}\delta\mathbb{A}\right)$$

and, thus, it transgresses to $\mathrm{tr}\left(\not{D}_{\mathbb{A}}^{-1}\delta\mathbb{A}\right)$. It is easy to see that the latter quantity coincides with $\mu_{\mathbb{A}}$. This heuristics can be made precise via ζ-function regularization, see Sect. 4 in [593]. Note that the proof of Proposition 9.3.3 presented here has the advantage of holding for any regularization. ♦

From now on, let us limit our attention to the case $M = \mathrm{S}^4$ and $G = \mathrm{SU}(n)$ with $n > 2$. Then, by Remark 9.3.2, the principal $\mathrm{SU}(n)$-bundle P is trivial. Using (9.2.3), together with $\pi_1(\mathrm{SU}(n)) = 0 = \pi_2(\mathrm{SU}(n))$, from the exact sequence (9.2.7) we read off

$$\pi_1(\mathscr{G}) \cong \pi_5(\mathrm{SU}(n)) = \mathbb{Z}\,. \tag{9.3.31}$$

By Proposition 9.2.3 and by the fact that $\pi_4(\mathrm{SU}(n)) = 0$ for $n > 2$, we also have

$$\pi_1(\mathscr{M}^{\mathrm{p}}) \cong \pi_0(\tilde{\mathscr{G}}) = 0\,. \tag{9.3.32}$$

Moreover, since $\pi_0(\mathscr{M}^{\mathrm{p}}) = 0$ for any fixed isomorphism class of principal bundles P, the Hurewicz Theorem implies

$$H^1_{\mathbb{Z}}(\tilde{\mathscr{G}}) \cong \pi_1(\tilde{\mathscr{G}}), \quad H^2_{\mathbb{Z}}(\mathscr{M}^{\mathrm{p}}) \cong \pi_2(\mathscr{M}^{\mathrm{p}}). \tag{9.3.33}$$

By exactness of (9.2.6), formula (9.3.31) implies $\pi_1(\tilde{\mathscr{G}}) = \mathbb{Z}$. Now, by the first equation in (9.3.33), in the case under consideration, a nontrivial anomaly will occur unless $[\mu_{\mathbb{A}}]$ vanishes identically for some reasons. By (9.3.26), the second isomorphism in (9.3.33) implies that, in the case under consideration, the transgression yields an isomorphism

$$H^2_{\mathbb{Z}}(\mathscr{M}^{\mathrm{p}}) \cong H^1_{\mathbb{Z}}(\tilde{\mathscr{G}}). \tag{9.3.34}$$

By Proposition 9.3.3, this isomorphism identifies the first Chern class of the determinant line bundle with the anomaly.

Following Atiyah and Singer [41], we further proceed as follows. Consider the action of $\tilde{\mathscr{G}}$ on $P \times \mathscr{C}^{\mathrm{p}}$, given by

$$(p, \mathbb{A}) \mapsto (\vartheta_\rho(p), \mathbb{A}^{(\rho)}),$$

where $p \to \vartheta_\rho(p)$ denotes the vertical automorphism of P defined by $\rho \in \tilde{\mathscr{G}}$. Since this action is free, it yields a principal $\tilde{\mathscr{G}}$-bundle $P \times \mathscr{C}^{\mathrm{p}}$ over

$$\mathscr{P} = (P \times \mathscr{C}^{\mathrm{p}})/\tilde{\mathscr{G}}.$$

Since the action of G on $P \times \mathscr{C}^{\mathrm{p}}$ induced from the right principal action on P commutes with the action of $\tilde{\mathscr{G}}$, it descends to a free action on $\mathscr{P}$ and, thus, it defines a principal G-bundle

$$\mathscr{P} \to M \times \mathscr{M}^{\mathrm{p}}.$$

We endow $P \times \mathscr{C}^{\mathrm{p}}$ with a natural metric as follows. For $(p, \mathbb{A}) \in P \times \mathscr{C}^{\mathrm{p}}$, via a standard Kaluza–Klein construction, the metrics on M and G together with the connection $\mathbb{A}$ yield a metric on $\mathrm{T}_p P$ which we combine with the natural L^2-metric on $\mathrm{T}_{\mathbb{A}}\mathscr{C}^{\mathrm{p}}$ to the product metric at $(p, \mathbb{A})$. By construction, the latter is $G \times \tilde{\mathscr{G}}$-invariant. Thus, it descends to a G-invariant metric on $\mathscr{P}$. Taking the orthogonal complement of the canonical vertical distribution on $\mathscr{P}$ with respect to this metric, we obtain a connection τ on $\mathscr{P}$. Analyzing this orthogonality condition, one easily finds (Exercise 9.3.3)

$$\tau_{[(p,\mathbb{A})]} = [\mathbb{A}_p + Z_{\mathbb{A}}], \quad [(p, \mathbb{A})] \in \mathscr{P}, \tag{9.3.35}$$

with Z given by (8.4.16).

Remark 9.3.5 The pair $(\mathscr{P}, \tau)$ is universal in the following sense [41]: assume Q is a principal G-bundle over $M \times X$, with X compact and $Q_{\restriction M \times x} \cong P$ for every $x \in X$, endowed with a fibre connection τ^Q on $Q_{\restriction M \times x}$ for every x, which is continuous with respect to x. Then, there exists a morphism $\Phi : Q \to \mathscr{P}$ inducing τ^Q from τ. Conversely, any mapping $\varphi : X \to \mathscr{M}^{\mathrm{p}}$ provides a fibre connection by pulling back $(\mathscr{P}, \tau)$ via $\mathrm{id} \times \varphi : M \times X \to M \times \mathscr{M}^{\mathrm{p}}$. ♦

First, let us calculate the curvature of τ. Recall that in the case under consideration P is trivial.[29] Thus, we can view $\mathbb{A}$ as a 1-form on $M = \mathrm{S}^4$ and, consequently, τ is represented by a $\mathfrak{g}$-valued 1-form on $M \times \mathscr{M}^{\mathrm{p}}$. Consequently, we represent its curvature by a $\mathfrak{g}$-valued 2-form Ω on $M \times \mathscr{M}^{\mathrm{p}}$. Clearly, Ω is given by its form components $\Omega^{(2,0)}$, $\Omega^{(1,1)}$ and $\Omega^{(0,2)}$, where the first index refers to M and the second one to $\mathscr{M}^{\mathrm{p}}$. For convenience, in the lemma below, we represent Ω by a 2-form on $M \times \mathscr{C}^{\mathrm{p}}$. Since P is trivial, we can identify tangent vectors at $\mathscr{C}^{\mathrm{p}}$ with elements of $\Omega^1(M) \otimes \mathfrak{g}$.

Lemma 9.3.6 *The curvature $\Omega \in \Omega^2(M \times \mathscr{C}^{\mathrm{p}}) \otimes \mathfrak{g}$ of τ is given by*

$$\Omega^{(2,0)}_{(m,\mathbb{A})} = \mathbb{F}_m \,, \tag{9.3.36}$$

$$\Omega^{(1,1)}_{(m,\mathbb{A})}((X,0),(0,\alpha)) = -\alpha_m(X) \,, \tag{9.3.37}$$

$$\Omega^{(0,2)}_{(m,\mathbb{A})}((0,\alpha),(0,\beta)) = -2\big(\mathrm{G}_{\mathbb{A}}\mathrm{C}^*_\alpha\beta\big)_m \,, \tag{9.3.38}$$

where $X \in \mathrm{T}_m M$, $\alpha, \beta \in \mathrm{T}_{\mathbb{A}}\mathscr{C}^{\mathrm{p}} = \Omega^1(M) \otimes \mathfrak{g}$ fulfilling $D^*_{\mathbb{A}}\alpha = 0$, $\mathbb{F}$ is the curvature of $\mathbb{A}$ and C_α is given by (8.4.27).

Proof Equation (9.3.36) is obvious. To prove (9.3.37), extend $X \in \mathrm{T}_m M$ to a vector field $X \in \mathfrak{X}(M)$ and α to a Z-horizontal vector field (also denoted by α) on $\mathscr{C}^{\mathrm{p}}$, that is, $D^*_{\tilde{\mathbb{A}}}\alpha = 0$ for all $\tilde{\mathbb{A}}$ in $\mathscr{C}^{\mathrm{p}}$. Then, by the Structure Equation, there is only one non-vanishing term,

$$\Omega^{(1,1)}_{(m,\mathbb{A})}((X,0),(0,\alpha)) = -(0,\alpha)_{(m,\mathbb{A})}\big(\tau(X,0)\big) \,.$$

To calculate the right hand side, we represent $(0,\alpha)$ by the Z-horizontal curve $s \mapsto \mathbb{A} + s\alpha$ through $\mathbb{A}$ and calculate

$$\tau_{(m,\mathbb{A}+s\alpha)}(X,0) = (\mathbb{A} + s\alpha)_m(X) \,.$$

Thus,

$$(0,\alpha)_{(m,\mathbb{A})}\big(\tau(X,0)\big) = \frac{\mathrm{d}}{\mathrm{d}s}_{\restriction 0} \tau_{(m,\mathbb{A}+s\alpha)}(X,0) = \alpha_m(X) \,.$$

This yields (9.3.37). To prove (9.3.38), extend $\alpha, \beta \in \Omega^1(M) \otimes \mathfrak{g}$ to Z-horizontal vector fields on $\mathscr{C}^{\mathrm{p}}$. Then, using the Structure Equation and (8.4.32), we obtain

$$\begin{aligned}\Omega^{(0,2)}_{(m,\mathbb{A})}((0,\alpha),(0,\beta)) &= -\tau_{(m,\mathbb{A})}\big([(0,\alpha),(0,\beta)]\big)\\ &= -\big(\mathrm{G}_{\mathbb{A}}\mathrm{d}^*_{\mathbb{A}}\big([\alpha,\beta]\big)\big)_m\\ &= -2\big(\mathrm{G}_{\mathbb{A}}\mathrm{C}^*_\alpha\beta\big)_m \,.\end{aligned}$$

■

[29]The case of a nontrivial bundle P can also be dealt with, see [41].

Clearly, Ω given by Lemma 9.3.6 descends to a 2-form on $M \times \mathscr{M}^{\mathrm{p}}$, which we denote by the same letter. Next we apply the Atiyah–Singer Family Index Theorem to the fibration

$$M \times \mathscr{M}^{\mathrm{p}} \to \mathscr{M}^{\mathrm{p}} .$$

For $M = \mathrm{S}^4$ and $G = \mathrm{SU}(n)$, formula (5.8.68) takes the form

$$\mathrm{ch}(\mathrm{Ind}(\not{D}_{\mathbb{A}})) = \int_{\mathrm{S}^4} \mathrm{ch}(E) , \tag{9.3.39}$$

where $E = \mathscr{P} \times_G \mathbb{C}^n$ with $\mathrm{SU}(n)$ acting in the basic representation. In particular, this yields an explicit formula for the first Chern class c_1 of the determinant line bundle in terms of the Chern classes k_i of $\mathscr{P}$:

$$\mathsf{c}_1 = \int_{\mathrm{S}^4} k_3(\Omega)_{(4,2)} = -\frac{i}{24\pi^3} \int_{\mathrm{S}^4} \mathrm{tr}\left(\Omega^3_{(4,2)}\right) , \tag{9.3.40}$$

where the double index refers to taking the form degree 4 on M and 2 on $\mathscr{M}^{\mathrm{p}}$. Using Lemma 9.3.6, one obtains an explicit formula for c_1. Here, we are only interested in the transgression $[\mu_{\mathbb{A}}]$ of c_1. To calculate $[\mu_{\mathbb{A}}]$ explicitly, we use standard secondary cohomology class techniques, see [130]. Since the following observations hold for all Chern classes of $\mathscr{P}$, let us consider the general case. Denote

$$d_{2j} := \int_{\mathrm{S}^4} k_{j+2}(\Omega)_{(4,2j)} .$$

Lift the closed $2j$-forms d_{2j} from $\mathscr{M}^{\mathrm{p}}$ to $\mathscr{C}^{\mathrm{p}}$. Since $\mathscr{C}^{\mathrm{p}}$ is weakly contractible, the lifted forms are exact. That is, there exist $(2j-1)$-forms β_{2j-1} on $\mathscr{C}^{\mathrm{p}}$ such that

$$\pi^* d_{2j} = \delta\beta_{2j-1} ,$$

with δ denoting the differential on $\mathscr{C}^{\mathrm{p}}$. Moreover, by transgression as explained above, the restriction t_{2j-1} of β_{2j-1} to the orbit through a chosen reference connection $\mathbb{A}$ is a closed $(2j-1)$-form on $\tilde{\mathscr{G}}$. Applying the technique of secondary characteristic classes, one can calculate β_{2j-1} and t_{2j-1} in terms of differential forms, up to exact forms. In detail, the lift of $k_{j+n}(\Omega)$ from $M \times \mathscr{M}^{\mathrm{p}}$ to $\mathscr{P}$ coincides with the exterior differential of the secondary characteristic class[30] α_{2j+3} and, according to formula (3.1) in [130], this quantity is given by

$$\alpha_{2j+3}(\tau) = (j+2) \int_0^1 \mathrm{d}t \, k_{j+2}(\tau, \Omega_t, \dots, \Omega_t) , \tag{9.3.41}$$

where

[30] Often referred to as the Chern–Simons form.

$$\Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\tau, \tau]\,. \tag{9.3.42}$$

Let $\tilde{\alpha}_{2j+3}(\tau)$ be the lift of $\alpha_{2j+3}(\tau)$ to $P \times \mathscr{C}^{\mathrm{p}}$. Embedding $M \subset P$ via a global section and integrating, we obtain a $(2j-1)$-form on $\mathscr{C}^{\mathrm{p}}$,

$$\tilde{\beta}_{2j-1} := \int_{\mathrm{S}^4} \tilde{\alpha}_{2j+3}(\tau)\,.$$

Let $\tilde{t}_{2j-1}$ be its restriction to the orbit through $\mathbb{A}$. By construction,

$$\mathrm{d}\tilde{\beta}_{2j-1} = d_{2j}\,.$$

Moreover, $\tilde{t}_1$ is a transgression of c_1 and, thus, it represents the anomaly $[\mu_{\mathbb{A}}]$. Thus, it remains to calculate the restriction of

$$\int_{\mathrm{S}^4} \tilde{\alpha}_5(\tau)$$

to the fibre through $\mathbb{A}$. For that purpose, we need the restrictions $\hat{\tau}$ and $\hat{\Omega}_t$ of τ and Ω_t, respectively, to a chosen fibre. First, since the restriction to the fibres of a connection form on a principal bundle may be identified with the Maurer–Cartan form on the structure group, the restriction of τ to the fibre through $\mathbb{A}$ is given by

$$\hat{\tau} = \mathbb{A} + \eta\,, \tag{9.3.43}$$

where η is the Maurer–Cartan form on $\tilde{\mathscr{G}}$. The latter is a 1-form on $\tilde{\mathscr{G}}$ with values in the Lie algebra $\mathrm{L}\tilde{\mathscr{G}}$. Since P is trivial, it may be identified with a 1-form on $\tilde{\mathscr{G}}$ with values in $\Omega^0(M, \mathfrak{g})$. Next, by Lemma 9.3.6, we have $\hat{\Omega} = \mathbb{F}$ and, thus,

$$\hat{\Omega}_t = t\mathbb{F} + \frac{1}{2}(t^2 - t)[\mathbb{A} + \eta, \mathbb{A} + \eta]\,. \tag{9.3.44}$$

Proposition 9.3.7 *The gauge anomaly $[\mu_{\mathbb{A}}]$ can be represented by the following 1-form on $\tilde{\mathscr{G}}$:*

$$\mu_{\mathbb{A}} = -\frac{i}{24\pi^3}\int_{\mathrm{S}^4} \mathrm{tr}\left\{\eta \mathrm{d}\big(\mathbb{A} \wedge \mathrm{d}\mathbb{A} + \frac{1}{2}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\big)\right\}\,. \tag{9.3.45}$$

Proof In the computation below, we omit the symbol of the wedge product. We must calculate the $(4, 1)$-component of the restriction of the 5-form $\lambda_5(\tau) = \mathrm{tr}(\tau\Omega_t^2)$ to the chosen fibre. In analogy to (9.3.42), denote

$$\mathbb{F}_t = t\mathbb{F} + \frac{1}{2}(t^2 - t)[\mathbb{A}, \mathbb{A}]\,.$$

Using (9.3.43) and (9.3.44), together with the Bianchi identity, we obtain

$$\begin{aligned}
\hat{\lambda}_{4,1}(\mathbb{A}+\eta) &= \mathrm{tr}\big(\eta\mathbb{F}_t^2 + (t^2-t)(\mathbb{A}[\mathbb{A},\eta]\mathbb{F}_t + \mathbb{A}\mathbb{F}_t[\mathbb{A},\eta])\big) \\
&= \mathrm{tr}\big(\eta\mathbb{F}_t^2 + 2(t^2-t)\mathbb{A}[\mathbb{A},\eta]\mathbb{F}_t\big) \\
&= \mathrm{tr}\big(\eta\mathbb{F}_t^2 + 2(t^2-t)\big([\mathbb{A},\mathbb{A}]\eta\mathbb{F}_t + \mathbb{A}\eta[\mathbb{A},\mathbb{F}_t]\big)\big) \\
&= \mathrm{tr}\left\{\eta\big(\mathbb{F}_t^2 + 2(t-1)\big(t[\mathbb{A},\mathbb{A}]\mathbb{F}_t - \mathbb{A}[t\mathbb{A},\mathbb{F}_t]\big)\big)\right\} \\
&= \mathrm{tr}\left\{\eta\Big[\mathbb{F}_t^2 + 2(t-1)\Big(\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}_t - \mathrm{d}\mathbb{A}\Big)\mathbb{F}_t + \mathbb{A}\mathrm{d}\mathbb{F}_t\Big)\Big]\right\} \\
&= \mathrm{tr}\left\{\eta\Big[\mathbb{F}_t^2 + 2(1-t)\mathrm{d}(\mathbb{A}\mathbb{F}_t) + (t-1)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}_t^2\Big]\right\}.
\end{aligned}$$

Since $\mathbb{F}_0 = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((t-1)\mathbb{F}_t^2\big) = \mathbb{F}_t^2 + (t-1)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}_t^2,$$

we obtain

$$\int_0^1 \mathrm{d}t\,\hat{\lambda}_{4,1}(\mathbb{A}+\eta) = 2\int_0^1 \mathrm{d}t\ \mathrm{tr}\big(\eta(1-t)\mathrm{d}(\mathbb{A}\mathbb{F}_t)\big).$$

Thus, by (9.3.40) and (9.3.41), we obtain the following transgression of c_1:

$$\begin{aligned}
\tilde{t}_1 &= -\frac{6i}{24\pi^3}\int_{\mathrm{S}^4}\int_0^1 \mathrm{d}t\ \mathrm{tr}\big(\eta(1-t)\mathrm{d}(\mathbb{A}\mathbb{F}_t)\big) \\
&= -\frac{6i}{24\pi^3}\int_{\mathrm{S}^4}\mathrm{tr}\left\{\eta\mathrm{d}\Big(\mathbb{A}\int_0^1 \mathrm{d}t\,(1-t)\big(t\mathrm{d}\mathbb{A} + \frac{1}{2}t^2[\mathbb{A},\mathbb{A}]\big)\Big)\right\} \\
&= -\frac{i}{24\pi^3}\int_{\mathrm{S}^4}\mathrm{tr}\left\{\eta\mathrm{d}\Big(\mathbb{A}\mathrm{d}\mathbb{A} + \frac{1}{2}\mathbb{A}^3\Big)\right\}.
\end{aligned}$$

■

Remark 9.3.8

1. Recall that the Maurer–Cartan form fulfils $\eta(\xi_*) = \xi$ for any $\xi \in \mathrm{L}\mathscr{G}$. Thus, in local coordinates $\{x^\mu\}$ on M and with respect to a basis $\{t_a\}$ of $\mathfrak{g}$, we obtain

$$\mu_a = -\frac{i}{24\pi^3}\varepsilon^{\mu\nu\rho\sigma}\int_{\mathrm{S}^4}\mathrm{d}^4x\,\partial_\mu\,\mathrm{tr}\left\{t_a\big(\mathbb{A}_\nu\partial_\rho\mathbb{A}_\sigma + \frac{1}{2}\mathbb{A}_\nu\mathbb{A}_\rho\mathbb{A}_\sigma\right\}. \qquad (9.3.46)$$

   The first term is obviously a contraction with the totally symmetric tensor

$$D_{abc} = \mathrm{tr}\big(t_a(t_bt_c + t_ct_b)\big).$$

   Using

$$\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} = \frac{1}{4}\big(\mathbb{A}\wedge[\mathbb{A},\mathbb{A}] + [\mathbb{A},\mathbb{A}]\wedge\mathbb{A}\big)$$

one can check that the second term is a contraction with D_{abc} as well. Thus, unless D_{abc} vanishes identically, there is a nontrivial gauge anomaly. If $D_{abc} = 0$, then one speaks of a safe theory. Note that, for any unitary representation of G, the coefficients D_{abc} are imaginary. Thus, for all real or pseudo-real representations, these coefficients vanish and, consequently, no anomaly occurs for Lie algebras having only representations of that type. This happens for $\mathfrak{so}(2n+1)$, $\mathfrak{so}(4n)$ with $n \geq 2$, $\mathfrak{sp}(n)$ for $n \geq 3$, G_2, F_4, E_7 and E_8. In particular, this is true for $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Moreover, there are some Lie algebras for which the coefficients D_{abc} vanish even though they admit representations which are neither real nor pseudo-real. This happens for $\mathfrak{so}(4n+2)$ (except for $\mathfrak{so}(2) \cong \mathfrak{u}(1)$ and $\mathfrak{so}(6) \cong \mathfrak{su}(4)$) and for E_6. As a result, anomalies are only possible if G contains $\mathrm{SU}(n)$-factors with $n \geq 3$ or $\mathrm{U}(1)$-factors. Fortunately, for the case of the standard model where we have $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ the coefficients D_{abc} vanish, see Sect. 22.4 in Volume II of [656] for a detailed proof. Thus, the standard model is safe.

2. There is a calculus developed by Wess, Zumino, Stora and others [603, 664, 693, 695], which on the one hand led to a geometric understanding of BRST transformations and on the other hand turned out to be useful in anomaly calculations. Its rigorous mathematical meaning has been clarified by Kastler and Stora [359, 360], see also [85, 166, 165, 167]. Here, we only describe the basic structure and refer to the above papers for details. Let $\Omega^*(P, \mathfrak{g})$ be the vector space of $\mathrm{Ad}(G)$-equivariant $\mathfrak{g}$-valued forms on P. Consider

$$\Omega^{p,a} := \Omega^a(\tilde{\mathscr{G}}, \Omega^p(P, \mathfrak{g})) \cong \Omega^p(P, \mathfrak{g}) \otimes \Omega^a \tilde{\mathscr{G}} ,$$

and define

$$\Omega^{**} := \bigoplus_{p,a} \Omega^{p,a} .$$

Let d and $\hat{\delta}$ be the differentials of $\Omega^*(P, \mathfrak{g})$ and $\Omega^* \tilde{\mathscr{G}}$, respectively. Then,

$$\hat{\delta}^2 = \mathrm{d}^2 = 0 , \quad \mathrm{d}\hat{\delta} - \hat{\delta}\mathrm{d} = 0 ,$$

that is, $(\Omega^{**}, \mathrm{d}, \hat{\delta})$ is a double complex. We have an associated total complex (Ω^*, Δ) defined as follows. Take

$$\Omega^* := \bigoplus_n \Omega^n , \quad \Omega^n := \bigoplus_{p+a=n} \Omega^{p,a} ,$$

with n called the total grading. For $U \in \Omega^{p,a}$, define

$$sU := (-1)^p \hat{\delta} U , \quad \Delta := \mathrm{d} + s . \tag{9.3.47}$$

Then,

$$\Delta^2 = \mathrm{d}s + s\mathrm{d} = 0 ,$$

and, clearly, both d and s are nilpotent. Moreover, endow (Ω^*, Δ) with the following exterior product:

$$[\alpha \otimes \rho, \beta \otimes \sigma] := (-1)^{aq}[\alpha, \beta] \otimes (\rho \wedge \sigma)\,,$$

where $\alpha \otimes \rho \in \Omega^{p,a}$, $\beta \otimes \sigma \in \Omega^{q,b}$ and $[\alpha, \beta]$ denotes the standard exterior product on $\Omega^*(P, \mathfrak{g})$ defined by the commutator. Then, $(\Omega^*, \Delta, [\cdot, \cdot])$ becomes a graded differential Lie algebra. In this formalism, the infinitesimal gauge transformation (6.1.20) takes the form

$$s\omega = -(\mathrm{d}\eta + [\omega, \eta])\,, \tag{9.3.48}$$

and the Maurer–Cartan equation for the Maurer–Cartan form η reads

$$s\eta = -\frac{1}{2}[\eta, \eta]\,. \tag{9.3.49}$$

Here, clearly, $\omega \in \Omega^{1,0}$ and $\eta \in \Omega^{0,1}$. If one interprets s as the BRST operator, then these equations coincide with the BRST relations. Thus, the above structure provides a differential geometric setting for the BRST formalism.[31]
Note that, by the definition of the anomaly,

$$s\,\mu_{\mathbb{A}} = 0\,.$$

This property is referred to as the Wess–Zumino consistency condition. Wess, Zumino and Stora noticed that this condition can be used to calculate the anomaly, up to the correct coefficient, via a system of descent equations. This way, the calculation of the anomaly becomes related to a problem in local cohomology. In more detail, if we denote

$$\omega := \mathbb{A} + \eta\,, \quad \mathscr{F} := \Delta\omega + \frac{1}{2}[\omega, \omega]\,,$$

then, by (9.3.48) and (9.3.49), $\mathscr{F} = \mathbb{F}$. Then, by analogous arguments as in the proof of Proposition 9.3.7,

$$\Delta Q_{2n-1}(\omega) = P(\mathbb{F})\,, \tag{9.3.50}$$

where P is a symmetric invariant polynomial of $\mathrm{SU}(n)$ and the Q_{2n-1} are defined by the right hand side of (9.3.41) with $j = n - 2$. Now, expanding the Chern–Simons forms Q in powers of η and decomposing (9.3.50) in the above double complex yields the following system, referred to as the system of descent

[31] The question how to accommodate the anti-ghost fields in such a geometric setting is an old problem, see e.g. [86] for a discussion. In this paper, the anti-ghosts are introduced via a certain gauge group doubling procedure based upon the fibre product bundle construction.

equations,

$$\begin{aligned}
P(\mathbb{F}) - \mathrm{d}Q_{0,2n-1} &= 0 \\
sQ_{0,2n-1} + \mathrm{d}Q_{1,2n-2} &= 0 \\
sQ_{1,2n-2} + \mathrm{d}Q_{2,2n-3} &= 0 \\
&\dots \\
sQ_{2n-2,1} + \mathrm{d}Q_{2n-1,0} &= 0 \\
sQ_{2n-1,0} &= 0\,.
\end{aligned}$$

Note that, up to a normalization factor, the first equation expresses $\mathrm{d}Q_{0,2n-1}$ in terms of the Abelian anomaly in $2n$ dimensions. Solving the above system for the chain of Chern–Simons forms yields the gauge anomaly in $2n-2$ dimensions (up to the correct normalization). It is obtained by integrating the term $Q_{1,2n-2}$ over M. It is in this sense that some authors say the Abelian anomaly implies the gauge anomaly. For the solution theory of the system of descent equations we refer to [165] and further references therein.

3. Since the anomaly (9.3.45) satisfies the Wess–Zumino consistency condition, it is sometimes referred to as the consistent anomaly. The corresponding current obtained as the variation of the vacuum functional does not transform covariantly under gauge transformations. However, by adding a local polynomial in the gauge potentials, one can construct a covariant current and, then, the corresponding anomaly transforms covariantly. One finds

$$\tilde{\mu}_{\mathbb{A}} = -\frac{i}{8\pi^2}\int_{\mathrm{S}^4} \operatorname{tr}\left(\eta\mathbb{F}^2\right). \tag{9.3.51}$$

From the point of view of perturbation theory, the consistent and the covariant anomaly correspond to two different regularization procedures. In the first case, gauge invariance is lost in the regularization, in the second one it is maintained. In particular, the Fujikawa method explained above may be applied here as well. Within this approach, it is natural to use a gauge invariant regularization of the Jacobian corresponding to the transformation of the path integral measure. Thus, via this method one finds the covariant form (9.3.51) of the gauge anomaly. We refer to Chaps. 5 and 10 in [74] for a detailed discussion. ♦

(c) Global Anomalies:

The following example was analyzed by Witten [674]. Consider the case $M = \mathrm{S}^4$ and $G = \mathrm{SU}(2)$. Then, combining (9.2.3) with the exact homotopy sequence (9.2.7), we obtain

$$\pi_0(\mathscr{G}) = \pi_0(\mathscr{G}_m) = \pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2\,.$$

This means that $\mathscr{G}$ is not connected, that is, there are global gauge transformations which cannot be continuously deformed to the unit element of $\mathscr{G}$. As before, consider a single left-handed fermion doublet coupled to an SU(2)-gauge field. Let $\det(\not{D}_{\mathbb{A}})$

be the corresponding fermionic determinant. Then, since such a doublet may be viewed as being composed of two left-handed doublets,[32] the fermionic determinant of one left-handed doublet is given by $\left(\det(\not{D}_{\mathbb{A}})\right)^{\frac{1}{2}}$ up to the sign. The latter must be chosen by hand. Then, the determinant is invariant under infinitesimal gauge transformations. But, as was shown by Witten, it is odd under gauge transformations which cannot be continuously deformed to the unit element. That is, if ρ is such a transformation, then

$$\left(\det(\not{D}_{\mathbb{A}})\right)^{\frac{1}{2}} = -\left(\det(\not{D}_{\mathbb{A}^{(\rho)}})\right)^{\frac{1}{2}} . \tag{9.3.52}$$

This implies that the path integral of this theory is ill-defined.

Let us outline Witten's proof of Eq. (9.3.52). Recall that the Dirac operator $\not{D}_{\mathbb{A}}$ has a discrete spectrum consisting of real eigenvalues and to every eigenvalue λ there corresponds an eigenvalue $-\lambda$. To have a non-vanishing determinant, we assume that there are no zero modes. Otherwise, (9.3.52) is trivially true. We may choose the sign of $\left(\det(\not{D}_{\mathbb{A}})\right)^{\frac{1}{2}}$ e.g. by taking the product of positive eigenvalues. Now, consider the following continuous path in $\mathscr{C}$:

$$t \mapsto \mathbb{A}(t) := (1-t)\mathbb{A} + t\mathbb{A}^{(\rho)} , \quad t \in [0, 1] .$$

It clearly interpolates between $\mathbb{A}$ and $\mathbb{A}^{(\rho)}$. Consider the flow of the eigenvalues of $\not{D}_{\mathbb{A}(t)}$ as t varies from 0 to 1. Clearly, the spectra for $t = 0$ and $t = 1$ are the same, but the individual eigenvalues may rearrange on the way. It turns out that the Atiyah–Singer Index Theorem implies such a rearrangement. The simplest one is given by a single pair of eigenvalues $(\lambda(t), -\lambda(t))$ which cross at zero and change places as t runs from 0 to 1. Thus, in this simple case (9.3.52) follows. It is also a consequence of the Index Theorem that the number of positive eigenvalues which can become negative is always odd. This yields (9.3.52) in the general case.

We briefly explain the idea of the proof of the above statements. Let $\not{D}^{(5)}_{\mathbb{A}}$ be the Dirac operator on the 5-dimensional manifold $S^4 \times \mathbb{R}$ or, rather, on the conformal compactification $M = S^5$. Let ψ be a doublet of fermions on M carrying the tensor product representation of the spinor representation of $O(5)$ and the fundamental representation of $SU(2)$. Explicitly, view ψ as a two-component column vector of quaternions, let the spin group $Sp(2)$ act by multiplication from the left and let $Sp(1) \cong SU(2)$ act by diagonal multiplication from the right. This is a real representation and, thus,[33] $\not{D}^{(5)}_{\mathbb{A}}$ is a self-adjoint operator on M with a discrete spectrum consisting of real eigenvalues which are either zero or come in pairs $(\lambda, -\lambda)$. As t changes from 0 to 1, the number of zero-modes can only change whenever such a pair moves to or away from zero. Thus, the number of zero modes of $\not{D}^{(5)}_{\mathbb{A}}$ mod 2 is a topological invariant called the mod 2 index of $\not{D}^{(5)}_{\mathbb{A}}$. There is a corresponding

[32] Since the $\frac{1}{2}$-representation of $SU(2)$ is pseudo-real, a left-handed doublet can be mapped to a right-handed one. Note that the argument is still formal as long as one does not regularize the determinant.

[33] Keep in mind our conventions, see Definition 5.5.12.

mod 2 Index Theorem, see Part IV of [40]. Applying this theorem to the case of an instanton-like SU(2)-gauge potential varying adiabatically from $\mathbb{A}$ to $\mathbb{A}^{(\rho)}$ along the above defined path, one obtains that the number of zero-modes is equal to 1 mod 2. Combining this with the study of the eigenvalue flow of $\not{D}^{(5)}_{\mathbb{A}}$, see [38], one obtains the above statement.

The above arguments immediately extend to the case of n copies of Weyl fermions. If n is even, there is no problem, but, if n is odd, we have an anomaly. Moreover, the above anomaly clearly extends to any symplectic group $\mathrm{Sp}(n)$, because $\pi_4(\mathrm{Sp}(n)) = \mathbb{Z}_2$ for any n. On the other hand, $\pi_4(\mathrm{SU}(n)) = 0$ for $n > 2$ and $\pi_4(\mathrm{O}(n)) = 0$ for $n > 5$, that is, for these cases no global anomaly occurs. For an extension to massive fermions we refer to [45], for a generalization to higher SU(2) representations see [53]. We also refer to [379] for a slightly different proof circumventing a debatable argument in the proof of Witten and to [184] for a proof based on homotopy theory. Nowadays, there exist various studies including other groups and theories in higher dimensions, see e.g. [690] and further references therein.

Exercises

9.3.1 Confirm the transformation law (9.3.7).

9.3.2 Prove formula (9.3.11).

9.3.3 Prove formula (9.3.35).

9.4 Hamiltonian Quantum Gauge Theory on the Lattice

In the final sections, we use some basic tools from functional analysis for which we refer to the classical textbooks, see [82, 102, 354, 507, 529]. Our main objective is to show how to implement the classical gauge orbit type stratification on quantum level. For the time being, we are able to do this in the Hamiltonian approach only. For putting the discussion below into a broader perspective, the reader may wish to recall Remark 9.1.1. We proceed as follows:

(a) We formulate quantum gauge field theory on a finite lattice within the Hamiltonian approach. In particular, we construct the field algebra and define the observable algebra as the algebra of gauge-invariant operators factorized with respect to the ideal generated by the Gauß law. Next, we comment on the classification of irreducible representations of the observable algebra in terms of global colour charge. Finally, we comment on recent results concerning the extension to an infinite lattice.

(b) We present the concept of a costratified Hilbert space as proposed by Huebschmann and explain how it can be used to encode the classical stratification of the gauge orbit space on quantum level. For this purpose, we use the Hilbert space representation of the observable algebra constructed in Sect. 9.5. We illustrate the construction of the costratification for the case of a toy model.

In the subsequent two sections, we accomplish point (a). So, we consider a model of gauge theory with gauge group G in the Hamiltonian framework on a finite regular cubic lattice Λ in a chosen equal-time hypersurface $\mathbb{R}^3$ of spacetime M. For completeness, we also include fermionic matter fields although they will not be relevant for the discussion of the gauge orbit strata. For basic notions and results concerning lattice gauge theories, we refer to the classical papers [385, 386, 672] as well as to the textbooks [143, 233, 458, 536, 579, 580] and further references therein.

We use the standard notation common in lattice models. For $k = 0, 1, 2, 3$, we consider the set Λ^k of k-dimensional elements with a chosen orientation. In increasing order of k, such elements are called sites, links, plaquettes and cubes. In more detail:

1. $\Lambda^0 := \{x = a(n_1, n_2, n_3) \in \mathbb{R}^3 \,:\, n_i \in \mathbb{Z}, a \in \mathbb{R}_+\} \cap X$, where X is an open connected set in $\mathbb{R}^3$ and a is the lattice spacing.
2. Λ^1 is a subset of the set $\tilde{\Lambda}^1$ of all oriented links between nearest neighbours,[34]
$$\Lambda^1 \subset \tilde{\Lambda}^1 := \{\ell = (x, y) \in \Lambda^0 \times \Lambda^0 :\; y = x \pm a\, \mathbf{e}_i \text{ for some } i\},$$
with the property that for each pair of nearest neighbours x and y it contains either (x, y) or (y, x) but not both. Thus, the pair (Λ^0, Λ^1) is a directed graph. We assume that it is connected.
3. Λ^2 is a subset of the set of all oriented plaquettes,
$$\Lambda^2 \subset \{p = (\ell_1, \ell_2, \ell_3, \ell_4) \in \left(\tilde{\Lambda}^1\right)^4 :\; \mathrm{pr}_2\, \ell_i = \mathrm{pr}_1\, \ell_{i+1}\,,\; \mathrm{pr}_2\, \ell_4 = \mathrm{pr}_1\, \ell_1\},$$
for $i = 1, 2, 3$. Here, $\mathrm{pr}_k : \Lambda^0 \times \Lambda^0 \to \Lambda^0$ is the projection onto the k-th component.
4. Finally, oriented elementary cubes $c \in \Lambda^3$ are defined in an analogous way.

It is easy to see that a change of the chosen orientations induces an isomorphism of the field and observable algebras to be constructed below and leaves the Hamiltonian of the system invariant, cf. [369].

Now, given a classical gauge field model with compact gauge group G and a matter field of type (μ, σ) taking values in the finite-dimensional Hilbert space $F = F_s \otimes F_i$, its lattice approximation is obtained by restricting the matter field ψ to Λ^0 and by approximating the gauge potential $\mathbb{A}$ by its parallel transporters along the elements of Λ^1, that is, the lattice approximation of the classical configuration $(\mathbb{A}, \psi)$ is given by the following pair of mappings

$$\psi : \Lambda^0 \to F\,, \quad \hat{\ell}_{\mathbb{A}} : \Lambda^1 \to G\,. \tag{9.4.1}$$

Thus, the classical lattice configuration space is given by $\mathscr{F}_\Lambda \times \mathscr{C}_\Lambda$, where

$$\mathscr{F}_\Lambda := \prod_{x \in \Lambda^0} F\,, \quad \mathscr{C}_\Lambda := \prod_{\ell \in \Lambda^1} G\,. \tag{9.4.2}$$

[34]Here, $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^3$.

Note that the phase space of the gauge configuration space $\mathscr{C}_\Lambda$ is

$$\mathscr{P}_\Lambda := \prod_{\ell \in \Lambda^1} \mathrm{T}^* G \cong \prod_{\ell \in \Lambda^1} (G \times \mathfrak{g}^*) . \tag{9.4.3}$$

Remark 9.4.1 Since, on the lattice, continuity of the underlying space $\mathbb{R}^3$ is lost, any parallel transporter on a link can be continuously deformed to the trivial one. Thus, one can naively conclude that the possible nontrivial topological character of a gauge field configuration is lost on the lattice. However, one can show [413] that non-Abelian gauge fields with a sufficiently small action density carry a topological charge which, in the continuum limit, reproduces the instanton number. We also refer to [512] for a similar study in the context of a simplicial lattice. There, it is shown that for sufficiently small action densities one can construct a principal bundle which may be trivialized over the 4-dimensional dual cells of the lattice. Topologically nontrivial configurations of monopole type may be dealt with as well, see [365]. ♦

Next, one defines local lattice gauge transformations by restricting $\mathscr{G}$ to Λ^0, that is, a lattice gauge transformation is given by a mapping $\rho : \Lambda^0 \to G$ and, thus, the lattice approximation of $\mathscr{G}$ is given by

$$\mathscr{G}_\Lambda := \prod_{x \in \Lambda^0} G = G^{\Lambda^0} . \tag{9.4.4}$$

By, (7.1.8) and (1.8.6), $\mathscr{G}_\Lambda$ acts on $\mathscr{F}_\Lambda \times \mathscr{C}_\Lambda$ via

$$(\psi_x, \hat{\ell}_{\mathbb{A}}) \mapsto (\sigma(\rho(x))\psi_x\,,\ \rho(x_\ell)\,\hat{\ell}_{\mathbb{A}}\,\rho(y_\ell)^{-1}) , \tag{9.4.5}$$

for any $x \in \Lambda^0$ and $\ell = (x_\ell, y_\ell) \in \Lambda^1$. This is the classical kinematical model we start with. In our presentation, we limit our attention to fermionic matter fields only. For $G = \mathrm{SU}(3)$, $F_s = \mathbb{C}^4$ carrying the bispinor representation and $F_i = \mathbb{C}^3$ carrying the fundamental representation σ of SU(3), we obtain the classical lattice approximation of QCD.

We construct the quantum model along the lines of [271, 368, 369]. Let us start with the fermionic matter field. We equip $\mathscr{F}_\Lambda$ with the natural pointwise inner product $\langle \psi, \phi \rangle := \sum_{x \in \Lambda^0} \langle \psi(x), \phi(x) \rangle_F$, and define the quantum matter field algebra as the CAR-algebra

$$\mathfrak{F}_\Lambda := \mathrm{CAR}\big(\mathscr{F}_\Lambda\big) . \tag{9.4.6}$$

That is, to every classical matter field $\psi \in \mathscr{F}_\Lambda$ we associate a fermionic field $\mathfrak{a}(\psi) \in \mathfrak{F}_\Lambda$, and these quantum fields satisfy the CAR-relations,

$$[\mathfrak{a}(\psi), \mathfrak{a}(\chi)^*]_+ = \langle \psi, \chi \rangle \, \mathbb{1} , \quad [\mathfrak{a}(\psi), \mathfrak{a}(\chi)]_+ = 0 ,$$

for any $\psi, \chi \in \mathscr{F}_\Lambda$. Since Λ^0 is finite, $\mathfrak{F}_\Lambda$ is a full matrix algebra, hence up to unitary equivalence it has only one irreducible representation which we will denote by

$(\mathscr{H}^{\mathrm{f}}_{\Lambda}, \pi^{\mathrm{f}})$. Clearly, a natural choice is provided by the fermionic Fock representation of Jordan and Wigner [351].

In physics textbook notation, the matter field generator at x is given by

$$\Psi_\alpha(x) = \mathfrak{a}(\mathbf{f}_\alpha \cdot \delta_x)\,,$$

where $\{\mathbf{f}_\alpha\}$ is an orthonormal basis of $(F, \langle\cdot,\cdot\rangle)$ and $\delta_x : \Lambda^0 \to \mathbb{R}$ is the characteristic function of $\{x\}$. For a model with Dirac fermions, the spacetime component of F is $F_s = \mathbb{C}^4$, standing for the bispinor degrees of freedom, and the internal part F_i is a tensor product of some $\mathbb{C}^k$, carrying a representation σ of G, with some vector space describing flavour degrees of freedom. Neglecting the latter, the matter field generator at $x \in \Lambda_0$ is given by

$$\Psi_{\mu i}(x) = \mathfrak{a}((\boldsymbol{\varepsilon}_\mu \otimes \mathbf{e}_i) \cdot \delta_x)\,, \tag{9.4.7}$$

where $\{\boldsymbol{\varepsilon}_\mu\}$ and $\{\mathbf{e}_i\}$ are orthonormal bases in $\mathbb{C}^4$ and $\mathbb{C}^k$, respectively. We note that $\mathfrak{F}_\Lambda$ is generated as a C*-algebra by the set

$$\{\Psi_{\mu i}(x) \mid \mu = 1, \ldots, 4,\ i = 1, \ldots, k,\ x \in \Lambda^0\}.$$

Next, to quantize the classical gauge connections, we generalize the Schrödinger representation for a particle on the real line acting on the Hilbert space $L^2(\mathbb{R})$ as follows: for any $\varphi \in L^2(G)$, we define the bounded operators

$$(U_g\varphi)(h) := \varphi(g^{-1}h)\,, \quad \big(T_f\varphi\big)(h) := f(h)\varphi(h)\,, \tag{9.4.8}$$

where $g,\ h \in G$ and $f \in L^\infty(G)$. Here, U is the left regular unitary representation of G and T is the natural representation of $L^\infty(G)$ given by left multiplication. Clearly, T and U represent the position and momentum operator analogues, respectively. The pair $\pi_0 := (U, T)$ will be referred to as the generalized Schrödinger representation. Below, it will be interpreted in the language of C^*-algebras. Note that π_0 is irreducible in the sense that the commutant of $U_G \cup T_{L^\infty(G)}$ consists of the scalars. Also note that there is a natural ground state unit vector $\varphi_0 \in L^2(G)$ given by the constant function $\varphi_0(h) = 1$ for all $h \in G$.[35] Then, $U_g\varphi_0 = \varphi_0$, and, by irreducibility, φ_0 is cyclic with respect to the *-algebra generated by $U_G \cup T_{L^\infty(G)}$. By construction, π_0 fulfils the intertwining relation

$$U_g \circ T_f \circ U_g^* = T_{\lambda_g(f)}\,, \tag{9.4.9}$$

where

$$\lambda : G \to \mathrm{Aut}(C(G))\,, \quad \lambda_g(f)(h) := f(g^{-1}h)\,, \tag{9.4.10}$$

for any $g, h \in G$. This relation implies generalized commutation relations as follows. By (9.4.3), identifiying $\mathfrak{g}^* \cong \mathfrak{g}$, the classical canonically conjugate momenta, also

[35] Assuming that the Haar measure of G is normalized.

referred to as the colour electric fields, are given by elements of $\mathfrak{g}$. For $X \in \mathfrak{g}$, we define the associated momentum operator by

$$P_X : C^\infty(G) \to C^\infty(G)\,, \quad P_X\varphi := i\frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0} U(\mathrm{e}^{tX})\varphi\,. \tag{9.4.11}$$

Then, for any $f, \varphi \in C^\infty(G)$ and $X \in \mathfrak{g}$,

$$\big[P_X,\, T_f\big]\varphi = i\frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0} U(\mathrm{e}^{tX})T_f U(\mathrm{e}^{-tX})\varphi = i\frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0} T_{\lambda_{\exp(tX)}(f)}\varphi\,.$$

Denoting the right invariant vector field on G by X^R, we obtain

$$\big[P_X,\, T_f\big] = iT_{X^R(f)}\,. \tag{9.4.12}$$

For $G = \mathbb{R}$, this yields the standard Heisenberg commutation relations. Since $P_X = \mathrm{d}U(X)$, we obtain a representation of the Lie algebra $\mathfrak{g}$ on $L^2(G)$ which obviously fulfils $P_X\varphi_0 = 0$.

Remark 9.4.2 (*Generators*) As above, let σ be a faithful representation of G on $\mathbb{C}^k$, e.g. the fundamental representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ for QCD. Choose an orthonormal basis $\{\mathbf{e}_i\}$, $i = 1, \dots k$, of $\mathbb{C}^k$ and define the collection of functions $\sigma_{ij} \in C(G)$ by

$$\sigma_{ij}(g) := \langle \mathbf{e}_i, \sigma(g)\mathbf{e}_j\rangle\,. \tag{9.4.13}$$

Since the σ_{ij} are matrix elements of elements of G in the representation σ, they fulfil obvious relations reflecting the structure of G, see [368, 369] for details. Moreover, by (9.4.10),

$$\lambda_g(\sigma_{ij})(h) = \sum_m \langle \mathbf{e}_i, \sigma(g^{-1})\mathbf{e}_m\rangle \sigma_{mj}(h)\,. \tag{9.4.14}$$

Since σ is faithful, the algebra generated by the functions σ_{ij} with respect to pointwise multiplication separates the points in G, hence by the Weierstrass Theorem, it is a dense subalgebra of $C(G)$. Thus, the C*-algebra generated by the operators $\{T_{\sigma_{ij}} \mid i,\ j = 1, \dots, k\}$ is $T_{C(G)}$, that is, the algebra of multiplication operators by continuous functions on G.

Next, choose an orthonormal basis $\{t_a\}$ of $\mathfrak{g}$ and consider the corresponding basis $\{\sigma'_{im}(t_a)\}$ in $\mathrm{End}(\mathbb{C}^k)$. Then, the operators $E_a := P_{t_a}$ span all of $P_{\mathfrak{g}}$ and, thus, the unitary group they generate is all of $U_G \subset M(C^*(G))$, where $M(C^*(G))$ denotes the multiplier algebra of $C^*(G)$. From Example 3 in Sect 3 of [679] and [475], we also see that they generate $C^*(G)$ in the sense of Woronowicz. Associated with the generators E_a, we have the following set of $\mathrm{End}(\mathbb{C}^k)$-valued generators:

$$E_{ij} := \sum_a \sigma'_{ij}(t_a)E_a\,. \tag{9.4.15}$$

In terms of the above generators, the generalized commutation relations (9.4.12) read (Exercise 9.4.1)

$$\left[P_{t_a},\ T_{\sigma_{ij}}\right] = i \sum_m T_{\sigma'_{im}(t_a)\sigma_{mj}}\,. \tag{9.4.16}$$

Clearly, these relations may also be expressed in terms of the E_{ij}, see [369]. ♦

Now, the bosonic Hilbert space of the full system is defined by

$$\mathscr{H}^{\mathrm{b}}_{\Lambda} := L^2(\mathscr{C}_{\Lambda}) \cong \bigotimes_{\ell\in\Lambda^1} L^2(G)\,. \tag{9.4.17}$$

Clearly, $\pi_0 = (U, T)$ induces a representation on $\mathscr{H}_{\Lambda}$ denoted by $\pi^{\mathrm{b}} := (\hat{U}, \hat{T})$. In detail, for every $\ell \in \Lambda^1$, we define

$$\hat{T}^{(\ell)}_f := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes T^{(\ell)}_f \otimes \mathbb{1} \cdots \otimes \mathbb{1}\,, \tag{9.4.18}$$

and

$$\hat{U}^{(\ell)}_g := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes U^{(\ell)}_g \otimes \mathbb{1} \cdots \otimes \mathbb{1}\,, \tag{9.4.19}$$

where $T^{(\ell)}_f$ and $U^{(\ell)}_g$ are the multiplication and translation operators acting on the ℓ^{th} tensor product factor of $\mathscr{H}^{\mathrm{b}}_{\Lambda}$, respectively. Then, by Remark 9.4.2,

$$\{\hat{T}^{(\ell)}_{\sigma_{ij}(\ell)} \,:\, \ell \in \Lambda^1,\ i, j = 1, \ldots, k\}$$

and

$$\{\hat{E}_a(\ell) \,:\, \ell \in \Lambda^1,\ a = 1, \ldots, \dim \mathfrak{g}\}$$

generate the representation π^{b}. To summarize, we denote the total Hilbert space of the system by

$$\mathscr{H}_{\Lambda} := \mathscr{H}^{\mathrm{f}}_{\Lambda} \otimes \mathscr{H}^{\mathrm{b}}_{\Lambda} \tag{9.4.20}$$

and endow it with the tensor product representation

$$\pi := \pi^{\mathrm{f}} \otimes \pi^{\mathrm{b}}\,. \tag{9.4.21}$$

Next, we show how to implement the local gauge transformation (9.4.5) on quantum level. For the fermionic part we define

$$\alpha^{\mathrm{f}} : \mathscr{G}_{\Lambda} \to \mathrm{Aut}\left(\mathfrak{F}_{\Lambda}\right), \quad \alpha^{\mathrm{f}}_{\rho}(\mathfrak{a}(\psi)) := \mathfrak{a}(\sigma(\rho)\psi)\,. \tag{9.4.22}$$

As already noted, π^f is equivalent to the fermionic Fock representation. Thus, it is covariant with respect to α^{f}, that is, there is a (continuous) unitary representation $V^{\mathrm{f}} : \mathscr{G}_{\Lambda} \to \mathscr{U}(\mathscr{H}^{\mathrm{f}}_{\Lambda})$ such that

$$\pi^{\mathrm{f}}(\alpha^{\mathrm{f}}_{\rho}(F)) = V^{\mathrm{f}}_{\rho} \circ \pi^{\mathrm{f}}(F) \circ V^{\mathrm{f}}_{\rho^{-1}} \,, \tag{9.4.23}$$

for any $F \in \mathfrak{F}_{\Lambda}$. To implement the action $\mathscr{G}_{\Lambda}$ on the bosonic part, for any link $\ell = (x_\ell, y_\ell) \in \Lambda^1$, we define the unitary representation $V^{(\ell)} : \mathscr{G}_{\Lambda} \to \mathscr{U}(L^2(G))$ by

$$(V^{(\ell)}_{\rho}\varphi)(h) := \varphi(\rho(x_\ell)^{-1}\, h\, \rho(y_\ell)) \,. \tag{9.4.24}$$

Then, by definition, $\rho \to V^{(\ell)}_{\rho}$ is a homomorphism fulfilling $V^{(\ell)}_{\rho}\varphi_0 = \varphi_0$. Using this unitary representation, we define the local gauge transformations of the quantum observables from $U_G \cup T_{L^\infty(G)}$ by

$$T_f \mapsto V^{(\ell)}_{\rho} \circ T_f \circ \left(V^{(\ell)}_{\rho}\right)^{-1} = T_{V^{(\ell)}_{\rho} f} \,, \tag{9.4.25}$$

where $f \in L^\infty(G) \subset L^2(G)$, and

$$U_g \mapsto V^{(\ell)}_{\rho} \circ U_g \circ \left(V^{(\ell)}_{\rho}\right)^{-1} = U_{\rho(x_\ell) g\, \rho(x_\ell)^{-1}} \,, \tag{9.4.26}$$

for any $g \in G$. Moreover, since every operator $V^{(\ell)}_{\rho}$ preserves the space $C^\infty(G)$, (9.4.26) implies

$$V^{(\ell)}_{\rho} \circ P_X \circ \left(V^{(\ell)}_{\rho}\right)^{-1} = P_{\mathrm{Ad}(\rho(x_\ell))X} \,, \tag{9.4.27}$$

for any $X \in \mathfrak{g}$. To summarize, for the full system, we have the following unitary representation of $\mathscr{G}_{\Lambda}$ on $\mathscr{H}_{\Lambda}$:

$$V := V^{\mathrm{f}} \otimes V^{\mathrm{b}} \,, \quad V^{\mathrm{b}} := \bigotimes_{\ell \in \Lambda^1} V^{(\ell)} \,. \tag{9.4.28}$$

Remark 9.4.3 (*Gauge transformations of generators*) First, from (9.4.22) we read off the gauge transformation law for the fermionic generators $\Psi_{\mu i}(x)$ given by (9.4.7)[36]:

$$\left(V^{\mathrm{f}}_{\rho}\Psi\right)_i(x) = \sum_j \sigma\left(\rho(x)^{-1}\right)_{ij} \Psi_j(x) \,. \tag{9.4.29}$$

Next, by (9.4.24) and (9.4.25), the transformation law for the gauge generators $\sigma_{ij}(\ell)$ given by (9.4.13) reads as follows:

$$\left(V^{(\ell)}_{\rho}\sigma_{ij}(\ell)\right)(g) = \sum_{n,m} \sigma\left(\rho(x_\ell)^{-1}\right)_{in} \sigma_{nm}(\ell)(g)\, \sigma\left(\rho(y_\ell)\right)_{mj} \,. \tag{9.4.30}$$

The transformation law for the quantum gauge momentum operators is obtained by setting $X = t_a$, $a = 1, \ldots, \dim \mathfrak{g}$, in (9.4.27). ♦

[36] Since gauge transformations do not act on the bispinor degrees of freedom, we may suppress the index μ.

Exercise

9.4.1 Confirm formula (9.4.16).

9.5 Field Algebra and Observable Algebra

In this section, we construct the field algebra and the observable algebra of the model presented in the previous section. For functional analytic basics used below we refer to [82, 102, 507].

Note that the fermionic field algebra $\mathfrak{F}_\Lambda$ has already been identified as the C^*-algebra of canonical anti-commutation relations. Thus, it remains to construct a C^*-algebra for the bosonic part. By (9.4.9), the generalized Schrödinger representation $\pi_0 = (U, T)$ is a covariant representation of the C^*-dynamical system $(C(G), G, \lambda)$ with $\lambda : G \to \mathrm{Aut}\big(C(G)\big)$ defined by (9.4.10). Associated with this C^*-dynamical system, there is a natural crossed product C^*-algebra[37] $C(G) \rtimes_\lambda G$. Its representations are exactly the covariant representations of the C^*-dynamical system defined by λ. It is well known that $C(G) \rtimes_\lambda G$ is isomorphic to the algebra of compact operators on $L^2(G)$,

$$C(G) \rtimes_\lambda G \cong \mathfrak{K}\big(L^2(G)\big) , \tag{9.5.1}$$

see [531] and Theorem II.10.4.3 in [82]. In fact,

$$\pi_0\big(C(G) \rtimes_\lambda G\big) = \mathfrak{K}\big(L^2(G)\big) .$$

Since $\mathfrak{K}\big(L^2(G)\big)$ has a unique irreducible representation up to unitary equivalence, it follows that π_0 is the unique irreducible covariant representation of $(C(G), G, \lambda)$ (up to equivalence). Moreover, as φ_0 is cyclic for $\mathfrak{K}\big(L^2(G)\big)$, π_0 is unitarily equivalent to the GNS-representation of the vector state ω_0 given by $\omega_0(A) := (\varphi_0, \pi_0(A)\varphi_0)$ for $A \in C(G) \rtimes_\lambda G$.

Remark 9.5.1

1. For the convenience of the reader, let us give the definition of $C(G) \rtimes_\lambda G$. Take $L^1(G, C(G))$, defined as the $*$-algebra of $C(G)$-valued L^1-functions on G, endowed with multiplication given by the twisted convolution

$$(z \times w)(g') := \int_G z(g)\lambda_g(w(g^{-1}g'))dg ,$$

with the involution induced from the $*$-structure of $C(G)$,

$$z^*(g) = \lambda_g(z(g^{-1})^*) ,$$

[37] Also referred to as the generalized Weyl algebra.

and with the standard L^1-norm. Consider all its non-degenerate Hilbert space representations. Then, $C(G) \rtimes_\lambda G$ is defined as the completion of the algebra $L^1(G, C(G))$ in the sup-norm taken over all these representations. This way we obtain a C^*-algebra without unit. This algebra can be viewed as a skew tensor product of $C(G)$ with the group algebra $C^*(G)$ in the following sense: For each $u \in C(G)$ and $f \in L^1(G)$ denote by $u \otimes f$ the element of $L^1(G, C(G))$ given by $(u \otimes f)(g) := uf(g)$. Then, the linear span of such elements is dense in $L^1(G, C(G))$. It is easily seen that

$$f \to 1 \otimes f \tag{9.5.2}$$

is an isomorphism onto its image, which allows for identifying $C^*(G)$ with the corresponding subalgebra:

$$C^*(G) \subset C(G) \rtimes_\lambda G\,. \tag{9.5.3}$$

As already noted, $C^*(G)$ is a C^*-algebra generated by unbounded elements in the sense of Woronowicz. Consequently, $C(G) \rtimes_\lambda G$ is of this type, too. It is generated by elements (X, f) fulfilling the canonical commutation relations (9.4.12). We stress that both the (unbounded) generators X and the (bounded) generators f do not belong to the algebra, but are only affiliated in the C^*-sense. Moreover, note that–contrary to (9.5.2)–the mapping $u \to u \otimes 1$ does not preserve the algebraic structure of $C(G)$ and, whence, cannot be used to imbed $C(G)$ into $C(G) \rtimes_\lambda G$. Hence, $C(G)$ is not a subalgebra of $C(G) \rtimes_\lambda G$, but belongs to its multiplier algebra $M(C(G) \rtimes_\lambda G)$. Note that, clearly, the operators U_g are not compact but belong to the multiplier algebra as well. This is not a problem, because a state or representation on $C(G) \rtimes_\lambda G$ has a unique extension to its multiplier algebra, so will be fully determined on these elements. If one chose $C^*(U_G \cup T_{L^\infty(G)})$ as the field algebra instead of $C(G) \rtimes_\lambda G$, then this would contain many inappropriate representations, e.g. covariant representations for $\lambda : G \to \mathrm{Aut}\big(C(G)\big)$ where the implementing unitaries are discontinuous with respect to G.

2. The algebraic counterpart of $C(G) \rtimes_\lambda G$ is the following crossed product of Hopf algebras[38]:

$$C^\infty(G) \rtimes_\lambda \mathfrak{U}(\mathfrak{g})\,,$$

see [368] for details. Here, $\mathfrak{U}(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$. ♦

We combine the above building blocks into the field algebra

$$\mathfrak{A}_\Lambda := \mathfrak{F}_\Lambda \otimes \mathfrak{B}_\Lambda \tag{9.5.4}$$

with the bosonic part defined by

[38] This is an example of a Heisenberg double of Hopf algebras, c.f. [358].

$$\mathfrak{B}_\Lambda := \bigotimes_{\ell \in \Lambda^1} \big(C(G) \rtimes_\lambda G\big) . \tag{9.5.5}$$

This algebra is well defined as Λ^1 is finite, and the cross-norms are unique as all algebras involved are nuclear. Moreover, using (9.5.1), we obtain

$$\mathfrak{B}_\Lambda \cong \bigotimes_{\ell \in \Lambda^1} \mathfrak{K}(L^2(G)) . \tag{9.5.6}$$

Since $\mathfrak{F}_\Lambda$ is a full matrix algebra, $\mathfrak{A}_\Lambda$ is simple and, thus,

$$\mathfrak{A}_\Lambda \cong \mathfrak{K}(\mathscr{L}) , \tag{9.5.7}$$

where $\mathscr{L}$ is some generic infinite-dimensional separable Hilbert space. This shows that, for a finite lattice, there will be only one irreducible representation, up to unitary equivalence.[39] Moreover, since $\mathfrak{A}_\Lambda$ is simple, all representations are faithful. This implies the following.

Proposition 9.5.2 *The field algebra $\mathfrak{A}_\Lambda$ is faithfully and irreducibly represented by $\big(\mathscr{H}_\Lambda, \pi\big)$, that is,*

$$\pi(\mathfrak{A}_\Lambda) = \mathfrak{K}\big(\mathscr{H}_\Lambda\big) . \tag{9.5.8}$$

Note that $\pi\left(\mathfrak{A}_\Lambda\right)$ contains in its multiplier algebra the operators $\hat{T}^{(\ell)}_{\sigma_{ij}(\ell)}$ and $\hat{U}^{(\ell)}_g$ for all $\ell \in \Lambda^1$.

Finally, we define the (product) action of the gauge group $\mathscr{G}_\Lambda$ on $\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \mathfrak{B}_\Lambda$,

$$\alpha : \mathscr{G}_\Lambda \to \mathrm{Aut}\left(\mathfrak{A}_\Lambda\right) , \quad \alpha := \alpha^{\mathrm{f}} \otimes \alpha^{\mathrm{b}} . \tag{9.5.9}$$

Recall that the action $\alpha^{\mathrm{f}} : \mathscr{G}_\Lambda \to \mathrm{Aut}\left(\mathfrak{F}_\Lambda\right)$ has already been defined, see (9.4.22). To define the action $\alpha^{\mathrm{b}} : \mathscr{G}_\Lambda \to \mathrm{Aut}\left(\mathfrak{B}_\Lambda\right)$, recall that in the representation π it is given by $\rho \to \mathrm{Ad}(V_\rho)$, cf. (9.4.25) and (9.4.26). This action clearly preserves $\pi\left(\mathfrak{A}_\Lambda\right) = \mathfrak{K}(\mathscr{H}_\Lambda)$ and, since $\rho \to V_\rho$ is strongly operator continuous, it defines a strongly continuous action α of $\mathscr{G}_\Lambda$ on $\pi\left(\mathfrak{A}_\Lambda\right)$ and, thus, on $\mathfrak{A}_\Lambda$, By construction, (π, V) is a covariant representation for the C*-dynamical system given by α. As $\mathscr{G}_\Lambda$ is locally compact, we can construct the crossed product $\mathfrak{A}_\Lambda \rtimes_\alpha \mathscr{G}_\Lambda$ whose representation space is built from all covariant representations of $\alpha : \mathscr{G}_\Lambda \to \mathrm{Aut}\left(\mathfrak{A}_\Lambda\right)$.

Let us describe the action α^{b} in detail. First, consider one building block $C(G) \rtimes_\lambda G$ of $\mathfrak{B}_\Lambda$ corresponding to a link $(x, y) \in \Lambda^1$. In view of (9.4.25) and (9.4.26) we define

$$\tau : \mathscr{G}_\Lambda \to \mathrm{Aut}(C(G)) , \quad (\tau_\rho u)(g) := u(\rho(x)^{-1} g \rho(y)) ,$$

[39]This may be viewed as a generalization of the classical von Neumann uniqueness theorem for irreducible representations of the canonical commutation relations on $\mathbb{R}$.

and

$$\beta : \mathscr{G}_\Lambda \to \mathrm{Aut}(L^1(G))\,, \quad (\beta_\rho f)(g) := f(\rho(x)^{-1} g \rho(x))\,.$$

By point 1 of Remark 9.5.1, $C(G) \rtimes_\lambda G$ is the closure of the space spanned by $L^1(G) \cdot C(G)$. Thus, the pair (τ, β) induces a representation on $C(G) \rtimes_\lambda G$ by

$$\theta_\rho(f \cdot u) := \beta_\rho(f) \cdot \tau_\rho(u)\,. \tag{9.5.10}$$

Now, α^{b} is defined as the tensor product representation of the representations $\theta^{(\ell)}$ over all $\ell \in \Lambda^1$, with every $\theta^{(\ell)}$ defined by (9.5.10).

Given the action α, we can derive the lattice counterpart of the local Gauß law. In abstract terms, a Gauß law generator is, by definition, a nonzero element in the range of the derived action

$$\mathrm{d}\alpha : \mathfrak{g}_\Lambda \to \mathrm{Der}\left(\mathfrak{A}_\Lambda^\infty\right),$$

where $\mathfrak{g}_\Lambda$ is the Lie algebra of $\mathscr{G}_\Lambda$ and $\mathfrak{A}_\Lambda^\infty$ denotes the subalgebra of smooth elements with respect to the action. By (9.5.9),

$$\mathrm{d}\alpha(\mathbf{v}) = \mathrm{d}\alpha^{\mathrm{f}}(\mathbf{v}) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\alpha^{\mathrm{b}}(\mathbf{v})\,,$$

for any $\mathbf{v} \in \mathfrak{g}_\Lambda$. To calculate $\mathrm{d}\alpha$ explicitly, note that $\mathfrak{g}_\Lambda$ is spanned by elements of the form $\mathbf{v} = X \cdot \delta_x$ for $X \in \mathfrak{g}$ and $x \in \Lambda^0$. Using this, we calculate

$$\mathrm{d}\alpha^{\mathrm{f}}(X \cdot \delta_x)(\mathfrak{a}(\psi)) = \frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0}\, \mathfrak{a}(\exp(tX \cdot \delta_x)\psi) = \mathfrak{a}(\delta_x \cdot X\psi)\,. \tag{9.5.11}$$

Next, by (9.5.10), $\mathrm{d}\theta(\mathbf{v}) = \mathrm{d}\beta(\mathbf{v}) + \mathrm{d}\tau(\mathbf{v})$. For $u \in C^\infty(G)$, we calculate

$$\mathrm{d}\tau(X \cdot \delta_x)(u)(g) = \frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0}\, u\left(\mathrm{e}^{-tX} g\right) = -\left(X^R u\right)(g)\,, \tag{9.5.12}$$

where X^R is the right invariant vector field on G generated by $X \in \mathfrak{g}$. Correspondingly, for $f \in L^1(G) \cap C^\infty(G)$, we obtain

$$\mathrm{d}\beta(X \cdot \delta_x)(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}{}_{\restriction 0}\, f\left(\mathrm{e}^{-tX} g \mathrm{e}^{tX}\right) = (-R_g' X + L_g' X) f\,. \tag{9.5.13}$$

Now, note that $\alpha^{\mathrm{b}}(X \cdot \delta_x)$ affects only those links which contain x, that is, the nearest neighbours $(x, y_k^\pm) := (x, x \pm a\mathbf{e}_k)$, with $k = 1, 2, 3$, of x,

$$\mathrm{d}\alpha^{\mathrm{b}}(X \cdot \delta_x) = \sum_{(x, y_k^\pm)} \mathrm{d}\theta^{(x, y_k^\pm)}(X \cdot \delta_x)\,.$$

To summarize, we have

$$\mathrm{d}\alpha(X \cdot \delta_x) = \mathrm{d}\alpha^{\mathrm{f}}(X \cdot \delta_x) \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{(x, y_k^{\pm})} \big(\mathrm{d}\tau^{(x,y_k^{\pm})} + \mathrm{d}\beta^{(x,y_k^{\pm})}\big)(X \cdot \delta_x)\,, \tag{9.5.14}$$

with $\mathrm{d}\alpha^{\mathrm{b}}$, $\mathrm{d}\tau^{(x,y_k^{\pm})}$ and $\mathrm{d}\beta^{(x,y_k^{\pm})}$ given by (9.5.11), (9.5.12) and (9.5.13), respectively. Correspondingly, we have a local Gauß law at every lattice point $x \in \Lambda^0$ given by

$$\mathrm{d}\alpha(X \cdot \delta_x) = 0\,, \tag{9.5.15}$$

for every $X \in \mathfrak{g}$.

Remark 9.5.3 (*Local Gauß Law*) Recall that, in the representation $(\mathscr{H}_\Lambda, \pi)$, the gauge group $\mathscr{G}_\Lambda$ acts via the unitary representation V given by (9.4.28). Using the description of the field algebra in terms of generators provided by (9.4.7) and Remark 9.4.2, in the representation V the local Gauß law reads as follows (Exercise 9.5.1):

$$\sum_{(x, y_k^{\pm})} E_{ij}(x, y_k^{\pm}) = \mathrm{q}_{ij}(x)\,. \tag{9.5.16}$$

Here, q_{ij} is the local matter charge density. For $G = \mathrm{SU}(3)$, it reads

$$\mathrm{q}_{ij}(x) = \Psi_i^*(x)\Psi_j(x) - \frac{1}{3}\delta_{ij}\Psi_l^*(x)\Psi_l(x)\,. \tag{9.5.17}$$

♦

Now, we can define the observable algebra of the system.[40]

Definition 9.5.4 (*Observable algebra*) The observable algebra of the lattice gauge theory is defined by

$$\mathfrak{O}_\Lambda := \mathfrak{A}^{\mathscr{G}_\Lambda}/\{\mathfrak{J}_\Lambda \cap \mathfrak{A}^{\mathscr{G}_\Lambda}\}\,,$$

where $\mathfrak{A}^{\mathscr{G}_\Lambda} \subset \mathfrak{A}_\Lambda$ is the subalgebra of $\mathscr{G}_\Lambda$-invariant elements of $\mathfrak{A}_\Lambda$ and $\mathfrak{J}_\Lambda \subset \mathfrak{A}_\Lambda$ is the ideal[41] generated by $\mathfrak{g}_\Lambda$.

Recall that, under the representation π, the field algebra $\mathfrak{A}_\Lambda$ gets identified with the algebra $\mathfrak{K}(\mathscr{H}_\Lambda)$ of compact operators on $\mathscr{H}_\Lambda$, cf. Proposition 9.5.2. Under this identification, we have a unitary representation V of $\mathscr{G}_\Lambda$ on $\mathscr{H}_\Lambda$ and the subalgebra $\mathfrak{A}^{\mathscr{G}_\Lambda}$ can be viewed as the commutant $(\mathscr{G}_\Lambda)'$ of this representation in $\mathfrak{K}(\mathscr{H}_\Lambda)$.

Consider the closed subspace $\mathscr{H}^{\mathscr{G}_\Lambda} \subset \mathscr{H}_\Lambda$ consisting of $\mathscr{G}_\Lambda$-invariant vectors,

$$\mathscr{H}^{\mathscr{G}_\Lambda} := \{\Phi \in \mathscr{H}_\Lambda \mid V_\rho(\Phi) = \Phi \ \text{ for all } \ \rho \in \mathscr{G}_\Lambda\}\,. \tag{9.5.18}$$

---

[40] In [271], we have shown that the definition below coincides with the algebra obtained by the T-procedure of Grundling and Hurst, see [268, 269, 270].

[41] Let $\mathfrak{C}$ be the ideal in $\mathfrak{A}_\Lambda$ generated by the local Gauß laws (9.5.16). Then, $\mathfrak{J}_\Lambda$ is the ideal generated by $\mathfrak{C}$ in $C^*(\mathfrak{A}^{\mathscr{G}_\Lambda} \cup \mathfrak{C})$, see [271] for further details.

Theorem 9.5.5 *The observable algebra $\mathfrak{O}_\Lambda$ is isomorphic to the algebra of compact operators on $\mathscr{H}^{\mathscr{G}_\Lambda}$,*

$$\mathfrak{O}_\Lambda \cong \mathfrak{K}\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big) . \tag{9.5.19}$$

Proof Consider the direct sum decomposition

$$\mathscr{H}_\Lambda = \mathscr{H}^{\mathscr{G}_\Lambda} \oplus \big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp , \tag{9.5.20}$$

with $\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp$ denoting the orthogonal complement of $\mathscr{H}^{\mathscr{G}_\Lambda}$. Since $\mathscr{H}^{\mathscr{G}_\Lambda}$ is invariant under $\mathscr{G}_\Lambda$, by unitarity of V, the complement $\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp$ is invariant, too. Hence, with respect to the decomposition (9.5.20), any element of $\mathscr{G}_\Lambda$ has the block-diagonal form $\begin{bmatrix} \mathbb{1} & 0 \\ 0 & B \end{bmatrix}$ with some unitary operator B on $\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp$. First, we show

$$(\mathscr{G}_\Lambda)' = \left\{ \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{K}(\mathscr{H}_\Lambda) : [B, D] = 0 \text{ for all } \begin{bmatrix} \mathbb{1} & 0 \\ 0 & B \end{bmatrix} \in \mathscr{G}_\Lambda \right\} . \tag{9.5.21}$$

Indeed, an operator $\begin{bmatrix} C & E \\ F & D \end{bmatrix}$ belongs to $(\mathscr{G}_\Lambda)'$ iff for any $\begin{bmatrix} \mathbb{1} & 0 \\ 0 & B \end{bmatrix} \in \mathscr{G}_\Lambda$ it satisfies

$$E = EB , \quad BF = F , \quad BD = DB . \tag{9.5.22}$$

This implies that for every $\phi \in \mathscr{H}^{\mathscr{G}_\Lambda}$ we have $F\phi = BF\phi$ and hence $F\phi \in \mathscr{H}^{\mathscr{G}_\Lambda}$. On the other hand, $F\phi \in \big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp$, because F maps $\mathscr{H}^{\mathscr{G}_\Lambda}$ to $\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big)^\perp$. It follows that $F\phi = 0$ and hence $F = 0$. By analogy, $E = 0$. This proves (9.5.21).

Now, we decompose

$$\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} .$$

Since the restriction of a compact operator to a closed subspace is compact, we have $C \in \mathfrak{K}(\mathscr{H}^{\mathscr{G}_\Lambda})$. Moreover, $\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{I}_\Lambda$. This yields the direct sum decomposition

$$(\mathscr{G}_\Lambda)' = \mathfrak{K}\big(\mathscr{H}^{\mathscr{G}_\Lambda}\big) \oplus \big(\mathfrak{I}_\Lambda \cap (\mathscr{G}_\Lambda)'\big)$$

and hence the assertion. ■

We close this section by three remarks. For details we refer to [271, 272, 368, 369]. In sharp contrast to the Abelian case,[42] the local Gauß laws (9.5.16) are neither built from gauge invariant operators nor are they linear.[43] Thus, the question arises

[42]See [373–375]. In these papers, the observable algebra for Lattice QED and Lattice Scalar QED is analyzed in detail.

[43]The apparent linearity with respect to the colour electric fields E_{ij} on the left hand side is due to the fact that, in this formula, every E_{ij} is 'parallelly transported' to the point x. If we would like to

whether one can extract from Eq. (9.5.16) a gauge invariant and linear relation for each lattice point. These relations could then be summed up over all lattice points to produce a gauge invariant global Gauß law. This problem was solved in [368].

Remark 9.5.6 (*Global colour charge*) For concreteness, we limit our attention to the case of QCD, that is, $G = \mathrm{SU}(3)$. Recall that, for every $\ell \in \Lambda^1$, the colour electric fields $E_{ij}(\ell)$ generate a unitary representation of G. Using the CAR-relations for the fermionic generators $\Psi_i(x)$, one easily shows (Exercise 9.5.2) that, for every $x \in \Lambda^0$, the local charge density operators $\mathrm{q}_{ij}(x)$ generate a unitary representation of G, too. By construction,

$$[E_{ij}(\ell), E_{kl}(\ell')] = 0\,, \quad [\mathrm{q}_{ij}(x), \mathrm{q}_{kl}(x')] = 0\,,$$

for $\ell \neq \ell'$ and $x \neq x'$. Thus, let $\{F_a\}$ be a collection of commuting unitary representations of G on a Hilbert space $\mathscr{H}$ and let $\{\mathfrak{f}_a\}$ be the corresponding collection of derived representations of the Lie algebra $\mathfrak{g}$. If $\mathfrak{f}_1$ and $\mathfrak{f}_2$ belong to that collection, then so does $\mathfrak{f}_1 + \mathfrak{f}_2$. Such a collection of operators is an operator domain in the sense of Woronowicz, see [678]. We define an operator function on this domain, that is, a mapping $\mathfrak{f} \to \varphi(\mathfrak{f})$ satisfying $\varphi(U\mathfrak{f}U^{-1}) = U\varphi(\mathfrak{f})U^{-1}$ for any isometry U, as follows: for a given representation $\mathfrak{f}$, consider the corresponding representation F of G. Its restriction to the center Z of G acts as a multiple of the identity on each irreducible subspace $\mathscr{H}_\alpha$ of F,

$$F(z)_{\restriction \mathscr{H}_\alpha} = \chi_F^\alpha(z) \cdot \mathbb{1}_{\mathscr{H}_\alpha}\,, \quad z \in Z\,.$$

Obviously, χ_F^α is a character on Z and, therefore, $(\chi_F^\alpha(z))^3 = 1$. We identify the group of characters on $Z = \{\zeta \cdot \mathbb{1}_3 \mid \zeta^3 = 1,\ \zeta \in \mathbb{C}\}$ with the additive group $\mathbb{Z}_3 \cong \{-1, 0, 1\}$ by assigning to any character χ_F^α a number $k(\alpha) \in \{-1, 0, 1\}$ fulfilling

$$\chi_F^\alpha(\zeta \cdot \mathbb{1}_3) = \zeta^{k(\alpha)}\,.$$

We define

$$\mathfrak{f} \mapsto \varphi(\mathfrak{f}) := \sum_\alpha \varphi_\alpha(\mathfrak{f})\, \mathbb{1}_{\mathscr{H}_\alpha}\,, \tag{9.5.23}$$

with $\varphi_\alpha(\mathfrak{f})$ given by

$$\zeta^{\,\varphi_\alpha(\mathfrak{f})} = \chi_F^\alpha(\,\zeta \cdot \mathbf{1}_3)\,. \tag{9.5.24}$$

Since χ_F^α are characters, we have

$$\varphi(\mathfrak{f}_1 + \mathfrak{f}_2) = \varphi(\mathfrak{f}_1) + \varphi(\mathfrak{f}_2)\,. \tag{9.5.25}$$

Now, using the equivalence of each irreducible representation α of G with highest weight $(m(\alpha), n(\alpha))$ with the tensor representation in the space ${\mathbb{T}^{m(\alpha)}}_{n(\alpha)}(\mathbb{C}^3)$ of

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assign them to, say, the middle of the link they live on we would have to apply the parallel transport operator. This would produce the lattice approximation of the covariant divergence on that link.

$m(\alpha)$-contravariant, $n(\alpha)$-covariant, completely symmetric and traceless tensors over $\mathbb{C}^3$, we get

$$\chi_F^\alpha(z) = \zeta^{\,\varphi_\alpha(\mathfrak{f})} = \zeta^{\,m(\alpha)-n(\alpha)} ,$$

for $z = \zeta \cdot \mathbf{1}_3 \in Z$. Thus, we have

$$\varphi_\alpha(\mathfrak{f}) = (m(\alpha) - n(\alpha)) \mod 3 , \tag{9.5.26}$$

for every irreducible highest weight representation $(m(\alpha), n(\alpha))$. In [368] we have given an explicit construction of $\varphi(\mathfrak{f})$ in terms of the Casimir operators of $\mathfrak{f}$.

Applying φ to the local Gauß law (9.5.16) and using the additivity property (9.5.25), we obtain a gauge invariant equation for operators with eigenvalues in $\mathbb{Z}_3$:

$$\sum_{(x, y_k^\pm)} \varphi\big(E(x, y_k^\pm)\big) = \varphi(\mathrm{q}(x)) , \tag{9.5.27}$$

valid at every lattice site x. Moreover, it is easy to check that

$$\varphi(E(x, y)) + \varphi(E(y, x)) = 0 , \tag{9.5.28}$$

for every link (x, y). The quantity on the right hand side of (9.5.27) is the (gauge invariant) local colour charge density carried by the quark field. By definition, the sum of local colour charges over all lattice sites is referred to as the global colour charge[44] carried by the matter field:

$$\mathfrak{t}_\Lambda := \sum_{x \in \Lambda^0} \varphi(\mathrm{q}(x)) . \tag{9.5.29}$$

Now, let us extend the picture by assigning to each point of the boundary $\partial \Lambda$ of Λ exactly one external link and let us assume that gluons and colour electric fields may live on these links. Let us take the sum of equations (9.5.27) over all lattice sites $x \in \Lambda^0$. Then, by (9.5.28), all terms on the left hand side cancel, except for contributions coming from the boundary. Thus, the global Gauß law takes the following form:

$$\Phi_{\partial \Lambda} = \mathfrak{t}_\Lambda , \tag{9.5.30}$$

where

$$\Phi_{\partial \Lambda} = \sum_{x \in \partial \Lambda^0} \varphi(E(x, \infty))$$

[44] Or, triality.

is the global $\mathbb{Z}_3$-valued boundary flux of the colour electric field. In [369] we have proved that the inequivalent irreducible representations of the observable algebra[45] are labelled by the global colour charge. We also refer to [348] for an alternative proof. ♦

Remark 9.5.7 (*Dynamics*) Disregarding the fermion doubling problem,[46] the dynamics of the lattice system is governed by the Kogut–Susskind Hamiltonian, see [385, 386],

$$\begin{aligned} H = & \frac{\kappa^2}{2a} \sum_{\ell \in \Lambda^1} E_{ij}(\ell) E_{ji}(\ell) - \frac{1}{\kappa^2 a} \sum_{p \in \Lambda^2} \left(W(p) + \overline{W(p)} \right) \\ & - \frac{i}{2a} \sum_{\ell \in \Lambda^1} \overline{\Psi}_{\mu i}(x_\ell) \big(\gamma \cdot \mathbf{n}_\ell \big)_{\mu\nu} \sigma_{ij}(\ell) \Psi_{\nu j}(y_\ell) + h.c. \\ & + m \sum_{x \in \Lambda^0} \overline{\Psi}_{\mu i}(x) \Psi_{\mu i}(x) \,. \end{aligned} \tag{9.5.31}$$

Here, $W(p)$ denotes the Wilson loop operator associated with the plaquette $p = (\ell_1, \ell_2, \ell_3, \ell_4) \in \Lambda^2$,

$$W(p)(g_1, \ldots g_4) := \sigma_{i_1 i_2}(\ell_1)(g_1) \sigma_{i_2 i_3}(\ell_2)(g_2) \sigma_{i_3 i_4}(\ell_3)(g_3) \sigma_{i_4 i_1}(\ell_4)(g_4) \,,$$

γ denotes the $\mathrm{End}(\mathbb{C}^4)$-valued space vector $(\gamma^1, \gamma^2, \gamma^3)$ and $\mathbf{n}_\ell$ denotes the unit vector pointing from x_ℓ to y_ℓ. Moreover, h.c. means taking the Hermitean conjugate and $\overline{\Psi}_{\mu i} = \Psi^\dagger_{\nu i} \gamma^0_{\nu\mu}$. The constants κ and m denote the gauge coupling constant and the fermion mass, respectively. The coefficients in H are determined by the requirements that, in the naive continuum limit, H tends to the continuum Hamiltonian and that the commutation relations tend to the standard commutation relations of the continuum theory. Clearly, H is a gauge-invariant (unbounded) operator acting on $\mathscr{H}^{\mathscr{G}_\Lambda}$. By Stone's Theorem, this operator generates a one-parameter group of time evolution on $\mathscr{H}^{\mathscr{G}_\Lambda}$. ♦

Remark 9.5.8 (*Towards the thermodynamical limit*) In [271, 272], some steps were made towards an understanding of Hamiltonian gauge theory on an infinite lattice. The starting point is a natural generalization of the representation $(\mathscr{H}_\Lambda, \pi)$ constructed above to the infinite lattice. This representation is defined as the tensor

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[45] That is, the observable algebra $\mathfrak{O}_\Lambda$ extended in an appropriate way in order to include the boundary data.

[46] The naive Hamiltonian given by (9.5.31) leads to the well known fermion doubling problem, that is, the lattice fermion propagator has 16 poles (in four dimensions). Starting with an improvement proposed by Wilson [673], various concepts to cure this problem have been developed, see the textbook literature cited above. In [488] it was shown that the doubling problem can only be avoided by giving up one of a number of plausible requirements, including chiral invariance in the zero mass case, see [216] for a rigorous proof. This observation led to an intensive study of the lattice approximation of the Dirac operator. We refer to [414–416] and the textbooks [233, 536].

product of a fermionic and a bosonic part, where the fermionic part is a Fock representation of the CAR-algebra of the full lattice. The bosonic part is an infinite tensor product of the generalized Schrödinger representations (in the sense of von Neumann) for the individual links with respect to a natural reference vector, and a fixed enumeration of the links. On that Hilbert space, all the local field algebras, that is, the field algebras associated with finite sublattices, are naturally represented. Then, on a suitable C^*-algebra (containing all the local algebras) acting on that Hilbert space, the existence of a one-parameter group generated by the (infinite lattice version of) the Hamiltonian (9.5.31) is proven. This one-parameter group is the pointwise norm limit of the local time evolutions with respect to a sequence of finite sublattices, exhausting the full lattice. Moreover, the existence of regular gauge invariant ground states is shown but, for the time being, there is no uniqueness proof. ♦

Exercises

9.5.1 Prove formula (9.5.16).

9.5.2 Using the CAR-relations for the fermionic generators, show that the local charge density operators $\mathrm{q}_{ij}(x)$ generate a unitary representation of G, for every $x \in \Lambda^0$.

9.5.3 Check that the lattice Hamiltonian given by (9.5.31) is gauge invariant.

9.6 Including the Nongeneric Strata

In this section, we limit our attention to pure gauge theory on a finite lattice. In this situation, the classical configuration space is $\mathscr{C}_\Lambda = G^{\Lambda^1}$, acted upon by the group of local gauge transformations $\mathscr{G}_\Lambda = G^{\Lambda^0}$ via (9.4.5). Correspondingly, the classical phase space is given by the associated Hamiltonian Lie group action. According to Corollary 10.1.21 of Part I, the latter is given by the following data:

1. the symplectic manifold $\mathrm{T}^*\mathscr{C}_\Lambda$,
2. the action of $\mathscr{G}_\Lambda$ by the induced point transformations,
3. the natural momentum mapping $\mathscr{J}_\Lambda : \mathrm{T}^*\mathscr{C}_\Lambda \to \mathrm{L}\mathscr{G}_\Lambda$ defined by evaluating the elements of $\mathrm{T}^*\mathscr{C}_\Lambda$ on the Killing vector fields of the action of $\mathscr{G}_\Lambda$ on $\mathscr{C}_\Lambda$.

Since the Killing vector fields correspond to'unphysical' directions in $\mathscr{C}_\Lambda$, they should not be recognized by 'physical' momenta. Hence, the latter should be annihilated by $\mathscr{J}_\Lambda$. This condition corresponds to the local Gauß law in the continuum theory. As a consequence, the classical reduced phase space of the model is obtained by symplectic reduction at zero level,[47]

$$\mathscr{P}_\Lambda = \mathscr{J}_\Lambda^{-1}(0)\big/\mathscr{G}_\Lambda\,. \tag{9.6.1}$$

[47] Cf. Sect. 10.5 of Part I.

This is a stratified symplectic space, where the strata are given by the orbit type components, that is, the connected components of the orbit type subsets.

It is convenient to carry out the reduction (9.6.1) in two stages: first with respect to the pointed gauge group

$$\mathscr{G}_{x_0} = \{g \in \mathscr{G} : g(x_0) = \mathbb{1}\}$$

for some chosen site $x_0 \in \Lambda^0$, and then with respect to the residual action of $\mathscr{G}_\Lambda/\mathscr{G}_{x_0} \cong G$. The first stage is obtained by zero level reduction of the Hamiltonian Lie group action associated with the action of $\mathscr{G}_{x_0}$ on $\mathscr{C}_\Lambda$. Since the latter action is free and since 0 is a regular value of $\mathscr{J}_\Lambda$, we are in the realm of regular zero level reduction. Consequently, the symplectic quotient is given by the cotangent bundle of the quotient manifold $\mathscr{C}_\Lambda/\mathscr{G}_{x_0}$. Thus, the second stage boils down to zero level reduction of the Hamiltonian Lie group action associated with the residual action of $\mathscr{G}_\Lambda/\mathscr{G}_{x_0} \cong G$ on $\mathscr{C}_\Lambda/\mathscr{G}_{x_0}$. Since the first stage of the reduction is regular, it is at the second stage where a stratification may arise. Consequently, for studying the quantum significance of the stratification, it suffices to restrict attention to that stage.

Let us give a more convenient description of the quotient manifold $\mathscr{C}_\Lambda/\mathscr{G}_{x_0}$ in terms of a tree gauge. For that purpose, choose a maximal lattice tree $\mathscr{T}$, that is, a simply connected subset $\mathscr{T} \subset \Lambda^1$ such that every site belongs to some link in $\mathscr{T}$. One can check the following (Exercise 9.6.1).

1. For every site x there exists a unique path in $\mathscr{T}$ from x to the site x_0 chosen in the definition of $\mathscr{G}_{x_0}$. Given a lattice gauge potential $\{\hat{\ell}_{\mathbb{A}}\}$, one can use these unique paths to construct a gauge transformation ρ such that

$$\hat{\ell}^{(\rho)}_{\mathbb{A}} = \mathbb{1} \text{ for all } \ell \in \mathscr{T}. \tag{9.6.2}$$

2. Two lattice gauge potentials satisfying (9.6.2) are conjugate under $\mathscr{G}_\Lambda$ iff they differ by a constant gauge transformation. In particular, no two such elements are conjugate under $\mathscr{G}_{x_0}$.

Via a numbering $\ell_1, \dots, \ell_N$ of the links in $\Lambda^1 \setminus \mathscr{T}$, every element $(g_1, \dots, g_N) \in G^N$ defines a mapping $\Lambda^1 \to G$ by assigning the members g_i to the corresponding off-tree links ℓ_i and $\mathbb{1}$ to all links in $\mathscr{T}$. This way, we obtain an embedding $G^N \to \mathscr{C}_\Lambda$ whose image coincides with the subset defined by (9.6.2). Thus, by composing this embedding with the natural projection to classes $\mathscr{C}_\Lambda \to \mathscr{C}_\Lambda/\mathscr{G}_{x_0}$, we obtain a diffeomorphism $G^N \cong \mathscr{C}_\Lambda/\mathscr{G}_{x_0}$ which is equivariant with respect to the action of G on G^N by diagonal inner automorphisms on G^N and the residual action of $\mathscr{G}_\Lambda/\mathscr{G}_{x_0} \cong G$ on $\mathscr{C}_\Lambda/\mathscr{G}_{x_0}$.

As a result, the second stage of the reduction (9.6.1) is equivalent to zero level symplectic reduction of the Hamiltonian Lie group action associated with the action of G on G^N by diagonal inner automorphisms,

$$\psi_g(g_1, \dots, g_N) = (g g_1 g^{-1}, \dots, g g_N g^{-1}).$$

Let us write down the corresponding data explicitly under the identification $\mathrm{T}^*G^N \cong \left(\mathrm{T}^*G\right)^N \cong \left(G \times \mathfrak{g}^*\right)^N$. The symplectic form ω is componentwise given by formula I/(8.3.8). According to Example I/10.1.25, the induced action of G reads

$$\Psi_g(g_1, \ldots, g_N, \xi_1, \ldots, \xi_N) = (gg_1g^{-1}, \ldots, gg_Ng^{-1}, \mathrm{Ad}^*(g)\xi_1, \ldots, \mathrm{Ad}^*(g)\xi_N)\,,$$

and the momentum mapping is given by

$$\mathscr{J}(g_1, \ldots, g_N, \xi_1, \ldots, \xi_N) = \sum_{i=1}^{N} \xi_i - \mathrm{Ad}^*(g_i)\xi_i\,. \tag{9.6.3}$$

As noted before, the corresponding reduced phase space

$$\mathscr{P} = \mathscr{J}^{-1}(0)/G$$

is a stratified symplectic space with the strata given by the orbit type components of $\mathscr{P}$. Thus, denoting the set of orbit type components by $\mathtt{T}$, we have a disjoint decomposition

$$\mathscr{P} = \bigcup_{\tau \in \mathtt{T}} \mathscr{P}_\tau \tag{9.6.4}$$

satisfying the frontier condition, that is, for all $\tau, \tau' \in \mathtt{T}$,

$$\mathscr{P}_\tau \cap \overline{\mathscr{P}_{\tau'}} \neq \varnothing \quad \text{implies} \quad \mathscr{P}_\tau \subset \overline{\mathscr{P}_{\tau'}}\,.$$

The partial ordering of orbit types defined in Sect. 8.2 extends to a partial ordering of $\mathtt{T}$ in an obvious way. Since the closures of distinct connected components of an orbit type subset do not intersect,

$$\mathscr{P}_\tau \subset \overline{\mathscr{P}_{\tau'}} \quad \text{iff} \quad \tau \leq \tau'\,. \tag{9.6.5}$$

Clearly, the orbit types appearing in $\mathscr{P}$ form a subset of the set of orbit types of the lifted action Ψ on T^*G^N. By [509], the latter coincides with the set of orbit types of the original action on the base space G^N. Thus, it is enough to know the orbit types of the latter. As an illustration, let us give an example [124].

Example 9.6.1 (*Orbit types of the diagonal action of G on G^N for $G = \mathrm{SU}(3)$*) Let Z denote the center of G. For $N = 1$, the action has three orbit types. Let $g \in G$.

1. If g has three distinct eigenvalues, $G_g \cong \mathrm{U}(1)^2$ and g lies in the generic stratum.
2. If g has two distinct eigenvalues, $G_g \cong \mathrm{U}(2)$.
3. If g has a single eigenvalue, it belongs to Z and $G_g = \mathrm{SU}(3)$.

For $N \geq 2$, the action has five orbit types. Let $\mathbf{g} := (g_1, \ldots, g_N) \in G^N$.

1. If the g_i have no common eigenspace, $G_{\mathbf{g}} = Z$ and $\mathbf{g}$ lies in the generic stratum.
2. If the g_i have exactly one common 1-dimensional eigenspace, $G_{\mathbf{g}} \cong \mathrm{U}(1)$.
3. If the g_i have three common 1-dimensional eigenspaces, $G_{\mathbf{g}} \cong \mathrm{U}(1) \times \mathrm{U}(1)$.
4. If the g_i have a 2-dimensional common eigenspace, $G_{\mathbf{g}} \cong \mathrm{U}(2)$.
5. Otherwise, all g_i belong to Z and $G_{\mathbf{g}} = \mathrm{SU}(3)$. ♦

There are two strategies for implementing the stratified structure on quantum level.

1. Quantization after reduction: perform the singular symplectic reduction at zero level and develop a quantum theory on the stratified space so obtained.
2. Reduction after quantization: start with the quantum theory as described in the previous section and develop a reduction procedure on quantum level.

A closer look at (9.6.3) shows that it is hard to perform the reduction to the zero level set on the classical level explicitly. For the study of toy models, including the investigation of the topological structure of the lattice gauge orbit space, we refer to [125, 202].

Below, we will follow the second strategy. To implement the stratified structure on quantum level, we will use the concept of costratified Hilbert space developed by Huebschmann [325, 326]. We start with the Hilbert space representation $\mathscr{H}^{\mathscr{G}_\Lambda}$ of the observable algebra constructed in Sect. 9.4, cf. formula (9.5.18). Let

$$\mathscr{H}_N = L^2(G)^{\otimes N} = L^2(G^N)$$

and

$$\mathscr{H} := \mathscr{H}_N^G = \{\varphi \in \mathscr{H}_N : \psi_g^* \varphi = \varphi \text{ for all } g \in G\}\,.$$

We have a natural isomorphism $\mathscr{H}^{\mathscr{G}_\Lambda} \cong \mathscr{H}$. Thus, we may take $\mathscr{H}$ as the Hilbert space of the quantum system.

Definition 9.6.2 A costratification of $\mathscr{H}$ associated with the stratification (9.6.4) is an assignment of a closed subspace $\mathscr{H}_\tau \subset \mathscr{H}$ to every $\tau \in \mathrm{T}$ such that $\tau \leq \tau'$ implies $\mathscr{H}_\tau \subset \mathscr{H}_{\tau'}$.

Our definition is adapted to the model under consideration. For the general concept, see [325]. Now, the idea is that $\mathscr{H}_\tau$ should consist of wave functions localized at $\mathscr{P}_\tau$. To make this precise, we must relate the elements of $\mathscr{H}$ to functions on $\mathscr{P}$. This will be accomplished in two steps. First, we use the Segal–Bargmann transformation for compact Lie groups developed by Hall [278] to obtain an isomorphism of $\mathscr{H}$ with the Hilbert space

$$\mathscr{H}^{\mathbb{C}} := HL^2\big((G_{\mathbb{C}})^N\big)^G$$

of G-invariant holomorphic functions which are square integrable with respect to a certain measure given below. Here, $G_{\mathbb{C}}$ is the complexification of G. The benefit of this will be that the elements of $\mathscr{H}^{\mathbb{C}}$ are true functions on $(G_{\mathbb{C}})^N$ and not just classes [282]. In a second step, we will relate these elements to functions on $\mathscr{P}$.

By the tensor product structure of the Hilbert spaces involved, for the discussion of the Segal–Bargmann transformation we may restrict attention to one copy of G. Let ρ_t be the heat kernel of the Laplace operator[48] on G with respect to a chosen Ad-invariant inner product on $\mathfrak{g}$. Since ρ_t is invariant under inner automorphisms, according to the Peter–Weyl Theorem, it can be expanded with respect to the characters χ_π of the irreducible representations π of G. The expansion coefficients are given by

$$\rho_t(g) = \sum_{\pi \in \hat{G}} \dim V_\pi \, \mathrm{e}^{-\zeta_\pi t/2} \, \chi_\pi(g) \,, \quad g \in G \,, \tag{9.6.6}$$

where ζ_π is the eigenvalue of the second Casimir operator of the representation π [600, p. 38]. Since every irreducible representation of G extends uniquely to a holomorphic representation of $G_\mathbb{C}$, the characters χ_π may be analytically continued. Thus, replacing each χ_π in (9.6.6) by its analytic continuation, we obtain a candidate for the analytic continuation of ρ_t. It can be shown that the corresponding series is convergent and holomorphic, indeed. Let us denote the analytic continuation of ρ_t so obtained by the same symbol. Now, the Segal–Bargmann transformation of G is defined by

$$C_t : L^2(G) \to \mathrm{Hol}(G_\mathbb{C}) \,, \quad C_t(\varphi)(g) := \int_G \rho_t(g'^{-1}g)\varphi(g')\mathrm{d}g' \,, \tag{9.6.7}$$

where $\mathrm{d}g'$ denotes the Haar measure on G and $\mathrm{Hol}(G_\mathbb{C})$ is the space of holomorphic functions on $G_\mathbb{C}$. For the proof of the following theorem, see [278].

Theorem 9.6.3 (Hall) *For every $t > 0$, there exists a measure ν_t on $G_\mathbb{C}$ such that C_t is a unitary mapping from $L^2(G, \mathrm{d}g)$ onto $HL^2(G_\mathbb{C}, \nu_t)$. This measure is given by*

$$\nu_t(g) = \int_G \mu_t(g'g)\mathrm{d}g' \,, \tag{9.6.8}$$

where μ_t is the heat kernel of the Laplace operator on $G_\mathbb{C}$.[49] ■

Next, recall that we may identify T^*G with $G \times \mathfrak{g}^*$ and, using the inner product, with $G \times \mathfrak{g}$. The latter may be identified with $G_\mathbb{C}$ via the polar decomposition isomorphism

$$\Phi : G \times \mathfrak{g} \to G_\mathbb{C} \quad \Phi(g, Y) := g\,\mathrm{e}^{iY} \,. \tag{9.6.9}$$

According to [279], under this mapping, the measure ν_t takes the form

$$\nu_t(g) = (\pi t)^{-\dim(G)/2} \mathrm{e}^{-|\delta|^2 t} \mathrm{e}^{-\frac{1}{t}|Y|^2} \eta(Y)\, \mathrm{d}g\, \mathrm{d}Y \,, \tag{9.6.10}$$

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[48] That is, the second Casimir operator.

[49] Here, the Lie algebra $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$ is viewed as a real Lie algebra endowed with the natural inner product obtained by identifying the real vector space $\mathfrak{g}_\mathbb{C}$ with the orthogonal direct sum $\mathfrak{g} \oplus \mathfrak{g}$.

where η is the $\mathrm{Ad}(G)$-invariant function on $G_{\mathbb{C}}$ given by

$$\eta(g\mathrm{e}^{\mathrm{i}Y}) := \sqrt{\det\left(\frac{\sin\left(\mathrm{ad}(Y)\right)}{\mathrm{ad}(Y)}\right)} \tag{9.6.11}$$

and δ denotes half the sum of positive roots of $\mathfrak{g}$. The Segal–Bargmann transformation takes a very simple explicit form when applied to representative functions. Recall that a representative function on G is a linear combination of functions of the form

$$G \to \mathbb{C}, \quad g \mapsto \langle \xi, \pi(g)v \rangle ,$$

where π is some irreducible representation of G on a complex vector space V and $v \in V$, $\xi \in V^*$. The characters χ_π are examples of that type with the additional property of being G-invariant. Since every irreducible complex representation of G extends uniquely to a holomorphic representation of $G_{\mathbb{C}}$, every representative function φ has an analytic continuation $\varphi^{\mathbb{C}}$ to $G_{\mathbb{C}}$.

Proposition 9.6.4 (Huebschmann) *Let φ be a representative function on G associated with the irreducible representation of highest weight λ. Then,*

$$C_t(\varphi) = \frac{\varphi^{\mathbb{C}}}{\sqrt{c_{t,\lambda}}}, \quad c_{t,\lambda} = (t\pi)^{\dim(G)/2} \mathrm{e}^{t|\lambda+\delta|^2} .$$

Proof See Theorem 6.5 in [327]. ∎

In terms of the highest weight λ, the eigenvalue ζ_π of the second Casimir operator of the irreducible representation π is given by

$$\zeta_\pi \equiv \zeta_\lambda = |\delta|^2 - |\lambda + \delta|^2 , \tag{9.6.12}$$

see for example [294, Sect. V.1].

Remark 9.6.5 (*Kähler Structure*) Let $\mathsf{J} : \mathrm{T}G_{\mathbb{C}} \to \mathrm{T}G_{\mathbb{C}}$ be the natural complex structure on the manifold $G_{\mathbb{C}}$ defined by multiplication with the imaginary unit i. Under the identification of the Lie algebra of $G_{\mathbb{C}}$ with $\mathfrak{g} \oplus \mathfrak{g}$, it is given by

$$\mathsf{J}(A, B) = (-B, A) , \quad A, B \in \mathfrak{g} .$$

Via the isomorphism Φ, we can transport J to a complex structure on T^*G,

$$\mathsf{J}^{\mathrm{T}^*G} := (\Phi_*)^{-1} \circ \mathsf{J} \circ \Phi_* .$$

One easily calculates (Exercise 9.6.2)

$$\Phi_*(g,Y) = \begin{pmatrix} \cos(\mathrm{ad}(Y)) & \frac{1-\cos(\mathrm{ad}(Y))}{\mathrm{ad}(Y)} \\ -\sin(\mathrm{ad}(Y)) & \frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)} \end{pmatrix} \tag{9.6.13}$$

and, thus,

$$\mathsf{J}^{\mathrm{T}^*G}(g,Y) = \frac{1}{\sin(\mathrm{ad}(Y))} \begin{pmatrix} 1-\cos(\mathrm{ad}(Y)) & 2\frac{\cos(\mathrm{ad}(Y))-1}{\mathrm{ad}(Y)} \\ \mathrm{ad}(Y) & \cos(\mathrm{ad}(Y))-1 \end{pmatrix}. \tag{9.6.14}$$

Combining this with the natural symplectic structure ω on T^*G, cf. Example 8.3.4 in Part I, we obtain a Kähler structure on T^*G. Using (9.6.13), one can check (Exercise 9.6.2) that the canonical 1-form θ of T^*G reads

$$\theta = \mathrm{Im}\big(\overline{\partial}\kappa\big), \quad \kappa(g,Y) = |Y|^2, \tag{9.6.15}$$

where $\overline{\partial}$ is the Dolbeault operator of the complex structure $\mathsf{J}^{\mathrm{T}^*G}$,

$$\overline{\partial}\kappa = (\mathrm{d}\kappa)^{(0,1)} = \Phi^*\left(\big((\Phi^{-1})^*\mathrm{d}\kappa\big)^{(0,1)}\right).$$

Thus, κ is a potential of the Kähler structure. It can be shown that the Hilbert space $HL^2(G_{\mathbb{C}}, \nu_t)$ may also be obtained via half-form Kähler quantization with respect to this Kähler structure [281]. ♦

Remark 9.6.6 (*Holomorphic Peter–Weyl Theorem*) In [327], Huebschmann has proved a Peter–Weyl Theorem for the Hilbert space $HL^2(G_{\mathbb{C}})$. He has called this the Holomorphic Peter–Weyl Theorem. Combining it with the ordinary Peter–Weyl Theorem for $L^2(G)$ and computing the norms of the analytic continuations of representative functions, one finds that the assignment $\varphi \mapsto \varphi^{\mathbb{C}}/\sqrt{c_{t,\lambda}}$ uniquely extends to a unitary isomorphism $L^2(G) \to HL^2(G_{\mathbb{C}})$. In view of Proposition 9.6.4, this provides an alternative proof of Theorem 9.6.3. Conversely, the Holomorphic Peter–Weyl Theorem is a consequence of Theorem 9.6.3 and Proposition 9.6.4. ♦

By applying the Segal–Bargmann transformation to every copy of G, we obtain a unitary isomorphism

$$C_t : \mathscr{H}_N \to HL^2\big((G_{\mathbb{C}})^N\big).$$

Using bi-invariance of the Haar measure on G and the fact that the irreducible characters χ_π are invariant under inner automorphisms of G, one can check that C_t is equivariant with respect to the actions of G on $\mathscr{H}_N$ and $HL^2\big((G_{\mathbb{C}})^N\big)$ induced by diagonal conjugation on G^N and $(G_{\mathbb{C}})^N$, respectively. Hence, C_t restricts to a unitary isomorphism of the subspaces of invariants, denoted by the same letter,

$$C_t : \mathscr{H} \to \mathscr{H}^{\mathbb{C}}.$$

As a result, via the isomorphisms C_t, wave functions are represented by holomorphic functions on $(G_{\mathbb{C}})^N \cong \mathrm{T}^*G^N$. As already noted, the elements of $\mathscr{H}^{\mathbb{C}}$ are true functions on $(G_{\mathbb{C}})^N$ and not just classes. This completes the first step in the process of relating the elements of $\mathscr{H}$ with functions on $\mathscr{P}$.

In the second step, we must now clarify how to interpret elements of $\mathscr{H}^{\mathbb{C}}$ as functions on $\mathscr{P}$. In the case $N = 1$, we observe that $\mathscr{J}(g, Y) = 0$ implies that, up to conjugacy, (g, Y) may be chosen from $T \times \mathfrak{t}$, where $T \subset G$ is a maximal toral subgroup and $\mathfrak{t} \subset \mathfrak{g}$ the corresponding Lie subalgebra. Hence,

$$\mathscr{P} \cong (T \times \mathfrak{t})^W \cong T_{\mathbb{C}}/W\,,$$

where $W = \mathrm{N}_G(T)/T$ is the Weyl group. On the other hand, restriction to $T_{\mathbb{C}}$ defines a unitary isomorphism

$$\mathscr{H}^{\mathbb{C}} = HL^2(G_{\mathbb{C}})^G \cong HL^2(T_{\mathbb{C}})^W$$

with the measure on T being obtained from (9.6.10) by integration over the conjugation orbits in $G_{\mathbb{C}}$, thus yielding an analogue of Weyl's integration formula for $HL^2(G_{\mathbb{C}})^G$.

In the case $N > 1$, the argument is more involved. First, we construct a quotient of $G_{\mathbb{C}}^N$ on which the elements of $\mathscr{H}^{\mathbb{C}}$ define functions. Consider the action of $G_{\mathbb{C}}$ on $(G_{\mathbb{C}})^N$ by diagonal conjugation. For $\mathbf{a} \in G_{\mathbb{C}}^N$, let $G_{\mathbb{C}} \cdot \mathbf{a}$ denote the corresponding orbit. Since $G_{\mathbb{C}}$ is not compact, $G_{\mathbb{C}} \cdot \mathbf{a}$ need not be closed. If a holomorphic function on $(G_{\mathbb{C}})^N$ is invariant under the action of G by diagonal conjugation, then it is invariant under the action of $G_{\mathbb{C}}$ by diagonal conjugation, i.e. it is constant on the orbit $G_{\mathbb{C}} \cdot \mathbf{a}$ for every $\mathbf{a} \in (G_{\mathbb{C}})^N$. Being continuous, it is then constant on the closure $\overline{G_{\mathbb{C}} \cdot \mathbf{a}}$. As a consequence, it takes the same value on two orbits whenever their closures intersect. This motivates the following definition. Two elements $\mathbf{a}, \mathbf{b} \in (G_{\mathbb{C}})^N$ are orbit closure equivalent if there exist $\mathbf{c}_1, \ldots, \mathbf{c}_r \in (G_{\mathbb{C}})^N$ such that

$$\overline{G_{\mathbb{C}} \cdot \mathbf{a}} \cap \overline{G_{\mathbb{C}} \cdot \mathbf{c}_1} \neq \varnothing\,, \quad \overline{G_{\mathbb{C}} \cdot \mathbf{c}_1} \cap \overline{G_{\mathbb{C}} \cdot \mathbf{c}_2} \neq \varnothing\,, \ \ldots\,, \quad \overline{G_{\mathbb{C}} \cdot \mathbf{c}_r} \cap \overline{G_{\mathbb{C}} \cdot \mathbf{b}} \neq \varnothing\,.$$

Clearly, orbit closure equivalence defines an equivalence relation on $(G_{\mathbb{C}})^N$. Let $(G_{\mathbb{C}})^N /\!\!/ G_{\mathbb{C}}$ denote the topological quotient.[50] By construction, the elements of $\mathscr{H}^{\mathbb{C}}$ descend to continuous functions on $(G_{\mathbb{C}})^N /\!\!/ G_{\mathbb{C}}$.

Second, following [291], we recall how the orbit closure quotient $(G_{\mathbb{C}})^N /\!\!/ G_{\mathbb{C}}$ is related to the reduced phase space $\mathscr{P}$. Via the polar decomposition isomorphism Φ, we can view the momentum mapping as a mapping

$$\mathscr{J} : (G_{\mathbb{C}})^N \to \mathfrak{g}^*$$

and we can view $\mathscr{P}$ as the quotient of $\mathscr{J}^{-1}(0) \subset (G_{\mathbb{C}})^N$ by the action of G. Since G is compact, $G_{\mathbb{C}}$ is linear algebraic. Then, $(G_{\mathbb{C}})^N$ is an affine variety in some complex vector space V, the action of G on $(G_{\mathbb{C}})^N$ by diagonal conjugation is the restriction

[50] The notation is motivated by the fact that the quotient provides a categorical quotient of $(G_{\mathbb{C}})^N$ by $G_{\mathbb{C}}$ in the sense of geometric invariant theory [461].

of a representation of G on V to $(G_{\mathbb{C}})^N$ and the momentum mapping is the restriction to $(G_{\mathbb{C}})^N$ of the mapping

$$\tilde{\mathscr{J}} : V \to \mathfrak{g}^* , \quad \tilde{\mathscr{J}}(v)(A) := \frac{1}{2\mathrm{i}} \langle v, Av \rangle ,$$

where $\langle \cdot , \cdot \rangle$ is an appropriate G-invariant scalar product on V and A acts on v by the induced representation of the Lie algebra. In this situation, the level set $\mathscr{J}^{-1}(0)$ has the following properties [361].

1. For all $\mathbf{a} \in (G_{\mathbb{C}})^N$, one has $\overline{G_{\mathbb{C}} \cdot \mathbf{a}} \cap \mathscr{J}^{-1}(0) \neq \varnothing$.
2. For all $\mathbf{a} \in (G_{\mathbb{C}})^N$, the orbit $G_{\mathbb{C}} \cdot \mathbf{a}$ is closed iff $(G_{\mathbb{C}} \cdot \mathbf{a}) \cap \mathscr{J}^{-1}(0) \neq \varnothing$.
3. For all $\mathbf{a} \in \mathscr{J}^{-1}(0)$, one has $(G_{\mathbb{C}} \cdot \mathbf{a}) \cap \mathscr{J}^{-1}(0) = G \cdot \mathbf{a}$.

Properties 2 and 3 ensure that $\mathscr{J}^{-1}(0)$ is what is known in geometric invariant theory as a Kempf–Ness set. Using properties 1–3, one can prove the following [291].

Theorem 9.6.7 *The natural inclusion mapping $\mathscr{J}^{-1}(0) \to (G_{\mathbb{C}})^N$ induces a homeomorphism $\mathscr{P} \to (G_{\mathbb{C}})^N /\!\!/ G_{\mathbb{C}}$.* ∎

As a consequence, via the homeomorphism of Theorem 9.6.7, the elements of $\mathscr{H}^{\mathbb{C}}$ can be interpreted as functions on $\mathscr{P}$. Thus, it makes sense to take the restriction of such an element to a subset of $\mathscr{P}$.

Definition 9.6.8 A wave function $\varphi \in \mathscr{H}^{\mathbb{C}}$ is said to be localized at the stratum $\mathscr{P}_\tau$ if it is orthogonal to all wave functions χ which vanish at $\mathscr{P}_\tau$.

Following this concept of localization, as the closed subspace $\mathscr{H}_\tau^{\mathbb{C}} \subset \mathscr{H}^{\mathbb{C}}$ consisting of the wave functions which are localized at the stratum $\mathscr{P}_\tau$ we obtain the orthogonal complement of the closed subspace

$$\mathscr{V}_\tau^{\mathbb{C}} := \{\chi \in \mathscr{H}^{\mathbb{C}} : \chi_{\restriction \mathscr{P}_\tau} = 0\} .$$

It follows that we have an orthogonal decomposition

$$\mathscr{H}^{\mathbb{C}} = \mathscr{H}_\tau^{\mathbb{C}} \oplus \mathscr{V}_\tau^{\mathbb{C}} .$$

Finally, the inverse of the isomorphism C_t maps the subspaces $\mathscr{H}_\tau^{\mathbb{C}}$ to subspaces $\mathscr{H}_\tau$ of $\mathscr{H}^G$.

Proposition 9.6.9 *The assignment of $\mathscr{H}_\tau$ to $\tau \in \mathscr{T}$ is a costratification of $\mathscr{H}^G$.*

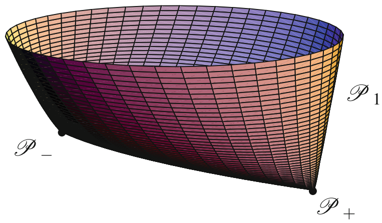
Proof By (9.6.5), if $\tau \leq \tau'$, then $\mathscr{P}_\tau \subset \overline{\mathscr{P}_{\tau'}}$. Since holomorphic functions are continuous, this implies $\mathscr{V}_\tau^{\mathbb{C}} \supset \mathscr{V}_{\tau'}^{\mathbb{C}}$ and thus $\mathscr{H}_\tau^{\mathbb{C}} \subset \mathscr{H}_{\tau'}^{\mathbb{C}}$. ∎

Exercises

9.6.1 Prove the statements 1 and 2 about maximal lattice trees on p. 744.

9.6.2 Prove formulae (9.6.13) and (9.6.15) in Remark 9.6.5.

Fig. 9.1 The reduced phase space $\mathscr{P}$ for $G = \mathrm{SU}(2)$ and $N = 1$



9.7 A Toy Model

In this section, we discuss the example $G = \mathrm{SU}(2)$ and $N = 1$ in some detail, cf. [328]. This corresponds to the toy model of a lattice consisting of one plaquette, because here every tree contains three of the four links. Alternatively, it may be viewed as a Hamiltonian $\mathrm{SU}(2)$-gauge theory on a circle after reduction by the pointed gauge group.

First, we determine the stratification of the reduced phase space

$$\mathscr{P} = \mathscr{J}^{-1}(0)/\mathrm{SU}(2)\,.$$

As already noted, in the case $N = 1$, the condition $\mathscr{J}(g, Y) = 0$ implies that up to conjugacy, g and Y may be chosen from a maximal toral subgroup $T \subset \mathrm{SU}(2)$ and the corresponding Lie subalgebra $\mathfrak{t} \subset \mathfrak{su}(2)$, respectively. Hence, $\mathscr{P} \cong (T \times \mathfrak{t})^W$, where $W = \mathrm{N}_{\mathrm{SU}(2)}(T)/T$ is the Weyl group of $\mathrm{SU}(2)$. If we choose T and $\mathfrak{t}$ to consist of the diagonal matrices in $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$, respectively, W acts on $T \times \mathfrak{t}$ by simultaneous permutation of the entries. The stabilizer of $(x, Y) \in T \times \mathfrak{t}$ is W in case $(x, Y) = (\pm\mathbb{1}, 0)$ and trivial otherwise. Hence, there are two orbit types and three orbit type connected components, given by $\mathscr{P}_+$ consisting of (the class of) $(\mathbb{1}, 0)$, $\mathscr{P}_-$ consisting of (the class of) $(-\mathbb{1}, 0)$, and $\mathscr{P}_1$ consisting of all the rest. Clearly, $\mathscr{P}_1$ is the principal stratum. Hence, in this simple example, there are only two secondary strata and these strata consist of isolated points. The stratified space $\mathscr{P}$ is depicted in Fig. 9.1. This space is known as the canoe.

Next, we choose bases in the relevant Hilbert spaces and determine the Segal–Bargmann transformation. The Schrödinger Hilbert space is $\mathscr{H} = L^2(G)^G$, the subspace of $L^2(G)$ consisting of the functions which are invariant under inner automorphisms. The holomorphic Hilbert space is $\mathscr{H}^{\mathbb{C}} = HL^2(G_{\mathbb{C}})^G$, the subspace of $HL^2(G_{\mathbb{C}})$ consisting of the functions which are invariant under conjugation by elements of G. Let χ_n denote the character of the irreducible representation of G of spin $n/2$. Then, the analytic continuation $\chi_n^{\mathbb{C}}$ is the character of the corresponding representation of $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. To find explicit formulae, recall that the representation $\mathrm{d}\pi_n$ of spin $n/2$ of $\mathfrak{su}(2)$ reads

$$\mathrm{d}\pi_n\big(\mathrm{diag}(\mathrm{i}, -\mathrm{i})\big) = \mathrm{diag}(\mathrm{i}n, \mathrm{i}(n-2), \ldots, \mathrm{i}(-n+2), -\mathrm{i}n)\,.$$

It follows that

$$\pi_n\big(\operatorname{diag}(e^{ix}, e^{-ix})\big) = \operatorname{diag}\left(e^{inx}, e^{i(n-2)x}, \dots, e^{-i(n-2)x}, e^{-inx}\right).$$

Hence, the restrictions of χ_n to T and of $\chi_n^{\mathbb{C}}$ to $T \times \mathfrak{t} \cong T_{\mathbb{C}}$ are given by, respectively,

$$\chi_n\big(\operatorname{diag}(e^{ix})\big) = \frac{\sin\big((n+1)x\big)}{\sin(x)}, \quad x \in \mathbb{R}, \tag{9.7.1}$$

and

$$\chi_n^{\mathbb{C}}\big(\operatorname{diag}(z, z^{-1})\big) = z^n + z^{n-2} + \cdots + z^{-n}, \quad z \in \mathbb{C}^*. \tag{9.7.2}$$

By the Peter–Weyl Theorem, the χ_n form an orthonormal basis in $\mathscr{H}$. By Theorem 9.6.3 and Proposition 9.6.4, then the $\chi_n^{\mathbb{C}}$ form an orthogonal basis in $\mathscr{H}^{\mathbb{C}}$.

Next, we determine the Segal–Bargmann transform of χ_n and the eigenvalues of the Laplacian. Since every invariant scalar product on $\mathfrak{su}(2)$ is proportional to the (negative definite) trace form, we have

$$|Y|^2 = -\frac{1}{2\beta^2}\operatorname{tr}(Y^2), \quad Y \in \mathfrak{su}(2),$$

for some positive number β.

Lemma 9.7.1 *The Segal–Bargmann transformation reads*[51]

$$C_\hbar(\chi_n) = (\hbar\pi)^{-3/4} e^{-\hbar\beta^2(n+1)^2/2}\chi_n^{\mathbb{C}} \tag{9.7.3}$$

and the eigenvalues of the second Casimir operator of the irreducible representation with spin $n/2$ are given by

$$\zeta_n = -\beta^2 n(n+2). \tag{9.7.4}$$

Proof According to Proposition 9.6.4,

$$C_\hbar(\chi_n) = \frac{\chi_n^{\mathbb{C}}}{\sqrt{c_{\hbar,\lambda}}}, \quad c_{\hbar,\lambda} = (\hbar\pi)^{n/2} e^{\hbar|\lambda+\delta|^2}.$$

To determine the factors $c_{\hbar,n}$, recall that the root system of $\mathfrak{su}(2)$ consists of the two roots α and $-\alpha$, given by $\alpha\big(Y\big) = 2y$, where $Y = \operatorname{diag}(iy, -iy) \in \mathfrak{t}$, $y \in \mathbb{R}$. Hence, $\delta = \frac{1}{2}\alpha$. The highest weight of the irreducible representation of spin $n/2$ is $\lambda_n = \frac{n}{2}\alpha$. Relative to the invariant scalar product on $\mathfrak{t}^*$ induced by that on $\mathfrak{t}$, the two roots α and $-\alpha$ have norm $|\alpha|^2 = 4\beta^2$. Hence $|\delta|^2 = \beta^2$ and $|\lambda_n + \delta|^2 = \beta^2(n+1)^2$. As a result,

$$c_{\hbar,n} = (\hbar\pi)^{3/2} e^{\hbar\beta^2(n+1)^2}.$$

[51] We now write $\hbar$ instead of t.

The formula for the eigenvalues ζ_n follows from (9.6.12). ■

Now, we are in a position to determine the subspaces $\mathscr{H}_\pm \subset \mathscr{H}$ associated with the secondary strata $\mathscr{P}_\pm$. By definition, under the Segal–Bargmann transformation they are mapped to the orthogonal complements of the subspaces $\mathscr{V}_\pm^{\mathbb{C}}$ of functions vanishing on $\mathscr{P}_\pm$.

Lemma 9.7.2 *The subspaces $\mathscr{V}_+^{\mathbb{C}}$ and $\mathscr{V}_-^{\mathbb{C}}$ are spanned by, respectively,*

$$\chi_n^{\mathbb{C}} - (n+1)\chi_0^{\mathbb{C}}\,, \quad n = 1, 2, 3, \dots\,, \tag{9.7.5}$$

$$\chi_n^{\mathbb{C}} + (-1)^n \frac{n+1}{2} \chi_1^{\mathbb{C}}\,, \quad n = 0, 2, 3, \dots\,. \tag{9.7.6}$$

Proof Under the identification $\mathscr{P} = \mathscr{J}^{-1}(0)/\mathrm{SU}(2) = (T \times \mathfrak{t})/W = T_{\mathbb{C}}/W$, the strata $\mathscr{P}_\pm$ correspond to the isolated points $\pm\mathbb{1}$ in $T_{\mathbb{C}}$. Hence, $\mathscr{V}_\pm^{\mathbb{C}} \subset \mathscr{H}^{\mathbb{C}}$ is defined by the condition $\psi(\pm\mathbb{1}) = 0$. By (9.7.2), we have

$$\chi_n^{\mathbb{C}}(\pm\mathbb{1}) = (\pm 1)^n (n+1)\,.$$

Hence, all the functions given in (9.7.5) belong to $\mathscr{V}_+^{\mathbb{C}}$ and all the functions given in (9.7.6) belong to $\mathscr{V}_-^{\mathbb{C}}$. Linear independence is obvious. That the system (9.7.5) spans $\mathscr{V}_+^{\mathbb{C}}$ follows by observing that, together with $\chi_0^{\mathbb{C}}$, it spans $\mathscr{H}^{\mathbb{C}}$. Similarly, the system (9.7.6) spans $\mathscr{V}_-^{\mathbb{C}}$, because together with $\chi_1^{\mathbb{C}}$, it spans $\mathscr{H}^{\mathbb{C}}$ as well. ■

To determine $\mathscr{H}_\pm$, we turn back to $\mathscr{H}$ and take the orthogonal complement there. Up to a factor, under the inverse of the Segal–Bargmann transformation, the basis elements (9.7.5) and (9.7.6) are mapped, respectively, to

$$\mathrm{e}^{\hbar\beta^2(n+1)^2/2}\chi_n - (n+1)\mathrm{e}^{\hbar\beta^2/2}\chi_0\,, \qquad n = 1, 2, 3, \dots\,, \tag{9.7.7}$$

$$\mathrm{e}^{\hbar\beta^2(n+1)^2/2}\chi_n - \frac{n+1}{2}\mathrm{e}^{2\hbar\beta^2}\chi_1\,, \qquad n = 0, 2, 3, \dots\,. \tag{9.7.8}$$

Proposition 9.7.3 *The subspaces $\mathscr{H}_\pm$ have dimension* 1. *They are spanned by the normalized vectors*

$$\varphi_\pm := \frac{1}{N}\sum\nolimits_{n=0}^{\infty}(\pm 1)^n\,(n+1)\,\mathrm{e}^{-\hbar\beta^2\,(n+1)^2/2}\,\chi_n\,, \quad N^2 = \sum_{n=1}^{\infty} n^2\,\mathrm{e}^{-\hbar\beta^2\,n^2}\,. \tag{9.7.9}$$

Proof Clearly, the series on the right hand side converges and its limit is normalized. Both the vector φ_+ together with the system (9.7.7) and the vector φ_- together with the system (9.7.8) span $L^2(G)^G$. Finally, a straightforward computation shows that φ_+ is orthogonal to all the vectors in (9.7.7) and φ_- is orthogonal to all the vectors in (9.7.8). ■

Remark 9.7.4

1. The transition probability $|\langle\varphi_+, \varphi_-\rangle|^2$ between the states defined by φ_+ and φ_- has the physical interpretation of a tunneling probability between the strata $\mathscr{P}_+$ and $\mathscr{P}_-$. It can be expressed in terms of the θ-constant $\theta_3(Q) = \sum_{k=-\infty}^{\infty} Q^{k^2}$ as

$$\langle\varphi_+, \varphi_-\rangle = -\frac{\theta_3'\big(-\mathrm{e}^{-\hbar\beta^2}\big)}{\theta_3'\big(\mathrm{e}^{-\hbar\beta^2}\big)}\,.$$

Figure 9.2 shows $|\langle\varphi_+, \varphi_-\rangle|^2$ as a function of the combined constant $\hbar\beta^2$. As one would expect, the tunneling probability vanishes in the semiclassical limit $\hbar \to 0$.

2. According to (9.6.6) and (9.7.4), the heat kernel on $G = \mathrm{SU}(2)$ is given by

$$\rho_t = \sum_{n=0}^{\infty}(n+1)\mathrm{e}^{-t\beta^2 n(n+2)/2}\chi_n\,.$$

Comparison with (9.7.9) shows that $\rho_\hbar = \mathrm{e}^{\hbar\beta^2} N\varphi_+$. More generally, using the analytic continuation of ρ_t to $G_\mathbb{C}$, for every $g \in G_\mathbb{C}$, we can define a function $\varphi_g^{(t)}$ on $G_\mathbb{C} = \mathrm{SL}(2,\mathbb{C})$ by $\varphi_g^{(t)}(h) := \overline{\rho_t(gh^{-1})}$, for any $h \in G_\mathbb{C}$. Then,

$$C_\hbar(\varphi_\pm) = \frac{\mathrm{e}^{-\hbar\beta^2}}{N}\varphi_{\pm\mathbb{1}}^{(\hbar)}\,.$$

According to [278], the functions $\varphi_g^{(\hbar)}$ admit an interpretation as coherent states on $G_\mathbb{C}$. Within the bounds imposed by the uncertainty relation, they are optimally localized at the phase space point g. Thus, the states spanning $\mathscr{H}_\pm$ are optimally localized at the points forming the corresponding strata.

Expressing the transition probability $|\langle\varphi_+, \varphi_-\rangle|^2$ in terms of the coherent states $\varphi_\mathbb{1}^{(\hbar)}$ and $\varphi_{-\mathbb{1}}^{(\hbar)}$, we obtain the identity

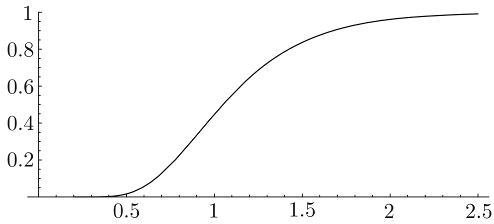
$$|\langle\varphi_+, \varphi_-\rangle|^2 = \frac{\big|\big\langle\varphi_\mathbb{1}^{(\hbar)}, \varphi_{-\mathbb{1}}^{(\hbar)}\big\rangle\big|^2}{\|\varphi_\mathbb{1}^{(\hbar)}\|^2\,\|\varphi_{-\mathbb{1}}^{(\hbar)}\|^2}\,.$$

The quantity on the right hand side is referred to as the overlap of the coherent states $\varphi_\mathbb{1}^{(\hbar)}$ and $\varphi_{-\mathbb{1}}^{(\hbar)}$. It was studied for arbitrary pairs of group elements in more general situations in a series of papers by Thiemann and collaborators [619]. ♦

Next, we discuss the eigenvalue problem of the Hamiltonian (9.5.31) for the lattice at hand and determine the transition probabilities between the energy eigenstates and the states $\psi_\pm$ associated with the strata. If for simplicity we put $a = 1$, the Hamiltonian reads

$$H = -\frac{\hbar^2\kappa^2}{2}\Delta - \frac{2}{\kappa^2}\chi_1\,, \tag{9.7.10}$$

Fig. 9.2 Tunneling probability $|\langle\varphi_+, \varphi_-\rangle|^2$ as a function of $\hbar\beta^2$



where Δ is the Laplacian on SU(2) and κ denotes the coupling constant. A core is given by the subspace $C^\infty(G)^G$.

For $\kappa \to \infty$, that is, in the strong coupling limit, the eigenvalue problem reduces to that of the Laplacian. Hence, in this case, according to (9.7.4), H has the non-degenerate eigenvalues

$$E_n = \frac{\hbar^2\kappa^2\beta^2}{2}\, n(n+2)$$

corresponding to the eigenvectors χ_n. To discuss the eigenvalue problem for finite κ, we pass from $L^2(G)^G$ to $L^2[0,\pi]$ using the unitary isomorphism (Exercise 9.7.1)

$$\psi \mapsto \tilde{\psi} := \sqrt{\frac{2}{\pi}}\, \sin(x)\, \psi\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})\big)\,. \tag{9.7.11}$$

According to (9.7.1), the characters are mapped to the functions

$$\tilde{\chi}_n(x) = \sqrt{\frac{2}{\pi}}\, \sin\big((n+1)x\big)\,.$$

The subspace $\{\tilde{\psi} : \psi \in C^\infty(G)^G\} \subset L^2[0,\pi]$ is a core for the transformed Hamiltonian $\tilde{H}$ and on this core, $\tilde{H}$ is given by (Exercise 9.7.2)

$$\tilde{H} = -\frac{\hbar^2\kappa^2\beta^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1\right) - \frac{4}{\kappa^2}\cos(x)\,. \tag{9.7.12}$$

One can check that $\tilde{H}$ is still symmetric on the larger subspace

$$\{\tilde{\psi} \in L^2[0,\pi] : \tilde{\psi}(0) = \tilde{\psi}(\pi) = 0\}\,, \tag{9.7.13}$$

so we may take the latter as a core (Exercise 9.7.3).

Now, consider the stationary Schrödinger equation $\tilde{H}\tilde{\psi} = E\tilde{\psi}$. Dividing by the factor $-\hbar^2\kappa^2\beta^2/2$ and substituting $y = (x-\pi)/2$, we obtain the Mathieu equation

$$f''(y) + (a - 2q\cos(2y))f(y) = 0 \tag{9.7.14}$$

with the parameters

$$a = \frac{8E}{\hbar^2\kappa^2\beta^2} + 4\,, \quad q = \frac{16}{\hbar^2\beta^2\kappa^4}\,, \tag{9.7.15}$$

and with f being a Whitney smooth function on the interval $[-\pi/2, 0]$ satisfying the boundary conditions

$$f(-\pi/2) = f(0) = 0\,. \tag{9.7.16}$$

For the theory of the Mathieu equation and its solutions, the Mathieu functions, we refer to [22, 436, 440].[52] All we need here is that for certain characteristic values of the parameter a, depending analytically on q and usually being denoted by $b_{2n+2}(q)$, $n = 0, 1, 2, \ldots$, solutions satisfying (9.7.16) exist. Given $a = b_{2n+2}(q)$, the corresponding solution is unique up to a complex factor and can be chosen to be real-valued. It is usually denoted by $\mathrm{se}_{2n+2}(y; q)$, where 'se' stands for 'sine elliptic'. For given $q \geq 0$, define functions $\tilde{\chi}_n^{(q)} \in L^2[0, \pi]$ by

$$\tilde{\chi}_n^{(q)}(x) = (-1)^{n+1}\sqrt{2/\pi}\;\mathrm{se}_{2n+2}\big((x-\pi)/2; q\big)\,, \quad n = 0, 1, 2, \ldots\,.$$

Since $\mathrm{se}_{2n+2}(y; 0) = \sin((2n+2)y)$, the factor $(-1)^{n+1}$ ensures that $\chi_n^{(0)} = \chi_n$. Using the results of Sects. 20.2 and 20.5 in [4], we obtain the following.

Proposition 9.7.5 *The functions $\tilde{\chi}_n^{(q)}$, $n = 0, 1, 2, \ldots$, form an orthonormal eigenbasis of $\tilde{H}$ with the non-degenerate eigenvalues $E_n = \frac{\hbar^2\kappa^2\beta^2}{2}\left(\frac{b_{2n+2}(q)}{4} - 1\right)$.* ∎

Finally, we discuss the transition probabilities

$$P_n^{\pm} := \left|\langle\chi_n^{(q)}|\varphi_{\pm}\rangle\right|^2$$

between the energy eigenstates $\chi_n^{(q)}$ and the states $\varphi_{\pm}$ spanning $\mathscr{H}_{\pm}$. Using the Fourier decomposition of se_{2n+2}, we obtain

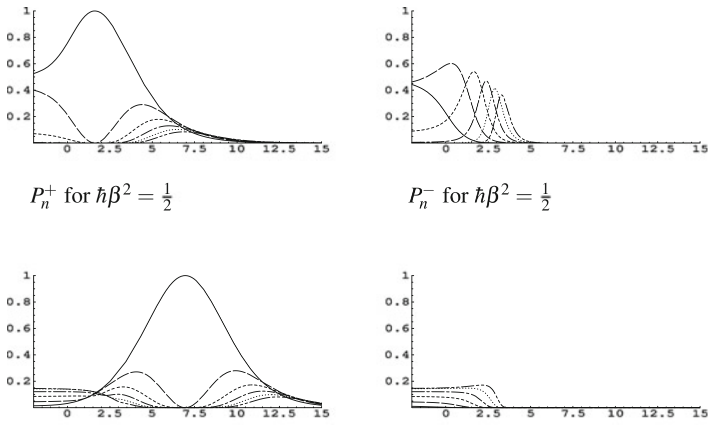
$$\langle\chi_n^{(q)}|\varphi_{\pm}\rangle = \frac{(-1)^n}{N}\sum_{k=0}^{\infty}(\mp 1)^k\,(k+1)\,\mathrm{e}^{-\hbar\beta^2(k+1)^2/2}\,B_{2k+2}^{2n+2}(q)\,,$$

where $B_{2k+2}^{2n+2}(q)$ are the Fourier coefficients, see [4, Sect. 20.2].

The transition probabilities $P_n^{\pm}$ depend on the parameters $\hbar$, β^2 and κ only via the combinations $\hbar\beta^2$ and $q = 16/(\hbar^2\beta^2\kappa^4)$. For illustration, they are displayed for $n = 0, \ldots, 5$ in Fig. 9.3 as functions of q for two specific values of $\hbar\beta^2$, thus treating $\hbar^2\beta^2\kappa^4$ and $\hbar\beta^2$ as independent parameters.[53]

We observe that the transition probability P_0^+ between φ_+ and the ground state $\chi_0^{(q)}$ has a dominant peak moving to smaller values of κ as $\hbar\beta^2$ decreases. In other

[52] Mathieu functions have already appeared in Example 9.8.9 of Part 1. Note that the Mathieu equation also arises as the Schrödinger equation of the quantum planar pendulum, yet with different boundary conditions [137], see also [13, 52, 519]. For a discussion of the relation between our system



P_n^+ for $\hbar\beta^2 = \frac{1}{2}$ $\quad$ P_n^- for $\hbar\beta^2 = \frac{1}{2}$

P_n^+ for $\hbar\beta^2 = \frac{1}{32}$ $\quad$ P_n^- for $\hbar\beta^2 = \frac{1}{32}$

Fig. 9.3 Expectation values $P_n^{\pm}$ for $n = 0$ (*continuous line*), 1 (*long dash*), 2 (*short dash*), 3 (*long-short dash*), 4 (*dotted line*) and 5 (*long-short-short dash*), plotted over $\log(q/16) = -2\log(\hbar\beta\kappa^2)$

words, for a certain value of the coupling constant, depending on $\hbar\beta^2$, the state φ_+ spanning $\mathscr{H}_+$ is very close to the ground state. The two states do not coincide completely though, because the Fourier coefficients of φ_+, given by (9.7.9), do not satisfy the recurrence relations for $B_{2k+2}^{2n+2}(q)$, given in [4, Sect. 20.2], for none of the values of q. This observation should be compared with an earlier result of Emmrich and Römer [185]. These authors considered Schrödinger quantum mechanics on a double cone and showed that the vacuum state concentrates around the singularity. Thus, the nongeneric strata seemingly carry information about the spectral measure of the Hamiltonian of a gauge theory.

Remark 9.7.6

1. One can derive explicit approximate formulae for $P_n^{\pm}$ in the strong and weak coupling limit, cf. [328].
2. Let us discuss the extension problem which arises by quantization on the principal stratum, see also [542] for further details. While naive quantization after reduction on all of T^*G fails, because of the presence of singularities in $\mathscr{P}$, it can be carried out on the part of T^*G where regular cotangent bundle reduction applies, that is, on the submanifold made up by the cotangent bundle of the unreduced principal stratum $G\backslash\{\pm\mathbb{1}\}$. For this submanifold, symplectic reduction leads to the cotangent bundle of the quotient manifold. In the parameterization of the quotient of G by inner automorphisms by the closed interval $[0, \pi]$, this quotient manifold corresponds to the open interval $]0, \pi[$. Since the parameterization is an

and the quantum planar pendulum, both on classical and quantum level, we refer to Remarks 2.2, 5.2 and 5.4 in [328].

[53]The plots were generated by numerical integration using the Mathematica function `MathieuS`.

isometry when scaled via β, canonical quantization of the kinetic energy yields the symmetric operator

$$-\frac{\hbar^2\kappa^2\beta^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{9.7.17}$$

on the Hilbert space $L^2[0,\pi]$ having as domain the compactly supported smooth functions on the open interval $]0,\pi[$. To arrive at a quantum theory of the entire system, one has to determine the self-adjoint extensions of the operator (9.7.17). This is the problem studied in [185] where the classical configuration space is a double cone. When the classical configuration space arises by reduction, as in the system under consideration, the extension problem can be solved by reduction after quantization, because this determines the kinetic energy operator uniquely. This was already observed in [680] in the context of quantization by Rieffel induction. Indeed, in our situation, up to the shift by a constant which can be obtained by the metaplectic correction, the first term in (9.7.12) defined on the core (9.7.13) is a self-adjoint extension of (9.7.17). In fact, this is the Friedrichs extension. ♦

Exercises

9.7.1 Use Weyl's Integration Formula to prove that (9.7.11) defines a unitary isomorphism.

9.7.2 Derive formula (9.7.12). *Hint.* Apply both sides of (9.7.12) to $\tilde{\chi}_n$ and use (9.7.4). Alternatively, one may use the formula for the radial part of the Laplacian on a compact group [294, Sect. II.3.4].

9.7.3 Show that (9.7.13) defines a core for $\tilde{H}$.

Appendix A
Field Restriction and Field Extension

Consider right $\mathbb{K}$-vector spaces with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We use the obvious subfield embeddings $\mathbb{R} \subset \mathbb{C}$ and $\mathbb{R} \subset \mathbb{H}$ as well as the embedding

$$\mathbb{C} \to \mathbb{H}, \quad x + \mathrm{i}y \mapsto x\mathbf{1} + y\mathbf{i}.$$

First,we discuss field restriction. For a $\mathbb{K}$-vector space V and a subfield $\mathbb{L} \subset \mathbb{K}$, we let $V_{\mathbb{L}}$ denote the $\mathbb{L}$-vector space obtained from V by field restriction, that is, by restricting multiplication by scalars to the subfield $\mathbb{L}$. The same notation will be used for vector bundles. One has

$$\dim(V_{\mathbb{L}}) = \dim(V)\dim_{\mathbb{L}}(\mathbb{K})$$

and a similar relation between the ranks in the case of vector bundles. Note that in the case $\mathbb{K} = \mathbb{H}$ and $\mathbb{L} = \mathbb{C}$, scalars keep multiplying from the right. That is, scalar multiplication by $z \in \mathbb{C}$ of an element $v\mathbf{q}$, where $v \in V$ and $\mathbf{q} \in \mathbb{H}$, yields $v\mathbf{q}z$.

Clearly, $\mathbb{C}^n_{\mathbb{R}} \cong \mathbb{R}^{2n}$ and $\mathbb{H}^n_{\mathbb{R}} \cong \mathbb{R}^{4n}$ as real vector spaces, and $\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{C}^{2n}$ as complex vector spaces. Throughout the book, the following concrete isomorphisms are used: $\mathbb{R}^{2n} \to \mathbb{C}^n_{\mathbb{R}}$ given by

$$(x_1, \dots, x_{2n}) \mapsto (x_1 + x_2\mathrm{i}, \dots, x_{2n-1} + x_{2n}\mathrm{i}), \tag{A.1}$$

$\mathbb{R}^{4n} \to \mathbb{H}^n_{\mathbb{R}}$ given by sending $(x_1, \dots, x_{4n})$ to

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, \ \dots, \ x_{4n-3} + x_{4n-2}\mathbf{i} + x_{4n-1}\mathbf{j} + x_{4n}\mathbf{k}), \tag{A.2}$$

$\mathbb{C}^{2n} \to \mathbb{H}^n_{\mathbb{C}}$ given by

$$(z_1, \dots, z_{2n}) \mapsto \big(z_1 + \mathbf{j}z_2, \dots, z_{2n-1} + \mathbf{j}z_{2n}\big). \tag{A.3}$$

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We point out that by further field restriction to $\mathbb{R}$, the isomorphism (A.3) yields a real vector space isomorphism $\mathbb{C}^{2n}_{\mathbb{R}} \to \mathbb{H}^n_{\mathbb{R}}$. Composition of the latter with the isomorphism $\mathbb{R}^{4n} \to \mathbb{C}^{2n}_{\mathbb{R}}$ given by (A.1) yields the isomorphism $\mathbb{R}^{4n} \to \mathbb{H}^n_{\mathbb{R}}$ given by sending $(x_1, \dots, x_{4n})$ to

$$(x_1 + x_2\mathbf{i} + x_3\mathbf{j} - x_4\mathbf{k}\,, \ \dots\ , \ x_{4n-3} + x_{4n-2}\mathbf{i} + x_{4n-1}\mathbf{j} - x_{4n}\mathbf{k})\,. \tag{A.4}$$

In particular, this isomorphism does not coincide with the one defined by (A.2). The isomorphisms (A.1)–(A.3) induce subalgebra embeddings

$$\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{2n}(\mathbb{R})\,, \quad \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{4n}(\mathbb{R})\,, \quad \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{2n}(\mathbb{C})\,. \tag{A.5}$$

The latter are obtained by replacing the entries by blocks according to, respectively,

$$A_{ij} + B_{ij}\mathrm{i} \ \mapsto\ \begin{bmatrix} A_{ij} & -B_{ij} \\ B_{ij} & A_{ij} \end{bmatrix} \tag{A.6}$$

$$A_{ij} + B_{ij}\mathbf{i} + C_{ij}\mathbf{j} + D_{ij}\mathbf{k} \ \mapsto\ \begin{bmatrix} A_{ij} & -B_{ij} & -C_{ij} & -D_{ij} \\ B_{ij} & A_{ij} & -D_{ij} & C_{ij} \\ C_{ij} & D_{ij} & A_{ij} & -B_{ij} \\ D_{ij} & -C_{ij} & B_{ij} & A_{ij} \end{bmatrix} \tag{A.7}$$

$$Z_{ij} + \mathbf{j}W_{ij} \ \mapsto\ \begin{bmatrix} Z_{ij} & -\overline{W}_{ij} \\ W_{ij} & \overline{Z}_{ij} \end{bmatrix}, \tag{A.8}$$

where $A_{ij}, B_{ij}, C_{ij}, D_{ij} \in \mathbb{R}$ and $Z_{ij}, W_{ij} \in \mathbb{C}$, and where $\overline{Z_{ij}}$ denotes the complex conjugate number. These subalgebra embeddings restrict to Lie subgroup embeddings

$$\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})\,, \quad \mathrm{GL}(n, \mathbb{H}) \to \mathrm{GL}(4n, \mathbb{R})\,, \quad \mathrm{GL}(n, \mathbb{H}) \to \mathrm{GL}(2n, \mathbb{C})\,.$$

Since $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{H})$ are connected, their images are contained in the identity component of $\mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{GL}(4n, \mathbb{R})$, respectively.

Next, we discuss field restriction of scalar products and Hermitean fibre metrics. If h is a scalar product on a complex or a quaternionic vector space V, then

$$\mathsf{h}_{\mathbb{R}}(v, w) := \mathrm{Re}\big(\mathsf{h}(v, w)\big)\,, \quad v, w \in V\,, \tag{A.9}$$

defines a scalar product on the realification $V_{\mathbb{R}}$. Similarly, if h is a scalar product on a quaternionic vector space V, then

$$\mathsf{h}_{\mathbb{C}}(v, w) := \mathrm{Co}\big(\mathsf{h}(v, w)\big)\,, \quad v, w \in V\,, \tag{A.10}$$

defines a scalar product on the complexification $V_{\mathbb{C}}$. Here,

$$\mathrm{Co}(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) := x_1 + \mathrm{i}x_2\,, \quad x_1, \dots, x_4 \in \mathbb{R}\,,$$

is the complex part of a quaternion. One can check that the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n_{\mathbb{R}}$ defined by (A.1) is isometric with respect to the standard scalar product on $\mathbb{R}^{2n}$ and the scalar product $\mathsf{h}_{\mathbb{R}}$ on $\mathbb{C}^n_{\mathbb{R}}$ obtained from the standard scalar product h on $\mathbb{C}^n$. An analogous statement holds for the isomorphisms $\mathbb{R}^{4n} \cong \mathbb{H}^n_{\mathbb{R}}$ defined by (A.2) and $\mathbb{C}^{2n} \cong \mathbb{H}^n_{\mathbb{C}}$ defined by (A.3). It follows that the corresponding subalgebra embeddings (A.5) restrict, respectively, to Lie subgroup embeddings

$$j_n^{\mathrm{U,O}} : \mathrm{U}(n) \to \mathrm{O}(2n)\,, \quad j_n^{\mathrm{Sp,O}} : \mathrm{Sp}(n) \to \mathrm{O}(4n)\,, \quad j_n^{\mathrm{Sp,U}} : \mathrm{Sp}(n) \to \mathrm{U}(2n)\,. \tag{A.11}$$

Now, consider vector bundles. Clearly, if h is a Hermitean fibre metric on a complex or quaternionic vector bundle E, then (A.9) defines fibrewise a Euclidean structure $\mathsf{h}_{\mathbb{R}}$ on the realification $E_{\mathbb{R}}$. If h is a Hermitean fibre metric on a quaternionic vector bundle E, then (A.10) defines fibrewise a Hermitean fibre metric $\mathsf{h}_{\mathbb{C}}$ on the complexification $E_{\mathbb{C}}$.

Lemma A.1 *Let* $(\mathbb{K}, \mathbb{L}) = (\mathbb{C}, \mathbb{R})$, $(\mathbb{H}, \mathbb{R})$ *or* $(\mathbb{H}, \mathbb{C})$ *and let* E *be a* $\mathbb{K}$*-vector bundle of rank* n *over a topological space* B *endowed with a fibre metric* h.

1. *For* $b \in B$, *if* $u = (u_1, \ldots, u_n)$ *is an* h*-orthonormal frame in the fibre* E_b, *then*

$$\tilde{u} = \begin{cases} (u_1, \mathrm{i}u_1, \ldots, u_n, \mathrm{i}u_n) & (\mathbb{K}, \mathbb{L}) = (\mathbb{C}, \mathbb{R}) \\ (u_1, u_1\mathbf{i}, u_1\mathbf{j}, u_1\mathbf{k}, \ldots, u_n, u_n\mathbf{i}, u_n\mathbf{j}, u_n\mathbf{k}) & (\mathbb{K}, \mathbb{L}) = (\mathbb{H}, \mathbb{R}) \\ (u_1, u_1\mathbf{j}, \ldots, u_n, u_n\mathbf{j}) & (\mathbb{K}, \mathbb{L}) = (\mathbb{H}, \mathbb{C}) \end{cases}$$

   is an $h_{\mathbb{L}}$*-orthonormal frame in the fibre* $(E_{\mathbb{L}})_b$.
2. *The mapping*

$$O(E) \to O(E_{\mathbb{L}})\,, \quad u \mapsto \tilde{u}\,,$$

   is a vertical morphism of principal bundles with Lie group homomorphism given by (A.11).

Proof Point 1 is proved by direct inspection. Point 2 follows from the equation

$$(u \cdot a)^{\tilde{}} = \tilde{u} \cdot j(a)$$

for all $a \in \mathrm{U}(n)$ in case $\mathbb{K} = \mathbb{C}$ or $a \in \mathrm{Sp}(n)$ in case $\mathbb{K} = \mathbb{H}$. Here, j denotes the corresponding embedding in (A.11). ■

Now, we turn to the discussion of field extension. Let $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{C})$, $(\mathbb{R}, \mathbb{H})$ or $(\mathbb{C}, \mathbb{H})$ and let V be an $\mathbb{L}$-vector space. Since $\mathbb{L}$ is a subfield of $\mathbb{K}$, we can view $\mathbb{K}$ as a vector space over $\mathbb{L}$ with scalars acting by multiplication from the left. Since, in addition, $\mathbb{L}$ is commutative, we can form the tensor product of $\mathbb{L}$-vector spaces

$$V_{\mathbb{K}} := V \otimes_{\mathbb{L}} \mathbb{K}\,.$$

For every $k' \in \mathbb{K}$, the mapping $V \times \mathbb{K} \to V_{\mathbb{K}}$ defined by $(v, k) \mapsto v \otimes (kk')$ is $\mathbb{L}$-bilinear and hence induces an $\mathbb{L}$-linear endomorphism of $V_{\mathbb{K}}$ which maps $v \otimes k$ to $v \otimes (kk')$. Thus, letting k' run through $\mathbb{K}$, we obtain a mapping

$$V_{\mathbb{K}} \times \mathbb{K} \to V_{\mathbb{K}}, \quad (v \otimes k, k') \mapsto v \otimes (kk').$$

This mapping, taken as the multiplication by scalars on $V_{\mathbb{K}}$, combines with the additive structure of $V_{\mathbb{K}}$ to a $\mathbb{K}$-linear structure on $V_{\mathbb{K}}$, thus turning $V_{\mathbb{K}}$ into a $\mathbb{K}$-vector space. This vector space is called the complexification of V in case $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{C})$ and the quaternionification of V in case $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{H})$ or $(\mathbb{C}, \mathbb{H})$. Multiplication by scalars will be written in the form

$$(v \otimes k)k' := v \otimes (kk'), \quad k, k' \in \mathbb{K}.$$

Clearly, if $\{\mathbf{e}_i\}$ is a basis in V, then $\{\mathbf{e}_i \otimes 1\}$ is a basis in $V_{\mathbb{K}}$. Therefore, $V_{\mathbb{K}}$ has the same dimension as V. Since $(vl) \otimes 1 = v \otimes l = (v \otimes 1)l$ for all $l \in \mathbb{L}$ and $v \in V$, the mapping

$$V \to V_{\mathbb{K}}, \quad v \mapsto v \otimes 1,$$

is $\mathbb{L}$-linear and embeds V into $V_{\mathbb{K}}$ as a linear subspace over $\mathbb{L}$. In the case $V = \mathbb{L}^n$, the vector space $V_{\mathbb{K}}$ may be identified with $\mathbb{K}^n$ via the natural isomorphism $\mathbb{L}^n_{\mathbb{K}} \to \mathbb{K}^n$ defined by $(l_1, \ldots, l_n) \otimes k \mapsto (l_1 k, \ldots, l_n k)$.

The concept of field extension carries over to vector bundles as follows. Given an $\mathbb{L}$-vector bundle E over a topological space B, by viewing $\mathbb{K}$ as above as a left $\mathbb{L}$-vector space, we can take the tensor product of $\mathbb{L}$-vector bundles

$$E_{\mathbb{K}} := E \otimes_{\mathbb{L}} (B \times \mathbb{K})$$

and endow each fibre $E_b \otimes_{\mathbb{L}} \mathbb{K}$ with the $\mathbb{K}$-linear structure of $(E_b)_{\mathbb{K}}$. Then, for every local frame $e_1, \ldots, e_n$ in E, the local sections $e_1 \otimes 1, \ldots, e_n \otimes 1$ of $E_{\mathbb{K}}$ form a local frame in $E_{\mathbb{K}}$. Hence, $E_{\mathbb{K}}$ is a locally trivial $\mathbb{K}$-vector bundle and it has the same rank as E. It is called the complexification of E in case $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{C})$ and the quaternionification of E in case $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{H})$ or $(\mathbb{C}, \mathbb{H})$. The following statements have their origin in corresponding statements about vector spaces.

(a) The mapping $E \to E_{\mathbb{K}}$ defined by $e \mapsto e \otimes 1$ is an $\mathbb{L}$-linear vector bundle morphism and embeds E into $E_{\mathbb{K}}$ as a vertical subbundle over $\mathbb{L}$.
(b) Given two complex vector bundles E and E' over B and B', respectively, and an $\mathbb{L}$-vector bundle morphism $F : E \to E'$, there exists a unique $\mathbb{K}$-vector bundle morphism $F_{\mathbb{K}} : E_{\mathbb{K}} \to E'_{\mathbb{K}}$ such that $F_{\mathbb{K}}(e \otimes k) = F(e) \otimes k$ for all $e \in E$ and $k \in \mathbb{K}$. This morphism is referred to as the $\mathbb{K}$-linear extension of F. It projects to the same mapping $B \to B'$ as F.
(c) Given a real vector bundle E, the mappings

$$E \oplus E \to (E_{\mathbb{C}})_{\mathbb{R}}\,, \qquad (v, w) \mapsto v \otimes 1 + w \otimes \mathrm{i}\,,$$
$$E^{\oplus 4} \to (E_{\mathbb{H}})_{\mathbb{R}}\,, \quad (v_1, v_2, v_3, v_4) \mapsto v_1 \otimes 1 + v_2 \otimes \mathbf{i} + v_3 \otimes \mathbf{j} + v_4 \otimes \mathbf{k}\,,$$

are vertical isomorphisms of real vector bundles. Given a complex vector bundle E, the mapping

$$E \oplus \overline{E} \to (E_{\mathbb{H}})_{\mathbb{C}}\,, \quad (v, w) \mapsto v \otimes 1 + w \otimes \mathbf{j}\,,$$

is a vertical isomorphism of complex vector bundles. Here, $\overline{E}$ denotes the complex conjugate vector bundle, cf. Sect. 4.4.

(d) Given a real vector bundle E, the mapping

$$E_{\mathbb{C}} \to \overline{E_{\mathbb{C}}}\,, \quad e \otimes z \mapsto e \otimes \bar{z} \tag{A.12}$$

is a vertical isomorphism of complex vector bundles.

Finally, we discuss field extension of scalar products and Hermitean fibre metrics. We use the fact that for every scalar product h on an $\mathbb{L}$-vector space V, there exists a unique scalar product $\mathsf{h}_{\mathbb{K}}$ on $V_{\mathbb{K}}$ such that

$$\mathsf{h}_{\mathbb{K}}(v \otimes k, v' \otimes k') = k^{\dagger}\mathsf{h}(v, v')k' \quad \text{for all } v, v' \in V\,,\ k, k' \in \mathbb{K}\,.$$

Accordingly, given a Hermitean fibre metric h on a $\mathbb{K}$-vector bundle E over a topological space B, there exists a unique Hermitean fibre metric $\mathsf{h}_{\mathbb{K}}$ on $E_{\mathbb{K}}$ such that

$$\mathsf{h}_{\mathbb{K}}(e \otimes k, e' \otimes k') = k^{\dagger}\mathsf{h}(e, e')k' \quad \text{for all } e, e' \in E_b\,,\ b \in B\,,\ k, k' \in \mathbb{K}\,. \tag{A.13}$$

Let us observe the following. It is clear that the $\mathbb{L}$-linear subspace embedding $V \to V_{\mathbb{K}}$ given by $v \mapsto v \otimes 1$ is isometric with respect to h and $\mathsf{h}_{\mathbb{K}}$. In the case $V = \mathbb{L}^n$ with h being the standard scalar product on $\mathbb{L}^n$, one can check that $\mathsf{h}_{\mathbb{K}}$ corresponds to the standard scalar product on $\mathbb{K}^n$ under the natural isomorphism $\mathbb{L}^n_{\mathbb{K}} \cong \mathbb{K}^n$. It follows that the $\mathbb{L}$-subalgebra embedding $\mathrm{M}_n(\mathbb{L}) \to \mathrm{M}_n(\mathbb{K})$ induced by the inclusion relation $\mathbb{L} \subset \mathbb{K}$ restricts to a Lie subgroup embedding of the corresponding isometry group. Thus, we obtain Lie subgroup embeddings

$$j_n^{\mathrm{O,U}} : \mathrm{O}(n) \to \mathrm{U}(n)\,, \quad j_n^{\mathrm{O,Sp}} : \mathrm{O}(n) \to \mathrm{Sp}(n)\,, \quad j_n^{\mathrm{U,Sp}} : \mathrm{U}(n) \to \mathrm{Sp}(n)\,. \tag{A.14}$$

We have the following analogue of Lemma A.1.

Lemma A.2 *Let* $(\mathbb{L}, \mathbb{K}) = (\mathbb{R}, \mathbb{C})$, $(\mathbb{R}, \mathbb{H})$ *or* $(\mathbb{C}, \mathbb{H})$ *and let* E *be a* $\mathbb{L}$*-vector bundle over a topological space* B *endowed with a fibre metric* h.

1. If $u = (u_1, \ldots, u_n)$ *is an* h*-orthonormal frame in the fibre* E_b*, then*

$$\tilde{u} = (u_1 \otimes 1, \ldots, u_n \otimes 1)$$

is an $\mathsf{h}_{\mathbb{K}}$*-orthonormal frame in the fibre* $(E_{\mathbb{K}})_b$.

2. *The mapping*

$$O(E) \to O(E_{\mathbb{K}}) , \quad u \mapsto \tilde{u} ,$$

is a vertical morphism of principal bundles with Lie group homomorphism given by (A.14).

Proof Point 1 is obvious from (A.13). Point 2 is due to the fact that for every $a \in \mathrm{M}_n(\mathbb{L})$ which is an isometry of the standard scalar product on $\mathbb{L}^n$, one has

$$\left(u_i a^i_j\right) \otimes 1 = (u_i \otimes a^i_j) = (u_i \otimes 1) a^i_j = (u_i \otimes 1) \left(j^{\mathbb{L},\mathbb{K}}_n(a)\right)^i_j \ .$$

■

Appendix B
The Conformal Group of the 4-Sphere

Consider the embedded submanifold $\mathrm{S}^4 \subset \mathbb{R}^5$ endowed with the standard metric g_0 obtained by restricting the Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^5$. In standard coordinates $z_0, \ldots, z_4$ of $\mathbb{R}^5$ corresponding to the canonical basis $\{\mathbf{e}_0, \ldots, \mathbf{e}_4\}$ it is given by $\|\mathbf{z}\|^2 = 1$. First, we show that the stereographic projection mappings yield a conformal identification of S^4 with $\mathbb{H} \cup \{\infty\}$, where $\mathbb{H} \cong \mathbb{R}^4$ is endowed with the Euclidean metric. That is, $(\mathrm{S}^4, \mathsf{g}_0)$ is locally conformally flat.

First, recall from Example 1.1.22 that S^4 is diffeomorphic to the quaternionic projective space $\mathbb{HP}^1$. Under the natural vector space isomorphism $\mathbb{H} \cong \mathbb{R}^4$, this diffeomorphism is given by

$$\mathbb{HP}^1 \to \mathrm{S}^4 \subset \mathbb{R} \times \mathbb{H} \cong \mathbb{R}^5 \,, \quad [(\mathbf{q}_1, \mathbf{q}_2)] \mapsto \mathbf{z} = (\|\mathbf{q}_1\|^2 - \|\mathbf{q}_2\|^2, 2\mathbf{q}_2\overline{\mathbf{q}}_1) \,. \tag{B.1}$$

Under this mapping, $[(1, 0)]$ is sent to $\mathbf{e}_0$ (north pole), $[(0, 1)]$ is sent to $-\mathbf{e}_0$ (south pole) and the stereographic projection mappings[1] take the following form[2]:

$$\varphi_{n,s} : U_{n,s} = \mathrm{S}^4 \setminus \{\pm \mathbf{e}_0\} \to \mathbb{H} \cong \mathbb{R}^4 \,, \quad \varphi_n(\mathbf{z}) := \overline{\mathbf{q}_1 \mathbf{q}_2^{-1}} \,, \quad \varphi_s(\mathbf{z}) := \mathbf{q}_2 \mathbf{q}_1^{-1} \,. \tag{B.2}$$

Indeed, using $\|\mathbf{q}_1\|^2 + \|\mathbf{q}_2\|^2 = 1$, from (B.1) we read off

$$(0, \mathbf{q}_2\mathbf{q}_1^{-1}) = \left(0, \frac{2\mathbf{q}_2\overline{\mathbf{q}}_1}{1 + (\|\mathbf{q}_1\|^2 - \|\mathbf{q}_2\|^2)}\right) = \frac{\mathbf{z} - z_0\mathbf{e}_0}{1 + z_0} = \varphi_s(\mathbf{z}) \,.$$

Similarly,

$$\left(0, \overline{\mathbf{q}_1\mathbf{q}_2^{-1}}\right) \mapsto \frac{\mathbf{z} - z_0\mathbf{e}_0}{1 - z_0} = \varphi_n(\mathbf{z}) \,.$$

[1] Cf. Example I/1.1.9.

[2] Here, n refers to the upper sign and s refers to the lower sign.

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Lemma B.1 *The stereographic projection mappings $\varphi_{n,s}$ from S^4 to the Euclidean space $\mathbb{R}^4$ are conformal. If we choose the orientation of $\mathbb{R}^4 \subset \mathbb{R}^5$ defined by $+\mathbf{e}_0$ and the orientation of $\mathrm{S}^4 \subset \mathbb{R}^5$ defined by the radial vector field pointing outwards, then φ_s is orientation preserving, whereas φ_n is orientation reversing.*

Proof Let $\mathbf{X}, \mathbf{Y} \in \mathrm{T}_{\mathbf{z}}\mathrm{S}^4 \subset \mathbb{R}^5$, that is, $\langle \mathbf{X}, \mathbf{z}\rangle = 0 = \langle \mathbf{Y}, \mathbf{z}\rangle$. Then,

$$\varphi'_{n,s}(\mathbf{X}) = \frac{(1 \mp z_0)\mathbf{X} - X_0(\mathbf{e}_0 \mp \mathbf{z})}{(1 \mp z_0)^2}\,,$$

and thus

$$\begin{aligned}\langle \varphi'_{n,s}(\mathbf{X}), \varphi'_{n,s}(\mathbf{Y})\rangle &= \frac{\langle (1 \mp z_0)\mathbf{X} - X_0(\mathbf{e}_0 \mp \mathbf{z}), (1 \mp z_0)\mathbf{Y} - Y_0(\mathbf{e}_0 \mp \mathbf{z})\rangle}{(1 \mp z_0)^4} \\ &= \frac{\langle \mathbf{X}, \mathbf{Y}\rangle}{(1 \mp z_0)^2}\,.\end{aligned}$$

Since $1 \mp z_0 = \frac{2}{1+\|\varphi_{n,s}(\mathbf{z})\|^2}$ and $\mathsf{g}_{\mathbf{z}}(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y}\rangle$, we obtain

$$\mathsf{g}_{\mathbf{z}}(\mathbf{X}, \mathbf{Y}) = \frac{4}{(1 + \|\varphi_{n,s}(\mathbf{z})\|^2)^2}\langle \varphi'_{n,s}(\mathbf{X}), \varphi'_{n,s}(\mathbf{Y})\rangle\,.$$

The second statement is a consequence of the following identity (Exercise B.1) for the canonical volume forms on $\mathbb{R}^4$ and S^4, respectively, corresponding to the above defined orientations:

$$\varphi^*_{n,s}(\mathsf{v}_{\mathbb{R}^4}) = \mp \frac{1}{(1 \mp z_0)^4}(\mathsf{v}_{\mathrm{S}^4})_{\restriction U_{n,s}}\,. \tag{B.3}$$

■

We conclude that (U_s, φ_s) and $(U_n, \overline{\varphi_n})$ constitute an oriented atlas of S^4 consisting of conformal local charts. One may choose one of the stereographic projection mappings, say φ_s, and extend it to a diffeomorphism

$$\varphi : \mathrm{S}^4 \cong \mathbb{H}\mathrm{P}^1 \to \mathbb{H} \cup \{\infty\} \tag{B.4}$$

by sending the southpole $-\mathbf{e}_0$ to $\{\infty\}$. This yields a conformal identification.

Under this identification, the proper (that is, orientation preserving) conformal group of S^4 is given by

$$\mathrm{C}_0(\mathrm{S}^4, [\mathsf{g}_0]) = \mathrm{SL}(2, \mathbb{H})/\{\pm \mathbf{1}\}\,. \tag{B.5}$$

Its universal covering group is $\widetilde{\mathrm{C}}_0(\mathrm{S}^4, [\mathrm{g}_0]) = \mathrm{SL}(2, \mathbb{H})$. Here, $\mathrm{SL}(2, \mathbb{H})$ denotes the group of (2×2)-matrices with quaternionic entries and determinant equal to 1. We present the algebraic part of the proof of this fact. First, let us recall the definition of the determinant: the representation of $\mathbb{H}$ on $\mathbb{C}^2$ given by

$$\mathbf{1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{j} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{k} \mapsto \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$

naturally lifts to an injective homomorphism of algebras,

$$\tau_n : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{2n}(\mathbb{C}) .$$

One defines

$$\det_{\mathbb{H}}(g) := \det(\tau_n(g)) , \quad g \in \mathrm{GL}(n, \mathbb{H}) . \tag{B.6}$$

Then, one easily checks the following (Exercise B.2):

$$\det_{\mathbb{H}}(g) \geq 0 , \quad \det_{\mathbb{H}}(gh) = \det_{\mathbb{H}}(g)\det_{\mathbb{H}}(h) . \tag{B.7}$$

In particular, for $n = 2$ one has[3]

$$\det_{\mathbb{H}}(g) = \det(\mathbf{ad} - \mathbf{aca}^{-1}\mathbf{b}) , \tag{B.8}$$

where $g = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{H}$. Now, consider the natural left action of $\mathrm{SL}(2, \mathbb{H})$ on $\mathbb{H}^2$,

$$\mathrm{SL}(2, \mathbb{H}) \times \mathbb{H}^2 \to \mathbb{H}^2 , \quad (k, (\mathbf{q}_1, \mathbf{q}_2)) \mapsto (\mathbf{aq}_1 + \mathbf{bq}_2, \mathbf{cq}_1 + \mathbf{dq}_2) ,$$

where $k = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{H})$. Clearly, this action projects onto a left transitive action of $\mathrm{SL}(2, \mathbb{H})$ on $\mathbb{H}\mathrm{P}^1 \cong \mathbb{H} \cup \{\infty\}$,

$$\Psi : \mathrm{SL}(2, \mathbb{H}) \times \mathbb{H}\mathrm{P}^1 \to \mathbb{H}\mathrm{P}^1 , \quad \Psi_k[(\mathbf{q}_1, \mathbf{q}_2)] = [(\mathbf{aq}_1 + \mathbf{bq}_2, \mathbf{cq}_1 + \mathbf{dq}_2)] .$$

1. Let $\mathbf{q}_1 \neq 0$. Then, denoting $\mathbf{x} = \mathbf{q}_2\mathbf{q}_1^{-1}$, we obtain

$$[(\mathbf{aq}_1 + \mathbf{bq}_2, \mathbf{cq}_1 + \mathbf{dq}_2)] = [(\mathbf{1}, (\mathbf{c} + \mathbf{dx})(\mathbf{a} + \mathbf{bx})^{-1})] .$$

2. Let $\mathbf{q}_1 = 0$. Under the isomorphism (B.4), this point corresponds to $\{\infty\}$. For $\mathbf{b} \neq 0$, $\{\infty\}$ is sent to $[(\mathbf{1}, \mathbf{db}^{-1})]$ and for $\mathbf{b} = 0$, $\{\infty\}$ is a fixed point.

---

[3] A priori, the following formula holds for $\mathbf{a} \neq 0$ only. But, if we declare conjugation by zero to be the identity, then this formula remains true for $\mathbf{a} = 0$ as well.

To summarize, $\mathrm{SL}(2, \mathbb{H})$ acts on $\mathbb{HP}^1 \cong \mathbb{H} \cup \{\infty\}$ via fractional linear transformations (or Möbius transformations[4]),

$$\mathbf{x} \mapsto (\mathbf{c} + \mathbf{dx})(\mathbf{a} + \mathbf{bx})^{-1}\,, \tag{B.9}$$

with the transformation law for $\{\infty\}$ specified under point 2.[5] It is easy to show that the kernel of this action is $\{\pm\mathbf{1}\}$ (Exercise B.3). Thus, $\mathrm{SL}(2, \mathbb{H})/\{\pm\mathbf{1}\}$ acts effectively on $\mathbb{HP}^1$. It is also easy to see that its building blocks have the following geometrical interpretation (Exercise B.4):

1. $\mathbf{x} \mapsto \mathbf{dxa}^{-1}$ with $\mathbf{a} \neq 0 \neq \mathbf{d}$: SO(4)-rotations and dilations,
2. $\mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$: translations,
3. $\mathbf{x} \mapsto \mathbf{x}^{-1}$: proper inversions.

Lemma B.2 *The Möbius transformations* (B.9) *are conformal.*

Proof Since the stereographic projection mappings $\varphi_{s,n}$ are conformal, it is enough to show that the mapping (B.9) is conformal with respect to the metric induced by the quaternionic norm. For $\mathbf{x}, \mathbf{y} \in \mathbb{H}$ and $\mathbf{b} \neq 0$, we calculate[6]

$$\begin{aligned}
&\|(\mathbf{c} + \mathbf{dy})(\mathbf{a} + \mathbf{by})^{-1} - (\mathbf{c} + \mathbf{dx})(\mathbf{a} + \mathbf{bx})^{-1}\| \\
&\qquad = \|\big[(\mathbf{c} + \mathbf{dy}) - \mathbf{db}^{-1}(\mathbf{a} + \mathbf{by})\big](\mathbf{a} + \mathbf{by})^{-1} \\
&\qquad\quad - \big[(\mathbf{c} + \mathbf{dx}) - \mathbf{db}^{-1}(\mathbf{a} + \mathbf{bx})\big](\mathbf{a} + \mathbf{bx})^{-1}\| \\
&\qquad = \|\mathbf{c} - \mathbf{db}^{-1}\mathbf{a}\|\,\|(\mathbf{a} + \mathbf{by})^{-1} - (\mathbf{a} + \mathbf{bx})^{-1}\| \\
&\qquad = \frac{\|\mathbf{c} - \mathbf{db}^{-1}\mathbf{a}\|\,\|(\mathbf{a} + \mathbf{by}) - (\mathbf{a} + \mathbf{bx})\|}{\|\mathbf{a} + \mathbf{by}\|\,\|\mathbf{a} + \mathbf{bx}\|} \\
&\qquad = \frac{\|\mathbf{y} - \mathbf{x}\|\,\|}{\|\mathbf{a} + \mathbf{by}\|\,\|\mathbf{a} + \mathbf{bx}\|}\,.
\end{aligned}$$

For $\mathbf{b} = 0$, the calculation is trivial. ■

To finish the proof of (B.5) it now remains to show that all conformal transformations of S^4 are given by (B.9). This fact is proven in the literature under various regularity conditions on the mapping, see e.g. Sect. 15 in [168]. In this complete version, the above statement is usually referred to as the Liouville Theorem.

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[4]Cf. [517, 518, 670] for an exhaustive discussion.

[5]The latter also follows from (B.9) by rewriting $(\mathbf{c} + \mathbf{dx})(\mathbf{a} + \mathbf{bx})^{-1} = (\mathbf{cx}^{-1} + \mathbf{d})(\mathbf{ax}^{-1} + \mathbf{b})^{-1}$ and taking the limit $\mathbf{x} \to \infty$.

[6]The trick in the calculation below is taken from [670].

Exercises

B.1 Prove formula (B.3).

B.2 Prove the formulae (B.7) and (B.8).

B.3 Show that the kernel of the fractional linear transformation (B.9) is $\{\pm\mathbf{1}\}$.

B.4 Verify the geometrical meaning of the building blocks of the action (B.9) given prior to Lemma B.2.

Appendix C
Simple Lie Algebras. Root Diagrams

We introduce the basic notions of root theory of simple Lie algebras in a rather operational spirit. For a presentation of the theory, we refer to [170], [329].

Let $\mathfrak{L}$ be a complex simple Lie algebra. By definition, a Cartan subalgebra of $\mathfrak{L}$ is a maximal Abelian subalgebra. Given a Cartan subalgebra $\mathfrak{L}_0$, we may decompose $\mathfrak{L}$ into the common eigenspaces of the endomorphisms $\mathrm{ad}(B)$ with $B \in \mathfrak{L}_0$. The common eigenspaces are labelled by the eigenvalue functionals α assigning to $B \in \mathfrak{L}_0$ the corresponding eigenvalue $\alpha(B)$. The nonzero eigenvalue functionals are referred to as the roots of $\mathfrak{L}$ relative to $\mathfrak{L}_0$. They form the root system $\mathscr{W} \subset \mathfrak{L}_0^*$. Given $\alpha \in \mathscr{W}$, the corresponding common eigenspace $\mathfrak{L}_\alpha$ has dimension one. It is called the root subspace of α and its elements are called the root vectors of α. As a result, we obtain a direct sum decomposition into vector subspaces

$$\mathfrak{L} = \mathfrak{L}_0 \oplus_{\alpha \in \mathscr{W}} \mathfrak{L}_\alpha \,, \tag{C.1}$$

where

$$[B, \mathbf{e}_\alpha] = \alpha(B)\mathbf{e}_\alpha \tag{C.2}$$

for all $\mathbf{e}_\alpha \in \mathfrak{L}_\alpha$ and $B \in \mathfrak{L}_0$.

The restriction to $\mathfrak{L}_0$ of the Killing form is negative definite. Thus, we may use a negative multiple of it to define a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{L}_0$. The latter induces, in turn, a scalar product $\langle \cdot, \cdot \rangle_*$ on $\mathfrak{L}_0^*$. We normalize these scalar products by the requirement

$$\langle \alpha, \alpha \rangle_* = 2$$

for the longest root α. Via the isomorphism $\mathfrak{L}_0 \cong \mathfrak{L}_0^*$ defined by $\langle \cdot, \cdot \rangle$, to every $\alpha \in \mathscr{W}$ there corresponds an element $\mathbf{h}_\alpha \in \mathfrak{L}_0$, called the Cartan element of α. By definition, $\langle \mathbf{h}_\alpha, B \rangle = \alpha(B)$ for all $B \in \mathfrak{L}_0$.

Let $\ell = \mathrm{rank}(\mathfrak{L})$. A subsystem $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathscr{W}$ is called a system of simple roots if it is a basis in $\mathfrak{L}_0^*$ and if for all $\alpha \in \mathscr{W}$ one has $\alpha = \pm \sum_i n_i \alpha_i$ with non-negative integers n_i. According to the sign, one speaks of positive and negative roots relative to Π. One can show that systems of simple roots exist. Given such

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a system, one may choose root vectors $\mathbf{e}_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$ and $\mathbf{f}_{\alpha_i} \in \mathfrak{L}_{-\alpha_i}$, $i = 1, \ldots, \ell$, so that the relations

$$[\mathbf{h}_{\alpha_i}, \mathbf{h}_{\alpha_j}] = 0\,, \quad [\mathbf{h}_{\alpha_i}, \mathbf{e}_{\alpha_j}] = \mathrm{A}_{ij}\,\mathbf{e}_{\alpha_i}\,, \qquad [\mathbf{h}_{\alpha_i}, \mathbf{f}_{\alpha_j}] = -\mathrm{A}_{ij}\,\mathbf{f}_{\alpha_i}\,, \tag{C.3}$$

$$[\mathbf{e}_{\alpha_i}, \mathbf{f}_{\alpha_j}] = -\delta_{ij}\,\mathbf{h}_{\alpha_i}\,, \quad \langle \mathbf{e}_{\alpha_i}, \mathbf{f}_{\alpha_j} \rangle = -\delta_{ij}$$

hold for all i, j. The matrix

$$\mathrm{A}_{ij} := 2\langle \alpha_i, \alpha_j \rangle_* / \langle \alpha_i, \alpha_i \rangle_* \tag{C.4}$$

is called the Cartan matrix. Every positive root can be written as a sum $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_n}$ in such a way that every partial sum is a root. Then,

$$\mathrm{ad}(\mathbf{e}_{\alpha_{i_n}}) \circ \cdots \circ \mathrm{ad}(\mathbf{e}_{\alpha_{i_2}})\mathbf{e}_{\alpha_{i_1}} \in \mathfrak{L}_\alpha\,.$$

An analogous statement holds for the negative roots. Thus, the vectors $\mathbf{h}_{\alpha_i}$, $\mathbf{e}_{\alpha_i}$ and $\mathbf{f}_{\alpha_i}$ generate $\mathfrak{L}$. In addition, one may choose root vectors $\mathbf{e}_\alpha \in \mathfrak{L}_\alpha$ for the remaining roots such that for any $\alpha, \beta \in \mathscr{W}$ one has

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \begin{cases} 0 & \alpha + \beta \notin \mathscr{W}\,, \\ N_{\alpha,\beta}\,\mathbf{e}_{\alpha+\beta} & \alpha + \beta \in \mathscr{W} \end{cases}$$

with

$$N_{\alpha,\beta}^2 = \frac{1}{2}(r_{\beta,\alpha} + 1) q_{\beta,\alpha} \langle \beta, \beta \rangle_*\,, \tag{C.5}$$

where $q_{\beta,\alpha}$ and $r_{\beta,\alpha}$ are the largest non-negative integers such that

$$\alpha - r_{\beta,\alpha}\beta, \ldots, \alpha + q_{\beta,\alpha}\beta \in \mathscr{W}\,.$$

The Cartan matrix may be represented by a diagram, known as the Dynkin diagram, as follows. As a matter of fact, the simple roots α_i can have at most two different lengths. In the Dynkin diagram, they are represented by circles, where the circle is filled in case α_i is short and unfilled in case it is long. The circles representing α_i and α_j are connected by $\mathrm{A}_{ij}\mathrm{A}_{ji}$ edges (no summation). Figure C.1 shows the Dynkin diagrams for the classical complex simple Lie algebras. Given the Dynkin diagram, one can reconstruct the Cartan matrix and from the Cartan matrix one can reconstruct $\mathfrak{L}$ up to isomorphy.

A semisimple Lie subalgebra $\mathfrak{L}'$ of $\mathfrak{L}$ is called regular if there exists a Cartan subalgebra $\mathfrak{L}_0$ of $\mathfrak{L}$ such that $\mathfrak{L}'$ is invariant under $\mathrm{ad}(B)$ for all $B \in \mathfrak{L}_0$. In this case, there exists a subspace $\mathfrak{L}_0' \subset \mathfrak{L}_0$ and a subset $\mathscr{W}' \subset \mathscr{W}$ such that

$$\mathfrak{L}' = \mathfrak{L}_0' \bigoplus_{\alpha' \in \mathscr{W}'} \mathfrak{L}_{\alpha'}\,.$$

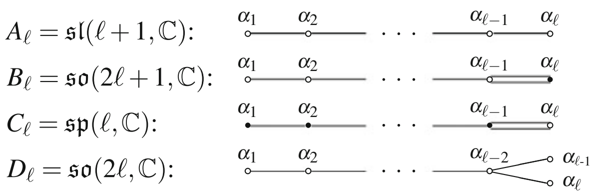

Fig. C.1 Dynkin diagrams of the classical complex simple Lie algebras

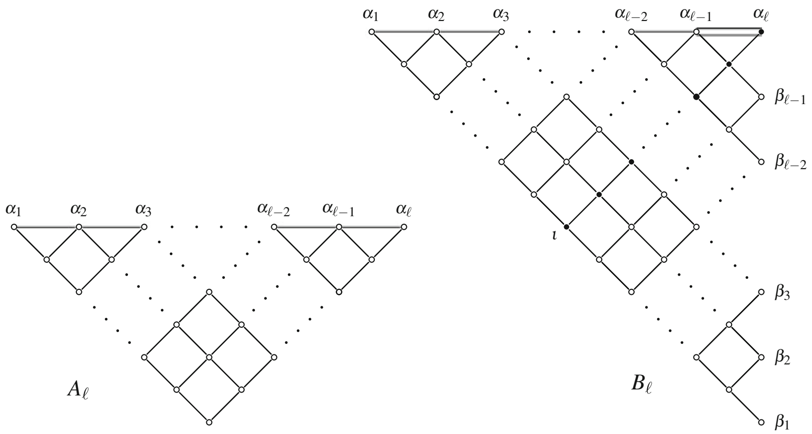

Fig. C.2 Root diagrams of the simple Lie algebras $A_\ell = \mathfrak{sl}(\ell+1,\mathbb{C})$ and $B_\ell = \mathfrak{so}(2\ell+1,\mathbb{C})$

In Sect. 7.9, we consider the restriction of the adjoint representation of a complex semisimple Lie algebra $\mathfrak{L}$ to a regular subalgebra $\mathfrak{L}' \subset \mathfrak{L}$ and decompose it into irreducible components. For that purpose, it is convenient to exploit a natural ordering in the set of positive roots. This allows for extending the Dynkin diagram to a diagram of the positive roots. The latter can be represented in the form of a triangle whose upper side coincides with the Dynkin diagram, see [643] for further details. Since we need the root diagrams for the series A_ℓ and B_ℓ only, we limit our attention to these series, see Fig. C.2.

In the root diagram of A_ℓ, the circle at the intersection of the lines starting from α_i and α_j, $i \le j$, corresponds to the root $\alpha(i,j) = \alpha_i + \ldots + \alpha_j$. In the normalization chosen above,

$$\langle \alpha_i, \alpha_j \rangle_* = \begin{cases} 2 & i = j\,, \\ -1 & |i-j| = 1\,, \\ 0 & |i-j| \ge 2\,. \end{cases}$$

Using this, one can easily calculate the scalar products between all $\alpha(i,j)$, see [643].

In the root diagram of B_ℓ, the roots contained in the triangle defined by $(\alpha_1, \alpha_\ell, \iota)$ have the same form as the roots in the A_ℓ-lattice. In the triangle $(\alpha_\ell, \beta_1, \iota)$, the circle at the intersection of the lines starting from β_i and β_j, $i \leq j$, corresponds to the root

$$\beta(i,j) = \alpha_i + \ldots + \alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_n)\,, \quad \beta_i = \alpha_i + 2(\alpha_{i+1} + \ldots \alpha_n)\,.$$

As an example, let us consider the decomposition of the restriction of the adjoint representation of A_ℓ to the regular subalgebra $A_2 \subset A_\ell$. This is used in Sect. 7.9. By virtue of (C.3)–(C.5), we obtain the decomposition

$$A_\ell = A_2 \oplus \mathfrak{c} \oplus \bigoplus_{i=1}^{\ell-2} \left(\mathfrak{p}_i \oplus \bar{\mathfrak{p}}_i\right), \tag{C.6}$$

where $\mathfrak{c}$ is the centralizer of A_2 in A_ℓ and where $\mathfrak{p}_i$ and $\bar{\mathfrak{p}}_i$ carry the basic representation of A_2. For the calculation of the centralizer, one can use a theorem of Dynkin [170] which states that the centralizer of a regular semisimple subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is the direct sum of a regular semisimple subalgebra $\tilde{\mathfrak{g}}$ and a regular Abelian subalgebra $\mathfrak{g}_0$, fulfilling

$$\mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{h}) - \mathrm{rank}(\tilde{\mathfrak{g}}) = \dim \mathfrak{g}_0\,, \quad [\tilde{\mathfrak{g}}, \mathfrak{g}_0] = 0\,. \tag{C.7}$$

In our case, this implies

$$\mathfrak{c} = A_{\ell-3} \oplus \mathfrak{g}_0\,, \quad \dim \mathfrak{g}_0 = 1\,. \tag{C.8}$$

In the basis consisting of the Cartan elements $\mathbf{h}_{\alpha_i}$, one obtains $\mathfrak{g}_0 = \mathbb{C}\tilde{B}$, where

$$\tilde{B} = \frac{2(\ell-2)}{\ell+1}\left(\mathbf{h}_{\alpha_1} + 2\mathbf{h}_{\alpha_2} + 3\mathbf{h}_{\alpha_3} + \frac{3}{\ell-2}\sum_{j=4}^{\ell}(\ell+1-j)\,\mathbf{h}_{\alpha_j}\right).$$

Up to a multiplicative constant, this form of $\tilde{B}$ follows from the definition of the centralizer and from the second relation in (C.8). The constant is fixed by the requirement $\langle\tilde{B}, \mathbf{h}_{\alpha_3}\rangle = 2$. It is easy to see that then $\langle\tilde{B}, h_{\alpha_i}\rangle = 0$ for all $i \neq 3$. In the root diagram of A_ℓ in Fig. C.2, the subalgebra A_2 corresponds to the small triangle on the left which is made up by the roots α_1, α_2 ad $\alpha_1 + \alpha_2$ and the subalgebra $A_{\ell-3}$ corresponds to the large triangle on the right which is generated by the roots $\alpha_4, \ldots, \alpha_\ell$. The remaining part of the diagram is a rectangle which can be divided into $\ell - 2$ lines containing three circles each. The root vectors of the roots in these lines span the subspaces $\mathfrak{p}_i$ and the root vectors of the corresponding negative roots span the subspaces $\overline{\mathfrak{p}}_i$.

Finally, we stress that the graphical method presented here gives more information than the mere tables of subalgebras. We do not only get the types and multiplicities of irreducible representations for the restriction of the adjoint representation to a subalgebra but also their explicit realization on the root vectors. This information is needed for calculating the scalar field potentials in Sect. 7.9.

Appendix D
ζ-Function Regularization

Let P be a symmetric positive operator on $\mathbb{R}^n$ and let p be the associated quadratic form,

$$p(\mathbf{x}) := \langle \mathbf{x}, P\mathbf{x} \rangle \,, \quad \mathbf{x} \in \mathbb{R}^n \,,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Recall the Gaussian integral

$$\int_{\mathbb{R}^n} \mathrm{d}\mathbf{x} \, \exp\big(- \pi p(\mathbf{x}) \big) = (\det(P))^{-\frac{1}{2}} \,.$$

In various branches of mathematics and physics, one wishes to generalize this formula to the case of operators on infinite-dimensional Hilbert spaces. For that purpose, a regularization method for the determinant is needed. Below, we explain the simplest and most convenient one.

Let P be a symmetric, positive operator with a discrete spectrum on the infinite-dimensional Hilbert space $\mathscr{H}$. Then, the ζ-function associated with P is defined as

$$\zeta_P(z) = \sum_{k=1}^{\infty} \lambda_k^{-z} \,, \tag{D.1}$$

where λ_k are the nonzero eigenvalues of P. Clearly, a priori, this formula makes only sense for values of z for which the above series converges. For other values, $\zeta(z)$ is defined by analytic continuation, see [581] for details. If $\zeta(z)$ can be analytically continued to $z = 0$, then $\zeta_P'(0)$ is well defined and we can put

$$\det_\zeta(P) := \exp\big(- \zeta_P'(0) \big) \,. \tag{D.2}$$

This is motivated by the formal calculation

$$\zeta_P'(z) = - \sum_{k=1}^{\infty} \ln(\lambda_k) \, \lambda_k^{-z} \,, \tag{D.3}$$

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and, thus, $\zeta_P'(0) = -\sum_{k=1}^{\infty} \ln(\lambda_k)$. This also shows that (D.2) reduces to the ordinary definition of the determinant in the finite-dimensional case.

One important class of operators for which the above regularization works is the class of elliptic operators of order r on an n-dimensional compact manifold. Then, the series (D.1) converges for $\mathrm{Re}(z) > \frac{n}{r}$ and $\zeta(z)$ may be analytically continued to a meromorphic function of z having no singularity at the origin.

The above result generalizes to the case when P is not necessarily positive, but is invertible and has a positive definite symbol. Then, all but a finite number of eigenvalues $\lambda_1, \ldots, \lambda_l$ lie in some cone about the positive axis and, for $k > l$, the quantities $\lambda_k^{-z} = \exp(-z \ln \lambda_k)$ are well defined using the cut along the negative real axis, see [575]. Then, one defines

$$\det{}_\zeta(P) := \lambda_1 \ldots \lambda_l \exp\Big(-\sum_{k>l} \lambda_k^{-z}\Big) . \tag{D.4}$$

If P has zero modes, then, roughly speaking, one has to restrict the domain of definition of P to the space orthogonal to the kernel of P, see Chap. X of [480] for details.

Appendix E
K-Theory and Index Bundles

K-theory is a (generalized) cohomology theory for vector bundles defined as follows, see [29, 288, 335]. Let X be a compact topological space[7] and let $V(X)$ be the set of isomorphism classes of complex vector bundles over X. Clearly, the set $V(X)$ is an Abelian semigroup with respect to the operation of taking the direct sum. It has a zero element given by the zero-dimensional bundle. Let $F(X)$ be the free Abelian group generated by $V(X)$ and let $E(X)$ be the subgroup of $F(X)$ generated by elements of the form

$$V + W - (V \oplus W)\,. \tag{E.1}$$

Then, we define the K-group (or Grothendieck group) of X by

$$K(X) := F(X)/E(X)\,. \tag{E.2}$$

By construction, $K(X)$ is an Abelian group and the elements of $K(X)$ are equivalence classes[8] fulfilling $[V] + [W] = [V \oplus W]$. Clearly, any element $\xi \in K(X)$ may be represented as a linear combination with integer coefficients and, thus,

$$\xi = \sum_{i=1}^{p_1} n_i[U_i] - \sum_{i=1}^{p_2} m_i[V_i]\,, \quad n_i > 0\,,\ m_i > 0\,.$$

Then, using (E.1), we obtain

$$\xi = \left[\bigoplus_{i=1}^{p_1} U_i^{\oplus n_i}\right] - \left[\bigoplus_{i=1}^{p_2} V_i^{\oplus m_i}\right] \equiv [W_1] - [W_2]\,,$$

---

[7] More generally, X can be locally compact, see [29].

[8] Some authors call them virtual bundles.

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showing that any element of $K(X)$ may be represented as the difference of two elements represented by vector bundles. Using the fact that X is compact, one of the two bundles W_1 and W_2 may be assumed to be trivial. Indeed, it can be shown that under this assumption, there exists a bundle W_3 such that $W_2 \oplus W_3$ is trivial. This implies

$$[W_1 \oplus W_3] - [W_2 \oplus W_3] = [W_1] - [W_2] .$$

Next, we note that two bundles V_1 and V_2 define the same element in $K(X)$ iff there exists a trivial bundle N such that

$$V_1 \oplus N = V_2 \oplus N . \tag{E.3}$$

This condition is referred to as stable equivalence of the vector bundles V_1 and V_2. Clearly, if (E.3) holds, then (E.1) implies $[V_1] + [N] = [V_2] + [N]$ and, thus $[V_1] = [V_2]$. The proof of the converse statement is a simple exercise which we leave to the reader.

Finally, we endow $K(X)$ with a natural ring structure:

$$[V] \cdot [W] := [V \otimes W] . \tag{E.4}$$

It is easy to show that for homotopy equivalent spaces X and Y the rings $K(X)$ and $K(Y)$ are isomorphic.

Recall from Sect. 4.7 that the Chern character $\mathrm{ch}(V)$ of a vector bundle V has the following properties:

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W) , \quad \mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W) .$$

Thus, it extends uniquely to a homomorphism of rings:

$$\mathrm{ch} : K(X) \to H^*_{\mathbb{Q}}(X) , \quad \mathrm{ch}([V]) := \mathrm{ch}(V) . \tag{E.5}$$

Now, consider the following setting relevant for the study of families of Fredholm operators. We formulate it in the context of Dirac operators as needed in the Family Index Theorem.

1. Let $\pi : M \to Y$ be a fibre bundle endowed with a fibre metric on the canonical vertical distribution $\mathrm{V}M$ and a connection, that is, a splitting of $\mathrm{T}M$ into $\mathrm{V}M$ and a complementary horizontal distribution.
2. Let $\mathscr{E} = \{\mathscr{E}_y\}$ be a family of Dirac bundles over $M_y := \pi^{-1}(y)$, $y \in Y$.

Let V and W be complex vector bundles over Y and denote $V_y := V_{\upharpoonright M_y}$, $W_y := W_{\upharpoonright M_y}$. Let $P = \{P_y\}$ be a family of Fredholm operators over Y, that is, for every $y \in Y$,

$$P_y : L^2(V_y) \to L^2(W_y)$$

is a Fredholm operator. If the subspaces $\ker(P_y)$ and $\operatorname{coker}(P_y)$ have constant dimension and thus combine to vector bundles over Y, the index bundle of P is the element of $K(Y)$ defined by

$$\operatorname{Ind}(P) := [\ker(P)] - [\operatorname{coker}(P)] . \tag{E.6}$$

In the general case, where the dimensions of $\ker(P_y)$ and $\operatorname{coker}(P_y)$ may jump, the index bundle is defined as follows. Let $y_0 \in Y$. Then,

$$\dim \ker(P_{y_0}) \geq \dim \ker(P_y) ,$$

for y sufficiently close to y_0. That is, dim ker is semi-continuous. The same is true for dim coker. In fact, one can prove that their difference remains constant. The basic idea for proving this is the following.[9] Let $P : H \to H$ be a family of Fredholm operators, let $\mathbf{e}_0, \mathbf{e}_1, \ldots$ be an orthonormal basis of H and let $H_n \subset H$ be the subspace spanned by the vectors $\{\mathbf{e}_i\}$ with $i \geq n$. Let $p^{(n)}$ be the orthogonal projector[10] onto the Hilbert subspace H_n. Define

$$P^{(n)} := p^{(n)} \circ P .$$

Using the compactness of Y, one can prove that there exists a (sufficiently large) number n such that $\operatorname{im}\left(P^{(n)}\right) = H_n$, for all $y, y' \in Y$, and

$$\dim \ker \left(P^{(n)}_y\right) = \dim \ker \left(P^{(n)}_{y'}\right) .$$

Thus, one can define

$$\operatorname{Ind}(P) := \operatorname{Ind}\left(P^{(n)}\right) . \tag{E.7}$$

Finally, one proves that this quantity is independent of n and of the choice of the basis. This construction extends to Fredholm operators acting between different Hilbert spaces. As in the case of the index of a single operator, one proves that $\operatorname{Ind}(P)$ is a homotopy invariant.

[9]See [29] or [83] for a pedagogical presentation.

[10]This is a self-adjoint Fredholm operator. Thus, it has index 0.

Appendix F
Determinant Line Bundles

There is a huge literature on this subject starting from the classical paper by Quillen [525], see [72, 79, 80, 210, 211, 593] and further references therein.

To start with, let V and W be finite-dimensional complex vector spaces with $\dim V = \dim W = n$ and let $P : V \to W$ be a homomorphism. Consider the complex lines

$$\mathrm{Det}(V) := \bigwedge^n V\,, \quad \mathrm{Det}(W) := \bigwedge^n W\,.$$

The determinant line of P is defined by

$$\mathrm{Det}(P) := \mathrm{Det}(V^*) \otimes \mathrm{Det}(W)\,, \tag{F.1}$$

and the Quillen determinant $\det(P) \in \mathrm{Det}(P)$ is defined as

$$\det(P) := \mathbf{e}_1^* \wedge \ldots \wedge \mathbf{e}_n^* \otimes (\bigwedge^n P)(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n)\,, \tag{F.2}$$

where $\{\mathbf{e}_k\}$ is any basis in V and $\{\mathbf{e}_k^*\}$ is its dual. For $V = W$, the determinant defined by (F.2) coincides with the classical determinant of the endomorphism $P \in \mathrm{End}(V)$, because in this case we have a natural isomorphism

$$\mathrm{Det}(V^*) \otimes \mathrm{Det}(V) \to \mathbb{C} \tag{F.3}$$

induced by the canonical pairing $\mathrm{Det}(V^*) \times \mathrm{Det}(V) \to \mathbb{C}$. The latter implies the identity

$$(\bigwedge^n P)(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n) = \det(P)\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n\,.$$

The canonical isomorphism (F.3) also implies the canonical isomorphism

$$\mathrm{Det}(P \circ Q) \cong \mathrm{Det}(P) \otimes \mathrm{Det}(Q)\,, \tag{F.4}$$

for $P \in \mathrm{Hom}(V, W)$ and $Q \in \mathrm{Hom}(U, V)$. Clearly, $\det(P)$ vanishes if P has a nontrivial kernel and is nonzero otherwise. Using the exact sequence

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$$0 \to \ker(P) \to V \xrightarrow{P} W \to \operatorname{coker}(P) \to 0$$

and the above multiplicative property, we obtain a natural isomorphism

$$\operatorname{Det}(V^*) \otimes \operatorname{Det}(W) \cong (\operatorname{Det} \ker(P))^* \otimes (\operatorname{Det} \operatorname{coker}(P)) \,. \tag{F.5}$$

Next, consider a parameter space Y, vector bundles V and W over Y and a vertical vector bundle morphism $P : V \to W$. The latter gives rise to a family of homomorphisms

$$P_y : V_y \to W_y$$

varying smoothly with $y \in B$. Then, the above construction yields the complex line $\operatorname{Det}(P_y) := \operatorname{Det}(V_y^*) \otimes \operatorname{Det}(W_y)$ for every $y \in B$, that is, we obtain a complex line bundle

$$\pi : \operatorname{Det}(P) \to Y \,, \tag{F.6}$$

which will be referred to as the determinant line bundle of P. Correspondingly, the determinants $\det(P_y)$ combine to a section $\det(P)$ in $\operatorname{Det}(P)$. In view of (F.5), it is tempting to generalize the above constructions to Fredholm operators acting between infinite-dimensional Hilbert spaces, cf. Definition 5.7.8. In that case, the fibres of $\operatorname{Det}(P)$ are defined by the right hand side of (F.5).[11] Clearly, in general, the dimensions of $\ker(P)$ and $\operatorname{coker}(P)$ may jump, so that one has to show that these fibres piece together to form a smooth vector bundle. This is done in terms of K-theory over Y, see Appendix E. Let $\xi \in K(Y)$. One defines

$$\operatorname{Det}(\xi) := (\operatorname{Det} V)^* \otimes (\operatorname{Det} W) \,,$$

where $[V] - [W]$ is any representative of ξ. It is easy to show that the line bundle $\operatorname{Det}(\xi)$ is independent of the choice of the representative in $K(Y)$. Thus, by (F.5), for a family P of Fredholm operators the corresponding element of $K(Y)$ is the index bundle

$$\operatorname{Ind}(P) = [\ker(P)] - [\operatorname{coker}(P)] \,,$$

and one defines

$$\operatorname{Det}(P) := \operatorname{Det}(\operatorname{Ind}(P)) \,. \tag{F.7}$$

That is, the determinant bundle of P is the top exterior power of the index bundle. It can be shown, see Proposition 3.42 in [83] for a detailed proof, that the set

$$\bigcup_{y \in Y} \big(\operatorname{Det} \ker(P_y)\big)^* \otimes \big(\operatorname{Det} \ker(P_y^*)\big)$$

[11] The left hand side of (F.5) no longer makes sense.

can be taken as a standard representative of the isomorphism class $\mathrm{Det}(P)$. This shows, in particular, that the definition (F.7) boils down to (F.6) in the finite-dimensional case. Generalizing the results of Quillen [525][12] it has been shown by Bismut and Freed [79] that, for families of twisted Dirac operators, there is a canonical section $\det(P)$ of $\mathrm{Det}(P)$ over components of Y where P has index zero. Over the connected components where the index is nonzero, $\det(P) = 0$ by definition. The section $\det(P)$ is referred to as the Quillen determinant of the family P. If $\mathrm{Det}(P)$ is trivial, then there exists a non-vanishing section $\sigma : Y \to \mathrm{Det}(P)$ and we may represent $\det(P)$ by an ordinary $\mathbb{C}$-valued function $\det_{\mathbb{C}}(P)$ on Y via

$$\det{}_{\mathbb{C}}(P)(y) := \frac{\det(P_y)}{\sigma(y)} .$$

For a global section σ to exist, the first Chern class $\mathsf{c}_1(\mathrm{Det}(P)) \in H^2_{\mathbb{Z}}(Y)$ must vanish. This cohomology class can be calculated using the Atiyah-Singer Family Index Theorem, see Remark 5.8.16. In general, $\mathrm{Det}(P)$ is nontrivial leading e.g. to anomalies in gauge theories, see Sect. 9.3.

Moreover, generalizing results of Quillen, Bismut and Freed proved that $\mathrm{Det}(P)$ carries a natural metric and a connection. Clearly, the curvature of the latter may be used to explicitly calculate the first Chern class of $\mathrm{Det}(P)$. We describe these structures in some detail[13]: let $\pi : M \to Y$ be a fibration of manifolds endowed with the structure described in Remark 5.8.16 and let $\mathscr{E} = \mathscr{S} \otimes E$ be a vector bundle over M also endowed with the structure described there. Then, we have a family of Dirac operators $\{\mathrm{D}_y\}$ over Y which, according to the grading of $\mathscr{E}$, splits into two families $\mathrm{D}^{\pm}_y : H^{\pm}_y \to H^{\mp}_y$, where $H^{\pm}_y$ are appropriate Hilbert spaces of sections of $\mathscr{E}^{\pm}_y$. The spaces $H^{\pm}_y$ fit together to form a continuous Hilbert bundle $H \to Y$. The square of D is, pointwise on Y, given by

$$\mathrm{D}^2 = \begin{bmatrix} \mathrm{D}^-\mathrm{D}^+ & 0 \\ 0 & \mathrm{D}^+\mathrm{D}^- \end{bmatrix}$$

and, by Theorem 5.7.17, D^2 is a family of Fredholm operators with index zero. Moreover, D^2 is non-negative and, by Proposition 5.7.11, it has a discrete spectrum. The same is true for $\mathrm{D}^-\mathrm{D}^+$ and $\mathrm{D}^+\mathrm{D}^-$, respectively. Now, take $P = \mathrm{D}^+$ and consider its determinant bundle $\mathrm{Det}(\mathrm{D}^+)$, which is defined by (F.7).

First, we give a more explicit description of $\mathrm{Det}(\mathrm{D}^+)$. This substantiates the remark after definition (F.7) for the case under consideration. Let a be a positive real number and let $U^a \subset Y$ be the subset on which a is not an eigenvalue of $\mathrm{D}^-\mathrm{D}^+$.[14] Let $H^{\pm}_a$ be the sum of eigenspaces corresponding to eigenvalues less than a. Clearly, the vector spaces $H^{\pm}_a$ are finite-dimensional and, by elliptic regularity, they consist of smooth fields. Since the spectrum of $\mathrm{D}^-\mathrm{D}^+$ is discrete, the sets U^a form an open

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[12]Quillen considered the case of Cauchy-Riemann operators over a Riemann surface.

[13]Our presentation is along the lines of Freed [211].

[14]For simplicity of notation, we omit the index y.

cover of Y. Moreover, it can be shown, see e.g. Sect. 9.2 in [72], that the spaces $H_a^{\pm}$ fit together to smooth finite-dimensional vector bundles of locally constant rank over U^a. Thus, we can define a line bundle $\mathscr{L}^a \to U^a$ by

$$\mathscr{L}^a := \mathrm{Det}\big(H_a^+\big)^* \otimes \mathrm{Det}\big(H_a^-\big)\,.$$

By the isomorphism (F.5), its fibres may be viewed as

$$\mathscr{L}_y^a \cong \mathrm{Det}\ \ker(\mathrm{D}_y^+)^* \otimes \mathrm{Det}\ \ker(\mathrm{D}_y^-)\,.$$

Clearly, for $b > a$, we have a decomposition $H_b^{\pm} = H_a^{\pm} \oplus H_{(a,b)}^{\pm}$ over the open set $U_a \cap U_b$. This implies $\mathscr{L}^b \cong \mathscr{L}^a \otimes \mathscr{L}^{(a,b)}$, where

$$\mathscr{L}^{(a,b)} = \mathrm{Det}\big(H_{(a,b)}^+\big)^* \otimes \mathrm{Det}\big(H_{(a,b)}^-\big)\,.$$

The isomorphism $\mathrm{D}_{(a,b)}^+ := \big(\mathrm{D}^+\big)_{\restriction H_{(a,b)}^+} :\ H_{(a,b)}^+ \to H_{(a,b)}^-$ induces an isomorphism

$$\det\big(\mathrm{D}_{(a,b)}^+\big) :\ \mathrm{Det}\big(H_{(a,b)}^+\big) \to \mathrm{Det}\big(H_{(a,b)}^-\big)$$

and, thus, a nonzero section of $\mathscr{L}^{(a,b)}$. Using this isomorphism, we can define a family of canonical smooth isomorphisms

$$\mathscr{L}^a \to \mathscr{L}^b = \mathscr{L}^a \otimes \mathscr{L}^{(a,b)}\,, \quad s \mapsto s \otimes \det\big(\mathrm{D}_{(a,b)}^+\big) \tag{F.8}$$

over $U^a \cap U^b$. These can be used to patch the bundles $\mathscr{L}^a$ together to form a line bundle $\mathscr{L} \to Y$. This is the determinant line bundle for the case under consideration, $\mathscr{L} = \mathrm{Det}(\mathrm{D}^+)$. Correspondingly, the Quillen determinant $\det(\mathrm{D}^+)$ is obtained as follows. Over connected components where the index of D^+ vanishes, one has $\dim\big(H_a^+\big) = \dim\big(H_a^-\big)$. There, every $\mathscr{L}^a$ has a canonical section

$$\det\big(\mathrm{D}_a^+\big) : \mathrm{Det}\big(H_a^+\big) \to \mathrm{Det}\big(H_a^-\big)\,,$$

where D_a^+ denotes the restriction of D^+ to H_a^+. By the multiplicativity property of determinants, $\det\big(\mathrm{D}_a^+\big)$ is identified with $\det\big(\mathrm{D}_b^+\big)$ via the isomorphism (F.8). Putting $\det(\mathrm{D}^+) = 0$ over components where the index is different from zero, we obtain a global section $\det(\mathrm{D}^+)$ of $\mathrm{Det}(\mathrm{D}^+)$ which is called the Quillen determinant.

Next, we construct the Quillen metric on $\mathscr{L}$. For any $a > 0$, the L^2-metric on $H^{\pm}$ induces fibre metrics on the subbundles $H_a^{\pm}$. Thus, by linear algebra, we obtain a fibre metric g^a on $\mathscr{L}^a$. By (F.8), for $b > a$, we have

$$\mathsf{g}^b = \mathsf{g}^a \parallel \mathrm{D}_{(a,b)}^+ \parallel^2 = \mathsf{g}^a \prod_{a<\lambda_i<b} \lambda_i\,.$$

Now, by the properties of D^-D^+, we can apply the ζ-function regularization to its determinant, see Appendix D. By (D.2) and (D.3),

$$\det_\zeta\left((D^-D^+)_{\restriction\lambda>a}\right) = \det_\zeta\left((D^-D^+)_{\restriction\lambda>b}\right)\prod_{a<\lambda_i<b}\lambda_i\,.$$

Thus, if we put

$$\hat{\mathsf{g}}^a := \mathsf{g}^a\cdot\det_\zeta\left((D^-D^+)_{\restriction\lambda>a}\right)\,, \tag{F.9}$$

then $\hat{\mathsf{g}}^a$ and $\hat{\mathsf{g}}^b$ coincide on $U^a\cap U^b$. Thus, the fibre metrics $\hat{\mathsf{g}}^a$ patch together to yield a fibre metric $\hat{\mathsf{g}}$ which is referred to as the Quillen metric.

Finally, we outline the construction of the connection, see [72, 79] or [211] for details. The connection ∇ on $\mathscr{E}$ introduced at the beginning clearly induces connections on the smooth bundles $H_a^\pm\to U^a$ which are unitary with respect to the restricted fibre metrics. Again, by linear algebra, we have induced connections ∇^a on $\mathscr{L}^a$ which are unitary with respect to g^a. For $b>a$, via the isomorphism (F.8), we have

$$\nabla^b\sigma = \nabla^a\sigma\otimes\det\left(D^+_{(a,b)}\right) + \sigma\otimes\nabla\left(\det\left(D^+_{(a,b)}\right)\right)\,,$$

for any section σ over $U^a\cap U^b$. By a standard calculation [211],

$$\nabla\left(\det\left(D^+_{(a,b)}\right)\right) = \mathrm{tr}\left(\left((D^+)^{-1}\nabla D^+\right)_{\restriction a<\lambda<b}\right)\det\left(D^+_{(a,b)}\right).$$

This implies

$$\nabla^b = \nabla^a + \mathrm{tr}\left(\left((D^+)^{-1}\nabla D^+\right)_{\restriction a<\lambda<b}\right).$$

Now, one proceeds as in the case of the metric, defining

$$\hat{\nabla}^a := \nabla^a + \mathrm{tr}_\zeta\left(\left((D^+)^{-1}\nabla D^+\right)_{\restriction\lambda>a}\right)\,, \tag{F.10}$$

where $\mathrm{tr}_\zeta\left(\left((D^+)^{-1}\nabla D^+\right)_{\restriction\lambda>a}\right)$ is the ζ-function regularization of the trace [211]. By an obvious additivity property of the regularized traces, $\hat{\nabla}^a$ and $\hat{\nabla}^b$ agree on $U^a\cap U^b$. Thus, these connections patch together to a unitary connection $\nabla^{\mathscr{L}}$ on $\mathscr{L}$.

Appendix G
Eilenberg–MacLane Spaces

Let A be an Abelian group and let n be a positive integer. A pathwise connected CW-complex X is called an Eilenberg–MacLane space of type $K(A, n)$ iff $\pi_n(X) = A$ and $\pi_i(X) = 0$ for $i \neq n$. Eilenberg–MacLane spaces exist for any choice of A and n and are unique up to homotopy equivalence.[15] The simplest example of an Eilenberg–MacLane space is the 1-sphere S^1, which is of type $K(\mathbb{Z}, 1)$. Note that Eilenberg–MacLane spaces are, apart from very special examples, infinite dimensional.

Assume A to be commutative also in the case $n = 1$. Due to the Universal Coefficient Theorem, Hom $(H_n(K(A, n)), A)$ is isomorphic to a subgroup of $H^n_A(K(A, n))$. Due to the Hurewicz Theorem, $H_n(K(A, n)) \cong \pi_n(K(A, n)) = A$. It follows that $H^n_A(K(A, n))$ contains elements which correspond to isomorphisms

$$H_n(K(A, n)) \to A \,.$$

Such elements are called characteristic. If $\gamma \in H^n_A(K(A, n))$ is characteristic, then for any CW-complex X, the mapping

$$[X, K(A, n)] \to H^n_A(X)\,, \quad f \mapsto f^*\gamma\,, \tag{G.1}$$

is a bijection [104, Sect. VII.12]. In this sense, Eilenberg–MacLane spaces provide a link between homotopy and cohomology.

Let us construct models for the Eilenberg–MacLane spaces $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_g, 1)$ and derive the integer-valued cohomology of these spaces. Consider the natural free action of $U(1)$ on the sphere S^∞ which is induced from the natural action of $U(1)$ on $S^{2n-1} \subset \mathbb{C}^n$. The orbit space of this action is the infinite complex projective space $\mathbb{C}P^\infty$. Moreover, by viewing $\mathbb{Z}_g$ as a subgroup of $U(1)$, this action gives rise to a natural free action of $\mathbb{Z}_g$ on S^∞. The orbit space of the latter is the infinite lens space L^∞_g. Thus, one has the principal bundles

[15] In case $n = 1$, an Eilenberg–MacLane space exists for any group.

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$$\mathrm{S}^\infty \xrightarrow{\mathrm{U}(1)} \mathbb{C}\mathrm{P}^\infty\,, \quad \mathrm{S}^\infty \xrightarrow{\mathbb{Z}_g} \mathrm{L}^\infty_g\,. \tag{G.2}$$

Due to $\pi_i(\mathrm{S}^\infty) = 0$, for every i, the corresponding exact homotopy sequences yield

$$\pi_i(\mathbb{C}\mathrm{P}^\infty) = \pi_{i-1}(\mathrm{U}(1)) = \begin{cases} \mathbb{Z} & i = 2\,, \\ 0 & i \neq 2\,, \end{cases}$$

$$\pi_i(\mathrm{L}^\infty_g) = \pi_{i-1}(\mathbb{Z}_g) = \begin{cases} \mathbb{Z}_g & i = 1\,, \\ 0 & i > 1\,. \end{cases}$$

As a consequence, $\mathbb{C}\mathrm{P}^\infty$ is a model of $K(\mathbb{Z}, 2)$ and L^∞_g is a model of $K(\mathbb{Z}_g, 1)$. In particular,

$$H^i_{\mathbb{Z}}(K(\mathbb{Z}, 2)) = H^i_{\mathbb{Z}}(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} & i \text{ even}, \\ 0 & i \text{ odd}, \end{cases} \tag{G.3}$$

see [104, Chap. VI, Proposition 10.2], and

$$H^i_{\mathbb{Z}}(K(\mathbb{Z}_g, 1)) = H^i_{\mathbb{Z}}(\mathrm{L}^\infty_g) = \begin{cases} \mathbb{Z} & i = 0\,, \\ \mathbb{Z}_g & i \neq 0, \text{ even} \\ 0 & i \neq 0, \text{ odd}, \end{cases} \tag{G.4}$$

see [665, Sect. II.7.7]. We notice that the vanishing of all homotopy groups of S^∞ also implies that the principal bundles (G.2) are universal for $\mathrm{U}(1)$ and $\mathbb{Z}_g$, respectively. Hence, $\mathbb{C}\mathrm{P}^\infty$ and L^∞_g are models of $\mathrm{BU}(1)$ and $\mathrm{B}\mathbb{Z}_g$, respectively. This is used in the proofs of Theorems 4.8.1 and 4.8.3.

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